**MATHEMATICAL ENGINEERING
TECHNICAL REPORTS**

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METR 2006–01 January 2006

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C^k-continuity of Stationary Subdivision Schemes

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Abstract

This paper derives a new necessary and sufficient condition for the smoothness of stationary subdivision surface schemes.

First, we give a new necessary and sufficient condition for C^1-continuity of stationary subdivision. Tangent plane continuity is equivalent to the convergence of difference vectors. Thus, using "normal subdivision matrix" [1], we derives a necessary and sufficient condition of tangent plane continuity for stationary subdivision at extraordinary points (including degree 6). Moreover, we derive a necessary and sufficient condition for C^1-continuity.

Second, we introduce new concepts such as the "j+1"-st difference vectors and the "j-th difference mesh", and construct procedure for generating j-th difference mesh from $j-1$-st difference mesh, and "$j+1$-st difference vectors" from j-th difference vectors. Here, we assume C^j-continuity at extraordinary points (including degree 6). Moreover, we define T^{j+1}-continuity for the subdivision surface as tangent plane continuity of the limit surface generated by j-th difference mesh. Then, we can derive a necessary and sufficient condition of T^{j+1}-continuity using "$j+1$-st normal subdivision matrix". Next, we assume T^k-continuity at extraordinary points (including degree 6). Then, we can derive a necessary and sufficient condition for C^k-continuity.

In this paper, we assume affine invariance. However, we can remove this assumption. So, the necessary and sufficient condition of C^k-continuity at extraordinary points (including degree 6) given in this paper is valid in a very general framework.

Moreover, the necessary and sufficient condition is valid for subdivision surfaces except extraordinary points (including 6). That is, we derive the necessary and sufficient condition of C^k-continuity for stationary subdivision schemes.

Key words: Subdivision surface, C^k-continuity, j-th difference vector, $j-1$-st difference mesh, j-th normal subdivision matrix

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1 Introduction

Subdivision [2–6] is a well-known method for geometric design and for computer graphics, because the subdivision makes smooth surfaces with arbitrary topology. A subdivision scheme is defined by subdivision matrices and a rule of change of connectivity. Many researchers study the condition of continuity of subdivision surfaces depending on subdivision matrices [5,7–14]. Moreover, multiresolution analysis [15–20] derived by subdivision theory is extremely useful on mesh editing. Multiresolution analysis is a wevelet transform on semi-regular meshes. In the framework, we want basis functions of subdivision and corresponding wavelet basis functions to be smooth. The wavelet basis functions are linear combination of basis functions of subdivision. As above, we want to derive necessary and sufficient conditions of C^k-continuity for subdivision schemes.

For smoothness of stationary subdivision schemes at extraordinary points, Reif [8] derived a sufficient condition for C^1-continuity. Moreover, Prautzch [9] derived some conditions for C^k-continuity. However, they are not necessary and sufficient condition.

Zorin [7] derived a necessary and sufficient condition for C^k-continuity with some assumptions. However, his condition is not intuitive, because the condition is described in terms of subdivision matrix S_k and eigen basis functions and parametric map.

In this paper, on the other hand, we derive a necessary and sufficient condition for C^k-continuity with affine invariance. Our condition is described in terms of "j-th normal subdivision matrix" instead of subdivision matrix S_k. Here, j-th normal subdivision matrix subdivides j-th difference vectors. And j-th difference vectors converge to j-th derivatives at the extraordinary point. So, the necessary and sufficient condition for j-th normal subdivision matrix is intuitive. And we use only linear algebra for our analysis. So, this analysis can be understood easily.

Moreover, we can remove our assumption: affine invariance. Thus, we can derive a necessary and sufficient condition of C^k-continuity for stationary subdivision with no assumption.

In this paper, we assume subset of vertices of meshes are not dense. If there is a dense set of vertices and the dense set is not C^1-continuous, then any subdivision scheme can not generate C^1-continuous surface. Moreover, we assume that control points span at least plane. If all control points are on a line, by linearity of subdivision, then any subdivision scheme can not generate surface.

2 Ordinary Subdivision

In this section, we review ordinary subdivisions in general.

2.1 Subdivision Matrix

A subdivision scheme is defined by subdivision matrices and a rule of connectivity change. The subdivision scheme, when it is applied to 2-manifold irregular meshes, generates smooth surfaces at the limit. Fig. 1 is an example of the Loop subdivision. In this figure, (a) is an original mesh; subdividing (a), we get (b); subdividing (b) once more, we get (c); subdividing infinite times, we get the smooth surface (d). We call (d) the subdivision surface. Here, a face is divided into four new faces. This is a change of connectivity. In this paper, the change of connectivity is fixed to this type, but other types of connectivity change can be discussed similarly.

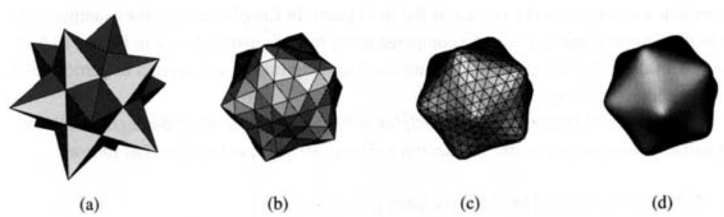

Fig. 1. Loop subdivision [17].

Next, let us consider how to change the positions of the old vertices, and how to decide the positions of the new vertices. They are specified by matrices called "subdivision matrices". The subdivision matrices are defined at vertices and they depend on degree k of the vertex (the degree is the number of edges connected to the vertex). For example, Fig. 2 denotes a vertex v_0^j which has five edges. Let $v_1^j, v_2^j, \cdots, v_5^j$ be the vertices at the other terminals of the five edges. Then, subdivision matrix S_5 is defined as follows:

$$\begin{pmatrix} v_0^{j+1} \\ v_1^{j+1} \\ \vdots \\ v_5^{j+1} \end{pmatrix} = S_5^j \begin{pmatrix} v_0^{j} \\ v_1^{j} \\ \vdots \\ v_5^{j} \end{pmatrix},$$

where v_0^{j+1} is the new locations of the vertex v_0^j after the $j+1$-st subdivision, while $v_1^{j+1}, \cdots, v_5^{j+1}$ are the newly generated vertices.

Here, the subdivision matrix S_5^j is a square matrix. The superscript j means the j-th step of the subdivision. Here, neighbor vertices of a vertex v are called vertices on

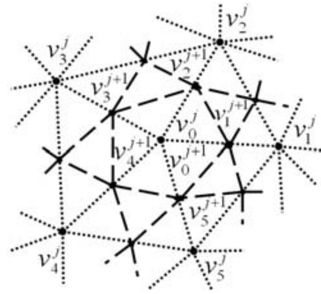

Fig. 2. subdivision matrix.

the 1-disc of v. The subdivision matrix is generally defined not only on vertices in the 1-disc, but also on other vertices around, $\cdots$. Here, we discuss only subdivision matrices that depend on vertices in the 1-disc. However, we can discuss other subdivision matrices, similarly. In this paper, we assume that the subdivision matrix is independent of j. A subdivision scheme of this type is called "stationary".

In this way, the subdivision matrix is written for a vertex. However, since a newly generated vertex is computed by two subdivision matrices at the ends of the edge, the two subdivision matrices must generate the same location of the vertex. So, the subdivision matrices have this kind of restriction.

For example, subdivision matrices S_k $(k \geq 3)$ for the Loop subdivision are

$$S_k = \begin{pmatrix} 1-k\beta & \beta & \beta & \beta & \beta & \beta & \cdots & \beta \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & \cdots & 0 & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & \cdots & 0 \\ \frac{3}{8} & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & & \\ \frac{3}{8} & \frac{1}{8} & 0 & 0 & \cdots & 0 & \frac{1}{8} & \frac{3}{8} \end{pmatrix},$$

where k is the degree of the associated vertex, and

$$\beta = \frac{1}{k}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\left(\frac{2\pi}{k}\right)\right)^2\right).$$

The degree k of a vertex is at least two. A vertex whose degree is two is a boundary vertex. The degree of a vertex of 2-manifold meshes is at least three. In this paper, we do not discuss boundaries of meshes. So, we assume that the degree is at least three.

As seen above, a stationary subdivision scheme is defined by subdivision matrices

S_k $(k \geq 3)$. Then, by the theorem 2.1 in [10], the limit surface of subdivision $f : |K| \to \mathbf{R}^3$ is the following parametric surface:

$$f[p](y) = \sum_i v_i \phi_i(y),$$
$$v_i \in \mathbf{R}^3,\ \phi_i(y) \in \mathbf{R},\ y \in |K|, p = (v_0, v_1, \cdots),$$

where K is a complex, $|K|$ is a topological space, that is, the mesh, i is an index of a vertex, v_i is the position of the i-th vertex, $p = (v_0, v_1, \cdots)$, and $\phi_i(y)$ is the weight function with the i-th vertex. Moreover, the weight function $\phi_i(y)$ is dependent on the subdivision matrices. If the sum of each row of the subdivision matrix is 1, vertices at each stage of the subdivision is affine combinations of the original vertices. Therefore,

$$\forall y \in |K|,\ \sum_i \phi_i(y) = 1.$$

So, weight functions make affine combinations, too. If the combination is not affine, it is not invariant under the translation of the coordinates systems, and hence we usually consider only affine combinations. Therefore, in what follows we assume that the sum of elements in each row of the subdivision matrix is equal to 1.

Here, we denote $\phi(y) = (\phi_0(y), \phi_1(y), \cdots)$. Then, $\phi(y)$ decides a set of representable surfaces. Then, the set is spanned by $\phi(y)$. So, we call the weight functions basis functions. The limit surface of the subdivision is a point in such a functional space.

2.2 C^0-continuity

First, we discuss convergence for stationary subdivision at extraordinary points. This problem was solved already.

Now, the subdivision scheme is written as:

$$p^{j+1} = S_k p^j,$$

where $p^j = (v_0^j, v_1^j, \cdots, v_k^j)^\top$. So, in order for the limit position $\lim_{j\to\infty} v_0^j$ to exist for arbitrary p^0, there must be $\lim_{j\to\infty} S_k^j$. Therefore, we could change the problem of C^0-continuity to the problem of convergence of $\lim_{j\to\infty} S_k^j$.

So,

$$p^\infty = S_k^\infty p^0.$$

In order for the limit surface of subdivision to be C^0-continuous at the extraordinary point,

$$p^\infty = \begin{pmatrix} v_0^\infty \\ v_0^\infty \\ \vdots \end{pmatrix}.$$

Here, we can easily solve this problem. Now, clearly, S_k has an eigen value $\lambda_1 = 1$ with right eigen vector $(1, \cdots, 1)^\top$ from affine invariance. Let $S_k = V_0^{-1} H V_0$, where H is the Jordan normal form. λ_1 has a single cyclic subspace of size 1, because v_0^∞ must be bounded and rank $S_k^\infty = 1$. Thus,

$$\begin{aligned} p^\infty &= S_k^\infty p^0 \\ &= \left(\begin{array}{c|c} 1 & \\ \vdots & * \\ 1 & \end{array}\right) \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array}\right) V_0 p^0 = \left(\begin{array}{c|c} 1 & \\ \vdots & 0 \\ 1 & \end{array}\right) V_0 p^0. \end{aligned}$$

Here, $|\lambda_i| < \lambda_1,\ i = 2, 3, \cdots$, because rank of p^∞ must be 1. So, the condition is that S_k has the first eigen value $\lambda_1 = 1$ with a single cyclic subspace of size 1 and $|\lambda_i| < \lambda_1,\ i = 2, 3, \cdots$.

Then, on edges and faces, clearly, subdivision surfaces are C^0-continuous. So, this condition is valid for subdivision surfaces.

3 C^1-continuity

In this section, we derive a sufficient and necessary condition of C^1-continuity for stationary subdivision with affine invariance. Here, we assume C^0-continuity at v_0^∞.

3.1 Normal subdivision matrix

At the $j + 1$-st step of the subdivision, new vertices are computed by subdivision matrix S_k as

$$
\begin{pmatrix} v_0^{j+1} \\ v_1^{j+1} \\ \vdots \\ v_k^{j+1} \end{pmatrix} = S_k \begin{pmatrix} v_0^{j} \\ v_1^{j} \\ \vdots \\ v_k^{j} \end{pmatrix}.
$$

Then, we define a matrix Δ as

$$
\Delta = \begin{pmatrix} 1 & 0 \cdots & & & \\ -1 & 1 & 0 & \cdots & \\ -1 & 0 & 1 & 0 & \cdots \\ \vdots & & & \ddots & \end{pmatrix}.
$$

Using a matrix $D'_k = \Delta S_k \Delta^{-1}$, we get

$$
\begin{pmatrix} v_0^{j+1} \\ v_1^{j+1} - v_0^{j+1} \\ \vdots \\ v_k^{j+1} - v_0^{j+1} \end{pmatrix} = D'_k \begin{pmatrix} v_0^{j} \\ v_1^{j} - v_0^{j} \\ \vdots \\ v_k^{j} - v_0^{j} \end{pmatrix}.
$$

Note that the sum of each row of S_k is 1, because the subdivision scheme is affine invariant. Therefore, the first element v_0^j does not affect elements $v_1^{j+1} - v_0^{j+1}, v_2^{j+1} - v_0^{j+1}, \cdots, v_k^{j+1} - v_0^{j+1}$. So, we denote the vector consisting of the elements $v_1^j - v_0^j, v_2^j - v_0^j, \cdots, v_k^j - v_0^j$ as d^j (see Fig. 3) and the associated submatrix of D'_k as D_k:

$$
D'_k = \left(\begin{array}{c|c} a & * \\ \hline 0 & \\ \vdots & D_k \\ 0 & \end{array}\right).
$$

Then, $d^{j+1} = D_k d^j$. We call this the difference scheme.

Moreover, we denote the column vector which is a set of x elements of d^j as d_x^j. Similarly, we denote the column vectors corresponding to y elements, z elements as d_y^j, d_z^j.

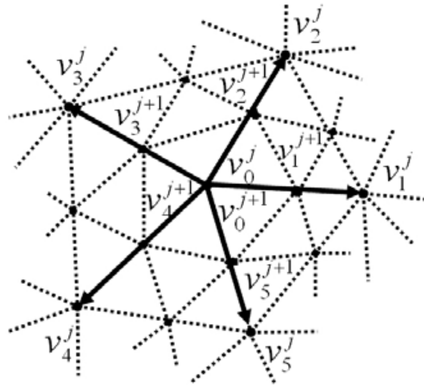

Fig. 3. Difference vectors. A row of d^j is a difference vector $v_i^j - v_0^j$. So, difference vectors converge to first derivatives at v_0^∞.

Here, using $u_1, u_2 \in \mathbf{R}^k$, we define a matrix ΛD_k as

$$\Lambda D_k(u_1 \wedge u_2) = D_k u_1 \wedge D_k u_2,$$

where $\wedge$ is the wedge product. Then,

$$\Lambda D_k(d_y^j \wedge d_z^j) = D_k d_y^j \wedge D_k d_z^j.$$

Now, we define $N^j = (d_y^j \wedge d_z^j, d_z^j \wedge d_x^j, d_x^j \wedge d_y^j)$. Then,

$$N^{j+1} = \Lambda D_k \; N^j,$$

where a row of N^j is a cross product of between $v_i^j - v_0^j$ and $v_l^j - v_0^j$, that is, normal on the neighborhood of v_0^j. So, we see that the matrix ΛD_k subdivides normals of faces which connects a vertex whose degree is k.

Note that N^j contains normals of unreal faces in the mesh (Real face can be written as $(v_i^j - v_0^j) \times (v_{i+1}^j - v_0^j)$ or $(v_k^j - v_0^j) \times (v_1^j - v_0^j)$. Otherwise, the row of N^j is unreal face.).

Therefore, we call this matrix the “normal subdivision matrix”.

3.2 Tangent plane continuity

Now, we easily see that difference vectors converge to first derivatives at v_0^∞. So, the limit surface of subdivision is tangent plane continuous at v_0^∞ if and only if all rows of N^∞ (these are normals) point to the same direction (This is direction of normal at v_0^∞. Here, normal n and $-n$ are called same direction).

Now, let $\Lambda D_k = V_1^{-1} A V_1$, where A is the Jordan normal form, V_1 is a regular matrix.

Therefore, in order to be tangent plane continuous,
$\lim_{j\to\infty}(\Lambda D_k)^j = V_1^{-1}\lim_{j\to\infty}(A)^j V_1$ must be rank 1.

Now, we consider maximal element of A^∞. Let $\Lambda_i, i = 1, 2, \cdots$ be eigen values of A, where $|\Lambda_i| \geq |\Lambda_{i+1}|$. Here, a Jordan cell of Λ_i can be written as:

$$\begin{pmatrix} \Lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \Lambda_i & 1 \\ & & & \Lambda_i \end{pmatrix}.$$

Let l be the size of this Jordan cell. Then,

$$\begin{pmatrix} \Lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \Lambda_i & 1 \\ & & & \Lambda_i \end{pmatrix}^n = \begin{pmatrix} \Lambda_i^n & {}_n\mathrm{C}_1\Lambda_i^{n-1} & \cdots & {}_n\mathrm{C}_{l-1}\Lambda_i^{n-l+1} \\ & \ddots & \ddots & \vdots \\ & & \Lambda_i^n & {}_n\mathrm{C}_1\Lambda_i^{n-1} \\ & & & \Lambda_i^n \end{pmatrix}.$$

Thus, the maximal element (on absolute value) of this Jordan cell is ${}_n\mathrm{C}_{l-1}\Lambda_i^{n-l+1}$, where n is a large positive integer.

Here, we can consider the maximal element of A^n.

3.2.1 The case of $|\Lambda_1| > |\Lambda_2|$

In this case,

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} \Lambda_1^n \left(\begin{array}{c|c} \frac{1}{\lim_{n\to\infty}\Lambda_1^n}\lim_{n\to\infty} *^n & 0 \\ \hline 0 & 0 \end{array} \right),$$

where $*$ is the Jordan Block of Λ_1. Let l_m be the maximal size of all Jordan cells of $*$. Then, the maximal element of $*^n$ is ${}_n\mathrm{C}_{l_m-1}\Lambda_1^{n-l_m+1}$, where n is a large positive integer. Other elements of $*^n$ are sufficiently small for the maximal element ${}_n\mathrm{C}_{l_m-1}\Lambda_1^{n-l_m+1}$ at $n \to \infty$.

If the Jordan cell with size l_m is unique, then the maximal element is unique. Let j be a positive integer such that the maximal element exists at j-th row of A^n. Then,

$$\begin{aligned}
N^\infty &= \lim_{j\to\infty} (\Lambda D_k)^j N^0 \\
&= V_1^{-1} \lim_{j\to\infty} (A)^j V_1 N^0 \\
&= V_1^{-1} \lim_{n\to\infty} ({}_n\mathrm{C}_{l_m-1} \Lambda_1^{n-l_m+1}) \cdot \left(\begin{array}{ccc|c|ccc}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{array}\right) V_1 N^0 \\
&= \lim_{n\to\infty} ({}_n\mathrm{C}_{l_m-1} \Lambda_1^{n-l_m+1}) \cdot \left(0 \middle| \cdots \middle| 0 \middle| V_j \middle| 0 \middle| \cdots \middle| 0 \right) V_1 N^0 \\
&= \lim_{n\to\infty} ({}_n\mathrm{C}_{l_m-1} \Lambda_1^{n-l_m+1}) \cdot \left(0 \middle| \cdots \middle| 0 \middle| V_j \middle| 0 \middle| \cdots \middle| 0 \right) \begin{pmatrix} vn_1^0 \\ vn_2^0 \\ \vdots \end{pmatrix} \\
&= \lim_{n\to\infty} ({}_n\mathrm{C}_{l_m-1} \Lambda_1^{n-l_m+1}) \begin{pmatrix} V_{1j} \cdot vn_j^0 \\ V_{2j} \cdot vn_j^0 \\ \vdots \end{pmatrix},
\end{aligned}$$

where V_j is j-th column of V_1^{-1}, V_{ij} is i-th element of V_j, vn_s^0 is s-th row of $V_1 N^0$. Now, rank $N^\infty = 1$. So, if Λ_1 is real positive, then the limit surface of subdivision is tangent plane continuous.

Note that vn_j^0 is not necessarily non-zero vector, that is, $\exists N^0, vn_j^0 = (0, 0, 0)$. Then, the limit surface of subdivision is not tangent plane continuous. However, the set of such N^0 (such control points p^0) is not dense set. Therefore, we discuss C^k-continuity of subdivision schemes except such case.

If the Jordan cell with size l_m is not unique, then the maximal element is not unique. Then, rank $V_1^{-1} A^\infty \neq 1$. So, the limit surface of subdivision is not tangent plane continuous.

3.2.2 *The case of* $|\Lambda_1| = |\Lambda_2| > |\Lambda_3|$

In this case,

$$\lim_{n\to\infty} A^n = \Lambda_1^\infty \left(\begin{array}{c|c|c} \frac{1}{\Lambda_1^\infty} \lim_{n\to\infty} *^n & 0 & 0 \\ \hline 0 & \frac{\Lambda_2^\infty}{\Lambda_1^\infty} \frac{1}{\Lambda_2^\infty} \lim_{n\to\infty} \acute{*}^n & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

where $*$ is the Jordan Block of Λ_1, $\acute{*}$ is the Jordan Block of Λ_2. Let $\acute{l}$ be the maximal size of all Jordan cells of $*$ and $\acute{*}$. If the Jordan cell with size $\acute{l}$ is unique, then we can discuss similarly. Otherwise, rank $V_1^{-1} A^\infty \neq 1$. So, the limit surface of subdivision is not tangent plane continuous.

Moreover, in other cases (for example, $|\Lambda_1| = |\Lambda_2| = |\Lambda_3| > |\Lambda_4|$), we can discuss similarly.

Note that N^j includes normals for unreal faces (for example, $(v_1^j - v_0^j) \times (v_3^j - v_0^j)$). However, we must not consider their convergence. Because, if normals for real faces (for example, $(v_1^j - v_0^j) \times (v_2^j - v_0^j)$) converge to the same direction, then normals for unreal faces converge to the direction.

3.3 C^1*-continuity*

In previous subsection, we discuss the condition of tangent plane continuity. Thus, here, we assume tangent plane continuity, and we discuss the condition of C^1-continuity.

Let $\mathcal{R}^1 = \{i | i\text{-th row of } N^\infty \text{ is normal of real face}\}$ (Remember the definition of N^0. N^0 includes unreal faces.). $\forall i \in \mathcal{R}^1$, N_i^∞ must point to the same direction including sign if and only if the limit surface is C^1-continuous, where N_i^∞ is i-th row of N^∞.

3.3.1 *The Case of All* V_{ij} *are real*

Note that there is a row of N^∞ which is $(v_1^\infty - v_0^\infty) \times (v_k^\infty - v_0^\infty)$. Here, normal of real face is $(v_k^\infty - v_0^\infty) \times (v_1^\infty - v_0^\infty)$. So, if the element of V_j corresponding to $(v_1^\infty - v_0^\infty) \times (v_2^\infty - v_0^\infty)$ is non-negative, then the element of V_j corresponding to $(v_1^\infty - v_0^\infty) \times (v_k^\infty - v_0^\infty)$ must be non-positive. If the element of V_j corresponding to $(v_1^\infty - v_0^\infty) \times (v_2^\infty - v_0^\infty)$ is non-positive, then the element of V_j corresponding to $(v_1^\infty - v_0^\infty) \times (v_k^\infty - v_0^\infty)$ must be non-negative.

Therefore, we define "proper sign" of N^∞ as same sign of normal of real face.

So, $\forall i \in \mathcal{R}^1$, all V_{ij} must be proper sign.

If $\forall i \in \mathcal{R}^1$, all V_{ij} are proper sign and non-zero, then the limit surface is C^1-continuous.

If $\exists i \in \mathcal{R}^1, V_{ij} = 0$, then the real face corresponding to N_i^∞ vanishes. See Fig. 4.

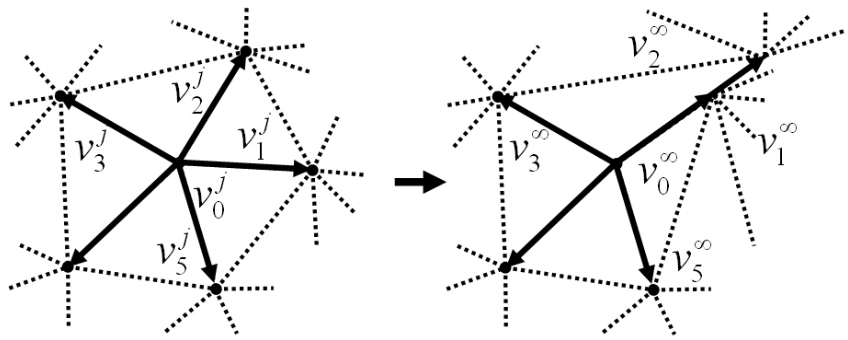

Fig. 4. Vanished face. $(v_1^j - v_0^j) \times (v_2^j - v_0^j)$ is the normal of real face (v_0^j, v_1^j, v_2^j). If the element of V_j corresponding to $(v_1^\infty - v_0^\infty) \times (v_2^\infty - v_0^\infty)$ is 0, then the triangle $(v_0^\infty, v_1^\infty, v_2^\infty)$ vanishes.

First, if $\forall i \in \mathcal{R}, V_{ij} = 0$, then the surface is not C^1-continuous. In order to be C^1-continuous, at least three faces must not vanish.

If $\exists i \in \mathcal{R}^1, V_{ij} = 0$, then there are three types of vanished faces. First type is, like Fig. 4, $(v_i^\infty - v_0^\infty) = a(v_{i+1}^\infty - v_0^\infty)$, where $a \in \mathbf{R}$ is positive.

Second type corresponds to the case of a is negative. Third type corresponds to the case of $a = 0$ or $(v_{i+1}^\infty - v_0^\infty)$ vanishes (that is, $0 \cdot (v_i^\infty - v_0^\infty) = (v_{i+1}^\infty - v_0^\infty)$).

Here, we can check the type. Now, we assume triangle $(v_0^\infty, v_{i-1}^\infty, v_i^\infty)$ does not vanish. Then, if the element of V_j corresponding to $(v_{i-1}^\infty - v_0^\infty) \times (v_{i+1}^\infty - v_0^\infty)$ and elements of V_j of other non-vanished real face are proper sign, then a is positive. If not proper sign, then a is negative. If the element of V_j corresponding to $(v_{i-1}^\infty - v_0^\infty) \times (v_{i+1}^\infty - v_0^\infty)$ is 0, then $(v_{i+1}^\infty - v_0^\infty)$ vanishes.

If the triangle $(v_0^\infty, v_{i-1}^\infty, v_i^\infty)$ vanishes, we can consider similarly for triangle $(v_0^\infty, v_{i-2}^\infty, v_{i-1}^\infty)$.

In order to be C^1-continuous at v_0^∞, a must be positive. If $a < 0$, then the neighborhood of v_0^∞ is not homeomorphic to disc, that is, the structure of 2-manifold is broken at v_0^∞. The neighborhood of the third type is similarly not homeomorphic.

However, even if $a > 0$, the neighborhood of v_1^∞ is not homeomorphic to disc (See Fig. 4), because the length of $v_2^\infty - v_0^\infty$ is not equal to that of $v_1^\infty - v_0^\infty$. For arbitrary p^0 (control points), the length of $v_2^\infty - v_0^\infty$ is not necessarily equal to that of $v_1^\infty - v_0^\infty$. So, if $a > 0$, the subdivision surface is not necessarily C^1-continuous at v_1^∞ (The subdivision surface is C^1-continuous at v_0^∞).

Therefore, in order for the subdivision scheme to be C^1-continuous at arbitrary extraordinary points, there must be no vanished faces, that is $\forall i \in \mathcal{R}^1$, all V_{ij} must be proper sign and non-zero.

3.3.2 The Case of $\exists i, V_{ij}$ is not real

If $\exists i', V_{i'j}$ are not real, then vn_j^0 has the factor $\bar{V}_{i'j}$ ($\bar{V}_{i'j}$ is the complex conjugate of $V_{i'j}$), because $\forall i, N_i^\infty$ must be real.

Thus, $\forall i, V_{ij}$ have the factor $V_{i'j}$, because $\forall i, N_i^\infty$ must be real.

So, $\forall i, \frac{V_{ij}}{V_{i'j}}$ is real. Then, as is the case of $\forall i, V_{ij}$ are real, we can consider C^1-continuous.

Here, we can see that if the limit surface of subdivision is C^1-continuous at all extraordinary points (including degree 6), then the limit surface of subdivision is C^1-continuous, because, on edges and faces, the limit surface is C^1-continuous (Normals of two faces which connect to a edge are same direction).

Therefore, this condition is valid for subdivision surfaces.

4 C^k-continuity

Now, we get the condition for C^1-continuity. So, we assume C^1-continuity at extraordinary points (degree k and degree 6). In this section, we get the condition for C^2-continuity at extraordinary point (degree k). Similarly, if we assume C^{k-1}-continuity at extraordinary points (degree k and degree 6), we can get the condition for C^k-continuity at extraordinary point (degree k).

4.1 Second difference vectors

In the case of tangent plane continuity, we made difference vectors d^j and difference scheme $d^{j+1} = D_k d^j$. Similarly, in the case of C^2-continuity, we make second difference vectors and second difference scheme.

Second difference vectors are generated by first difference vectors in 2-disc (Here, 1-disc means triangles connecting with v_0^j. 2-disc means triangles connecting with vertices in 1-disc. Similarly, 1-ring means edges in 1-disc except edges connecting with v_0^j.). See Fig. 5. We can get (first) difference vectors in 2-disc.

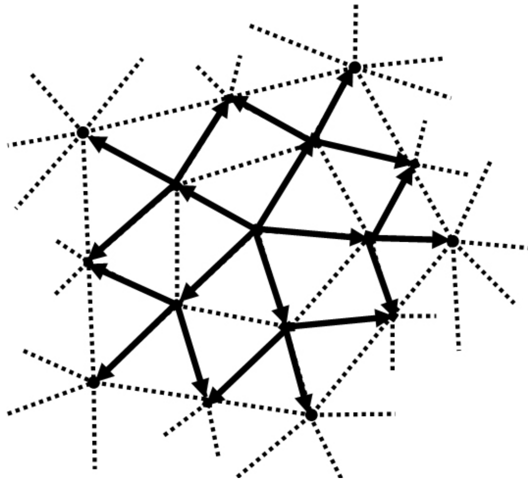

Fig. 5. Difference vectors in 2-disc. Arrows denote difference vectors in 2-disc. We can assume all degrees of vertices on 1-ring are six, because we can see the subdivided mesh as new original mesh.

Now, we make second difference vectors. Let a second difference vector be $u - v$, where v is a difference vector in 1-disc, u is a difference vectors in 2-disc except those in 1-disc. Here, we can make first difference mesh M_d^1. Position vectors of vertices of M_d^1 are difference vectors in 2-disc, and the connectivity of M_d^1 is generated by connectivity of difference vectors in 2-disc. See Fig. 6.

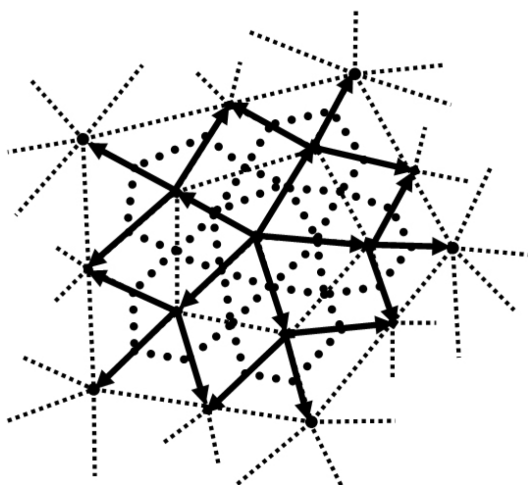

Fig. 6. Connectivity of first difference mesh M_d^1. Circle dashed lines denote edges of M_d^1. M_d^1 consists of central k-gon and hexagons and triangles between them. Position vectors of vertices of M_d^1 are difference vectors. So, u is a vertex of central k-gon, v is a vertex of hexagons except u. Then, $v - u$ is a second difference vector.

The central k-gon of M_d^1 converge to a flat face including the origin. Other hexagons of M_d^1 converge to flat faces at the neighborhood of the origin. So, second difference vectors converge to first derivatives at the origin of M_d^1 (See Fig. 7).

On the other hand, we can see that on the original mesh, the central k-gon converge to tangent plane at v_0^∞, other hexagons converge to tangent planes at the neighbor-

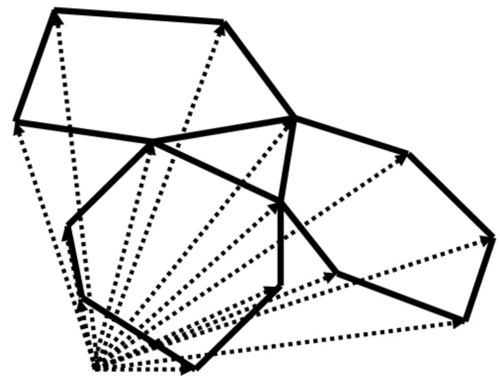

Fig. 7. Three faces of first difference mesh M_d^1. Dashed lines denote (first) difference vectors. Real lines denote edges of M_d^1. At the limit, central k-gon and hexagons are flat, because we assumed C^1-continuity. Then, these flat faces are tangent planes at the origin and its neighborhood of M_d^1.

hood of v_0^∞. So, second difference vectors converge to second derivatives at v_0^∞. Thus, we can consider C^2-continuity from second difference vectors.

Here, we can see that the limit surface generated by M_d^1 at the origin corresponds to $\partial f(y)$ at v_0^∞, where $f(y)$ is the limit surface of subdivision. The k-gon and hexagons of M_d^1 converge to flat faces, because we assumed C^1-continuity at v_0^∞, and converge to vertices, because we assumed C^0-continuity at v_0^∞. So, convergence to a plane of second difference vectors means tangent plane continuity of $\partial f(y)$ at v_0^∞. Moreover, C^1-continuity of the limit surface generated by M_d^1 at the origin means C^1-continuity of $\partial f(y)$ at v_0^∞, that is, C^2-continuity of the limit surface of subdivision at v_0^∞.

4.2 T^2-continuity

Here, we define a term on smoothness.

Definition 4.1
The limit surface of subdivision is T^k-continuous at v_0^∞ if and only if the limit surface of subdivision is C^{k-1}-continuous at extraordinary points (degree k and degree 6) and the limit surface generated from M_d^{k-1} is tangent plane continuous at the origin.

In this case, we consider $k = 2$. Therefore, the limit surface of subdivision is T^2-continuous if and only if second difference vectors span a plane at the limit (We assumed C^1-continuity of the limit surface of subdivision at v_0^∞.).

Here, we consider the condition for T^2-continuous. d^j is difference vectors in 1-disc. Let d_2^j be second difference vectors. Now, we can get a matrix $D_{2,k}$ such that

$$\begin{pmatrix} d^{j+1} \\ \hline d_2^{j+1} \end{pmatrix} = D_{2,k} \begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}.$$

(As in the case of difference scheme, we use matrices corresponding to matrix Δ.) We name this second difference scheme.

Here, we define matrix $D'_{2,k}$ as

$$\begin{pmatrix} d^{j+1} \\ \hline d_2^{j+1} \end{pmatrix} = \left(\begin{array}{c|c} D_k & 0 \\ \hline * & D'_{2,k} \end{array}\right) \begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}.$$

Here, we define "second normal subdivision matrix" $\Lambda D_{2,k}$ as

$$N_2^{j+1} = \Lambda D_{2,k} N_2^j,$$

where $N_2^j = (\begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}_y \wedge \begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}_z, \begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}_z \wedge \begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}_x, \begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}_x \wedge \begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}_y)$.

$\begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}_i$ is i elements of $\begin{pmatrix} d^j \\ \hline d_2^j \end{pmatrix}$.

Let $\Lambda D_{2,k} = V_2^{-1} B V_2$, where B is Jordan normal form. So,

$$\begin{aligned} N_2^{\infty} &= (\Lambda D_{2,k})^{\infty} N_2^0 \\ &= V_2^{-1} B^{\infty} V_2 N_2^0. \end{aligned}$$

Now, we define a matrix T, T is Jordan normal form, as

$$\begin{aligned} N_2^{\infty} &= V_2^{-1} B^{\infty} V_2 N_2^0 \\ &= V_2^{-1} \left(\begin{array}{c|c} T & 0 \\ \hline 0 & 0 \end{array}\right)^{\infty} V_2 N_2^0. \end{aligned}$$

Here, all rows of $((d_2^{\infty})_x \wedge (d_2^{\infty})_y, (d_2^{\infty})_y \wedge (d_2^{\infty})_z, (d_2^{\infty})_z \wedge (d_2^{\infty})_x)$ point to same direction if and only if the limit surface generated from M_d^1 is tangent plane continuous, that is, the limit surface of subdivision is T^k-continuous (Remember tangent plane continuity of the limit surface of subdivision).

So, the maximal rank of T^∞ must be $2k+2$, because rank of rows corresponding to $d^\infty \wedge d^\infty$ of N_2^∞ is 1 and rank of rows corresponding to $d_2^\infty \wedge d_2^\infty$ of N_2^∞ must be 1 and maximal rank of rows corresponding to $d^\infty \wedge d_2^\infty$ of N_2^∞ is $2k$. Here,

$$V_2^{-1} = \left(\begin{array}{c} * \\ \hline *' \\ \hline G \end{array} \right),$$

where G corresponds to rows $d_2^\infty \wedge d_2^\infty = ((d_2^\infty)_x \wedge (d_2^\infty)_y, (d_2^\infty)_y \wedge (d_2^\infty)_z, (d_2^\infty)_z \wedge (d_2^\infty)_x)$ of N_2^∞, $*$ corresponds to rows $d_1^\infty \wedge d_1^\infty$ of N_2^∞, $*'$ corresponds to $d_1^\infty \wedge d_2^\infty$ of N_2^∞.

Now, we define set of non-zero column of G as $\mathcal{C}$ ($\forall c \in \mathcal{C}, c \neq \begin{pmatrix} 0 \\ \vdots \end{pmatrix}$). Here, we consider rows of T^∞ corresponding to $\mathcal{C}$. Because, we want to consider maximal column of F, where

$$V_2^{-1} B^\infty = \left(\begin{array}{c} * \\ \hline *' \\ \hline F \end{array} \right).$$

Let $\Lambda_{T,i}, i = 1, 2, \cdots$ be eigen values of T corresponding to $\mathcal{C}$, where $|\Lambda_{T,i}| \geq |\Lambda_{T,i+1}|$.

4.2.1 *The case of* $|\Lambda_{T,1}| > |\Lambda_{T,2}|$

In this case,

$$\begin{aligned} B^\infty &= \left(\begin{array}{c|c} T^\infty & 0 \\ \hline 0 & 0 \end{array} \right) \\ &= \Lambda_{T,1}^\infty \left(\begin{array}{c|c|c} \frac{1}{\Lambda_{T,1}^\infty} * & 0 & 0 \\ \hline 0 & \frac{1}{\Lambda_{T,1}^\infty} T_1^\infty & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \end{aligned}$$

where T_1 is Jordan Block of $\Lambda_{T,1}$, $*$ is Jordan Blocks of other eigen values of T.

Here, $\Lambda_{T,1}$ is maximal eigen value corresponding to $\mathcal{C}$. So, eigen values of $*$ whose magnitudes are larger than $|\Lambda_{T,1}|$ corresponds to zero-column of G. Therefore, $*$ does not affect $d_2^\infty \wedge d_2^\infty$.

So, we consider maximal element of $(T_1)^\infty$. Let l be the maximal size of all Jordan cells of T_1. Let $J_{c,i}, i = 1, 2, 3, \cdots, q$ be Jordan cells of T_1 whose sizes are l ($q \in$ **N**).

(Here, we assume that the first rows of $J_{c,i}, i = 1, 2, 3, \cdots, q$ correspond to $\mathcal{C}$. Otherwise, we can see that new size of the Jordan cell is old size minus 1. Similarly, we can get maximal elements of T_1.)

Then, on T_1^n, the maximal element (on absolute value) of all Jordan cells of T_1 is ${}_n\mathrm{C}_{l-1}\Lambda_{T,1}^{n-l+1}$, where n is a large positive integer. Here, T_1^n has q maximal elements.

At $n \to \infty$, ${}_n\mathrm{C}_{l-1}\Lambda_{T,1}^{n-l+1}$ is sufficiently larger than other elements which corresponds to $\mathcal{C}$. Therefore, we can ignore effects of the other elements.

Let $r_i, i = 1, 2, \cdots, q$ be rows of $\mathcal{C}$ corresponding to ${}_n\mathrm{C}_{l-1}\Lambda_{T,1}^{n-l+1}$. Then, maximal rows of F are ${}_n\mathrm{C}_{l-1}\Lambda_{T,1}^{n-l+1} \cdot r_i, i = 1, 2, \cdots, q$. So, for arbitrary N_2^0, if $q = 1$, the rank of $d_2^\infty \wedge d_2^\infty$ is 1. Then, if $\Lambda_{T,1}$ is real positive, the limit surface generated by M_d^1 is tangent plane continuous.

If $q \neq 1$, then $\forall i, r_i$ and r_1 must be linear dependent. Otherwise, the rank of $d_2^\infty \wedge d_2^\infty \neq 1$. If $\forall i, r_i$ and r_1 are linear dependent and $\Lambda_{T,1}$ is real positive, the limit surface generated by M_d^1 is tangent plane continuous. Otherwise, the rank of $d_2^\infty \wedge d_2^\infty \neq 1$.

Note that N_2^j includes normals for unreal faces of M_d^1. However, we must not consider their convergence. Because, if normals for real faces of M_d^1 converge to the same direction, then normals for unreal faces of M_d^1 converge to the direction.

4.2.2 *The case of* $|\Lambda_{T,1}| = |\Lambda_{T,2}| > |\Lambda_{T,3}|$

In this case,

$$B^{\infty} = \left(\begin{array}{c|c} T^{\infty} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$= \Lambda_{T,1}^{\infty} \left(\begin{array}{c|c|c|c} \frac{1}{\Lambda_{T,1}^{\infty}} * & 0 & 0 & 0 \\ \hline 0 & \frac{1}{\Lambda_{T,1}^{\infty}} T_1^{\infty} & 0 & 0 \\ \hline 0 & 0 & \frac{\Lambda_{T,2}^{\infty}}{\Lambda_{T,1}^{\infty}} \frac{1}{\Lambda_{T,2}^{\infty}} T_2^{\infty} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

where T_1 is Jordan Block of $\Lambda_{T,1}$, T_2 is Jordan Block of $\Lambda_{T,2}$, $*$ is Jordan Blocks of other eigen values of T. Let $\acute{l}$ be the maximal size of all Jordan cells of T_1 and T_2. Then, let $\acute{J}_{c,i}, i = 1, 2, \cdots, q$ be Jordan cells of T_1 and T_2 whose sizes are $\acute{l}$ ($q \in \mathbf{N}$).

(Here, we assume that the first rows of $\acute{J}_{c,i}, i = 1, 2, \cdots, q$ correspond to $\mathcal{C}$ in a similar way of first case.)

Then, on T_1^n and T_2^n, the maximal element (on absolute value) is ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,1}^{n-\acute{l}+1}$ or ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,2}^{n-\acute{l}+1}$, where n is a large positive integer. There are q maximal elements.

At $n \to \infty$, ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,1}^{n-\acute{l}+1}$ and ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,2}^{n-\acute{l}+1}$ are sufficiently larger than other elements which corresponds to $\mathcal{C}$. Therefore, we can ignore effects of the other elements.

Let $r_i, i = 1, 2, \cdots, q$ be rows of $\mathcal{C}$ corresponding to ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,1}^{n-\acute{l}+1}$ or ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,2}^{n-\acute{l}+1}$. Then, maximal rows of F are ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,1}^{n-\acute{l}+1} \cdot r_i$ and ${}_n\mathrm{C}_{\acute{l}-1}\Lambda_{T,2}^{n-\acute{l}+1} \cdot r_j$. So, for arbitrary N_2^0, if $q = 1$, the rank of $d_2^{\infty} \wedge d_2^{\infty}$ is 1. Then, if $\Lambda_{T,i}$ corresponding to r_1 is real positive, the limit surface generated by M_d^1 is tangent plane continuous.

If $q \neq 1$, then $\forall i, r_i$ and r_1 must be linear dependent. Otherwise, the rank of $d_2^{\infty} \wedge d_2^{\infty} \neq 1$. Here, if $\Lambda_{T,1}$ is real, then $\Lambda_{T,2}$ is $-\Lambda_{T,1}$ or not real. So, if $\forall i, r_i$ and r_1 are linear dependent and $\forall i$, r_i corresponds $\Lambda_{T,1}$ ($\Lambda_{T,2}$) and $\Lambda_{T,1}$ ($\Lambda_{T,2}$) is real positive, the limit surface generated by M_d^1 is tangent plane continuous. Otherwise, for arbitrary N_2^0, the rank of $d_2^{\infty} \wedge d_2^{\infty}$ is not necessarily 1.

In other cases (for example, $|\Lambda_{T,1}| = |\Lambda_{T,2}| = |\Lambda_{T,3}| > |\Lambda_{T,4}|$), we can discuss similarly.

4.3 C^2-continuity

Now, we can assume T^k-continuity at v_0^{∞}. Next, we derive the condition of C^k-continuity at v_0^{∞}.

Here, the limit surface of subdivision is C^k-continuous at v_0^∞ if and only if the limit surface of subdivision is C^{k-1}-continuous at extraordinary points (degree k and degree 6) and the limit surface generated from M_d^{k-1} is C^1-continuous at the origin.

So, in this case of $k = 2$, we consider C^1-continuity for the limit surface generated from M_d^1.

Let $\mathcal{R}^2 = \{i | i$-th row of N_2^∞ is normal of real face of $M_d^1\}$. See Fig. 8. For u of central k-gon, triangles $(u, v_1, v_2), (u, v_2, v_3)$ are real faces of M_d^1. The triangles $(u, v_0, v_1), (u, v_3, v_4)$ are also real faces of M_d^1. However, it is sufficient that we check only direction of normals of real faces $(u, v_1, v_2), (u, v_2, v_3)$, because we assumed C^1-continuity at v_0^∞ and vertices on 1-ring of v_0^∞.

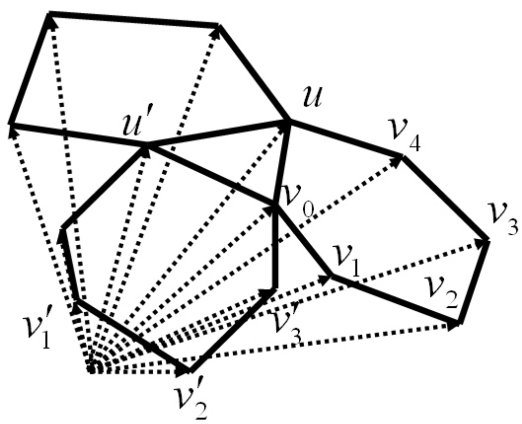

Fig. 8. Real faces of M_d^1. For a vertex u of central k-gon, triangles $(u, v_1, v_2), (u, v_2, v_3)$ are real faces of M_d^1. Similarly, for u' triangles $(u', v_1', v_2'), (u', v_2', v_3')$ are real faces of M_d^1.

Let r_{ij} be the i-th element of the column vector r_j.

4.3.1 *The Case of* $\forall j, \forall i, r_{ij}$ *are real*

Here, as in the case of C^1-continuity, we define "proper sign" of r_j as same sign of normal of real face of M_d^1.

So, if $q = 1$, and $\forall i \in \mathcal{R}^2, r_{i1}$ are proper sign, then the limit surface generated by M_d^1 is C^1-continuous, that is, the subdivision surface is C^2-continuous.

As in the case of C^1-continuity, we can consider the case of $\exists i, r_{i1} = 0$.

In the case of $q \neq 1$. Similarly, if $\forall i \in \mathcal{R}^2, r_{i1}$ are proper sign, then the limit surface generated by M_d^1 is C^1-continuous, that is, the subdivision surface is C^2-continuous, because $\forall j, r_j$ and r_1 is linear dependent.

Moreover, we can consider the case of $\exists i, r_{i1} = 0$ similarly.

4.4 The Case of $\exists j, \exists i, r_{ij}$ is not real

If $\exists j', \exists i', r_{i'j'}$ is not real, then $\forall i, r_{ij'}$ must have the factor $r_{i'j'}$, because all rows of N_2^∞ must be real for arbitrary N_2^0 (As is the case of C^1-continuous, the corresponding row of $V_2 N_2^0$ must have the factor $\bar{r}_{i'j'}$ which is the complex conjugate of $r_{i'j'}$ for arbitrary N_2^0).

So, $\forall i, \frac{r_{ij'}}{r_{i'j'}}$ is real. Then, we can see that there is the new row $\frac{1}{r_{i'j'}} r_{j'}$ whose elements are real. $\forall j$, we can consider similarly.

Therefore, as is the case of $\forall j, \forall i, r_{ij}$ are real, we can consider C^2-continuous.

Here, as in the case of C^1-continuity, if the limit surface generated by M_d^1 is C^1-continuous at all extraordinary points (including degree 6), then the limit surface generated by M_d^1 is C^1-continuous. Then, the limit surface of subdivision is C^2-continuous, because the limit surface generated by M_d^1 at the origin corresponds to $\partial f(y)$ at v_0^∞, where $f(y)$ is the limit surface of subdivision.

Therefore, this condition is valid for subdivision surfaces.

4.5 Convergence to j-th derivatives

In previous subsection, we derived the necessary and sufficient condition for subdivision surfaces to be C^2-continuous. We used the relation that the limit surface of subdivision is C^k-continuous at v_0^∞ if and only if the limit surface of subdivision is C^{k-1}-continuous at extraordinary points (degree k and degree 6) and the limit surface generated by M_d^{k-1} is C^1-continuous at the origin. In this subsection, we explain this relation.

We can see that second difference vectors converge to linear combinations of second derivatives at v_0^∞. See Fig. 9.

Let ∂_1 be the differentiation of direction $v_1^\infty - v_0^\infty$ and vertex v^j be linear combination of v_2^j and v_3^j, where $(v^j - v_1^j)//(v_1^j - v_0^j)$. Let $f(y)$ be the limit surface of subdivision. Then,

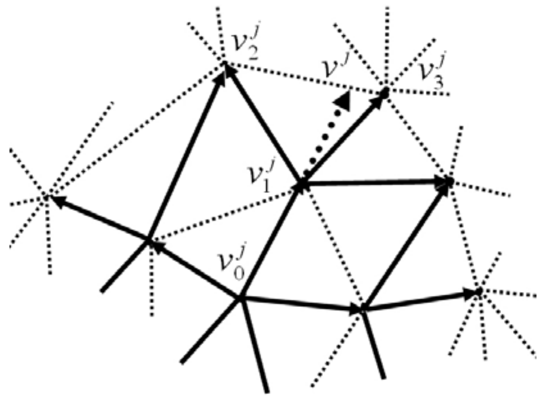

Fig. 9. Convergence to second derivatives. v^j is linear combination of v_2^j and v_3^j, where $(v^j - v_1^j)//(v_1^j - v_0^j)$. $\lim_{j\to\infty}(v^j - v_1^j - v_1^j + v_0^j)$ is the second derivative of direction $v_1^\infty - v_0^\infty$. So, second difference vectors converge to linear combinations of second derivatives.

$$\begin{aligned}
\partial_1^2 f(v_0^\infty) &= \lim_{h\to 0} \frac{\partial_1 f(v_0^\infty + h) - \partial_1 f(v_0^\infty)}{h} \\
&= \lim_{h\to 0} \frac{\lim_{t\to 0} \frac{f(v_0^\infty + h + t) - f(v_0^\infty + h)}{t} - \lim_{t\to 0} \frac{f(v_0^\infty + t) - f(v_0^\infty)}{t}}{h} \\
&= \lim_{j\to\infty} (v^j - v_1^j - v_1^j + v_0^j) \\
&= \lim_{j\to\infty} (\alpha v_2^j + \beta v_3^j - 2v_1^j + v_0^j) \\
&= \alpha \lim_{j\to\infty} (v_2^j - 2v_1^j + v_0^j) + \beta \lim_{j\to\infty} (v_3^j - 2v_1^j + v_0^j),
\end{aligned}$$

where $\alpha, \beta \in \mathbf{R}, \alpha + \beta = 1$. Therefore, second difference vectors converge to second derivatives at v_0^∞.

Similarly, j-th difference vectors, which are difference vectors between $j-1$-st difference vectors, converge to linear combinations of j-th derivatives at v_0^∞.

The second derivatives must span a plane, because, for arbitrary p^0 (control points), the subdivision scheme must be C^2-continuous. So, the subdivision scheme must be T^2-continuous, because second difference vectors converge to linear combinations of second derivatives. Therefore, the limit surface generated by M_d^1 is C^1-continuous at the origin if and only if the limit surface of subdivision is C^2-continuous (Here, we assumed C^1-continuity at extraordinary points).

So, we could discuss the necessary and sufficient condition for subdivision surfaces to be C^j-continuous at extraordinary points. In the analysis of smoothness, similarly, we use j-th normal subdivision matrix on $j-1$-st difference mesh.

In the analysis, we use only linear algebra. So, this analysis is intuitive and can be easily computable.

5 Conclusion

In this paper, we proposed a new analysis of smoothness for stationary subdivision schemes.

We assume affine invariance for subdivision. Most of existing subdivision schemes satisfy this assumption.

Our first contribution is the necessary and sufficient condition for stationary subdivision surfaces to be C^1-continuous. First, we made difference vectors which converge to first derivatives. Next, we defined normal subdivision matrix which subdivides difference vectors. So, we could discuss the convergence of difference vectors as analysis of Jordan cells of normal subdivision matrix.

Our second contribution is the necessary and sufficient condition for stationary subdivision surfaces to be C^k-continuous. First, we defined first difference mesh M_d^1 and second difference vectors which converge to second derivatives. Similarly, we can define $k-1$-th difference mesh and k-th difference vectors which converge to k-th derivatives.

Moreover, we define T^k-continuity of the subdivision surface at the extraordinary point as C^{k-1}-continuity of the subdivision surface at extraordinary points (the extraordinary point and extraordinary point with degree 6) and tangent plane continuity of the limit surface generated by M_d^{k-1} at the origin.

Then, we could define second normal subdivision matrix similarly. Thus, we could derive the necessary and sufficient condition to be T^2-continuous. Here, if the subdivision surface is T^k-continuous and the limit surface generated by M_d^{k-1} is C^1-continuous, then the subdivision surface is C^k-continuous. So, we can derive the necessary and sufficient condition to be C^k-continuous. In this paper, we derived the condition to be C^2-continuous.

In this paper, we assumed affine invariance. However, this assumption can be removed easily. Therefore, we could derive the necessary and sufficient condition for stationary subdivision surfaces to be C^k-continuous with no assumption.

Acknowledgments

This work is supported by the 21st Century COE Program on Information Science Strategic Core of the Ministry of Education, Science, Sports and Culture of Japan.

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