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Making Bipartite Graphs DM-irreducible

Satoru Iwata* Jun Kato† Yutaro Yamaguchi‡

Abstract

The Dulmage–Mendelsohn decomposition (or the DM-decomposition) gives a unique partition of the vertex set of a bipartite graph reflecting the structure of all the maximum matchings therein. A bipartite graph is said to be DM-irreducible if its DM-decomposition consists of a single component. For connected bipartite graphs, this is equivalent to the condition that every edge is contained in some perfect matching.

In this paper, we focus on the problem of making a given bipartite graph DM-irreducible by adding edges. When the input bipartite graph is balanced (i.e., the both sides have the same number of vertices) and has a perfect matching, this problem is equivalent to making a directed graph strongly connected by adding edges, for which the minimum number of additional edges was characterized by Eswaran and Tarjan (1976).

We provide a simple solution to the general case. Specifically, we present a combinatorial algorithm for finding a minimum number of additional edges to make a given bipartite graph DM-irreducible, which also leads to a constructive proof for a min-max duality. Our algorithm requires only $\mathrm{O}(|V| \cdot |E|)$ time for bipartite graphs with vertex set V and edge set E, and just utilizes two fundamental techniques on graphs: the result of Eswaran and Tarjan on making a directed graph strongly connected, and finding edge-disjoint paths between two vertices in a directed graph.

We also show that our problem can be formulated as a special case of the bisupermodular covering problem introduced by Frank and Jordán (1995). This provides an alternative proof to our min-max characterization.

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1 Introduction

The *Dulmage–Mendelsohn decomposition* [3, 4] (or the *DM-decomposition*) of a bipartite graph gives a unique partition of the vertex set, which reflects the structure of all the maximum matchings therein (see Section 2.2 for the details). A bipartite graph is said to be *DM-irreducible* if its DM-decomposition consists of only one nonempty component.

In this paper, we focus on the following question: how many additional edges are necessary to make a given bipartite graph G DM-irreducible?

Problem (DMI)

Input: A bipartite graph $G = (V^+, V^-; E)$.

Goal: Find a minimum-cardinality set F of additional edges such that $G+F$ is DM-irreducible.

Throughout this paper, for an input bipartite graph $G = (V^+, V^-; E)$, we define $n := \max\{|V^+|, |V^-|\}$ and $m := |E|$, and denote by $\mathrm{opt}(G)$ the optimal value of Problem (DMI), i.e., the minimum number of additional edges to make G DM-irreducible.

When G is *balanced* (i.e., $|V^+| = |V^-|$) and has a perfect matching, Problem (DMI) is equivalent to the problem of making a directed graph strongly connected by adding as few edges as possible (see Section 3.5). For the latter problem, Eswaran and Tarjan [5] gave a simple solution (Theorem 2.1).

A natural generalization of the strong connectivity augmentation is to find a smallest set of additional edges that make a given directed graph strongly k-vertex connected. In order to investigate this problem, Frank and Jordán [7] introduced a general framework of covering a crossing bisupermodular function by directed edges. They provided a min-max duality theorem and a polynomial-time algorithm relying on the ellipsoid method. Later, Végh and Benczúr [14] devised a combinatorial algorithm whose running time bound is pseudopolynomial, depending polynomially on the function values.

In this paper, we present a simple combinatorial algorithm for Problem (DMI), which runs in $\mathrm{O}(nm)$ time.

Theorem 1.1. *For a bipartite graph $G = (V^+, V^-; E)$ with $\max\{|V^+|, |V^-|\} = n$ and $|E| = m$, one can find in $\mathrm{O}(nm)$ time a minimum number of additional edges to make G DM-irreducible.*

Specifically, we give an algorithm for the case when G is balanced, and show that the unbalanced case can be reduced to the balanced case (see Section 3.3). Our algorithm gives a constructive proof of the following min-max duality on Problem (DMI) for the balanced case. For a one-side vertex set X in a bipartite graph G, we denote by $\Gamma_G(X)$ the set of vertices in the other side that are adjacent to some vertex in X. For a set S, a *subpartition* of S is a partition of some subset of S (i.e., a family of disjoint nonempty subsets of S). A subpartition $\mathcal{X}$ of S is said to be *proper* if $\mathcal{X} \neq \{S\}$.

Theorem 1.2. *Let $G = (V^+, V^-; E)$ be a bipartite graph with $|V^+| = |V^-| \geq 2$. Then the minimum number $\mathrm{opt}(G)$ of additional edges to make G DM-irreducible is equal to the maximum value of*

$$\tau_G(\mathcal{X}) := \sum_{X \in \mathcal{X}} (|X| - |\Gamma_G(X)| + 1), \tag{1}$$

taken over all proper subpartitions $\mathcal{X}$ of V^+ and of V^-.

Through a reduction to the balanced case, we obtain the following form in the unbalanced case (see Section 3.4).

Corollary 1.3. *Let $G = (V^+, V^-; E)$ be a bipartite graph with $|V^+| < |V^-|$. Then,*

$$\mathrm{opt}(G) = \max_{\mathcal{X}^+} \tau_G(\mathcal{X}^+),$$

where the maximum is taken over all subpartitions $\mathcal{X}^+$ of V^+.

In addition, we show that Problem (DMI) is a special case of the Frank–Jordán framework. As a consequence, Theorem 1.2 can also be derived from the min-max duality theorem of Frank and Jordán [7, Theorem 2.3]. The function values that appear in the reduction are bounded by $\mathrm{O}(n)$, and a direct application of the Végh–Benczúr algorithm runs in polynomial time (roughly bounded by $\mathrm{O}(n^7)$).

The DM-decomposition is known to be a useful tool in numerical linear algebra (see, e.g., [2]). A bipartite graph associated with a matrix is naturally defined by its nonzero entries, and its DM-decomposition gives the finest block-triangularization, which helps us to solve the system of linear equations efficiently. The finer decomposed, the finer from computational point of view. Hence the DM-irreducibility is not a desirable property in this context. There are, however, certain situations in which DM-irreducibility is rather preferable. For example, in game theory, the uniqueness of the utility profile in a subgame perfect equilibrium in a bargaining game is characterized by the DM-irreducibility. In control theory, the structural controllability is characterized in terms of the DM-irreducibility. We explicate these situations and possible applications of our result in Section 7.

The rest of this paper is organized as follows. In Section 2, we describe necessary definitions and known results on the DM-decomposition of bipartite graphs and on the strong connectivity of directed graphs. We then observe basic properties on the DM-irreducibility and our problem in Section 3. Section 4 is devoted to presenting our algorithm. The correctness of the algorithm also gives a constructive proof of Theorem 1.2. A key procedure in our algorithm is shown separately in Section 5. In Section 6, we formulate our problem in terms of the Frank–Jordán framework and apply their result to provide an alternative proof of Theorem 1.2. Finally, in Section 7, we discuss possible applications of our result in game theory and in control theory.

2 Preliminaries

2.1 Strong connectivity of directed graphs

Let $G = (V, E)$ be a directed graph. A sequence $P = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ is called a *path* (or, in particular, a *v_0–v_l path*) in G if $v_0, v_1, \ldots, v_l \in V$ are distinct and $e_i = v_{i-1}v_i \in E$ for each $i \in \{1, 2, \ldots, l\}$. For two vertices $u, w \in V$ (possibly $u = w$), we say that u is *reachable to* w (or, equivalently, w is *reachable from* u) in G and denote by $u \xrightarrow{G} w$ if there exists a u–w path in G. A directed graph is said to be *strongly connected* if every two vertices are reachable to each other (also from each other). A *strongly connected component* of G is a maximal induced subgraph of G that is strongly connected. The strongly connected components of a directed graph can be found in linear time with the aid of the depth first search [13].

Let $\mathcal{S} = \{V_1, V_2, \ldots, V_k\}$ be the partition of V according to the strongly connected components of G, i.e., for any two vertices $u, w \in V$, we have $u \xrightarrow{G} w$ and $w \xrightarrow{G} u$ if and only if $\{u, w\} \subseteq V_i$ for some i. For $V_i, V_j \in \mathcal{S}$, we denote by $V_i \succeq_G V_j$ if $u \xrightarrow{G} w$ for every pair of $u \in V_i$ and $w \in V_j$. Then the binary relation $\succeq_G$ is a partial order on $\mathcal{S}$. A strongly connected component of G is called a *source component* if its vertex set V_i is maximal with respect to $\succeq_G$ (i.e., there is no $V_j \in \mathcal{S} \setminus \{V_i\}$ with $V_j \succeq_G V_i$), and a *sink component* if minimal. Note that a strongly connected component is a source or sink component if and only if no edge enters or leaves it, respectively. The numbers of source and sink components of G are denoted by $s(G)$ and $t(G)$, respectively.

Eswaran and Tarjan [5] characterized the minimum number of additional edges to make a directed graph strongly connected, and proposed a linear-time algorithm for finding such additional edges as follows.

Theorem 2.1 (Eswaran–Tarjan [5, Section 2])**.** *Let $G = (V, E)$ be a directed graph that is not strongly connected. Then the minimum number of additional edges to make G strongly connected is equal to* $\max\{s(G), t(G)\}$*. Moreover, one can find such additional edges in* $\mathrm{O}(|V|+|E|)$ *time.*

2.2 DM-decomposition of bipartite graphs

Let $G = (V^+, V^-; E)$ be a bipartite graph with the vertex set V partitioned into the *left side* V^+ and the *right side* V^-. Throughout this paper, a bipartite graph is dealt with as a directed graph in which each edge is directed from left to right, i.e., $E \subseteq V^+ \times V^-$. An edge set $M \subseteq E$ is called a *matching* in G if $|\partial^+ M| = |\partial^- M| = |M|$, where $\partial^+ M := \{\, u \mid uw \in M \,\} \subseteq V^+$ and $\partial^- M := \{\, w \mid uw \in M \,\} \subseteq V^-$. A matching M is said to be *maximum* if $|M|$ is maximum, and *perfect* if $|M| = \min\{|V^+|, |V^-|\}$. A bipartite graph is said to be *perfectly matchable* if it has a perfect matching, and *matching covered* if every edge is contained in some perfect matching.

The DM-decomposition of a bipartite graph gives a unique partition of the vertex set, which reflects the structure of all the maximum matchings therein as follows. We define $X^+ := X \cap V^+$ and $X^- := X \cap V^-$ for a vertex set $X \subseteq V$, and $[k] := \{1, 2, \dots, k\}$ for a nonnegative integer k.

Theorem 2.2 (Dulmage–Mendelsohn [3, 4])**.** *Let $G = (V^+, V^-; E)$ be a bipartite graph. Then there exists a partition $(V_0; V_1, V_2, \dots, V_k; V_\infty)$ of the vertex set V such that*

1. *either $|V_0^+| > |V_0^-|$ or $V_0 = \emptyset$,*
2. *$|V_i^+| = |V_i^-| > 0$ for each $i \in [k]$,*
3. *either $|V_\infty^+| < |V_\infty^-|$ or $V_\infty = \emptyset$,*
4. *the induced subgraph $G[V_i]$ is matching covered for each $i \in [k] \cup \{0, \infty\}$, and*
5. *every maximum matching in G is a union of perfect matchings in $G[V_i]$ ($i \in [k] \cup \{0, \infty\}$).*

We here define the *DM-decomposition* $(V_0; V_1, V_2, \dots, V_k; V_\infty)$ of a bipartite graph $G = (V^+, V^-; E)$, which satisfies the conditions in Theorem 2.2 (see also, e.g., [9, 11]). Define a set function $f_G \colon 2^{V^+} \to \mathbb{Z}$ as

$$f_G(X^+) := |\Gamma_G(X^+)| - |X^+| \quad (X^+ \subseteq V^+), \tag{2}$$

where recall $\Gamma_G(X^+) = \{\, w \mid \exists e = uw \in E \colon u \in X^+ \,\} \subseteq V^-$. It is well-known that f_G is submodular, and hence all the minimizers of f_G form a distributive lattice $\mathcal{L}(f_G)$ with respect to the set union and intersection (see, e.g., [8, Lemma 2.1]). For a maximal chain (inclusion-wise monotone sequence) $X_0^+ \subsetneq X_1^+ \subsetneq \cdots \subsetneq X_k^+$ in $\mathcal{L}(f_G)$, define $V_i := V_i^+ \cup V_i^-$ for each $i \in [k] \cup \{0, \infty\}$ as follows:

$$\begin{aligned}
V_0^+ &:= X_0^+, & V_0^- &:= \Gamma_G(X_0^+), & \\
V_i^+ &:= X_i^+ \setminus X_{i-1}^+, & V_i^- &:= \Gamma_G(X_i^+) \setminus \Gamma_G(X_{i-1}^+) & (i \in [k]),\\
V_\infty^+ &:= V^+ \setminus X_k^+, & V_\infty^- &:= V^- \setminus \Gamma_G(X_k^+). &
\end{aligned}$$

It is known that the resulting partition of V with the following partial order $\sqsubseteq$ is unique (i.e., does not depend on the choice of a maximal chain in $\mathcal{L}(f_G)$):

$$V_i \sqsubseteq V_j \iff \left[V_j^+ \subseteq X^+ \in \mathcal{L}(f_G) \implies V_i^+ \subseteq X^+ \right] \quad (i, j \in [k] \cup \{0, \infty\}).$$

Moreover, while V^+ and V^- do not seem symmetric in the above definition, it is also known that essentially the same partially-ordered partition is obtained by interchanging the roles of V^+ and of V^-, in which, e.g., V_0 and V_∞ are interchanged and the direction of $\sqsubseteq$ is reversed.

The DM-decomposition is known to be obtained as follows. Take an arbitrary maximum matching $M \subseteq E$ in G. Construct the *auxiliary graph* $G(M) := G + \overline{M}$ with respect to M, where $\overline{M} := \{ \bar{e} := wu \mid e = uw \in M \} \subseteq V^- \times V^+$ denotes the set of reverse edges. The set of vertices reachable from some vertex in $V^+ \setminus \partial^+ M$ in $G(M)$ is V_0, and the set of vertices reachable to some vertex in $V^- \setminus \partial^- M$ in $G(M)$ is V_∞. The rest $V_* := V \setminus (V_0 \cup V_\infty)$ is partitioned according to the strongly connected components of $G_* := G(M)[V_*]$. The partial order $\sqsubseteq$ is defined by $\preceq_{G_*}$ on $\{ V_i \mid i \in [k] \}$ and so that V_0 and V_∞ are minimum and maximum elements, respectively. By this computation, one can easily see the following properties.

Observation 2.3. *Let $(V_0; V_1, V_2, \ldots, V_k; V_\infty)$ be the DM-decomposition of a bipartite graph $G = (V^+, V^-; E)$. Then, for any maximum matching $M \subseteq E$ in G, the auxiliary graph $G(M)$ satisfies the following conditions.*

- *No edge leaves V_0.*
- *No edge enters V_∞.*
- *Each source component of $G(M)[V_0]$ is a single vertex in $V^+ \setminus \partial^+ M$, and vice versa. Hence, $s(G(M)[V_0]) = |V^+| - |M|$.*
- *Each sink component of $G(M)[V_\infty]$ is a single vertex in $V^- \setminus \partial^- M$, and vice versa. Hence, $t(G(M)[V_\infty]) = |V^-| - |M|$.*

3 DM-irreducibility

A bipartite graph $G = (V^+, V^-; E)$ is said to be *DM-irreducible* if its DM-decomposition consists of only one nonempty component. In this section, we show several basic properties on the DM-irreducibility and Problem (DMI). Throughout this section, we assume $|V^+| \leq |V^-|$ by the symmetry.

3.1 Characterization of DM-irreducibility

We first characterize the DM-irreducibility in terms of the set function $f_G \colon 2^{V^+} \to \mathbb{Z}$ defined in (2), which is useful in the following discussions.

Lemma 3.1. *A bipartite graph $G = (V^+, V^-; E)$ with $|V^+| \leq |V^-|$ and $|V^-| \geq 2$ is DM-irreducible if and only if $f_G(X^+) \geq 1$ for every nonempty $X^+ \subseteq V^+$ with $|X^+| < |V^-|$.*

Proof. By Conditions 1–3 in Theorem 2.2, the DM-irreducibility of G is equivalent to $V_\infty = V$ when $|V^+| < |V^-|$, and to $V_1 = V$ when $|V^+| = |V^-|$. In the both cases, $X_0^+ = V_0^+ = \emptyset$ minimizes f_G, and $f_G(\emptyset) = 0$.

Suppose that $|V^+| < |V^-|$. Then, G is DM-irreducible if and only if $X_0^+ = \emptyset$ is a unique minimizer of f_G; equivalently, $f_G(X^+) \geq 1$ for every nonempty $X^+ \subseteq V^+$, which satisfies $|X^+| \leq |V^+| < |V^-|$.

Suppose that $|V^+| = |V^-| \geq 2$. Then, G is DM-irreducible if and only if f_G has exactly two minimizers $X_0^+ = \emptyset$ and $X_1^+ = V_1^+ = V^+$; equivalently, $f_G(V^+) = 0$ and $f_G(X^+) \geq 1$ for every nonempty $X^+ \subsetneq V^+$, which satisfies $|X^+| < |V^+| = |V^-|$. Note that the former condition is automatically satisfied by the latter condition as follows. For any nonempty $X^+ \subsetneq V^+$ with $|X^+| = |V^+| - 1$ (such X^+ exists because $|V^+| \geq 2$), the latter condition implies

$$1 \leq f_G(X^+) = |\Gamma_G(X^+)| - |X^+| \leq |V^-| - |X^+| = 1.$$

We then have $V^- \supseteq \Gamma_G(V^+) \supseteq \Gamma_G(X^+) = V^-$, and hence $\Gamma_G(V^+) = V^-$, which leads to $f(V^+) = |V^-| - |V^+| = 0$. $\square$

3.2 Weak duality

We now show the weak duality part of Theorem 1.2, i.e., $\mathrm{opt}(G) \geq \max_{\mathcal{X}} \tau_G(\mathcal{X})$.

Lemma 3.2. *Let $G = (V^+, V^-; E)$ be a bipartite graph with $|V^+| = |V^-|$. Then, for any edge set $F \subseteq (V^+ \times V^-) \setminus E$ such that $G + F$ is DM-irreducible and any proper subpartition $\mathcal{X}$ of V^+ or of V^-, we have $|F| \geq \tau_G(\mathcal{X})$.*

Proof. Fix an edge set $F \subseteq (V^+ \times V^-) \setminus E$ such that $G + F$ is DM-irreducible and a proper subpartition $\mathcal{X}$ of V^+. By Lemma 3.1, the DM-irreducibility of $G+F$ implies that $|\Gamma_{G+F}(X^+)| \geq |X^+| + 1$ for every $X^+ \in \mathcal{X}$. Hence,

$$|F(X^+,\, V^- \setminus \Gamma_G(X^+))| \geq |\Gamma_{G+F}(X^+)| - |\Gamma_G(X^+)| \geq |X^+| - |\Gamma_G(X^+)| + 1,$$

where $F(Y^+, Y^-) := F \cap (Y^+ \times Y^-)$ denotes the restriction of F to $Y^+ \times Y^-$ for $Y^+ \subseteq V^+$ and $Y^- \subseteq V^-$. For every distinct $X_1^+, X_2^+ \in \mathcal{X}$, since $X_1^+ \cap X_2^+ = \emptyset$ implies $F(X_1^+,\, V^- \setminus \Gamma_G(X_1^+)) \cap F(X_2^+,\, V^- \setminus \Gamma_G(X_2^+)) = \emptyset$, we see

$$|F| \geq \sum_{X^+ \in \mathcal{X}} |F(X^+, V^- \setminus \Gamma_G(X^+))| \geq \sum_{X^+ \in \mathcal{X}} \bigl(|X^+| - |\Gamma_G(X^+)| + 1\bigr) = \tau_G(\mathcal{X}).$$

We can handle the proper subpartitions of V^- in the same way by considering the interchanged bipartite graph $(V^-, V^+; \overline{E})$ and the set $\overline{F}$ of reverse edges, and thus we have done. $\square$

We prove the strong duality in Section 4.2, together with the correctness of our algorithm.

3.3 Reduction to the balanced case

A bipartite graph $G = (V^+, V^-; E)$ is said to be *balanced* if $|V^+| = |V^-|$. As remarked in Introduction, the unbalanced case of Problem (DMI) can be reduced to the balanced case. The next lemma gives such a reduction. That is, making an unbalanced bipartite graph G DM-irreducible by adding edges is equivalent to doing so the corresponding balanced bipartite graph G' defined in Lemma 3.3, where the set of usable additional edges is not changed.

Lemma 3.3. *For a bipartite graph $G = (V^+, V^-; E)$ with $|V^+| < |V^-|$, define a balanced bipartite graph $G' = (V^+ \cup Z^+, V^-; E')$ as follows: let Z^+ be a set of new vertices with $|Z^+| = |V^-| - |V^+|$ and $E' := E \cup (Z^+ \times V^-)$. Then, G is DM-irreducible if and only if so is G'.*

Proof. When $|V^-| \leq 1$, both G and G' are DM-irreducible. Assume $|V^-| \geq 2$ in what follows.

Consider the set functions $f_G\colon 2^{V^+} \to \mathbb{Z}$ and $f_{G'}\colon 2^{V^+ \cup Z^+} \to \mathbb{Z}$ defined in (2). By Lemma 3.1, G is DM-irreducible if and only if $f_G(X^+) \geq 1$ for every nonempty $X^+ \subseteq V^+$, and so is G' if and only if $f_{G'}(X^+) \geq 1$ for every nonempty $X^+ \subsetneq V^+ \cup Z^+$. By the definition of E', for every $X^+ \subseteq V^+ \cup Z^+$ with $X^+ \cap Z^+ \neq \emptyset$, we have $\Gamma_{G'}(X^+) = V^-$, which implies $f_{G'}(X^+) = |V^-| - |X^+| = |V^+ \cup Z^+| - |X^+|$. Hence, $f_{G'}(X^+) \geq 1$ for every $X^+ \subsetneq V^+ \cup Z^+$ with $X^+ \cap Z^+ \neq \emptyset$. Since $f_G(X^+) = f_{G'}(X^+)$ for every $X^+ \subseteq V^+$, the above two conditions for the DM-irreducibility of G and of G' are equivalent. $\square$

This reduction increases the size of the input graph. In particular, G' may have an essentially larger number of edges than G, i.e., $|E'| \neq \mathrm{O}(m)$. While our algorithm for Problem (DMI) shown in Section 4 only handles the balanced case, it runs in $\mathrm{O}(nm)$ time in terms of the original input size also for the unbalanced case through this reduction. See Section 4.3 for the details. The following observation is utilized in the analysis.

Observation 3.4. *For a bipartite graph $G = (V^+, V^-; E)$ with $|V^+| < |V^-|$, let $G' = (V^+ \cup Z^+, V^-; E')$ be the balanced bipartite graph defined in Lemma 3.3, $M' \subseteq E'$ a maximum matching in G', and $(V_0; V_1, V_2, \ldots, V_k; V_\infty)$ the DM-decomposition of G'. Then the following conditions hold.*

- *M' consists of a maximum matching in G and a perfect matching in $G'[Z^+ \cup V^-]$.*
- *Z^+ is included in a single strongly connected component of $G'(M') = G' + \overline{M'}$, which is a unique source component, and hence $s(G'(M')) = 1$.*
- *If $V_\infty \neq \emptyset$, then $Z^+ \subseteq V_\infty^+$, and hence $G' - V_\infty = G - V_\infty$. In particular, $G'[V_0] = G[V_0]$.*

3.4 Duality in the unbalanced case

In this section, we derive the min-max duality theorem for the unbalanced case (Corollary 1.3) from that for the balanced case (Theorem 1.2) through the reduction in Section 3.3. First, we see the following weak duality as a corollary of Lemma 3.2 via Lemma 3.3.

Corollary 3.5. *Let $G = (V^+, V^-; E)$ be a bipartite graph with $|V^+| < |V^-|$. Then, for any edge set $F \subseteq (V^+ \times V^-) \setminus E$ such that $G + F$ is DM-irreducible and any subpartition $\mathcal{X}^+$ of V^+, we have $|F| \geq \tau_G(\mathcal{X}^+)$.*

We now start to prove Corollary 1.3. Let $G = (V^+, V^-; E)$ be a bipartite graph with $|V^+| < |V^-|$. By Corollary 3.5, it suffices to construct a subpartition $\mathcal{X}^+$ of V^+ with $\tau_G(\mathcal{X}^+) = \mathrm{opt}(G)$. If $|V^-| = 1$, then G itself is DM-irreducible, and $\mathcal{X}^+ := \emptyset$ is indeed a subpartition of V^+ with $\tau_G(\mathcal{X}^+) = 0 = \mathrm{opt}(G)$. In what follows, we assume $|V^-| \geq 2$.

Let $G' = (V^+ \cup Z^+, V^-; E')$ be the balanced bipartite graph defined in Lemma 3.3. By Theorem 1.2, there exists a proper subpartition $\mathcal{Y}$ of $V^+ \cup Z^+$ or of V^- such that $\tau_{G'}(\mathcal{Y}) = \mathrm{opt}(G') = \mathrm{opt}(G)$. Suppose that $\mathcal{Y}$ is a proper subpartition of $V^+ \cup Z^+$. Since every vertex in Z^+ is adjacent to all the vertices in V^-, for each $X^+ \subseteq V^+ \cup Z^+$ with $X^+ \cap Z^+ \neq \emptyset$, we have $|X^+| - |\Gamma_{G'}(X^+)| + 1 = |X^+| - |V^-| + 1 \leq 0$. By the maximality of $\tau_{G'}(\mathcal{Y})$, we may assume that $\mathcal{Y}$ contains no such X^+, i.e., $\mathcal{Y}$ is a subpartition of V^+. We then obtain a desired subpartition $\mathcal{X}^+ := \mathcal{Y}$ of V^+ with $\tau_G(\mathcal{X}^+) = \tau_{G'}(\mathcal{Y}) = \mathrm{opt}(G)$.

Otherwise, $\mathcal{Y}$ is a nonempty proper subpartition of V^-. Suppose that $\mathcal{Y}$ contains two distinct elements $X^-, Y^- \in \mathcal{Y}$. By the definition of E', we have $\emptyset \neq Z^+ \subseteq \Gamma_{G'}(X^-) \cap \Gamma_{G'}(Y^-)$, which implies $|\Gamma_{G'}(X^- \cup Y^-)| = |\Gamma_{G'}(X^-) \cup \Gamma_{G'}(Y^-)| \leq |\Gamma_{G'}(X^-)| + |\Gamma_{G'}(Y^-)| - 1$. Hence,

$$\big(|X^-| - |\Gamma_{G'}(X^-)| + 1\big) + \big(|Y^-| - |\Gamma_{G'}(Y^-)| + 1\big) \leq |X^- \cup Y^-| - |\Gamma_{G'}(X^- \cup Y^-)| + 1.$$

This enables us to replace X^- and Y^- with $X^- \cup Y^-$ without reducing the value of $\tau_{G'}(\mathcal{Y})$. Thus, by the maximality of $\tau_{G'}(\mathcal{Y})$, we may assume $\mathcal{Y} = \{Y^-\}$ for some nonempty $Y^- \subsetneq V^-$. If $\Gamma_{G'}(Y^-) = V^+ \cup Z^+$, then $\tau_{G'}(\mathcal{Y}) = |Y^-| - |V^+ \cup Z^+| + 1 = |Y^-| - |V^-| + 1 \leq 0$, and hence $\mathcal{X}^+ := \emptyset$ is a desired subpartition of V^+. Otherwise, let $X^+ := V^+ \setminus \Gamma_G(Y^-) = (V^+ \cup Z^+) \setminus \Gamma_{G'}(Y^-) \neq \emptyset$ and $\mathcal{X}^+ := \{X^+\}$. We then see that

$$\begin{aligned}
\tau_G(\mathcal{X}^+) &= |X^+| - |\Gamma_G(X^+)| + 1 \\
&= \big(|V^+ \cup Z^+| - |\Gamma_{G'}(Y^-)|\big) - |\Gamma_{G'}(X^+)| + 1 \\
&= \big(|V^-| - |\Gamma_{G'}(X^+)|\big) - |\Gamma_{G'}(Y^-)| + 1 \\
&\geq |Y^-| - |\Gamma_{G'}(Y^-)| + 1 = \tau_{G'}(\mathcal{Y}),
\end{aligned}$$

which concludes that $\mathcal{X}^+$ is a desired subpartition of V^+.

3.5 Equivalence to strong connectivity under perfect matchability

From the computation of the DM-decomposition (see Section 2.2), we see the equivalence between the DM-irreducibility of a perfectly-matchable balanced bipartite graph and the strong connectivity of a directed graph as follows.

A balanced bipartite graph G with a perfect matching M is DM-irreducible if and only if the auxiliary directed graph $G(M) = G + \overline{M}$ is strongly connected. In addition, a directed graph

$G = (V, E)$ is strongly connected if and only if the balanced bipartite graph $\tilde{G} = (\tilde{V}^+, \tilde{V}^-; \tilde{E})$ defined as follows is DM-irreducible:

$$\tilde{V}^+ := \{\, v^+ \mid v \in V \,\}, \quad \tilde{V}^- := \{\, v^- \mid v \in V \,\},$$
$$\tilde{E} := \{\, u^+w^- \mid uw \in E \,\} \cup \{\, v^+v^- \mid v \in V \,\}.$$

Note that $\tilde{G}$ has a perfect matching $\tilde{M} := \{\, v^+v^- \mid v \in V \,\} \subseteq \tilde{E}$, and the DM-irreducibility of $\tilde{G}$ is equivalent to the strong connectivity of $\tilde{G}(\tilde{M})$, in which the two vertices $v^+ \in \tilde{V}^+$ and $v^- \in \tilde{V}^-$ derived from each vertex $v \in V$ must be contained in a single strongly connected component.

Hence, Problem (DMI) with the input bipartite graph balanced and perfectly matchable is equivalent to making a directed graph strongly connected by adding a minimum number of edges, which was solved by Eswaran and Tarjan [5] (cf. Theorem 2.1). Note that every strongly connected component of the auxiliary directed graph intersects both V^+ and V^- in this case, and one can choose, freely in each strongly connected component, the heads and the tails of additional edges to make a directed graph strongly connected. The strong duality (Theorem 1.2) under the perfectly-matchable assumption is confirmed as follows.

Lemma 3.6. *For a perfectly-matchable bipartite graph $G = (V^+, V^-; E)$ with $|V^+| = |V^-| \geq 2$, there exists a proper subpartition $\mathcal{X}$ of V^+ or of V^- such that $\tau_G(\mathcal{X}) = \mathrm{opt}(G)$.*

Proof. Let $M \subseteq E$ be a perfect matching in G, and $F \subseteq (V^+ \times V^-) \setminus E$ an optimal solution to G, i.e., a minimum-cardinality edge set such that $G(M) + F$ strongly connected. If $G(M)$ itself is strongly connected, then $\mathcal{X} := \emptyset$ is a desired proper subpartition of V^+ (and of V^-), i.e., $\tau_G(\mathcal{X}) = 0 = |F|$.

Otherwise, $|F| = \max\{s(G(M)),\ t(G(M))\}$ by Theorem 2.1. Define two subpartitions $\mathcal{X}^-$ of V^- and $\mathcal{X}^+$ of V^+ as follows (see also Fig. 1):

$$\mathcal{X}^- := \{\, X^- \mid G(M)[X] \text{ is a source component of } G(M) \,\},$$
$$\mathcal{X}^+ := \{\, X^+ \mid G(M)[X] \text{ is a sink component of } G(M) \,\},$$

where recall that $X^+ := X \cap V^+$ and $X^- := X \cap V^-$ for $X \subseteq V$. Since $G(M)$ is not strongly connected, we have $\mathcal{X}^- \neq \{V^-\}$ and $\mathcal{X}^+ \neq \{V^+\}$. We show that one of $\mathcal{X}^-$ and $\mathcal{X}^+$ is a desired proper subpartition by confirming $\tau_G(\mathcal{X}^-) = s(G(M))$ and $\tau_G(\mathcal{X}^+) = t(G(M))$.

Since any edge in $M \cup \overline{M}$ is contained in some strongly connected component of $G(M)$, distinct strongly connected components are connected only by edges in $E \setminus M \subseteq V^+ \times V^-$. Hence, for each source component $G(M)[X]$ of $G(M)$, since no edge can enter X in $G(M)$, we have $\Gamma_G(X^-) = X^+$, which implies $|\Gamma_G(X^-)| = |X^+| = |X^-|$. Similarly, for each sink component $G(M)[X]$ of $G(M)$, we have $|\Gamma_G(X^+)| = |X^-| = |X^+|$. Thus we see

$$\tau_G(\mathcal{X}^-) = \sum_{X^- \in \mathcal{X}^-} 1 = |\mathcal{X}^-| = s(G(M)) \quad \text{and} \quad \tau_G(\mathcal{X}^+) = \sum_{X^+ \in \mathcal{X}^+} 1 = |\mathcal{X}^+| = t(G(M)). \quad \square$$

4 Algorithm

In this section, we present an algorithm for Problem (DMI) that requires $\mathrm{O}(nm)$ time, where the input bipartite graph $G = (V^+, V^-; E)$ is assumed to be balanced (cf. Section 3.3) with $|V^+| = |V^-| = n$ and $|E| = m$. We first describe our algorithm in Section 4.1. Next, in Section 4.2, we show the optimality of the output, which also gives a constructive proof of the strong duality (Theorem 1.2). Finally, we analyze the running time of our algorithm in Section 4.3, where we also discuss the unbalanced case through the reduction in Section 3.3.

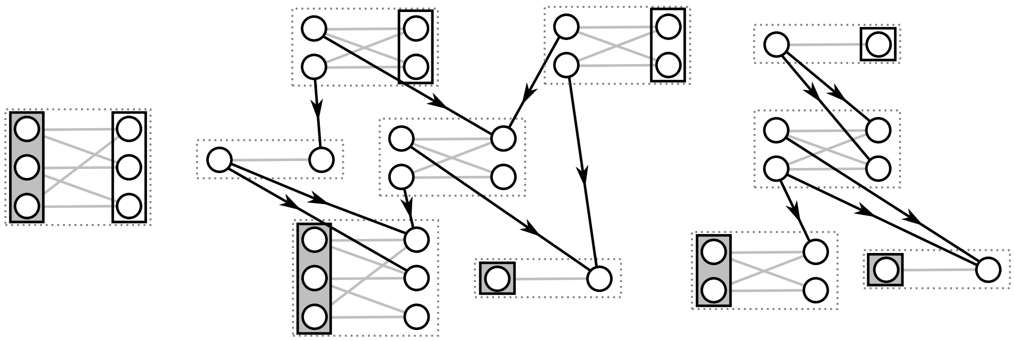

Figure 1: Proper subpartitions $\mathcal{X}^-$ (white squares) of V^- and $\mathcal{X}^+$ (gray squares) of V^+ with $\tau_G(\mathcal{X}^-) = s(G(M))$ and $\tau_G(\mathcal{X}^+) = t(G(M))$ when G has a perfect matching M.

4.1 Algorithm description

Let $(V_0; V_1, V_2, \ldots, V_k; V_\infty)$ be the DM-decomposition of G. If $V_0 = V_\infty = \emptyset$, then G has a perfect matching $M \subseteq E$. In this case, it suffices to find a minimum number of additional edges to make the auxiliary graph $G(M) = G + \overline{M}$ strongly connected, which can be done in linear time by Theorem 2.1. Recall that the strong duality is already seen in Lemma 3.6.

Otherwise, since $|V^+| = |V^-|$, both V_0 and V_∞ are nonempty, and hence G has no perfect matching. A possible strategy is to make G perfectly matchable by adding a perfect matching $N \subseteq (V^+ \setminus \partial^+ M) \times (V^- \setminus \partial^- M) \subseteq (V^+ \times V^-) \setminus E$ between the vertices exposed by some maximum matching $M \subseteq E$ in G. The resulting graph $\tilde{G} := G + N$ has a perfect matching $\tilde{M} := M \cup N$, and hence a minimum number of further additional edges to make $\tilde{G}$ DM-irreducible can be found in linear time. Thus we obtain a feasible solution, which may fail to be optimal.

We adopt a maximum matching $M \subseteq E$ in G whose restrictions to $G[V_0]$ and to $G[V_\infty]$ are both *eligible* perfect matchings defined as follows. This modification enables us to guarantee the optimality of the output with the aid of Lemma 3.2.

Definition 4.1. Let $H = (U^+, U^-; E)$ be a DM-irreducible unbalanced bipartite graph, and $M \subseteq E$ a perfect matching in H. When $|U^+| < |U^-|$, we say that M is *eligible* if there exists a subpartition $\mathcal{X}^-$ of U^- such that $\tau_H(\mathcal{X}^-) = |U^-| - |U^+| + s(H(M))$. Similarly, when $|U^+| > |U^-|$, we say so if there is a subpartition $\mathcal{X}^+$ of U^+ such that $\tau_H(\mathcal{X}^+) = |U^+| - |U^-| + t(H(M))$.

Note that this definition is symmetric, i.e., the eligibility of M when $|U^+| > |U^-|$ is equivalent to the eligibility of $\overline{M}$ in the interchanged bipartite graph $(U^-, U^+; \overline{E})$.

Procedure EPM for finding an eligible perfect matching will be described in Section 5.1. A formal description of the entire algorithm is now given as follows.

Algorithm DMI(G)

Input: A bipartite graph $G = (V^+, V^-; E)$ with $|V^+| = |V^-| = n$.

Output: An edge set $F \subseteq (V^+ \times V^-) \setminus E$ with $|F| = \mathrm{opt}(G)$ such that $G + F$ is DM-irreducible.

Step 0. Compute the DM-decomposition $(V_0; V_1, V_2, \ldots, V_k; V_\infty)$ of G.

Step 1. If $V_0 = V_\infty = \emptyset$, then set $N \leftarrow \emptyset$ and go to Step 4.

Step 2. Otherwise (i.e., if $V_0 \neq \emptyset \neq V_\infty$), find eligible perfect matchings $M_0 \subseteq E \cap (V_0^+ \times V_0^-)$ in $G[V_0]$ and $M_\infty \subseteq E \cap (V_\infty^+ \times V_\infty^-)$ in $G[V_\infty]$ by Procedure EPM.

Step 3. Take an arbitrary perfect matching $N \subseteq (V_0^+ \setminus \partial^+ M_0) \times (V_\infty^- \setminus \partial^- M_\infty)$.

Step 4. Let $\tilde{G} := G + N$, which has a perfect matching $\tilde{M} \subseteq E \cup N$. Find an edge set $\tilde{F} \subseteq (V^+ \times V^-) \setminus (E \cup N)$ with $|\tilde{F}| = \mathrm{opt}(\tilde{G})$ such that $\tilde{G}(\tilde{M}) + \tilde{F}$ is strongly connected, and return $F \leftarrow N \cup \tilde{F}$.

4.2 Optimality

In this section, we show that the output F of Algorithm DMI(G) is an optimal solution to Problem (DMI). Since the feasibility is obvious from Step 4 ($G+F=\tilde{G}+\tilde{F}$ is DM-irreducible), it suffices to confirm $|F| = \mathrm{opt}(G)$. To see this, we construct a proper subpartition $\mathcal{X}$ of V^+ or of V^- such that $\tau_G(\mathcal{X}) = |F|$. Since the weak duality is already shown in Lemma 3.2, this gives the optimality of F as well as a constructive proof of Theorem 1.2. Recall that the case when $V_0 = V_\infty = \emptyset$ is already seen in Section 3.5. In what follows, we discuss the case when $V_0 \neq \emptyset \neq V_\infty$.

In this case, our algorithm finds a maximum matching $M \subseteq E$ in G whose restrictions M_0 to $G[V_0]$ and M_∞ to $G[V_\infty]$ are both eligible perfect matchings in Steps 0 and 2 (cf. Condition 5 in Theorem 2.2 and the computation of the DM-decomposition in Section 2.2), adds to G a perfect matching $N \subseteq (V^+ \setminus \partial^+ M) \times (V^- \setminus \partial^- M)$ between the exposed vertices in Step 3, and finds an optimal solution $\tilde{F} \subseteq (V^+ \times V^-) \setminus (E \cup N)$ to $\tilde{G} = G + N$ in Step 4.

If $n = 1$, then $E = \emptyset$, $N = V^+ \times V^-$, and $\tilde{F} = \emptyset$. Then the output $F = V^+ \times V^-$ is a unique feasible solution, and hence optimal. In what follows, we assume $n \geq 2$. Then, as done in Section 3.5, it suffices to construct two proper subpartitions $\mathcal{X}^-$ of V^- and $\mathcal{X}^+$ of V^+ such that $\max\{\tau_G(\mathcal{X}^-), \tau_G(\mathcal{X}^+)\} = |F| = |N| + |\tilde{F}|$.

Note that $|N| = n - |M| = |V_0^+| - |V_0^-| = |V_\infty^-| - |V_\infty^+|$. The following claim implies $|\tilde{F}| = \max\{s(\tilde{G}(\tilde{M})), t(\tilde{G}(\tilde{M}))\}$ by Theorem 2.1, and hence

$$|F| = \max\{|V_\infty^-| - |V_\infty^+| + s(\tilde{G}(\tilde{M})),\ |V_0^+| - |V_0^-| + t(\tilde{G}(\tilde{M}))\}, \tag{3}$$

where $\tilde{M} := M \cup N$ is a perfect matching in $\tilde{G}$.

Claim 4.2. $\tilde{G}(\tilde{M})$ *is not strongly connected.*

Proof. By Observation 2.3, each exposed vertex $u \in V^+ \setminus \partial^+ M$ forms a source component of $G(M)$ which is reachable only to some vertices in V_0, and each $w \in V^- \setminus \partial^- M$ forms a sink component of $G(M)$ which is reachable only from some vertices in V_∞. Since each edge $uw \in N$ connects such source and sink components one by one, the two end vertices $u \in V^+$ and $w \in V^-$ form a new strongly connected component in $\tilde{G}(\tilde{M}) = G(M) + (N \cup \overline{N})$, which is reachable only to some vertices in V_0 and only from some in V_∞. Recall that $|V^+| = |V^-| = n \geq 2$, and hence $\tilde{G}(\tilde{M})$ has at least two distinct strongly connected components. $\square$

In what follows, we shall construct a subpartition $\mathcal{X}^-$ of V^- such that $\tau_G(\mathcal{X}^-) = |V_\infty^-| - |V_\infty^+| + s(\tilde{G}(\tilde{M}))$. By the symmetry, one can obtain a subpartition $\mathcal{X}^+$ of V^+ such that $\tau_G(\mathcal{X}^+) = |V_0^+| - |V_0^-| + t(\tilde{G}(\tilde{M}))$ in the same way (consider the interchanged bipartite graph $(V^-, V^+; \overline{E})$). By (3), unless $\mathcal{X}^- = \{V^-\}$ or $\mathcal{X}^+ = \{V^+\}$, these two subpartitions are desired ones.

Since no edge enters V_∞ in G as well as in $G(M)$ (see Observation 2.3) and M_∞ is an eligible perfect matching in $G_\infty := G[V_\infty]$, there exists a subpartition $\mathcal{X}_\infty^-$ of V_∞^- such that $\tau_G(\mathcal{X}_\infty^-) = \tau_{G_\infty}(\mathcal{X}_\infty^-) = |V_\infty^-| - |V_\infty^+| + s(G_\infty(M_\infty))$. Define

$$\mathcal{X}_*^- := \{\, X^- \mid G(M)[X] \text{ is a source component of } G(M) \text{ and } X \cap (V_0 \cup V_\infty) = \emptyset \,\},$$

and $\mathcal{X}^- := \mathcal{X}_\infty^- \cup \mathcal{X}_*^-$. When $\mathcal{X}^- \neq \{V^-\}$, the following claim completes the proof.

Claim 4.3. $\tau_G(\mathcal{X}^-) = |V_\infty^-| - |V_\infty^+| + s(\tilde{G}(\tilde{M}))$.

Proof. We first see $\tau_G(\mathcal{X}^-) = |V_\infty^-| - |V_\infty^+| + s(G(M) - V_0)$. Since no edge enters V_∞ in $G(M)$, the source components of $G(M) - V_0$ are partitioned into those of $G(M)[V_\infty] = G_\infty(M_\infty)$ and those of $G(M)$ disjoint from $V_0 \cup V_\infty$. Similarly to Section 3.5, we see $\tau_G(\mathcal{X}_*^-) = |\mathcal{X}_*^-|$, and hence $\tau_G(\mathcal{X}^-) = \tau_G(\mathcal{X}_\infty^-) + \tau_G(\mathcal{X}_*^-) = |V_\infty^-| - |V_\infty^+| + s(G(M) - V_0)$.

Thus it suffices to show $s(\tilde{G}(\tilde{M})) = s(G(M) - V_0)$. Since no edge leaves V_0 in $G(M)$ and each source component of $G(M)[V_0]$ is a single exposed vertex $u \in V_0^+ \setminus \partial^+ M$ with no entering edge, the source components of $G(M)$ are partitioned into those of $G(M) - V_0$ and of $G(M)[V_0]$. Hence, we have $s(G(M) - V_0) = s(G(M)) - s(G(M)[V_0])$. Each exposed vertex $u \in V_0^+ \setminus \partial^+ M$ is connected to some exposed vertex $w \in V_\infty^- \setminus \partial^- M$ by two edges in $N \cup \overline{N}$. As seen in the proof of Claim 4.2, these two vertices u and w form a new strongly connected component in $\tilde{G}(\tilde{M})$, which is no longer a source component unless w is isolated in $G(M)$, i.e, the sink component $G_\infty(M_\infty)[\{w\}]$ is also a source component of $G(M) - V_0$. Hence, whether some exposed vertices $w \in V_\infty^- \setminus \partial^- M$ are isolated or not, by adding $N \cup \overline{N}$ to $G(M)$, the number of source components decreases exactly by $s(G(M)[V_0])$. Thus, $s(\tilde{G}(\tilde{M})) = s(G(M)) - s(G(M)[V_0]) = s(G(M) - V_0)$. $\square$

Finally, we consider the case of $\mathcal{X}^- = \{V^-\}$. Since $\tau(\mathcal{X}_\infty^-) = |V_\infty^-| - |V_\infty^+| + s(G_\infty(M_\infty)) > 0$, we have $\mathcal{X}_\infty^- = \{V^-\}$ and $\mathcal{X}_*^- = \emptyset$. In this case, $V_\infty^- = V^-$ and $V_0^- = \emptyset$. Hence, each vertex $u \in V_0^+$ is isolated in $G(M)$, and is contained in a new sink component of $\tilde{G}(\tilde{M}) = G(M) + (N \cup \overline{N})$ consisting of two vertices. Since the DM-decomposition of G has no balanced component in this case, we have $t(\tilde{G}(\tilde{M})) = |V_0^+| = n - |M| \geq 1$, which leads to

$$|V_0^+| - |V_0^-| + t(\tilde{G}(\tilde{M})) = 2(n - |M|) \geq n - |M| + 1 \geq |V^-| - |\Gamma_G(V^-)| + 1 = \tau_G(\mathcal{X}_\infty^-).$$

Then the maximum in (3) is attained by the latter term, which is equal to $2|V_0^+|$. Thus, for a subpartition $\mathcal{X}^+ := \{\, \{u\} \mid u \in V_0^+ \,\} \neq \{V^+\}$ of V^+, we have $\tau_G(\mathcal{X}^+) = |F|$.

4.3 Running time analysis

In this section, we show that Algorithm DMI(G) runs in $\mathrm{O}(nm)$ time, where recall $m := |E|$.

In Step 0, we find a maximum matching M in G and compute the strongly connected components of the auxiliary graph $G(M)$ (see Section 2.2). The former can be done in $\mathrm{O}(nm)$ time even by a naïve augmenting-path algorithm (see, e.g., [12, Section 16.3]), and the latter in $\mathrm{O}(n + m)$ time with the aid of the depth first search. As shown in Section 5.3, it takes $\mathrm{O}(nm)$ time to find an eligible perfect matching, which is performed twice in Step 2. Step 3 requires $\mathrm{O}(n)$ time, and one can perform Step 4 in $\mathrm{O}(n + m)$ time by Theorem 2.1 (note that a perfect matching $\tilde{M}$ in $\tilde{G}$ is obtained by combining the perfect matching $N \cup M_0 \cup M_\infty$ in $G[V_0 \cup V_\infty]$ with a perfect matching M_* in $G - (V_0 \cup V_\infty)$, which is included in the maximum matching M in G found in Step 0). Thus the entire running time is bounded by $\mathrm{O}(nm)$.

We here discuss the case when the original input bipartite graph $G = (V^+, V^-; E)$ is unbalanced, where we assume that $|V^+| < |V^-| = n$ and $|E| = m$. In this case, we construct a balanced bipartite graph $G' = (V^+ \cup Z^+, V^-; E')$ as in Lemma 3.3, and obtain an optimal solution to G by Algorithm DMI(G'). According to the above analysis, it runs in $\mathrm{O}(n|E'|)$ time, where $|E'|$ is not necessarily bounded by $\mathrm{O}(m)$. We however can bound the running time by $\mathrm{O}(nm)$ as follows.

By Observation 3.4, a maximum matching $M' \subseteq E$ in G' consists of a maximum matching in G and a perfect matching in $G'[Z^+ \cup V^-]$. Hence, we can find a maximum matching in G' in $\mathrm{O}(nm)$ time just by doing so in G and adding an arbitrary perfect matching between the exposed vertices in $G'[Z^+ \cup V^-]$. In addition, since Z^+ is included in a single strongly connected component of $G'(M')$, we can regard Z^+ as a single vertex in computing the strongly connected component of $G'(M')$. This makes it possible to obtain the strongly connected components of $G'(M')$ in $\mathrm{O}(n + m)$ time, which concludes that Step 0 can be done in $\mathrm{O}(nm)$ time.

Since Step 4 is also done in $\mathrm{O}(n + m)$ time by the same argument, it suffices to bound the running time of Step 2 by $\mathrm{O}(nm)$. If $V_\infty = \emptyset$, then we do not reach Step 2. Otherwise, by Observation 3.4, we see $Z^+ \subseteq V_\infty^+$ and $G'[V_0] = G[V_0]$. Hence, one can find an eligible perfect matching in $G'[V_0]$ in $\mathrm{O}(nm)$ time by Procedure EPM. In addition, since no edge enters V_∞ in $G'(M')$ by Observation 2.3, the strongly connected component including Z^+ is a unique source component also in $G'(M')[V_\infty]$, and hence $s(G'(M')[V_\infty]) = 1$. This condition does not depend on the choice of M', which means that all the perfect matchings in $G'[V_\infty]$ is eligible. Hence, we do not need to use Procedure EPM for finding an eligible perfect matching in $G'[V_\infty]$, which concludes that Step 2 can be done in $\mathrm{O}(nm)$ time.

5 Finding Eligible Perfect Matchings

In this section, we show a procedure for finding an eligible perfect matching in a DM-irreducible unbalanced bipartite graph $H = (U^+, U^-; E)$, which plays a key role in Algorithm DMI. Since the definition of eligibility is symmetric (see Definition 4.1), we assume $|U^+| < |U^-|$ in this section. We describe an algorithm for finding an eligible perfect matching in Section 5.1. Sections 5.2 and 5.3 are devoted to its correctness proof and complexity analysis.

5.1 Algorithm description

To describe the procedure, we introduce an *augmented auxiliary graph*.

Definition 5.1. For a perfect matching $M \subseteq E$ in a DM-irreducible bipartite graph $H = (U^+, U^-; E)$ with $|U^+| < |U^-|$, an *augmented auxiliary graph* $\hat{H}(M)$ is constructed from $H(M) = H + \overline{M}$ as follows (see also Fig. 2). Let $S^- \subseteq U^-$ be a vertex set obtained by collecting one vertex in U^- from each source component of $H(M)$, and hence, $|S^-| = s(H(M))$. Add to $H(M)$ a new vertex r and an edge rv for each $v \in S^-$. That is, $\hat{H}(M) = (U \cup \{r\},\ E \cup \overline{M} \cup E_r)$, where $E_r := \{r\} \times S^-$.

Note that, since there may be several possible choices of S^-, an augmented auxiliary graph $\hat{H}(M)$ is not uniquely determined in general.

The procedure for finding an eligible perfect matching is now given as follows.

<u>**Procedure EPM**(H)</u>

Input: A DM-irreducible bipartite graph $H = (U^+, U^-; E)$ with $|U^+| < |U^-|$.

Output: An eligible perfect matching $M \subseteq E$ in H.

Step 0. Take an arbitrary perfect matching $M \subseteq E$ in H, and set $W \leftarrow U^- \setminus \partial^- M$.

Step 1. Construct an augmented auxiliary graph $\hat{H}(M) = (U \cup \{r\},\ E \cup \overline{M} \cup E_r)$, and set $\hat{H} = (\hat{U}, \hat{E}) \leftarrow \hat{H}(M)$.

Step 2. While $W \neq \emptyset$, do the followings.

Step 2.1. Take an exposed vertex $w \in W$, and update $W \leftarrow W \setminus \{w\}$.

Step 2.2. Find two edge-disjoint r–w paths in $\hat{H}$, or certify the nonexistence of such paths.

Step 2.3. If $\hat{H}$ has two edge-disjoint r–w paths, then let P be one of those r–w paths, and update $M \leftarrow (M \cup E(P)) \setminus M(P)$ and $\hat{E} \leftarrow (\hat{E} \cup \overline{E(P)}) \setminus (\overline{M(P)} \cup \{e_1\})$ (see Fig. 3), where we denote by $E(P) \subseteq E$ the set of edges that appear in P, by $M(P) \subseteq M$ the set of edges whose reverse edges appear in P, and by $e_1 \in E_r$ the first edge of P.

Step 3. Return the current perfect matching M.

The following lemma gives an important observation on Procedure EPM, whose proof is left to Section 5.2

Lemma 5.2. *At the beginning of each iteration of Step* 2, $\hat{H} = (\hat{U}, \hat{E})$ *is an augmented auxiliary graph* $\hat{H}(M)$, *which does not have two edge-disjoint* r–w *paths for any* $w \in (U^- \setminus \partial^- M) \setminus W$.

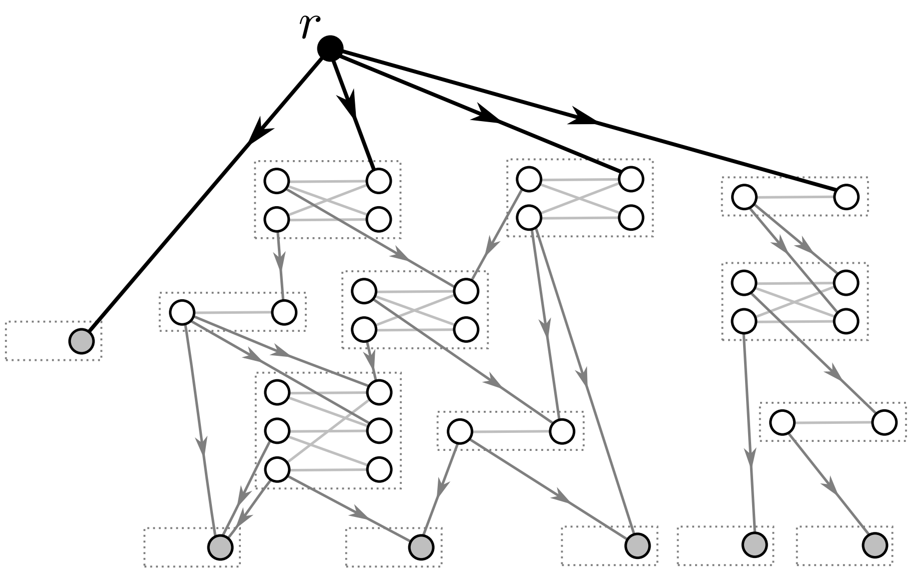

Figure 2: An augmented auxiliary graph $\hat{H}(M)$, where the gray vertices are exposed by M.

5.2 Correctness of EPM

Proof of Lemma 5.2

We first see that $\hat{H}$ is an augmented auxiliary graph with respect to M.

Claim 5.3. *After Step* 1, $\hat{H} = (\hat{U}, \hat{E})$ *is always an augmented auxiliary graph* $\hat{H}(M)$.

Proof. By Step 1, $\hat{H}$ is initialized as $\hat{H}(M)$. We show that, if the current perfect matching M and an augmented auxiliary graph $\hat{H} = \hat{H}(M) = (U \cup \{r\}, E \cup \overline{M} \cup E_r)$ are updated to M' and $\hat{H}'$, respectively, in Step 2.3, then $\hat{H}'$ is an augmented auxiliary graph $\hat{H}(M')$.

Let $v \in U^- \setminus \partial^- M'$ be the new exposed vertex, and then $e_1 = rv \in E_r$. Since $H(M') = H + \overline{M'}$ is obtained from $H(M) = H + \overline{M}$ by adding the edges in $\overline{E(P)}$ and removing those in $\overline{M(P)}$, it suffices to show that the source components of $H(M')$ coincide with those of $H(M)$ except for that containing v.

Let $X \subseteq U$ be the vertex set of a source component of $H(M)$ with $v \notin X$. Then, since no edge enters X in $\hat{H}$ except for one in $E_r \setminus \{e_1\}$, the r–w path P starting e_1 is disjoint from X. Hence, $H(M')[X] = H(M)[X]$ remains a source component in $H(M')$ as it is in $H(M)$.

Suppose to the contrary that $H(M')$ has another source component $H(M')[Y]$. If P is disjoint from Y, then $H(M)[Y] = H(M')[Y]$ is a source component of $H(M)$, and hence $v \in Y$, which however contradicts that P is disjoint from Y. Since $r \notin Y$, the r–w path P must enter Y at least once. If P leaves Y using an edge $e \in E \cup \overline{M}$, then the reverse edge $\bar{e}$ enters Y in $H(M')$, which contradicts that $H(M')[Y]$ is a source component. Thus P enters Y exactly once, and Y must contain the end w of P.

Since $\hat{H}$ has two edge-disjoint r–w paths, Y has an entering edge e in $\hat{H}$ that does not appear in P. If $e \in E \cup \overline{M}$, then e remains in $H(M')$ as an edge entering Y, a contradiction. Otherwise, $e \in E_r \setminus \{e_1\}$. This however contradicts that Y is disjoint from any source component of $H(M)$ that does not contain v. $\square$

When the procedure reaches Step 2 for the first time, we have $W = U^- \setminus \partial^- M$, and hence there is no choice of $w \in (U^- \setminus \partial^- M) \setminus W = \emptyset$. We inductively show that, at the beginning of each iteration of Step 2, $\hat{H}$ does not have two edge-disjoint r–w paths for any $w \in (U^- \setminus \partial^- M) \setminus W$. That is, we prove that, if this property holds at the beginning of some iteration of Step 2, then so does it at the end of the iteration (equivalently, at the beginning of the next iteration).

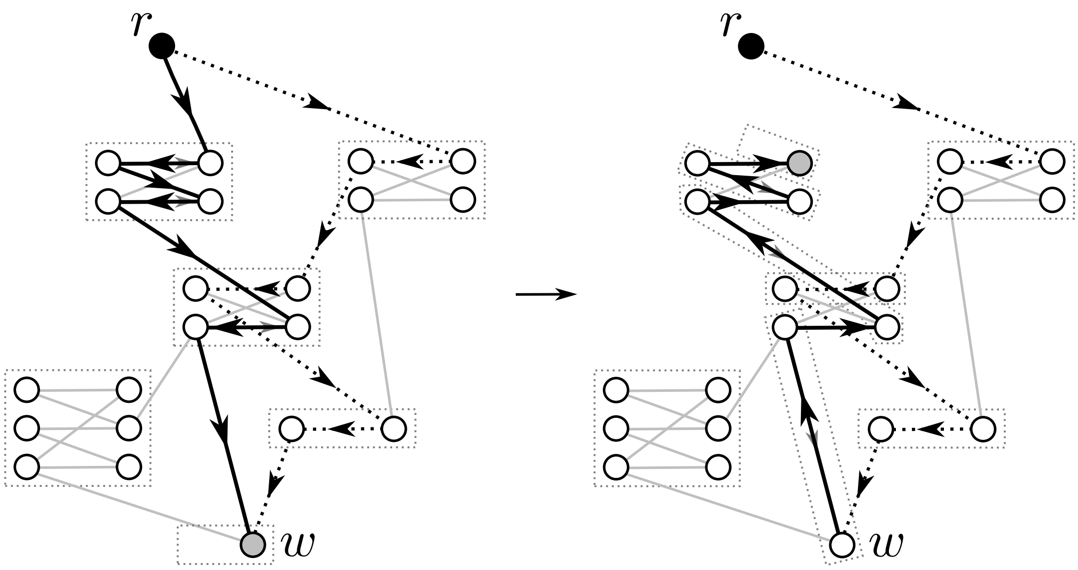

Figure 3: How $\hat{H}$ is updated in Step 2.3 of Procedure EPM along the bold r–w path P.

Let $w^* \in W$ be the exposed vertex chosen in Step 2.1, and $W' := W \setminus \{w^*\}$. If $\hat{H}$ does not have two edge-disjoint r–w^* paths, then M and $\hat{H}$ are not updated. In this case, combining with the induction hypothesis, we see that $\hat{H}$ does not have two edge-disjoint r–w paths for any $w \in ((U^- \setminus \partial^- M) \setminus W) \cup \{w^*\} = (U^- \setminus \partial^- M) \setminus W'$.

Suppose that $\hat{H} = (\hat{U}, \hat{E})$ has two edge-disjoint r–w^* paths, and M and $\hat{H}$ are updated to M' and $\hat{H}'$, respectively, in Step 2.3. Let $v^* \in U^- \setminus \partial^- M'$ be the new exposed vertex, i.e., $e_1 = rv^* \in E_r$. We then see $(U^- \setminus \partial^- M') \setminus W' = ((U^- \setminus \partial^- M) \setminus W) \cup \{v^*\}$, and show that $\hat{H}'$ does not have two edge-disjoint r–w paths, separately for $w = v^*$ and for $w \in (U^- \setminus \partial^- M) \setminus W$.

Claim 5.4. *$\hat{H}'$ does not have two edge-disjoint r–v^* paths.*

Proof. Since v^* is in a source component of $H(M)$ that does not contain w^*, its vertex set $X \subseteq U$ satisfies that $v^* \in X$, $w^* \notin X$, and X has no entering edge in $H(M)$. Hence, the r–w^* path P leaves X exactly once through an edge $e \in E \cup \overline{M}$. If $e \in \overline{M}$, then the reverse edge $\bar{e} \in M \subseteq E$ enters X in $H(M)$, a contradiction. Otherwise, $e \in E$, which implies that X has a unique entering edge $\bar{e} \in \overline{M'}$ in $\hat{H}'$. Then, v^* is not reachable from r in $\hat{H}' - \bar{e}$, and hence $\hat{H}'$ cannot have two edge-disjoint r–v^* paths. $\square$

In what follows, we show that $\hat{H}'$ does not have two edge-disjoint r–w paths for any $w \in (U^- \setminus \partial^- M) \setminus W$. Fix $w \in (U^- \setminus \partial^- M) \setminus W$. Then, by the induction hypothesis and Menger's theorem [10], there exists an edge $e_w \in \hat{E}$ such that w is not reachable from r in $\hat{H} - e_w$. One can choose such an edge so that $e_w \in \hat{E} \setminus E = \overline{M} \cup E_r$ as follows.

Claim 5.5. *Choose an edge $e_w \in \hat{E}$ so that the set Y_w of vertices that are not reachable from r in $\hat{H} - e_w$ contains w and is maximal. Then, $e_w \notin E$.*

Proof. By the definition, only e_w enters Y_w in $\hat{H}$. Suppose to the contrary that $e_w = uv \in E$ for some $u \in U^+ \setminus Y_w^+$ and $v \in Y_w^-$. Since M is a perfect matching in H, there exists an edge $e' = v'u \in \overline{M}$ as well as $uv' \in M$ for some $v' \in U^-$. If $v' \neq v$, then $v' \in U^- \setminus Y_w^-$. Since only e' enters $u \in U^+$ in $\hat{H}$, we can expand Y_w to $Y_w \cup \{u\}$ by rechoosing e_w as e', which contradicts the maximality of Y_w. Otherwise, $e_w = uv \in M$. Since only $e' = \bar{e}_w$ enters $u \in U^+$ in $\hat{H}$, every r–u path in $\hat{H}$ must intersect v, and hence any r–v path Q in $\hat{H}$ cannot traverse e_w. Such a path Q exists (since every vertex is reachable from r in $\hat{H}$ by the definition of an augmented auxiliary graph) and enters Y_w through an edge except for e_w in $\hat{H}$, a contradiction. $\square$

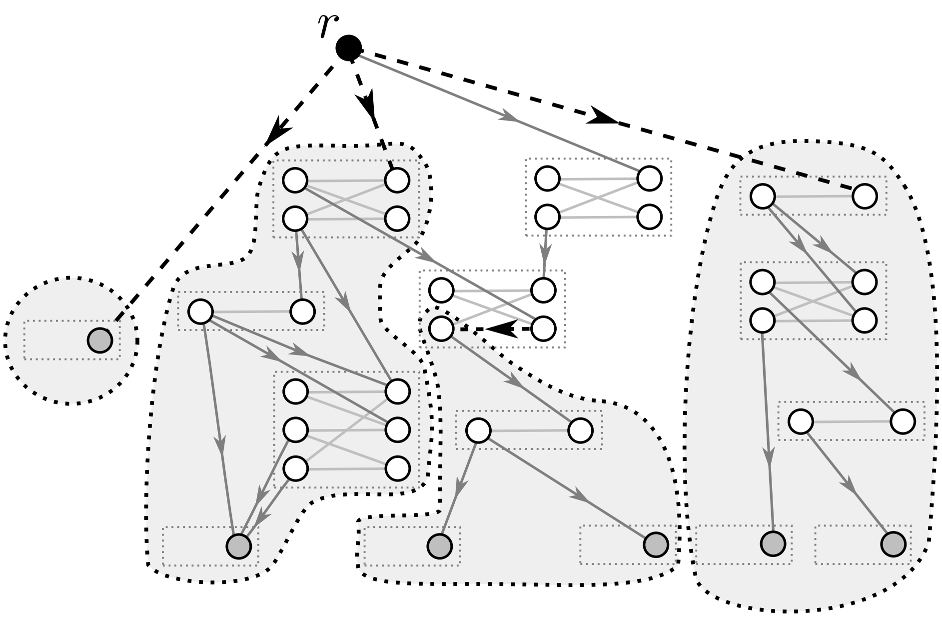

Figure 4: The subpartition of U induced by Y_w ($w \in U^- \setminus \partial^- M$).

If P is disjoint from Y_w, then $w \in Y_w$ is not reachable from r also in $\hat{H}' - e_w$, and hence $\hat{H}'$ cannot have two edge-disjoint r–w paths. Otherwise, P enters Y_w through the edge $e_w \in \overline{M} \cup E_r$, and leaves Y_w at most once through an edge e. Then, e_w is no longer in $\hat{H}'$, and Y_w has at most one new entering edge $\bar{e}$. This also concludes that $\hat{H}'$ cannot have two edge-disjoint r–w paths.

Eligibility of output

We here show that the output of Procedure EPM(H) is indeed an eligible perfect matching. Suppose that EPM(H) returns a perfect matching $M \subseteq E$ in H, and let $\hat{H} = (\hat{U}, \hat{E})$ be the augmented auxiliary graph $\hat{H}(M)$ when EPM(H) halts, where $\hat{U} = U \cup \{r\}$ and $\hat{E} = E \cup \overline{M} \cup E_r$. Then, by Lemma 5.2 and Menger's theorem [10], for any $w \in U^- \setminus \partial^- M$, there exists an edge $e_w \in \hat{E}$ such that w is not reachable from r in $\hat{H} - e_w$. Choose such an edge e_w as in Claim 5.5, i.e., so that the set Y_w of vertices that are not reachable from r in $\hat{H} - e_w$ is maximal. We then see the following property.

Claim 5.6. *For any exposed vertices $w_1, w_2 \in U^- \setminus \partial^- M$, either $Y_{w_1} = Y_{w_2}$ or $Y_{w_1} \cap Y_{w_2} = \emptyset$.*

Proof. Let $w_1, w_2 \in U^- \setminus \partial^- M$ be distinct vertices, and suppose to the contrary that $Y_{w_1} \neq Y_{w_2}$ and $Y_{w_1} \cap Y_{w_2} \neq \emptyset$. We then have $e_{w_1} \neq e_{w_2}$. If $Y_{w_1} \subsetneq Y_{w_2}$ or $Y_{w_2} \subsetneq Y_{w_1}$, then we can expand the included one to the including one by rechoosing e_{w_1} or e_{w_2} as the other one, respectively, which contradicts the maximality of Y_{w_1} and Y_{w_2}. Thus, $Y_{w_1} \setminus Y_{w_2} \neq \emptyset \neq Y_{w_2} \setminus Y_{w_1}$.

Suppose that no edge enters $Y_{w_1} \cap Y_{w_2} \neq \emptyset$ in $\hat{H}$. Then, $\hat{H}[Y_{w_1} \cap Y_{w_2}]$ has some source component of $\hat{H}[U] = H(M)$, which contradicts that E_r contains an edge from $r \notin Y_{w_1} \cap Y_{w_2}$ to each source component of $H(M)$.

Thus, $\hat{H}$ has an edge e entering $Y_{w_1} \cap Y_{w_2}$, which must be e_{w_1} or e_{w_2}. If e enters $Y_{w_1} \cup Y_{w_2}$, then $e_{w_1} = e = e_{w_2}$, a contradiction. Otherwise, assume that $e = e_{w_1}$ leaves $Y_{w_2} \setminus Y_{w_1}$ without loss of generality. In this case, since $r \xrightarrow{\hat{H}} w_2 \in Y_{w_1} \cup Y_{w_2}$, the other edge e_{w_2} must enter $Y_{w_1} \cup Y_{w_2}$. This implies $Y_{w_2} \supseteq Y_{w_1} \cup Y_{w_2}$, which contradicts $Y_{w_1} \setminus Y_{w_2} \neq \emptyset$. $\square$

By Claim 5.6, $\{ Y_w \mid w \in U^- \setminus \partial^- M \}$ is a subpartition of U (see Fig. 4). Let $Y :=$

$\bigcup_{w \in U^- \setminus \partial^- M} Y_w$, and define $\mathcal{X}^- := \mathcal{Y}^- \cup \mathcal{Z}^-$ as follows:

$$\mathcal{Y}^- := \{\, Y_w^- \mid w \in U^- \setminus \partial^- M \,\},$$

$$\mathcal{Z}^- := \{\, Z^- \mid H(M)[Z] \text{ is a source component of } H(M) \text{ and } Z \cap Y = \emptyset \,\},$$

where recall that $X^+ := X \cap U^+$ and $X^- := X \cap U^-$ for $X \subseteq U$. This $\mathcal{X}^-$ is indeed a subpartition of V^-, and we prove that $\mathcal{X}^-$ is a desired one, i.e., $\tau_H(\mathcal{X}^-) = |U^-| - |U^+| + s(H(M))$.

By the definition (1), we see $\tau_H(\mathcal{X}^-) = \tau_H(\mathcal{Y}^-) + \tau_H(\mathcal{Z}^-)$. We first calculate $\tau_H(\mathcal{Y}^-)$ by evaluating $|Y_w^-| - |\Gamma_H(Y_w^-)| + 1$ for each exposed vertex $w \in U^- \setminus \partial^- M$. Fix $w \in U^- \setminus \partial^- M$, and let $T_w^- := Y_w \cap (U^- \setminus \partial^- M)$. Then, by Claim 5.6, we have $Y_{w'} = Y_w$ for every $w' \in T_w^-$, and $\{\, T_w^- \mid w \in U^- \setminus \partial^- M \,\}$ is a partition of $U^- \setminus \partial^- M$. By Claim 5.5, we consider the following two cases separately: when $e_w \in E_r$ and when $e_w \in \overline{M}$.

Claim 5.7. *If $e_w \in E_r$, then $|Y_w^-| - |\Gamma_H(Y_w^-)| + 1 = |T_w^-| + 1$.*

Proof. In this case, no edge enters Y_w in $H(M)$. Hence, each strongly connected component of $H(M)[Y_w]$ is also one of $H(M)$. Since each sink component of $H(M)$ is a single vertex in $U^- \setminus \partial^- M$ (Observation 2.3) and any other strongly connected component of $H(M)$ is balanced, we see $|Y_w^-| = |Y_w^+| + |T_w^-|$. Since only the edge $e_w \in E_r$ enters Y_w in $\hat{H}$ and every vertex in Y_w is reachable in $\hat{H} - e_w$ to some vertex in $T_w^- \subseteq Y_w^-$, we see $\Gamma_H(Y_w^-) = Y_w^+$, and hence $|Y_w^-| - |\Gamma_H(Y_w^-)| + 1 = |T_w^-| + 1$. □

Claim 5.8. *If $e_w \in \overline{M}$, then $|Y_w^-| - |\Gamma_H(Y_w^-)| + 1 = |T_w^-|$.*

Proof. In this case, $e_w = vu \in \overline{M}$ for some $v \in U^- \setminus Y_w^-$ and $u \in Y_w^+$. Since only the edge e_w enters Y_w in $\hat{H}$ and M is a perfect matching in H, any $u' \in Y_w^+ \setminus \{u\} \subseteq U^+$ is matched with some $v' \in Y_w^- \setminus T_w^-$ by M, and vice versa. Hence, $|Y_w^-| = |Y_w^+| - 1 + |T_w^-|$. We observe $\Gamma_H(Y_w^-) = Y_w^+$ in the same way as the previous proof, and hence $|Y_w^-| - |\Gamma_H(Y_w^-)| + 1 = |T_w^-|$. □

Let $\alpha := \bigl|\{\, Y_w \mid w \in U^- \setminus \partial^- M \text{ with } e_w \in E_r \,\}\bigr|$. Then, by Claims 5.7 and 5.8, we see

$$\tau_H(\mathcal{Y}^-) = \sum_{Y_w^- \in \mathcal{Y}^-} |T_w^-| + \alpha = |U^- \setminus \partial^- M| + \alpha = |U^-| - |U^+| + \alpha.$$

Since the corresponding source component $H(M)[Z]$ is balanced for each $Z^- \in \mathcal{Z}^-$ (which is disjoint from $Y \supseteq U^- \setminus \partial^- M$), we see $\tau_H(\mathcal{Z}^-) = |\mathcal{Z}^-|$ (cf. Section 3.5). Hence, the following claim leads to $\alpha + \tau_H(\mathcal{Z}^-) = s(H(M))$, which completes the proof.

Claim 5.9. $\alpha = \bigl|\{\, Z \mid H(M)[Z] \text{ is a source component of } H(M) \text{ and } Z \cap Y \neq \emptyset \,\}\bigr|$.

Proof. We show that, for each $w \in U^- \setminus \partial^- M$, exactly one source component of $H(M)$ intersects Y_w if $e_w \in E_r$, and so does no source component if $e_w \in \overline{M}$. Since any strongly connected component of $H(M)[Y_w]$ is also one of $H(M)$ when $e_w \in E_r$, a unique source component intersecting Y_w is included in $H(M)[Y_w]$, and hence this is sufficient for the claim. Fix $w \in U^- \setminus \partial^- M$.

Suppose that $e_w = rv \in E_r$ for some $v \in S^- \cap Y_w^-$. By the definition of S^-, the vertex v is in a source component of $H(M)$. Suppose to the contrary that there exists another source component of $H(M)$ intersecting Y_w. Then, such a source component must be included in $H(M)[Y_w]$, and hence there exists another edge $rv' \in E_r$ with $v' \in Y_w^-$. This contradicts that only e_w enters Y_w^- in $\hat{H}$.

Suppose that $e_w = vu \in \overline{M}$ for some $v \in U^- \setminus Y_w^-$ and $u \in Y_w^+$, and to the contrary that there exists a source component $H(M)[Z]$ of $H(M)$ with $Z \cap Y_w \neq \emptyset$. Then, by the definition of S^-, there exists a vertex $v' \in S^- \cap Z$ with $e' = rv' \in E_r$. If $v' \in Y_w$, then e' enters Y_w in $\hat{H}$, which contradicts that only $e_w \neq e'$ enters Y_w. Otherwise, since $H(M)[Z]$ is strongly connected, for any vertex $z \in Z \cap Y_w \neq \emptyset$, there exists a v'–z path in $H(M)[Z]$. Such a path must traverse $e_w = vu$ (since only e_w enters Y_w), and hence $\{u, v\} \subseteq Z$. In this case, we can expand Y_w to $Y_w \cup Z \supsetneq Y_w$ by rechoosing e_w as e', which contradicts the maximality of Y_w. □

5.3 Running time analysis

In this section, we see that Procedure EPM(H) runs in $\mathrm{O}(nm)$ time, where $n := |U^-|$ and $m := |E|$ (note that $|U| = \mathrm{O}(n)$ since $|U^+| < |U^-|$). Since the isolated vertices in H can be ignored in the procedure (which are added to W in Step 0 and just discarded in Step 2.1), we may assume $n = \mathrm{O}(m)$.

In Step 0, a perfect matching $M \subseteq E$ in H can be found in $\mathrm{O}(nm)$ time even by a naïve augmenting-path algorithm (in fact, before calling this procedure, one has been obtained in the course of computing the DM-decomposition). In Step 1, since the strongly connected components of the auxiliary graph $H(M)$ are obtained in linear time, an augmented auxiliary graph $\hat{H}(M)$ is constructed in $\mathrm{O}(m)$ time. Since W is monotonically reduced in Step 2.1, the number of iterations of Step 2 is $|W| = \mathrm{O}(n)$. Step 2.2 can be done by performing the breadth first search twice (i.e., by a naïve augmenting-path algorithm originated by Ford and Fulkerson [6]), which requires $\mathrm{O}(m)$ time. The update of M an $\hat{H}$ along an r–w path P in Step 2.3 requires $\mathrm{O}(n)$ time. Thus we conclude that the total computational time is bounded by $\mathrm{O}(nm)$.

6 As Bisupermodular Covering

6.1 Definition and min-max duality

Let V^+ and V^- be finite sets. Two ordered pairs $(X^+, X^-), (Y^+, Y^-) \in 2^{V^+} \times 2^{V^-}$ are said to be *dependent* if both $X^+ \cap Y^+$ and $X^- \cap Y^-$ are nonempty, and *independent* otherwise. A family $\mathcal{F} \subseteq 2^{V^+} \times 2^{V^-}$ is called *crossing* if, for every two dependent members $(X^+, X^-), (Y^+, Y^-) \in \mathcal{F}$, both $(X^+ \cap Y^+, X^- \cup Y^-)$ and $(X^+ \cup Y^+, X^- \cap Y^-)$ are also in $\mathcal{F}$. A function $g\colon \mathcal{F} \to \mathbb{Z}_{\geq 0}$ on a crossing family $\mathcal{F} \subseteq 2^{V^+} \times 2^{V^-}$ is said to be *crossing bisupermodular* if

$$g(X^+ \cap Y^+, X^- \cup Y^-) + g(X^+ \cup Y^+, X^- \cap Y^-) \geq g(X^+, X^-) + g(Y^+, Y^-),$$

for every dependent pairs $(X^+, X^-), (Y^+, Y^-) \in \mathcal{F}$ with $g(X^+, X^-) > 0$ and $g(Y^+, Y^-) > 0$.

We say that a multiset F of directed edges in $V^+ \times V^-$ *covers* a crossing bisupermodular function $g\colon \mathcal{F} \to \mathbb{Z}_{\geq 0}$ if $|F(X^+, X^-)| \geq g(X^+, X^-)$ for every $(X^+, X^-) \in \mathcal{F}$, where $F(X^+, X^-)$ denotes the multiset obtained by restricting F into $X^+ \times X^-$.

Problem (FJ)

Input: A crossing bisupermodular function $g\colon \mathcal{F} \to \mathbb{Z}_{\geq 0}$ on a crossing family $\mathcal{F} \subseteq 2^{V^+} \times 2^{V^-}$.

Goal: Find a minimum-cardinality multiset F of directed edges in $V^+ \times V^-$ such that F covers g.

Frank and Jordán [7] showed a min-max duality on this problem as follows.

Theorem 6.1 (Frank–Jordán [7, Theorem 2.3]). *The minimum cardinality of a multiset F of directed edges in $V^+ \times V^-$ such that F covers a crossing bisupermodular function $g\colon \mathcal{F} \to \mathbb{Z}_{\geq 0}$ is equal to the maximum value of*

$$\sum_{(X^+, X^-) \in \mathcal{S}} g(X^+, X^-), \tag{4}$$

taken over all subfamilies $\mathcal{S} \subseteq \mathcal{F}$ such that every two distinct pairs in $\mathcal{S}$ are independent.

6.2 Formulation of Problem (DMI) as Problem (FJ)

We shall show that Problem (DMI) reduces to Problem (FJ). Let $G = (V^+, V^-; E)$ be a bipartite graph with $|V^+| = |V^-| = n \geq 2$ (recall that the unbalanced case reduces to the balanced case in Section 3.3). Define a family $\mathcal{F} \subseteq 2^{V^+} \times 2^{V^-}$ and a function $g\colon \mathcal{F} \to \mathbb{Z}_{\geq 0}$ by

$$\begin{aligned} \mathcal{F} &:= \{ (X^+, X^-) \mid \emptyset \neq X^+ \subseteq V^+,\ \emptyset \neq X^- \subseteq V^-,\ E(X^+, X^-) = \emptyset \}, \\ g(X^+, X^-) &:= \max\{0, |X^+| + |X^-| - n + 1\}. \end{aligned} \tag{5}$$

Then, $\mathcal{F}$ is crossing because $E(X^+ \cup Y^+, X^- \cap Y^-)$ and $E(X^+ \cap Y^+, X^- \cup Y^-)$ are included in $E(X^+, X^-) \cup E(Y^+, Y^-)$ for every $X^+, Y^+ \subseteq V^+$ and every $X^-, Y^- \subseteq V^-$, and g is crossing bisupermodular because the second part in the maximum is modular.

Claim 6.2. *An edge set $F \subseteq V^+ \times V^-$ covers g if and only if $G + F$ is DM-irreducible.*

Proof. ["Only if" part] Suppose that $F \subseteq V^+ \times V^-$ covers g. By Lemma 3.1, to see the DM-irreducibility of $G + F$, it suffices to show that $|\Gamma_{G+F}(X^+)| \geq |X^+| + 1$ for every nonempty $X^+ \subsetneq V^+$. Fix such X^+, and let $X^- := V^- \setminus \Gamma_{G+F}(X^+) \subseteq V^- \setminus \Gamma_G(X^+)$. If $X^- = \emptyset$, then $\Gamma_{G+F}(X^+) = V^-$, which implies $|\Gamma_{G+F}(X^+)| = |V^-| = |V^+| \geq |X^+| + 1$. Otherwise, $\emptyset \neq X^- \subseteq V^- \setminus \Gamma_G(X^+)$, and hence $(X^+, X^-) \in \mathcal{F}$. Since F covers g and $F(X^+, X^-) = \emptyset$, we have $0 \geq g(X^+, X^-) = \max\{0, |X^+| + |X^-| - n + 1\}$. This means $0 \geq |X^+| + |X^-| - n + 1 = |X^+| - |\Gamma_{G+F}(X^+)| + 1$, and hence $|\Gamma_{G+F}(X^+)| \geq |X^+| + 1$.

["If" part] Suppose that $G + F$ is DM-irreducible for $F \subseteq V^+ \times V^-$. Then, by Lemma 3.1, we have $|\Gamma_{G+F}(X^+)| \geq |X^+| + 1$ for every nonempty $X^+ \subsetneq V^+$. For any $(X^+, X^-) \in \mathcal{F}$, since $\Gamma_G(X^+) \cap X^- = \emptyset$, we have $|F(X^+, X^-)| \geq |\Gamma_{G+F}(X^+) \cap X^-|$. It is easy to observe that $|\Gamma_{G+F}(X^+) \cap X^-| \geq |\Gamma_{G+F}(X^+)| - |V^- \setminus X^-| \geq |X^+| + 1 + |X^-| - n$, which coincides with $g(X^+, X^-)$ when $g(X^+, X^-) > 0$. Thus F covers g. $\square$

Since parallel edges make no effect on the DM-decomposition, which is defined only by the adjacency relation (cf. the definition (2) of f_G), the minimum of $|F|$ for covering a crossing bisupermodular function g defined by (5) is attained by an edge "set" $F \subseteq V^+ \times V^-$. Thus, Problem (DMI) reduces to Problem (FJ). Since the values of g are bounded by $n + 1$, this problem is solved in polynomial time by the pseudopolynomial-time algorithm of Végh and Benczúr [14]. In addition, we can derive Theorem 1.2 from Theorem 6.1 as shown in the next section.

6.3 Proof of Theorem 1.2 via Theorem 6.1

We first show that the maximum value of (1) is at most the maximum value of (4). Fix a proper subpartition $\mathcal{X}$ of V^+, and define $X^- := V^- \setminus \Gamma_G(X^+)$ for each $X^+ \in \mathcal{X}$. Then, $(X^+, X^-) \in \mathcal{F}$ ($X^+ \in \mathcal{X}$) are pairwise independent (since $\mathcal{X}$ is a subpartition of V^+), and $g(X^+, X^-) = \max\{0, |X^+| - |\Gamma_G(X^+)| + 1\}$ by (5). Hence,

$$\tau_G(\mathcal{X}) = \sum_{X^+ \in \mathcal{X}} \bigl(|X^+| - |\Gamma_G(X^+)| + 1\bigr) \leq \sum_{X^+ \in \mathcal{X}} g(X^+, X^-).$$

We can handle the proper subpartitions of V^- in the same way, and hence $\max(1) \leq \max(4)$.

In order to prove the equality, it suffices to show that, for any pairwise-independent subfamily $\mathcal{S} \subseteq \mathcal{F}$, there exists a proper subpartition $\mathcal{Y}$ of V^+ or V^- such that

$$\tau_G(\mathcal{Y}) = \sum_{Y \in \mathcal{Y}} (|Y| - |\Gamma_G(Y)| + 1) \geq \sum_{(X^+, X^-) \in \mathcal{S}} g(X^+, X^-). \tag{6}$$

Since any pair $(X^+, X^-) \in \mathcal{F}$ with $g(X^+, X^-) = 0$ does not contribute to the right-hand side of (6), we assume that $g(X^+, X^-) > 0$ for every $(X^+, X^-) \in \mathcal{S}$ by removing redundant pairs if necessary. We then have $g(X^+, X^-) = |X^+| + |X^-| - n + 1 \le |X^+| - |\Gamma_G(X^+)| + 1$ for every $(X^+, X^-) \in \mathcal{S}$. Let $\mathcal{S}^* := \{\, X^* \mid (X^+, X^-) \in \mathcal{S} \,\}$ for $* = +$ and $-$.

Case 1. When $\mathcal{S}^*$ is a subpartition of V^* for $* = +$ or $-$.

By the symmetry, suppose that $\mathcal{S}^+$ is a subpartition of V^+. If $V^+ \notin \mathcal{S}^+$, then $\mathcal{Y} := \mathcal{S}^+$ is a desired proper subpartition of V^+. Otherwise, we have $\mathcal{S}^+ = \{V^+\}$. If $\mathcal{S}^- \neq \{V^-\}$, then $\Gamma_G(X^-) = \emptyset$ and $g(V^+, X^-) = |X^-| + 1$ for a unique element $X^- \in \mathcal{S}^-$, and hence it suffices to take $\mathcal{Y} := \mathcal{S}^-$. Otherwise, $\mathcal{S} = \{(V^+, V^-)\}$, and hence $E = E(V^+, V^-) = \emptyset$. Note that $g(V^+, V^-) = n + 1$, and recall that we assume $n \ge 2$. In this case, if we take a proper partition $\mathcal{Y} := \{\, \{u\} \mid u \in V^+ \,\}$ of V^+, then (6) is satisfied as follows:

$$\sum_{Y \in \mathcal{Y}} (|Y| - |\Gamma_G(Y)| + 1) = \sum_{u \in V^+} (1 - 0 + 1) = 2n \ge n + 1 = g(V^+, V^-).$$

Case 2. When $\mathcal{S}^*$ is not a subpartition of V^* for $* = +$ and $-$.

Since $X^+ \cap Y^+ = \emptyset$ or $X^- \cap Y^- = \emptyset$ for every distinct pairs $(X^+, X^-), (Y^+, Y^-) \in \mathcal{S}$, we have $|\mathcal{S}| \ge 3$. We shall show by induction on $|\mathcal{S}|$ that this case reduces to Case 1 by an *uncrossing* procedure.

We first observe that $V^+ \notin \mathcal{S}^+$ or $V^- \notin \mathcal{S}^-$. Suppose to the contrary that $V^+ \in \mathcal{S}^+$ and $V^- \in \mathcal{S}^-$. We then have $(V^+, X^-), (Y^+, V^-) \in \mathcal{S}$ for some $X^- \subseteq V^-$ and $Y^+ \subseteq V^+$. If $X^- = V^-$ or $Y^+ = V^+$, then $(V^+, V^-) \in \mathcal{S}$ cannot be independent from any other pair in $\mathcal{S} \subseteq \mathcal{F}$, which contradicts $|\mathcal{S}| \ge 3$. Otherwise (i.e., if $X^- \neq V^-$ and $Y^+ \neq V^+$), since $X^- \neq \emptyset \neq Y^+$ by the definition of $\mathcal{F}$, the two pairs $(V^+, X^-), (Y^+, V^-) \in \mathcal{F}$ cannot be independent, a contradiction. By the symmetry, we assume that $V^+ \notin \mathcal{S}^+$.

The following claim shows a successful uncrossing procedure.

Claim 6.3. *If distinct $X^+, Y^+ \in \mathcal{S}^+$ satisfy $X^+ \cap Y^+ \neq \emptyset$ and $X^+ \cup Y^+ \neq V^+$, then one can reduce $|\mathcal{S}|$ by replacing (X^+, X^-) and (Y^+, Y^-) with $(X^+ \cap Y^+, X^- \cup Y^-)$ without reducing the right-hand side of* (6).

Proof. We first see that $(X^+ \cap Y^+, X^- \cup Y^-) \in \mathcal{F}$. This follows from $X^+ \cap Y^+ \neq \emptyset$ and $E(X^+ \cap Y^+, X^- \cup Y^-) \subseteq E(X^+, X^-) \cup E(Y^+, Y^-) = \emptyset$.

Next, we confirm that $(X^+ \cap Y^+, X^- \cup Y^-)$ and each $(Z^+, Z^-) \in \mathcal{S} \setminus \{(X^+, X^-), (Y^+, Y^-)\}$ are independent. Since (Z^+, Z^-) is independent from both (X^+, X^-) and (Y^+, Y^-), at least one of $X^+ \cap Z^+$, $Y^+ \cap Z^+$, and $(X^- \cup Y^-) \cap Z^-$ is empty. This implies that $(X^+ \cap Y^+) \cap Z^+ = \emptyset$ or $(X^- \cup Y^-) \cap Z^- = \emptyset$.

Finally, we show that the right-hand side of (6) does not decrease by this replacement. Recall that $X^- \cap Y^- = \emptyset$ (since $X^+ \cap Y^+ \neq \emptyset$), both $g(X^+, X^-)$ and $g(Y^+, Y^-)$ are positive, and $X^+ \cup Y^+ \subsetneq V^+$. Thus we have the following inequalities, which complete the proof:

$$\begin{aligned}
&g(X^+ \cap Y^+, X^- \cup Y^-) \\
&\quad \ge |X^+ \cap Y^+| + |X^- \cup Y^-| - n + 1 \\
&\quad = \big(|X^+| + |Y^+| - |X^+ \cup Y^+|\big) + \big(|X^-| + |Y^-|\big) - n + 1 \\
&\quad = \big(|X^+| + |X^-| - n + 1\big) + \big(|Y^+| + |Y^-| - n + 1\big) + \big(n - 1 - |X^+ \cup Y^+|\big) \\
&\quad \ge g(X^+, X^-) + g(Y^+, Y^-).
\end{aligned}$$

□

There must be a pair that can be uncrossed by Claim 6.3 as follows, which completes the proof.

Claim 6.4. *There exist distinct $X^+, Y^+ \in \mathcal{S}^+$ such that $X^+ \cap Y^+ \neq \emptyset$ and $X^+ \cup Y^+ \neq V^+$.*

Proof. Suppose to the contrary that, for every distinct $X^+, Y^+ \in \mathcal{S}^+$, we have $X^+ \cap Y^+ = \emptyset$ or $X^+ \cup Y^+ = V^+$. Take distinct elements $X^+, Y^+ \in \mathcal{S}^+$ with $X^+ \cap Y^+ \neq \emptyset$, and distinct pairs $(Z_1^+, Z_1^-), (Z_2^+, Z_2^-) \in \mathcal{S}$ with $Z_1^- \cap Z_2^- \neq \emptyset$ (recall the case assumption that $\mathcal{S}^*$ is not a subpartition of V^* for $* = +$ and $-$). Then, $X^+ \cup Y^+ = V^+$ and $Z_1^+ \cap Z_2^+ = \emptyset$. We show that, for each $i \in \{1, 2\}$, exactly one of the following statements holds:

(a) $Z_i^+ = X^+$;

(b) $Z_i^+ = Y^+$;

(c) $Z_i^+ \supseteq X^+ \triangle Y^+ := (X^+ \setminus Y^+) \cup (Y^+ \setminus X^+)$.

Since $X^+ \setminus Y^+ \neq \emptyset \neq Y^+ \setminus X^+$ (otherwise, $X^+ = V^+$ or $Y^+ = V^+$, which contradicts that $V^+ \notin \mathcal{S}^+$), every possible pair of (a)–(c) leads to $Z_1^+ \cap Z_2^+ \neq \emptyset$, a contradiction.

Suppose that $Z_i \neq X^+$ and $Z_i \neq Y^+$, and we derive Condition (c). Since $X^+ \cup Y^+ = V^+$, we assume $Z_i^+ \cap X^+ \neq \emptyset$ without loss of generality. This implies $Z_i^+ \cup X^+ = V^+$, and hence $Z_i^+ \supseteq V^+ \setminus X^+ = Y^+ \setminus X^+$. Since $Y^+ \setminus X^+ \neq \emptyset$, we also have $Z_i^+ \cap Y^+ \neq \emptyset$. We then similarly see $Z_i^+ \supseteq X^+ \setminus Y^+$, which concludes that $Z_i^+ \supseteq X^+ \triangle Y^+$. □

7 Applications

We show possible applications of Problem (DMI) in game theory and in control theory (see [1] and [11, Section 6.4], respectively, for the details).

7.1 Bargaining in a two-sided market

Consider bargaining in a two-sided market with the seller set S and the buyer set B in which the tradable pairs are exogenously given as a bipartite graph $G = (S, B; E)$, where each edge in E represents a tradable pair. Each seller has an indivisible good and each buyer has money. The bargaining process is repeated as described in the next paragraph, and the utility received from a successful trade is defined as follows: for a prescribed constant $\delta \in (0, 1)$, if the trade is done at price p at period $t \in \{0, 1, 2, \ldots\}$, then the seller receives $\delta^t p$ and the buyer does $\delta^t(1 - p)$. Note that all the sellers share one utility function, and so do all the buyers.

The bargaining process is as follows (see [1, Section 2.2] for the precise formulation). All the sellers and all the buyers alternately offer prices in $[0, 1]$ for trade as the *proposers*. Each agent in the other side accepts exactly one offered price or rejects all of them as a *responder*, where the responders do not care with which specific proposer they trade. For each price p accepted by some responder, restrict ourselves to the subgraph induced by the agents offering or accepting the price p, and trade is done at price p according to a maximum matching in the subgraph. Note that there may be several possible choices of maximum matchings. If there are multiple possibilities, then one is chosen so that the set of matched agents is lexicographically minimum in terms of the agent indices given in advance. Note also that we are not concerned with which specific edges are used in the maximum matching, because the utility of each agent depends only on the price p and the period t. Remove all the agents who have traded from the graph, and repeat the above process for the remaining graph until it has no edge.

A *subgame perfect equilibrium* in such a repeated game is, roughly speaking, a strategy profile (i.e., in the above bargaining game, the offering prices and the responses to offered prices of all the agents at all the possible situations) in which every agent has no incentive to change his or her action at any possible situation. Corominas-Bosch [1] investigated the utility profile in each subgame perfect equilibrium in the above game, which is denoted by PEP for short (standing for a subgame Perfect Equilibrium Payoff). She captured a typical utility profile extending unique PEPs in several small markets, called it the *reference solution*, and characterized when the reference solution is indeed a PEP and moreover when it is a unique PEP.

Theorem 7.1 (Corominas-Bosch [1, Theorem 1]). *Consider the above bargaining game on a bipartite graph $G = (S, B; E)$.*

- *When G is unbalanced, the reference solution is a PEP if and only if G is DM-irreducible.*
- *When G is balanced, the reference solution is a PEP if and only if G is perfectly matchable.*

Theorem 7.2 (Corominas-Bosch [1, Proposition 6]). *Consider the above bargaining game on a bipartite graph $G = (S, B; E)$, and suppose that the game starts with the sellers' proposes. Then, the restriction of any PEP to G_0 is the reference solution to G_0, where $G_0 = (S_0, B_0; E_0)$ denotes the DM-irreducible component of G with $|S_0| > |B_0|$. In particular, if $|S| > |B|$ and G is DM-irreducible, then there exists a unique PEP, which is the reference solution.*

Based on the above characterizations, for the unbalanced case, our result gives a minimum number of additional tradable pairs to make such a bargaining game admit a unique PEP, which is the reference solution. On the other hand, for the balanced case, the uniqueness of a PEP is just guaranteed for the complete bipartite graphs [1, Proposition 5]. She also gave an example enjoying multiple PEPs, in which the bipartite graph is not DM-irreducible. What role the DM-decomposition of perfectly-matchable balanced bipartite graphs plays in such bargaining has been left as an interesting question.

7.2 Structural controllability of a linear system

Consider a linear time-invariant system (K, A, B) in a descriptor form

$$K\dot{x} = Ax + Bu$$

with state variable x and input variable u. Under the genericity assumption that the set of nonzero entries in K, A, and B are algebraically independent over $\mathbb{Q}$, the system (K, A, B) is said to be *structurally controllable* if the matrix pencil $A - sK$ is regular (i.e., $\det(A - sK) \neq 0$) and $[A - zK \mid B]$ is of row-full rank for every $z \in \mathbb{C}$.

For a matrix pencil $D(s)$, let $G(D(s))$ denote the associated bipartite graph. The both-side vertex sets are the row set and the column set of $D(s)$, respectively, and the edges correspond to the nonzero entries of $D(s)$.

Theorem 7.3 (Murota [11, Corollary 6.4.8]). *Let (K, A, B) be a linear time-invariant system in a descriptor form with nonsingular K. Under the genericity assumption, (K, A, B) is structurally controllable if and only if the following two conditions hold.*

- *The bipartite graph $G([A \mid B])$ has a perfect matching.*
- *The bipartite graph $G([A - sK \mid B])$ is DM-irreducible.*

This characterization enables us to check efficiently if a given linear system is structurally controllable. If it turns out not to be, then a natural question is how to modify the system to make it structurally controllable. If $G([A \mid B])$ admits a perfect matching, our result provides an answer to this question by identifying the minimum number of additional connections between the variables and the equations required to make the entire system structurally controllable.

It would be more desirable if one can extend this approach to the case in which $G[A \mid B]$ may not have a perfect matching. It is also interesting to deal with the case of singular K. These problems are left for future investigation.

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