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Realizing Symmetric Set Functions as Hypergraph Cut Capacity

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Abstract

A set function is a function defined on a set family. It is said to be symmetric if the value for each set coincides with that for its complement. A cut capacity function of an undirected graph or hypergraph is a fundamental example of symmetric set functions, and is also submodular if the capacity on each edge or hyperedge is nonnegative. Fujishige and Patkar (2001) provided necessary and sufficient conditions for set functions to be realized as cut capacity functions of several types of networks. In this paper, we consider the case of undirected hypergraphs, which was not dealt with in their previous work. We characterize symmetric set functions that can be realized as cut capacity functions of undirected hypergraphs, and present various types of "standard forms" of hypergraphs for the cut capacity realization of symmetric set functions.

1 Introduction

A cut capacity function of a network with nonnegative capacity is a fundamental example of submodular functions, which are set functions $f\colon 2^V \to \mathbb{R}$ satisfying

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$

for every $X, Y \subseteq V$. The submodularity is necessary for set functions to be realized as cut capacity functions of networks with nonnegative capacity, but is far from sufficient. When is a set function realized as a cut capacity function? As an answer to this natural question, Fujishige and Patkar [2] characterized set functions that can be realized as the cut capacity functions of the following types of networks:

1) a directed graph with nonnegative capacity on each arc,
2) an undirected graph with arbitrary or nonnegative capacity on each edge, and
3) a directed hypergraph with arbitrary or nonnegative capacity on each hyperarc, which has exactly one specified tail.

For a finite set V, a set function $f\colon 2^V \to \mathbb{R}$ is said to be symmetric if $f(X) = f(V \setminus X)$ holds for every $X \subseteq V$. In the case of an undirected graph or an undirected hypergraph, a cut capacity function is also symmetric regardless of the nonnegativity of edge capacity.

The symmetry of submodular functions is an important property from the viewpoint of optimization. It is well-known that one can minimize an arbitrary submodular function in polynomial time [3, 4, 10]. The fastest strongly polynomial algorithm currently known requires $\mathrm{O}(|V|^5\mathrm{EO} + |V|^6)$ time [7], where EO denotes the time to evaluate the function value. On the other hand, for minimizing a symmetric submodular function, Queyranne [8] provided a much faster algorithm, which runs in $\mathrm{O}(|V|^3\mathrm{EO})$ time.

Queyranne's algorithm generalizes the algorithm of Nagamochi and Ibaraki [6] for finding a minimum cut in an undirected graph with nonnegative capacity on each edge, which was also naturally extended to the case of an undirected hypergraph by Klimmek and Wagner [5]. Based on this fact, the class of cut capacity functions of undirected hypergraphs with nonnegative capacity on each hyperedge seems to be quite close to that of symmetric submodular functions.

In this paper, we first characterize symmetric set functions that can be realized as cut capacity functions of undirected hypergraphs with arbitrary capacity on each hyperedge. It in fact turns out that there exists such a realization for any symmetric set function $f\colon 2^V \to \mathbb{R}$ as long as $f(\emptyset) = 0$ (Theorem 3.1). Such realizations are not unique in general, since the number of possible hyperedges in a hypergraph with the vertex set V is $2^{|V|}$, which means that the degree of freedom of the cut capacity realization is $2^{|V|}$, whereas that of a symmetric set function $f\colon 2^V \to \mathbb{R}$ with $f(\emptyset) = 0$ is $2^{|V|-1} - 1$. To fill this gap, we define various types of "standard forms" of hypergraphs (see Section 3.2). A hypergraph with the vertex set V in a standard form can realize any symmetric set function $f\colon 2^V \to \mathbb{R}$ with $f(\emptyset) = 0$ as its cut capacity function by uniquely determining the capacity of each hyperedge. Our standard forms may be useful also for showing several properties of symmetric set functions (e.g., Proposition 4.7).

The rest of this paper is organized as follows. In Section 2, we give necessary definitions and describe related results due to Fujishige and Patkar [2]. Section 3 presents our results: the realizability of symmetric set functions as cut capacity functions of undirected hypergraphs, and various standard forms for such realization. Their proofs are provided in Section 4.

2 Preliminaries

2.1 Set functions

A function is called a *set function* if its domain is a set family. Throughout this paper, we consider only set functions defined on families of subsets of a finite set V, which is called the *ground set*. We say that a set function $f\colon 2^V \to \mathbb{R}$ is *symmetric* if $f(X) = f(V \setminus X)$ holds for every $X \subseteq V$.

For a finite set V and $i = 0, 1, \ldots, |V|$, let $\binom{V}{i}$ denote the family of i-element subsets of V, i.e., $\binom{V}{i} = \{\, X \subseteq V \mid |X| = i \,\}$. The following two propositions are special cases of the Möbius inversion formula (see, e.g., [1, 9]).

Proposition 2.1. *For a set function $f\colon 2^V \to \mathbb{R}$, there uniquely exists a family of set functions $f^{(i)}\colon \binom{V}{i} \to \mathbb{R}$ $(i = 0, 1, \ldots, |V|)$ such that*

$$f(X) = \sum_{Y \subseteq X} f^{(|Y|)}(Y)$$

holds for every $X \subseteq V$. Moreover, they are explicitly written as

$$f^{(i)}(X) = \sum_{Y \subseteq X} (-1)^{|X \setminus Y|} f(Y)$$

for each $i = 0, 1, \ldots, |V|$ and each $X \in \binom{V}{i}$.

Proposition 2.2. *For a symmetric set function $f\colon 2^V \to \mathbb{R}$ and a fixed element $r \in V$, there uniquely exists a family of set functions $f_r^{(i)}\colon \{\, X \mid r \in X \in \binom{V}{i} \,\} \to \mathbb{R}$ $(i = 1, 2, \ldots, |V|)$ such that*

$$f(X) = \sum_{Y\colon r \in Y \subseteq X} f_r^{(|Y|)}(Y)$$

holds for every $X \subseteq V$ with $r \in X$. Moreover, they are explicitly written as

$$f_r^{(i)}(X) = \sum_{Y\colon r \in Y \subseteq X} (-1)^{|X \setminus Y|} f(Y) = f^{(i)}(X) + f^{(i-1)}(X - r)$$

for each $i = 1, 2, \ldots, |V|$ and each $X \in \binom{V}{i}$ with $r \in X$.

2.2 Hypergraphs

Let V be a finite set. A pair of V and a family $\mathcal{E}$ of subsets of V is called an *undirected hypergraph*, where each element of V is called a *vertex* and of $\mathcal{E}$ a *hyperedge*. We call a hyperedge $E \in \mathcal{E}$ of size at least 2 with a specified tail $t_E \in E$ a *hyperarc*. In other words, a hyperarc is a pair $\vec{E} = (t_E, E - t_E) \in V \times 2^V$ such that $\{t_E\} \subsetneq E \subseteq V$. A pair of V and a set $\vec{\mathcal{E}}$ of hyperarcs is called a *directed hypergraph*. Although it may be more popular to assign exactly one specified head to each hyperedge, we adopt the present way following Fujishige and Patkar [2].

A pair $\mathcal{N} = (\mathcal{H}, c)$ of an undirected hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a function $c\colon \mathcal{E} \to \mathbb{R}$ is called an *undirected hypernetwork*. For an undirected hypernetwork $\mathcal{N}$, define the *cut capacity function* $\kappa_{\mathcal{N}}\colon 2^V \to \mathbb{R}$ as

$$\kappa_{\mathcal{N}}(X) := \sum_{E \in \mathcal{E}} \{\, c(E) \mid E \cap X \neq \emptyset \neq E \setminus X \,\} \tag{1}$$

for each $X \subseteq V$. Similarly, a pair $\vec{\mathcal{N}} = (\vec{\mathcal{H}}, c)$ of a directed hypergraph $\vec{\mathcal{H}} = (V, \vec{\mathcal{E}})$ and a function $c\colon \vec{\mathcal{E}} \to \mathbb{R}$ is called a *directed hypernetwork*, and the cut capacity function $\kappa_{\vec{\mathcal{N}}}\colon 2^V \to \mathbb{R}$ for $\vec{\mathcal{N}}$ is defined by

$$\kappa_{\vec{\mathcal{N}}}(X) := \sum_{\vec{E} \in \vec{\mathcal{E}}} \{\, c(\vec{E}) \mid \vec{E} = (t_E,\, E - t_E),\ t_E \in X,\ \text{and } E \setminus X \neq \emptyset \,\}$$

for each $X \subseteq V$. In both (undirected and directed) cases, the function c is called the *capacity function* of the hypernetwork ($\mathcal{N}$ and $\vec{\mathcal{N}}$, respectively).

Note that simple undirected and directed graphs are special cases of undirected and directed hypergraphs, respectively, with the size of each hyperedge and hyperarc exactly 2. When the underlying hypergraph $\mathcal{H}$ (or $\vec{\mathcal{H}}$) is a simple undirected (or directed) graph, we refer to the hypernetwork $\mathcal{N}$ (or $\vec{\mathcal{N}}$) simply as an undirected (or directed) *network*.

2.3 Previous work

For a finite domain $\mathcal{D}$, we often regard a function $f\colon \mathcal{D} \to \mathbb{R}$ as a vector in $\mathbb{R}^{\mathcal{D}}$ without notice. For example, $f \geq \mathbf{0}$ means that $f(X) \geq 0$ for every $X \in \mathcal{D}$. For a pair of functions $f\colon \mathcal{D} \to \mathbb{R}$ and $g\colon \mathcal{D} \to \mathbb{R}$, $f = g$ means that $f(X) = g(X)$ for every $X \in \mathcal{D}$.

Here we describe theorems due to Fujishige and Patkar [2] that are related to our results. These theorems correspond to the cases 2) and 3) in Introduction.

Theorem 2.3 (Fujishige and Patkar [2]). *A symmetric set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of an undirected network if and only if the following conditions hold:*

(i) $f(\emptyset) = 0$, *and*

(ii) $f^{(i)} = \mathbf{0}$ *for* $i = 3, 4, \ldots, |V|$.

Theorem 2.4 (Fujishige and Patkar [2]). *A symmetric set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of an undirected network with a nonnegative capacity function if and only if the following conditions hold:*

(i) $f(\emptyset) = 0$,

(ii) $f^{(i)} = \mathbf{0}$ *for* $i = 3, 4, \ldots, |V|$, *and*

(iii) $f^{(2)} \leq \mathbf{0}$, *which is equivalent to the submodularity of f under* (ii).

Theorem 2.5 (Fujishige and Patkar [2]). *A set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of a directed hypernetwork if and only if $f(\emptyset) = f(V) = 0$.*

Theorem 2.6 (Fujishige and Patkar [2])**.** *A set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of a directed hypernetwork with a nonnegative capacity function if and only if the following conditions hold:*

(i) $f(\emptyset) = f(V) = 0$,

(ii) $f^{(i)} \leq \mathbf{0}$ *for* $i = 2, 3, \ldots, |V|$, *and*

(iii) $f \geq \mathbf{0}$.

Remark 1. Their proof of Theorem 2.6 leads to a slightly more general result. When we assume that a directed hypernetwork contains only hyperarcs of size at most k ($2 \leq k \leq |V|$), the condition (ii) in Theorem 2.6 is replaced by the following conditions: $f^{(i)} \leq \mathbf{0}$ for $i = 2, 3, \ldots, k$ and $f^{(i)} = \mathbf{0}$ for $i = k+1, k+2, \ldots, |V|$. This generalized claim extends their characterization for simple directed graphs (i.e., $k = 2$) [2, Theorem 3.2].

3 Results

3.1 Realizability as hypergraph cut capacity

Analogously to Theorem 2.5, the following theorem provides a simple answer to the question: "when is a symmetric set function realized as a cut capacity function?"

Theorem 3.1. *A symmetric set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of an undirected hypernetwork if and only if $f(\emptyset) = 0$.*

We can claim a slightly stronger statement as follows. It should be noted that Theorem 3.2 also extends Theorem 2.3 in the same direction as the extension seen in Remark 1.

Theorem 3.2. *A symmetric set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of an undirected hypernetwork that contains only hyperedges of size at most k ($2 \leq k \leq |V|$) if and only if the following conditions hold:*

(i) $f(\emptyset) = 0$, *and*

(ii) $f^{(i)} = \mathbf{0}$ *for* $i = k+1, k+2, \ldots, |V|$.

Regarding the nonnegativity, it may seem possible to extend Theorem 2.4 to the case of undirected hypernetworks analogously to Theorem 2.6. However, a straightforward extension by using the submodularity and the decomposition by Proposition 2.1 fails to give a sufficient condition for symmetric set functions to be realized as the cut capacity functions of undirected hypernetworks with nonnegative capacity functions. The next theorem claims only one direction, i.e., shows a necessary condition for such a realization, and there exists a counterexample for the converse direction (see Remark 2).

Theorem 3.3. *If a symmetric set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of an undirected hypernetwork with a nonnegative capacity function, then the following conditions hold:*

(i) $f(\emptyset) = 0$,

(ii) $(-1)^i f^{(i)} \leq \mathbf{0}$ *for* $i = 1, 2, \ldots, |V|$,

(iii) $f \geq \mathbf{0}$, *and*

(iv) f *is submodular.*

Remark 2. As a counterexample for the sufficiency, let us define a hypernetwork $\tilde{\mathcal{N}} = (\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}}), \tilde{c})$ with $|V| = 5$ as follows: let $\tilde{\mathcal{E}} := \{\, E \subseteq V \mid |E| = 2 \text{ or } 4 \,\}$, fix two distinct vertices $u, v \in V$, and define $\tilde{c}$ as

$$\tilde{c}(E) := \begin{cases} -1 & (E = \{u,v\}) \\ 2 & (E \supsetneq \{u,v\}) \\ 1 & (E \supsetneq V \setminus \{u,v\}) \\ 0 & (\text{otherwise}) \end{cases}$$

for each $E \in \tilde{\mathcal{E}}$. Then, it is easy to check that the cut capacity function $\kappa_{\tilde{\mathcal{N}}}$ satisfies the four conditions in Theorem 3.3. Furthermore, we can see that there is no undirected hypernetwork $\mathcal{N} = ((V, \mathcal{E}), c)$ such that $c \geq \mathbf{0}$ and $\kappa_{\mathcal{N}} = \kappa_{\tilde{\mathcal{N}}}$ by Theorem 3.5 and Corollary 4.4.

Suppose to the contrary that there exists such an undirected hypernetwork $\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$. Since $\tilde{c}$ is unique for $\tilde{\mathcal{H}}$ to realize $\kappa_{\tilde{\mathcal{N}}}$ by Theorem 3.5, $\mathcal{E}$ contains some hyperedges of size 3 or 5. By Corollary 4.4, a hyperedge $\{x,y,z\} \subseteq V$ of size 3 with capacity 1 can be replaced by the $\binom{3}{2}$ hyperedges $\{x,y\}$, $\{y,z\}$, $\{z,x\}$ of size 2 with capacity 1/2 without changing the cut capacity function, and so can the hyperedge V with capacity 1 by the $\binom{5}{2}$ hyperedges $X \subseteq V$ of size 2 with capacity $-1/4$ and the $\binom{5}{4}$ hyperedges $Y \subseteq V$ of size 4 with capacity 1/2. Hence, we have

$$\begin{aligned} -1 \;&=\; \tilde{c}(\{u,v\}) \;=\; c(\{u,v\}) + \frac{1}{2}\sum_{w \in V \setminus \{u,v\}} c(\{u,v,w\}) - \frac{1}{4}c(V) \;\geq\; -\frac{1}{4}c(V), \\ 1 \;&=\; \tilde{c}(V - v) \;=\; c(V - v) + \frac{1}{2}c(V) \;\geq\; \frac{1}{2}c(V). \end{aligned}$$

Combining these inequalities, we obtain $-4 \geq -c(V) \geq -2$, a contradiction.

3.2 Standard forms of hypergraphs for cut realization

Fix a nonempty finite set V. Let $\mathcal{F}$ be the vector space consisting of all symmetric set functions $f\colon 2^V \to \mathbb{R}$ with $f(\emptyset) = 0$, i.e.,

$$\mathcal{F} = \{\, f \in \mathbb{R}^{2^V} \mid f(\emptyset) = 0 \text{ and } f(X) = f(V \setminus X) \ (\forall X \subseteq V) \,\}.$$

For an undirected hypernetwork $\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$, the construction (1) of the cut capacity function $\kappa_{\mathcal{N}} \in \mathcal{F}$ from the capacity function $c \in \mathbb{R}^{\mathcal{E}}$ defines a linear mapping $\varphi_{\mathcal{H}}\colon \mathbb{R}^{\mathcal{E}} \to \mathcal{F}$ ($c \mapsto \kappa_{\mathcal{N}}$). We say that a hypergraph $\mathcal{H}$ is *in a standard form* (*for cut realization*) if $\varphi_{\mathcal{H}}$ is bijective. Note that $\dim(\mathcal{F}) = 2^{|V|-1} - 1$ and $\dim(\mathbb{R}^{\mathcal{E}}) = |\mathcal{E}|$. Hence, $|\mathcal{E}| = 2^{|V|-1} - 1$ if $\mathcal{H}$ is in a standard form.

Here we present several types of standard forms. Note that Theorem 3.1 immediately follows from the correctness of each standard form, which is shown in Section 4.

Theorem 3.4. *Fix an arbitrary vertex $r \in V$, and let $\mathcal{E}_r$ be the set of all hyperedges of size at least 2 that contain r, i.e., $\mathcal{E}_r = \{\, E \mid r \in E \subseteq V$ and $|E| \geq 2 \,\}$. Then, $\mathcal{H}_r = (V, \mathcal{E}_r)$ is in a standard form, which we call a* rooted standard form (*or, in particular, the* r-standard form).

Theorem 3.5. *Let $\tilde{\mathcal{E}}$ be the set of all nonempty hyperedges of even size, i.e., $\tilde{\mathcal{E}} = \{\, E \mid \emptyset \neq E \subseteq V$ and $|E|$ is even$\,\}$. Then, $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$ is in a standard form, which we call the* even standard form.

Theorem 3.6. *Suppose that $|V|$ is odd, and let $\hat{\mathcal{E}}$ be the set of all majority hyperedges, i.e., $\hat{\mathcal{E}} = \{\, E \subseteq V \mid |E| \geq \lceil |V|/2 \rceil \,\}$. Then, for any $X \in \hat{\mathcal{E}}$, $\hat{\mathcal{H}}_X = (V, \hat{\mathcal{E}} \setminus \{X\})$ is in a standard form, which we call a* majority standard form.

Remark 3. Even when $|V|$ is even, there is a similar standard form. Let $\hat{\mathcal{E}}$ be an arbitrary maximal family of subsets $E \subseteq V$ with $|E| \geq |V|/2$ such that $E_1 \neq V \setminus E_2$ for every $E_1, E_2 \in \hat{\mathcal{E}}'$. Note that $\hat{\mathcal{E}}$ contains all majority hyperedges and exact half of hyperedges of size $|V|/2$, and hence $|\hat{\mathcal{E}}| = 2^{|V|-1}$. Then, for any $X \in \hat{\mathcal{E}}$ with $|X| = |V|/2$, $\hat{\mathcal{H}}_X = (V, \hat{\mathcal{E}} \setminus \{X\})$ is in a standard form. We give a proof of Theorem 3.6 in Section 4.2, which also implies this standard form.

In advance of proofs, here we confirm that the size of each hyperedge set defined in Theorems 3.4–3.6 is indeed equal to $2^{|V|-1} - 1$.

First, in the r-standard form, the hyperedge set is $\mathcal{E}_r = \{\, E \mid r \in E \subseteq V \text{ and } |E| \geq 2 \,\}$. Then, for each $X \subseteq V$, $X \in \mathcal{E}_r$ if and only if $\emptyset \neq X - r \subseteq V - r$. The number of such nonempty sets $X - r$ is equal to $2^{|V-r|} - 1 = 2^{|V|-1} - 1$, and hence $|\mathcal{E}_r| = 2^{|V|-1} - 1$.

Next, in the even standard form, the hyperedge set is $\tilde{\mathcal{E}} = \{\, E \mid \emptyset \neq E \subseteq V \text{ and } |E| \text{ is even} \,\}$. Since V is nonempty, we have

$$\begin{aligned} 2^{|V|} &= (1+1)^{|V|} + (1-1)^{|V|} = \sum_{X \subseteq V} \left(1 + (-1)^{|X|}\right) \\ &= 2\,|\{\, X \subseteq V \mid |X| \text{ is even} \,\}| = 2(|\tilde{\mathcal{E}}| + 1), \end{aligned}$$

which leads to $|\tilde{\mathcal{E}}| = 2^{|V|-1} - 1$.

Finally, in a majority standard form, the hyperedge set is obtained from $\hat{\mathcal{E}} = \{\, E \subseteq V \mid |E| \geq \lceil |V|/2 \rceil \,\}$ by removing exactly one hyperedge $X \in \hat{\mathcal{E}}$. For each $Y \subseteq V$, $Y \in \hat{\mathcal{E}}$ if and only if $V \setminus Y \notin \hat{\mathcal{E}}$, and hence $|\hat{\mathcal{E}}| = 2^{|V|-1}$. This implies $|\hat{\mathcal{E}} \setminus \{X\}| = 2^{|V|-1} - 1$.

4 Proofs

In this section, we consider only undirected hypernetworks, and we refer to them simply as hypernetworks.

4.1 Rooted standard forms (Proof of Theorem 3.4)

Since we have already confirmed the equality between the dimensions of $\mathbb{R}^{\mathcal{E}_r}$ and $\mathcal{F}$, it suffices to show the following lemma, which claims that we can construct the inverse mapping of $\varphi_{\mathcal{H}_r} \colon \mathbb{R}^{\mathcal{E}_r} \to \mathcal{F}$ $(c_r \to \kappa_{\mathcal{N}_r})$.

Lemma 4.1. *For any symmetric set function $f \colon 2^V \to \mathbb{R}$ with $f(\emptyset) = 0$, there exists a capacity function $c_r \colon \mathcal{E}_r \to \mathbb{R}$ such that $\kappa_{\mathcal{N}_r} = f$ for $\mathcal{N}_r = (\mathcal{H}_r = (V, \mathcal{E}_r), c_r)$, where $\mathcal{E}_r = \{\, E \mid r \in E \subseteq V \text{ and } |E| \geq 2 \,\}$, for an arbitrary fixed element $r \in V$.*

Proof. By using Proposition 2.2, let us define c_r as

$$c_r(E) := -f_r^{(|E|)}(E) = -\sum_{Y \colon r \in Y \subseteq E} (-1)^{|E \setminus Y|} f(Y) \tag{2}$$

for each $E \in \mathcal{E}_r$. We then obtain the cut capacity function $\kappa_{\mathcal{N}_r} = \varphi_{\mathcal{H}_r}(c_r)$ such that, for each $X \subsetneq V$ with $r \in X$ (note that $r \in E \cap X$ for each $E \in \mathcal{E}_r$),

$$\begin{aligned} \kappa_{\mathcal{N}_r}(X) &= \sum_{E \in \mathcal{E}_r} \{\, c_r(E) \mid E \setminus X \neq \emptyset \,\} \qquad (3) \\ &= -\sum_{E \in \mathcal{E}_r} \left\{ \sum_{Y \colon r \in Y \subseteq E} (-1)^{|E \setminus Y|} f(Y) \;\middle|\; E \setminus X \neq \emptyset \right\} \\ &= -\sum_{Y \colon r \in Y \subseteq V} f(Y) \sum_{E \in \mathcal{E}_r} \left\{ (-1)^{|E \setminus Y|} \;\middle|\; Y \subseteq E \text{ and } E \setminus X \neq \emptyset \right\} \\ &= f(X). \end{aligned}$$

This means $\kappa_{\mathcal{N}_r} = f$ since $\kappa_{\mathcal{N}_r}(V) = 0 = f(\emptyset) = f(V)$. In the rest of this proof, we confirm the last equality by evaluating the second summation for each Y with $r \in Y \subsetneq V$ (note that $f(V) = 0$), which is equal to -1 when $Y = X$ and 0 when $Y \neq X$.

Suppose that $Y = X \subsetneq V$. Then, $Y \subseteq E \in \mathcal{E}_r$ and $E \setminus X \neq \emptyset$ if and only if $E = X \cup Z$ for some nonempty $Z \subseteq V \setminus X$. Hence, the second summation is rewritten as

$$\sum_{Z\colon \emptyset \neq Z \subseteq V\setminus X} (-1)^{|Z|} = \sum_{Z \subseteq V\setminus X} 1^{|(V\setminus X)\setminus Z|}(-1)^{|Z|} - 1^{|V\setminus X|}(-1)^{|\emptyset|} = (1-1)^{|V\setminus X|} - (-1)^0 = -1.$$

Suppose that $Y \setminus X \neq \emptyset$. Then, $Y \subseteq E$ implies $E \setminus X = \emptyset$, and hence the second summation is similarly rewritten as $(1-1)^{|V\setminus Y|} = 0$.

Otherwise, we have $Y \subsetneq X \subsetneq V$. In this case, $Y \subseteq E \in \mathcal{E}_r$ and $E \setminus X \neq \emptyset$ if and only if $E = Y \cup Z$ for some $Z \subseteq V \setminus Y$ with $Z \setminus X \neq \emptyset$. Hence, the second summation is rewritten as

$$\sum_{Z \subseteq V\setminus Y} (-1)^{|Z|} - \sum_{Z \subseteq X\setminus Y} (-1)^{|Z|} = (1-1)^{|V\setminus Y|} - (1-1)^{|X\setminus Y|} = 0. \qquad \square$$

Remark 4. We can use Proposition 2.2 more directly to show Lemma 4.1. Let $\mathcal{E}'_r := \mathcal{E}_r \cup \{\{r\}\} = \{\, X \mid r \in X \subseteq V \,\}$, and define a symmetric set function $g\colon 2^V \to \mathbb{R}$ by

$$g(X) := \kappa_{\mathcal{N}_r}(\{r\}) - \kappa_{\mathcal{N}_r}(X) = \sum_{E \in \mathcal{E}_r} \{\, c_r(E) \mid E \subseteq X \,\}$$

for each $X \in \mathcal{E}'_r$, where the second equality follows from (3). Then, by Proposition 2.2, there exists a unique family of set functions $g_r^{(i)}\colon \{\, X \mid r \in X \in \binom{V}{i} \,\} \to \mathbb{R}$ ($i = 1, 2, \ldots, |V|$) such that

$$g(X) = \sum_{Y\colon r \in Y \subseteq X} g_r^{(|Y|)}(Y) = \sum_{E \in \mathcal{E}'_r} \{\, g_r^{(|E|)}(E) \mid E \subseteq X \,\}$$

holds for every $X \in \mathcal{E}'_r$. By the definitions, we have $g_r^{(1)}(\{r\}) = g(\{r\}) = 0$, which implies that

$$\begin{aligned}
c_r(E) &= g_r^{(|E|)}(E) \\
&= \sum_{X\colon r \in X \subseteq E} (-1)^{|E\setminus X|} g(X) \\
&= \kappa_{\mathcal{N}_r}(\{r\}) \sum_{X\colon r \in X \subseteq E} (-1)^{|E\setminus X|} - \sum_{X\colon r \in X \subseteq E} (-1)^{|E\setminus X|} \kappa_{\mathcal{N}_r}(X) \\
&= -\sum_{X\colon r \in X \subseteq E} (-1)^{|E\setminus X|} \kappa_{\mathcal{N}_r}(X) = -\kappa_{\mathcal{N}_r}^{(|E|)}(E)
\end{aligned}$$

for every $E \in \mathcal{E}_r$ (note that $|E| \geq 2$). Thus we obtain the inverse mapping of $\varphi_{\mathcal{H}_r}\colon c_r \mapsto \kappa_{\mathcal{N}_r}$.

Note again that Theorem 3.1 immediately follows from Theorem 3.4, and hence it also has been proven.

Since the construction (2) defines the inverse mapping of the linear bijection $\varphi_{\mathcal{H}_r}\colon \mathbb{R}^{\mathcal{E}_r} \to \mathcal{F}$, the capacity function c_r is always uniquely determined by $\varphi_{\mathcal{H}_r}^{-1}\colon \mathcal{F} \to \mathbb{R}^{\mathcal{E}_r}$. Based on this fact, the following corollary characterizes symmetric set functions that can be realized by rooted standard forms with nonnegative capacity functions.

Corollary 4.2. *For any $r \in V$, a symmetric set function $f\colon 2^V \to \mathbb{R}$ can be realized as the cut capacity function of a hypernetwork $\mathcal{N}_r = (\mathcal{H}_r, c_r)$ with $\mathcal{H}_r$ in the r-standard form and with a nonnegative capacity function c_r if and only if the following conditions hold:*

(i) *$f(\emptyset) = 0$, and*

(ii) *$f_r^{(i)} \leq \mathbf{0}$ for $i = 2, 3, \ldots, |V|$.*

4.2 Kernel of the cut capacity construction (Proofs of Theorems 3.5 and 3.6)

The next lemma shows the kernel of the linear mapping $\varphi_{\mathcal{H}^*}\colon \mathbb{R}^{2^V} \to \mathcal{F}$ with respect to the complete hypergraph $\mathcal{H}^* = (V, 2^V)$, which leads to other standard forms.

Lemma 4.3. *For a hypernetwork $\mathcal{N}^* = (\mathcal{H}^* = (V, 2^V), c^*)$, the following capacity functions form a basis of the kernel of $\varphi_{\mathcal{H}^*}\colon \mathbb{R}^{2^V} \to \mathcal{F}$ ($c^* \mapsto \kappa_{\mathcal{N}^*}$):*

- $c_0^*\colon 2^V \to \mathbb{R}$ *such that $c_0^*(\emptyset) = 1$ and $c_0^*(E) = 0$ for every nonempty $E \subseteq V$, and*
- $c_U^*\colon 2^V \to \mathbb{R}$ $\left(U \in \binom{V}{2k+1},\ k = 0, 1, \ldots, \lceil |V|/2 \rceil - 1\right)$ *such that*

$$c_U^*(E) = \begin{cases} \alpha_{k,|E|} & (E \subseteq U \text{ and } k+1 \le |E| \le 2k+1) \\ 0 & (\text{otherwise}), \end{cases} \tag{4}$$

where $\alpha_{k,i} \in \mathbb{R}$ ($k = 0, 1, \ldots, \lceil |V|/2 \rceil - 1,\ i = k+1, k+2, \ldots, 2k+1$) are defined as

$$\alpha_{k,i} := (-1)^{2k+1-i} \frac{\binom{k}{2k+1-i}}{\binom{2k}{2k+1-i}} = (-1)^{2k+1-i} \frac{k!(i-1)!}{(2k)!(i-(k+1))!}. \tag{5}$$

Proof. First of all, we have $\dim(\ker \varphi_{\mathcal{H}^*}) = 2^{|V|} - (2^{|V|-1} - 1) = 2^{|V|-1} + 1$ by the dimension theorem since Theorem 3.1 implies that $\varphi_{\mathcal{H}^*}$ is surjective. Let $K^* := \{c_0^*\} \cup \{\, c_U^* \mid U \in \binom{V}{2k+1},\ k = 0, 1, \ldots, \lceil |V|/2 \rceil - 1 \,\}$ be the family of such functions.

For each $k = 0, 1, \ldots, \lceil |V|/2 \rceil - 1$ and each $U \in \binom{V}{2k+1}$, we have $c_U^*(E) = 0$ for every $E \in 2^V$ with $|E| \ge |U|$ and $E \ne U$, and $c_U^*(U) = \alpha_{k,2k+1} = 1$. Based on this fact, it is easy to confirm that K^* is linearly independent. It is also easy to see that $|K^*| = 2^{|V|-1} + 1$, $c_0^* \in \ker \varphi_{\mathcal{H}^*}$, and $c_{\{v\}}^* \in \ker \varphi_{\mathcal{H}^*}$ for each $v \in V$. Hence, it suffices to show that $c_U^* \in \ker \varphi_{\mathcal{H}^*}$ for each $U \subseteq V$ with $|U| = 2k+1$ for each $k = 1, 2, \ldots, \lceil |V|/2 \rceil - 1$. Fix such k and U, and define a hypernetwork $\mathcal{N}_U^* := (\mathcal{H}^*, c_U^*)$. We show $\kappa_{\mathcal{N}_U^*} = \mathbf{0}$ below.

Let $\mathcal{E}_{U,i}^* := \binom{U}{i}$ for $i = k+1, k+2, \ldots, 2k+1$ and $\mathcal{E}_U^* := \bigcup_{i=k+1}^{2k+1} \mathcal{E}_{U,i}^*$. We then have $c_U^*(E) = 0$ for every $E \in 2^V \setminus \mathcal{E}_U^*$ by (4). Hence, for each $X \subseteq V$ with $k+1 \le |X \cap U| \le 2k$ (note that $X \cap E \ne \emptyset$ for each $E \in \mathcal{E}_U^*$), we have

$$\begin{aligned}
\kappa_{\mathcal{N}_U^*}(X) &= \sum_{E \in \mathcal{E}_U^*} \{\, c_U^*(E) \mid E \setminus X \ne \emptyset \,\} \\
&= \sum_{i=k+1}^{2k+1} \sum_{E \in \mathcal{E}_{U,i}^*} \{\, \alpha_{k,i} \mid E \setminus X \ne \emptyset \,\} \\
&= \sum_{i=k+1}^{2k+1} \alpha_{k,i} \left(|\mathcal{E}_{U,i}^*| - |\{\, E \in \mathcal{E}_{U,i}^* \mid E \subseteq X \,\}| \right) \\
&= \sum_{i=k+1}^{2k+1} \alpha_{k,i} \binom{2k+1}{i} - \sum_{i=k+1}^{|X \cap U|} \alpha_{k,i} \binom{|X \cap U|}{i} \;\; =: \;\; g(|X \cap U|).
\end{aligned}$$

We shall prove $g(j) = 0$ for $j = k+1, k+2, \ldots, 2k$.

First, we check the value of $g(2k)$:

$$\begin{aligned}
g(2k) &= \alpha_{k,2k+1}\binom{2k+1}{2k+1} + \sum_{i=k+1}^{2k} \alpha_{k,i}\left(\binom{2k+1}{i} - \binom{2k}{i}\right) \\
&= \sum_{i=k+1}^{2k+1} \alpha_{k,i}\binom{2k}{i-1} \\
&= \sum_{i=k+1}^{2k+1} (-1)^{2k+1-i}\frac{\binom{k}{2k+1-i}}{\binom{2k}{2k+1-i}}\binom{2k}{i-1} \\
&= \sum_{i=k+1}^{2k+1} (-1)^{2k+1-i}\binom{k}{2k+1-i} \\
&= \sum_{i'=0}^{k}(-1)^{i'}\binom{k}{i'} \;=\; (1-1)^k \;=\; 0.
\end{aligned}$$

Next, we confirm that $g(j+1) - g(j) = 0$ for $j = k+1, k+2, \ldots, 2k-1$, which inductively completes this proof:

$$\begin{aligned}
g(j+1) - g(j) &= \sum_{i=k+1}^{j} \alpha_{k,i}\binom{j}{i} - \sum_{i=k+1}^{j+1} \alpha_{k,i}\binom{j+1}{i} \\
&= -\sum_{i=k+1}^{j+1} \alpha_{k,i}\binom{j}{i-1} \\
&= -\alpha_{k,j+1}\sum_{i=k+1}^{j+1} \frac{\alpha_{k,i}}{\alpha_{k,j+1}}\binom{j}{i-1} \\
&= -\alpha_{k,j+1}\sum_{i=k+1}^{j+1} (-1)^{j+1-i}\frac{(i-1)!(j-k)!}{j!(i-(k+1))!}\frac{j!}{(i-1)!(j-(i-1))!} \\
&= -\alpha_{k,j+1}\sum_{i=k+1}^{j+1} (-1)^{j+1-i}\binom{j-k}{j+1-i} \;=\; 0. \quad \square
\end{aligned}$$

Since $\varphi_{\mathcal{H}^*} : \mathbb{R}^{2^V} \to \mathcal{F}$ is surjective by Theorem 3.1, Theorem 3.5 follows from the next corollary, which implies that the quotient space $\mathbb{R}^{2^V} / \ker \varphi_{\mathcal{H}^*}$ is isomorphic to $\mathbb{R}^{\tilde{\mathcal{E}}}$, where $\tilde{\mathcal{E}} = \{\, E \mid \emptyset \neq E \subseteq V \text{ and } |E| \text{ is even} \,\}$. This corollary can be seen by straightforward calculation using (4) in Lemma 4.3 repeatedly.

Corollary 4.4. *Let $\mathcal{N}^* = (\mathcal{H}^* = (V, 2^V), c^*)$ be a complete hypernetwork. For any positive integer k with $1 \leq k \leq \lceil |V|/2 \rceil - 1$ and any vertex subset $U \in \binom{V}{2k+1}$, $\tilde{c}^*_U : 2^V \to \mathbb{R}$ defined as follows is in the kernel of $\varphi_{\mathcal{H}^*} : \mathbb{R}^{2^V} \to \mathcal{F}$ ($c^* \mapsto \kappa_{\mathcal{N}^*}$):*

$$\tilde{c}^*_U(E) := \begin{cases} 1 & (E = U) \\ \beta_{k,|E|} & (\emptyset \neq E \subseteq U \text{ and } |E| \text{ is even}) \\ 0 & (\text{otherwise}), \end{cases}$$

where $\beta_{k,2k+1} = -1$ and $\beta_{k,i} \in \mathbb{R}$ ($i = 2k, 2k-1, \ldots, 2$) are recursively defined as

$$\beta_{k,i} := -\sum_{j=\lceil i/2 \rceil}^{\min\{i-1,\,k\}} \binom{2k+1-i}{2j+1-i}\beta_{k,2j+1}\alpha_{j,i}$$

by using the constants $\alpha_{j,i} \in \mathbb{R}$ $(j = 1, 2, \ldots, \lceil |V|/2 \rceil - 1,\ i = j+1, j+2, \ldots, 2j+1)$ *defined as* (5)*. Moreover,* $\{c_0^*\} \cup \{ c_{\{v\}}^* \mid v \in V \} \cup \{ \tilde{c}_U^* \mid U \in \binom{V}{2k+1}$ *and* $k = 1, 2, \ldots, \lceil |V|/2 \rceil - 1 \}$ *is a basis of* $\ker \varphi_{\mathcal{H}^*}$*.*

Next we prove Theorem 3.6. Since we have already confirmed $|\hat{\mathcal{E}}| = 2^{|V|-1}$, it suffices to show the following lemma similarly to the proof of Theorem 3.4 in Section 4.1.

Lemma 4.5. *Suppose that* $|V|$ *is odd, and let* $\hat{\mathcal{E}} := \{ E \subseteq V \mid |E| \geq \lceil |V|/2 \rceil \}$*. Then, for any symmetric set function* $f\colon 2^V \to \mathbb{R}$ *with* $f(\emptyset) = 0$*, there exists a capacity function* $\hat{c}_X \colon \hat{\mathcal{E}} \setminus \{X\} \to \mathbb{R}$ *such that* $\kappa_{\hat{\mathcal{N}}_X} = f$ *for* $\hat{\mathcal{N}}_X = (\hat{\mathcal{H}}_X = (V, \hat{\mathcal{E}} \setminus \{X\}), \hat{c}_X)$*, for an arbitrary fixed hyperedge* $X \in \hat{\mathcal{E}}$*.*

Proof. Since Lemma 4.3 claims that $c_V^* \colon 2^V \to \mathbb{R}$ (whose domain can be regarded as $\hat{\mathcal{E}}$ since $c_V^*(Y) \neq 0$ if and only if $|Y| \geq \lceil |V|/2 \rceil$) is in $\ker \varphi^*$, it suffices to consider just one $X \in \hat{\mathcal{E}}$.

Fix arbitrary $X \in \hat{\mathcal{E}}$ with $|X| = \lceil |V|/2 \rceil$, and let $\hat{\mathcal{E}}_X := \hat{\mathcal{E}} \setminus \{X\}$. We then have

$$\kappa_{\hat{\mathcal{N}}_X}(Y) = \sum_{E \in \hat{\mathcal{E}}_X} \{ \hat{c}_X(E) \mid E \setminus Y \neq \emptyset \} \tag{6}$$

for each $Y \subseteq V$ with $|Y| \geq \lceil |V|/2 \rceil$. Since X cuts all hyperedges in $\hat{\mathcal{E}}_X$ and each other $Y \subseteq V$ with $|Y| \geq \lceil |V|/2 \rceil$ cuts all hyperedges in $\hat{\mathcal{E}}_X$ of size at least $|Y|$ but $Y \in \hat{\mathcal{E}}_X$ itself, we can determine $\hat{c}_X \colon \hat{\mathcal{E}}_X \to \mathbb{R}$ from $\kappa_{\hat{\mathcal{N}}_X}$ recursively (in the ascending order of the size) as

$$\hat{c}_X(E) = \kappa_{\hat{\mathcal{N}}_X}(X) - \kappa_{\hat{\mathcal{N}}_X}(E) - \sum_{E' \in \hat{\mathcal{E}}_X} \{ \hat{c}_X(E') \mid E' \subsetneq E \} \tag{7}$$

for each $E \in \hat{\mathcal{E}}_X$. This construction (7) defines the inverse mapping of the linear mapping $\varphi_{\hat{\mathcal{H}}_X} \colon \hat{\mathcal{E}}_X \to \mathcal{F}$ $(\hat{c}_X \mapsto \kappa_{\hat{\mathcal{N}}_X})$ defined by (6). $\square$

Remark 5. Similarly to Remark 4, we can obtain a more direct expression of the construction (7) by applying the Möbius inversion formula over a partially ordered set $(\hat{\mathcal{E}}, \subseteq)$ (see, e.g., [1,9]) to a set function $g\colon \hat{\mathcal{E}} \to \mathbb{R}$ defined as

$$g(Y) := \kappa_{\hat{\mathcal{N}}_X}(X) - \kappa_{\hat{\mathcal{N}}_X}(Y) = \sum_{E \in \hat{\mathcal{E}}_X} \{ \hat{c}_X(E) \mid E \subseteq Y \}$$

for each $Y \in \hat{\mathcal{E}}$. Here we omit the straightforward calculation and describe only the result:

$$\hat{c}_X(E) = (-1)^{|E| - \lceil |V|/2 \rceil} \binom{|E| - 1}{|E| - \lceil |V|/2 \rceil} \kappa_{\hat{\mathcal{N}}_X}(X) - \sum_{Y \in \hat{\mathcal{E}}} \left\{ (-1)^{|E \setminus Y|} \kappa_{\hat{\mathcal{N}}_X}(Y) \;\middle|\; Y \subseteq E \right\}.$$

4.3 Decomposition of cut capacity functions (Proofs of Theorems 3.2 and 3.3)

The following lemma shows the decomposition of the cut capacity function of a hypernetwork into the i-th components $(i = 0, 1, \ldots, |V|)$ (see Proposition 2.1).

Lemma 4.6. *For any hypernetwork* $\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$*, we have*

$$\kappa_{\mathcal{N}}^{(i)}(X) = \begin{cases} 0 & (i = 0) \\ \displaystyle\sum_{E\colon X \subsetneq E \in \mathcal{E}} c(E) & (i \text{ is odd}) \\ -\left(\displaystyle\sum_{E\colon X \subsetneq E \in \mathcal{E}} c(E) + 2c(X) \right) & (i \text{ is even and } i > 0) \end{cases}$$

for each $i = 0, 1, \ldots, |V|$ *and each* $X \in \binom{V}{i}$*.*

Proof. The case of $i = 0$ is obvious. Fix an integer i with $1 \leq i \leq |V|$. By Proposition 2.1,

$$\kappa_{\mathcal{N}}^{(i)}(X) = \sum_{Y \subseteq X} (-1)^{|X \setminus Y|} \kappa_{\mathcal{N}}(Y) \tag{8}$$

for each $X \in \binom{V}{i}$. For each hyperedge $E \in \mathcal{E}$, let us count the contribution of its capacity $c(E)$ to the right-hand side of (8).

Case 1. Suppose that $E \subseteq V \setminus X$. Then, $c(E)$ does not contribute to $\kappa_{\mathcal{N}}(Y)$ for every $Y \subseteq X$.

Case 2. Suppose that $E \subseteq X$. Then, for each $Y \subseteq X$, $c(E)$ does not contribute to $\kappa_{\mathcal{N}}(Y)$ if and only if $E \subseteq Y$ or $Y \subseteq X \setminus E$, i.e., $Y = E \cup Z$ or $Y = Z$ for some $Z \subseteq X \setminus E$. Hence, the contribution of $c(E)$ to the right-hand side of (8) is

$$\left(\sum_{Y \subseteq X} (-1)^{|X \setminus Y|} - \sum_{Z \subseteq X \setminus E} (-1)^{|X \setminus (E \cup Z)|} - \sum_{Z \subseteq X \setminus E} (-1)^{|X \setminus Z|} \right) c(E)$$
$$= \begin{cases} \left(0 - (-1)^0 - (-1)^{|X|}\right) c(E) & (E = X) \\ 0 & (E \subsetneq X). \end{cases}$$

The former value is equal to $-2c(X)$ if $|X| = i$ is even, and 0 otherwise.

Case 3. Otherwise, we have $E \cap X \neq \emptyset \neq E \setminus X$. If $X \subsetneq E$, then $c(E)$ contributes $\kappa_{\mathcal{N}}(Y)$ for every nonempty $Y \subseteq X$, and hence the total contribution to the right-hand side of (8) is

$$\left(\sum_{Y \colon \emptyset \neq Y \subseteq X} (-1)^{|X \setminus Y|} \right) c(E) \;=\; \left((1-1)^{|X|} - (-1)^{|X|} \right) c(E) \;=\; \begin{cases} c(E) & (|X| = i \text{ is odd}) \\ -c(E) & (|X| = i \text{ is even}). \end{cases}$$

Otherwise, i.e., in the case of $X \setminus E \neq \emptyset$, $c(E)$ does not contribute to $\kappa_{\mathcal{N}}(Y)$ $(Y \subseteq X)$ if and only if $Y \subseteq X \setminus E$. Hence, the total contribution is 0 similarly to Case 2. $\square$

Let $\mathcal{F}' := \{\, f \in \mathbb{R}^{2^V} \mid f(X) = f(V \setminus X) \ (\forall X \subseteq V) \,\}$. Theorem 3.5 and Lemma 4.6 imply Proposition 4.7, which claims that the decomposition of symmetric set functions by Proposition 2.1 defines a linear bijection from $\mathcal{F}'$ to $\mathbb{R}^{\binom{V}{0}} \times \mathbb{R}^{\binom{V}{2}} \times \cdots \times \mathbb{R}^{\binom{V}{2\tilde{n}}}$, where $\tilde{n} := \lfloor |V|/2 \rfloor$.

Proposition 4.7. *Let V be a finite set. For any family of functions $\tilde{f}_{2i} \colon \binom{V}{2i} \to \mathbb{R}$ $(i = 0, 1, \ldots, \lfloor |V|/2 \rfloor)$, there uniquely exists a symmetric set function $f \colon 2^V \to \mathbb{R}$ such that $f^{(2i)} = \tilde{f}_{2i}$ for $i = 0, 1, \ldots, \lfloor |V|/2 \rfloor$.*

Proof. For any $f' \in \mathcal{F}'$, there uniquely exist $f \in \mathcal{F}$ and $\alpha \in \mathbb{R}$ such that $f' = f + \alpha \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^{2^V}$ denotes the all-one vector. We obviously have $\alpha = f'(\emptyset) = f'^{(0)}(\emptyset)$, and hence it suffices to show that the decomposition of symmetric set functions $f \in \mathcal{F}$ (i.e., with $f(\emptyset) = 0$) defines a linear bijection from $\mathcal{F}$ to $\mathbb{R}^{\binom{V}{2}} \times \mathbb{R}^{\binom{V}{4}} \times \cdots \times \mathbb{R}^{\binom{V}{2\tilde{n}}}$.

Recall that the linear mapping $\varphi_{\tilde{\mathcal{H}}} \colon \mathbb{R}^{\tilde{\mathcal{E}}} \to \mathcal{F}$ is bijective (Theorem 3.5), where $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$ and $\tilde{\mathcal{E}} = \{\, E \mid \emptyset \neq E \subseteq V \text{ and } |E| \text{ is even} \,\}$. Moreover, for any hypernetwork $\tilde{\mathcal{N}} = (\tilde{\mathcal{H}}, \tilde{c})$, $\tilde{c} \colon \tilde{\mathcal{E}} \to \mathbb{R}$ is recursively (in the descending order of the size) determined from $\kappa_{\tilde{\mathcal{N}}}^{(2i)} \in \mathbb{R}^{\binom{V}{2i}}$ $(i = 1, 2, \ldots, \tilde{n})$ as follows by Lemma 4.6:

$$\tilde{c}(E) = -\frac{1}{2} \left(\kappa_{\tilde{\mathcal{N}}}^{(|E|)}(E) + \sum_{E' \colon E \subsetneq E' \in \tilde{\mathcal{E}}} \tilde{c}(E') \right) \tag{9}$$

for each $E \in \tilde{\mathcal{E}}$. This means that the linear mapping $\varphi_{\tilde{\mathcal{H}}}^{\mathrm{even}} \colon \tilde{c} \mapsto (\kappa_{\tilde{\mathcal{N}}}^{(2)}, \kappa_{\tilde{\mathcal{N}}}^{(4)}, \ldots, \kappa_{\tilde{\mathcal{N}}}^{(2\tilde{n})})$ is also bijective, and hence the composition $\varphi_{\tilde{\mathcal{H}}}^{\mathrm{even}} \circ \varphi_{\tilde{\mathcal{H}}}^{-1} \colon \kappa_{\tilde{\mathcal{N}}} \mapsto (\kappa_{\tilde{\mathcal{N}}}^{(2)}, \kappa_{\tilde{\mathcal{N}}}^{(4)}, \ldots, \kappa_{\tilde{\mathcal{N}}}^{(2\tilde{n})})$ is a linear bijection from $\mathcal{F}$ to $\mathbb{R}^{\binom{V}{2}} \times \mathbb{R}^{\binom{V}{4}} \times \cdots \times \mathbb{R}^{\binom{V}{2\tilde{n}}}$. $\square$

We finally give proofs of Theorems 3.2 and 3.3.

The proof of Theorem 3.2 is almost done by Theorem 3.5 and Lemma 4.6. Take arbitrary $f \in \mathcal{F}$. If there exists a hypernetwork $\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$ such that $\kappa_{\mathcal{N}} = f$ and $|E| \leq k$ for every $E \in \mathcal{E}$, then we have $f^{(i)}(X) = \kappa_{\mathcal{N}}^{(i)}(X) = 0$ for each $i = k+1, k+2, \ldots, |V|$ and each $X \in \binom{V}{i}$ by Lemma 4.6. To the contrary, if we have $f^{(i)}(X) = 0$ for each $i = k+1, k+2, \ldots, |V|$ and each $X \in \binom{V}{i}$, then $\tilde{c}(E) = 0$ holds for every $E \in \tilde{\mathcal{E}}$ with $|E| \geq k+1$ in the even standard form $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$ (see the construction (9) of $\tilde{c}$ from $\kappa_{\tilde{\mathcal{N}}}$), which leads to a desired hypernetwork.

Theorem 3.3 is almost obvious, where we note that the condition (ii) follows from Lemma 4.6 and that the condition (iv) is a well-known property of cut capacity functions of hypergraphs with nonnegative capacity.

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