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Abstract

In this paper, we investigate the regulation properties pertaining to single-input multiple-output (SIMO) linear time-invariant (LTI) systems, in which the objective function of regulated response is minimized jointly with the control effort. We provide analytical closed-form expressions of the best achievable $\mathcal{H}_2$ optimal regulation performances against impulsive disturbance inputs for unstable/non-minimum phase continuous-time and discrete-time systems. We also modify the latter results by means the delta operator and show the continuity property, i.e., we show that the continuous-time solution can be completely recovered when the sampling period tends to zero in the delta-domain solution. We then apply the results to a magnetic bearing system and discuss relations between the sensor selection and the best achievable $\mathcal{H}_2$ performances.

Key words: Performance limitations, $\mathcal{H}_2$ optimal control, SIMO systems, unstable poles/zeros, delta operator, magnetic bearing system.

1 Introduction

The study on control performance limitations achievable by feedback is one of the important research topics in control theory. It has been paid much attention in the recent years as seen in a special issue of the IEEE Transactions on Automatic Control in August 2003, a book [13], and recent publications [1, 4, 5, 6, 8, 9, 14]. These researches have been initiated by the well known

Bode integral relation and they examined certain classical optimal problems under optimality criteria formulated in time or frequency domain, which have led to the analytical closed-form solutions of the best achievable performance including the tracking and regulation performances. The analytical closed-form expressions rather than numerical solutions are quite useful not only to understand feedback control systems but also to characterize a set of easily controllable plants in practical situations. Actually, the formulae can provide guidelines for plant design from the view point of control.

We here focus on the $\mathcal{H}_2$ optimal regulation performance, which is measured by minimizing the energy of control input, or by minimizing the energy of control input jointly with the energy of system output. We call the former the energy regulation problem and the latter the output regulation problem. Results on $\mathcal{H}_2$ energy regulation problem can be found in [9] for continuous-time systems and in [8] for discrete-time systems. Both results are conducted for unstable/non-minimum phase SISO/SIMO plants. Equivalent results in SISO systems but articulated in term of signal-to-noise ratio constrained channels are found in [3, 11]. Meanwhile, results on $\mathcal{H}_2$ output regulation problem of minimum phase SISO/MIMO systems are presented in [5]. Summarizing these existing results, the investigation for the minimum phase plant is almost complete, while the researches for unstable/non-minimum phase plants are not complete. Especially further investigations are required for the LTI discrete-time case. Note that the linear time-varying feedback stabilization has been discussed in [11].

This paper discusses the output regulation problem of unstable/non-minimum phase SIMO continuous-time and discrete-time systems. In the latter case, we provide a much more general expressions of the optimal performance than those given in [1]. Beyond that, we reformulate and solve the problem in terms of delta operator, see [12], and show its continuity properties. In other words, we can completely recover the continuous-time solution by taking the sampling time tends to zero. In order to confirm the effectiveness of the derived results in practical applications, where we have a single actuator but multiple sensors, we apply our results to a magnetic bearing system and discuss relations between the sensor selection and the best achievable $\mathcal{H}_2$ performances.

It is worth to point out that in the regulation problem, we have to exploit a certain function evaluated at infinity which is laid on the jω-axis (boundary of s-domain) but not on the unit circle (boundary of z-domain). It means that derivation process in discrete-time case is not parallel with that of continuous-time case. This is contrast with the tracking problem, where the derivations for the discrete-time are almost parallel to those for the continuous-time case, see [2] and [4]. We should also note that the final formula of the discrete-time regulation performance limit is different from others in the following sense: It includes a term of product of unstable poles, while for the other cases the limits include a term which consists of sum of unstable pole or reciprocal of non-minimum phase zeros.

The rest of this paper is organized as follows. In Section 2 we describe the problem formulation including the description of the standard unity feedback control system under consideration and some preliminary results. Sections 3 and 4 provide analytical closed-form expressions of the optimal regulation performance for continuous-time and discrete-time cases, respectively. In Section 5 we reformulate the problem in terms of delta operator and provide the solution and its continuity properties. An application of our results to a magnetic bearing systems is provided in Section 6. This paper concludes in Section 7.

Notation: We give a brief description of the notation used throughout this paper. We denote the real set by $\mathbb{R}$ and the complex set by $\mathbb{C}$. For any $c \in \mathbb{C}$, its complex conjugate is denoted by $\bar{c}$. For any vector u we shall use u^{T}, u^{H}, and $\|u\|$ as its transpose, conjugate transpose, and Euclidean norm, respectively. For any matrix $A \in \mathbb{C}^{m\times n}$, we denote its conjugate transpose by A^{H} and its column space by $\mathbb{R}[A]$. In s-domain analysis, i.e., continuous-time case, let the open right half plane be denoted by $\mathbb{C}_+ := \{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$ and the open left half plane by $\mathbb{C}_- := \{s \in \mathbb{C} : \mathrm{Re}\, s < 0\}$. And for any matrix function $f \in \mathbb{C}^{m\times n}$ we denote $f^{\sim}(s) = f^{\mathrm{T}}(-s)$. While in z-domain analysis, i.e., discrete-time case, the regions inside and outside unit circle are denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathbb{D}}^c := \{z \in \mathbb{C} : |z| > 1\}$, respectively. Here we denote $f^{\sim}(z) = f^{\mathrm{T}}(z^{-1})$. We denote by $\mathbb{R}\mathcal{H}_\infty$ the class of all stable and proper rational transfer function matrices. We define by $\hat{u}(s)$ the $\mathcal{L}$-transform of signal $u(t)$ and by $\hat{u}(z)$ the $\mathcal{Z}$-transform of sequence $u(k)$. The cardinality of a set S is denoted by $\#S$. In this paper, the space $\mathcal{L}_2$ is the Hilbert space with inner product

$$\langle f, g\rangle := \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f^{\mathrm{H}}(\mathrm{j}\omega)g(\mathrm{j}\omega)\,\mathrm{d}\omega \ ; & \text{for } s\text{-domain} \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\theta})g(\mathrm{e}^{\mathrm{j}\theta})\,\mathrm{d}\theta \ ; & \text{for } z\text{-domain.} \end{cases}$$

2 Problem Formulation and Preliminaries

2.1 Feedback Control Systems

We consider the standard unity feedback configuration of finite dimensional linear time-invariant systems depicted in Fig. 1. In this setup, P denotes the SIMO LTI plant, K the stabilizing compensator, W_v and W_y the stable/minimum phase weighting functions. We will use the same letters to denote their transfer matrices. The signals $d \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}^m$, and $z \in \mathbb{R}^{q+r}$ are the disturbance input, the plant input, the measurable output, and the weighted output to be minimized, respectively. The first and second elements of z are set for sensitivity reduction and disturbance attenuation, respectively. Hereafter, it will be assumed that all vectors and matrices

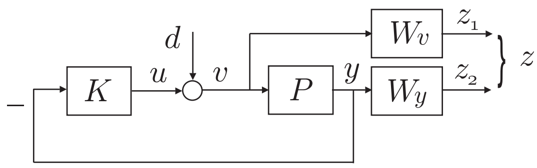

Figure 1: The regulation scheme.

involve in the sequel have compatible dimensions, and for simplicity their dimensions will be omitted. In subsequent analysis, the plant is given by

$$P = \begin{pmatrix} P_1 & P_2 & \dots & P_m \end{pmatrix}^{\mathrm{T}} \tag{1}$$

with P_i ($i = 1, \dots, m$) are scalar transfer functions. We assume that the system is initially at rest. For technical reasons, it is also assumed that the plant does not have non-minimum phase zeros and unstable poles at the same location. In the present work, we consider an impulse function as the disturbance signal.

For the plant rational transfer function P, its left and right coprime factorizations are given by

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{2}$$

where $N, M, \tilde{N}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$ and they satisfy the double Bezout identity

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I, \tag{3}$$

for some $X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$. All the stabilizing compensators K can be characterized by Youla parameterization

$$\begin{aligned} \mathcal{K} \;:=\; & \{K : K = (Y - MQ)(NQ - X)^{-1} \\ & = (Q\tilde{N} - \tilde{X})^{-1}(\tilde{Y} - Q\tilde{M});\ Q \in \mathbb{R}\mathcal{H}_\infty\}. \end{aligned} \tag{4}$$

A complex number z is said to be a zero of P if $P_i(z) = 0$. In addition, if z is lying either in $\mathbb{C}_+$ for s-domain or $\bar{\mathbb{D}}^{\mathrm{c}}$ for z-domain then z is said to be a non-minimum phase zero. P is said to be minimum phase if it has no non-minimum phase zero; otherwise, it is said to be non-minimum phase. On the other hand, a complex number $p \in \mathbb{C}$ is said to be a pole of P if $P(p)$ is unbounded. A pole p is said to be unstable if it lies in $\mathbb{C}_+$ or $\bar{\mathbb{D}}^{\mathrm{c}}$. P is said to be stable if it has no unstable pole; otherwise, unstable.

A transfer function N, not necessarily square, is called an inner if N is in $\mathbb{R}\mathcal{H}_\infty$ and $N^{\sim}N = I$ for all $s = \mathrm{j}\omega$ or $z = \mathrm{e}^{\mathrm{j}\theta}$. A transfer function M is called outer if M is in $\mathbb{R}\mathcal{H}_\infty$ and has a right inverse which is analytic in $\mathbb{C}_+$ or $\bar{\mathbb{D}}^{\mathrm{c}}$. For an arbitrary $P \in \mathbb{R}\mathcal{H}_\infty$,

$$P = \Theta_{\mathrm{i}}\Theta_{\mathrm{o}}, \tag{5}$$

where Θ_{i} is inner and Θ_{o} is outer, is defined as an inner-outer factorization of P. We call Θ_{i} the inner factor and Θ_{o} the outer factor.

2.2 Optimal Regulation Problem

We here consider an $\mathcal{H}_2$ regulation performance limitations for SIMO plants. Our interest is not on how to find a optimal compensator K which stabilizes the feedback control system and regulate the plant output y to zero under control input and sensitivity penalties. Rather, we are interested in relating the best achievable performance with some simple characteristics of the plant P. To this end, for continuous-time case we adopt the following performance index:

$$E_{\mathrm{c}} := \int_0^\infty \left(\|z(t)\|^2 + |u(t)|^2\right) \mathrm{d}t, \tag{6}$$

where $z(t)$ is the weighted output, i.e.,

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{L}^{-1}\{W_v(s)\hat{v}(s)\} \\ \mathcal{L}^{-1}\{W_y(s)\hat{y}(s)\} \end{pmatrix}.$$

Similarly for discrete-time case,

$$E_{\mathrm{d}} := \sum_{k=0}^{\infty} \left(\|z(k)\|^2 + |u(k)|^2\right), \tag{7}$$

where $z(k)$ is given by

$$z(k) = \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix} = \begin{pmatrix} \mathcal{Z}^{-1}\{W_v(z)\hat{v}(z)\} \\ \mathcal{Z}^{-1}\{W_y(z)\hat{y}(z)\} \end{pmatrix}.$$

It follows from the well-known Parseval identity that

$$E = \|\hat{z}_1\|_2^2 + \|\hat{z}_2\|_2^2 + \|\hat{u}\|_2^2$$

holds, where E stands either E_{c} for the continuous-time case or E_{d} for the discrete-time case. Note that if $W_v = 0$ and $W_y = 0$ the problem then reduces to an energy regulation one, which have been discussed in [9, 8].

Let the input and output sensitivity functions be defined by

$$S_{\mathrm{i}} := (1 + KP)^{-1}, \tag{8}$$

$$S_{\mathrm{o}} := (I + PK)^{-1}, \tag{9}$$

respectively. Then, it is immediate to obtain that

$$E = \|W_v S_{\mathrm{i}}\hat{d}\|_2^2 + \|W_y P S_{\mathrm{i}}\hat{d}\|_2^2 + \|KPS_{\mathrm{i}}\hat{d}\|_2^2. \tag{10}$$

Since d is an impulse signal, i.e., $\hat{d} = 1$, from Bezout identity (3) and Youla parameterization (4) we may express (10) as

$$\begin{aligned} E &= \|W_v(1 - KS_{\mathrm{o}}P)\hat{d}\|_2^2 + \|W_y S_{\mathrm{o}} P\hat{d}\|_2^2 + \|KS_{\mathrm{o}}P\hat{d}\|_2^2 \\ &= \|W_v(1 + Y\tilde{N} - MQ\tilde{N})\|_2^2 + \|W_y(X\tilde{N} - NQ\tilde{N})\|_2^2 + \|Y\tilde{N} - MQ\tilde{N}\|_2^2, \end{aligned}$$

from which we then want to determine the best achievable regulation performance with respect to all stabilizing compensators,

$$E^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W_v(1+Y\tilde{N}-MQ\tilde{N}) \\ W_y(X\tilde{N}-NQ\tilde{N}) \\ Y\tilde{N}-MQ\tilde{N} \end{bmatrix} \right\|_2^2 . \tag{11}$$

Our purpose in this paper is to derive analytical closed-form expressions on E^* for both continuous-time systems (E_{c}^*) and discrete-time systems (E_{d}^*) and to discuss the continuity property.

2.3 Integral Formulae

The following integral formulae play an important role in our subsequent derivation. We note that Lemma 1, i.e., Poisson-Jensen Formula, can be found, for instance, in [7] and Lemma 2, i.e., the continuous-time counterpart of Lemma 1, can be obtained by implementing bilinear transformation: $z = (1+s)/(1-s)$. The proof of Lemma 3 can be done by using Cauchy integral and residue theorems.

Lemma 1 (Poisson-Jensen Formula) *Let f is analytic in $\bar{\mathbb{D}}^{\mathrm{c}}$ and $d_k (k = 1, \ldots, n_d)$ be the zeros of f in $\bar{\mathbb{D}}^{\mathrm{c}}$, counting their multiplicities. If $z \in \bar{\mathbb{D}}^{\mathrm{c}}$ and $f(z) \neq 0$, then*

$$\log|f(z)| = \frac{1}{\pi}\int_0^\pi \mathrm{Re}\left(\frac{z\mathrm{e}^{\mathrm{j}\theta}+1}{z\mathrm{e}^{\mathrm{j}\theta}-1}\right)\log|f(\mathrm{e}^{\mathrm{j}\theta})|\,\mathrm{d}\theta - \sum_{k=1}^{n_d}\log\left|\frac{1-\bar{d}_k z}{z-d_k}\right|.$$

Lemma 2 *Let g is analytic in $\mathbb{C}_+$ and c_k $(k = 1, \ldots, n_c)$ be the zeros of g in $\mathbb{C}_+$, counting their multiplicities. If $s \in \mathbb{C}_+$ and $g(s) \neq 0$, then*

$$\log|g(s)| = \frac{2}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{1+\mathrm{j}\omega s}{s+\mathrm{j}\omega}\right)\frac{\log|g(\mathrm{j}\omega)|}{1+\omega^2}\,\mathrm{d}\omega - \sum_{k=1}^{n_c}\log\left|\frac{1-c_k}{1-\bar{c}_k}\frac{\bar{c}_k+s}{c_k-s}\right|.$$

Lemma 3 *If $f \in \mathbb{R}\mathcal{H}_\infty$, then*

$$\frac{1}{\pi}\int_0^\pi \mathrm{Re}\, f(\mathrm{e}^{\mathrm{j}\theta})\,\mathrm{d}\theta = f(\infty).$$

3 Continuous-time Case

This section is devoted to the continuous-time case, where an analytical closed-form expression of the performance limitation is derived. Suppose that the plant $P(s)$ is given by (1) and its coprime factorizations are represented by (2). We denote by p_k $(k = 1, \ldots, n_p)$ the unstable poles of $P(s)$. Then, without loss of generality we may set M as

$$M(s) := \prod_{k=1}^{n_p}\frac{s-p_k}{s+\bar{p}_k}. \tag{12}$$

Since M is a Blaschke product, by definition, $M = 1$ whenever P is stable. It is useful to point out that $M(\infty) = 1$.

In order for E in (10) to be finite, for continuous-time systems, it is necessary that $P\hat{d} \in \mathcal{L}_2$ and $W_v\hat{d} \in \mathcal{L}_2$. Since $d(t)$ is an impulse function so that $\hat{d}(s) = 1$, we need the following assumption.

Assumption 1 *$P(s)$ and $W_v(s)$ are strictly proper.*

We are now ready to provide the best achievable regulation performance.

Theorem 1 (Continuous-time) *Suppose that the plant $P(s)$ has unstable poles p_k $(k = 1, \ldots, n_p)$ and its coprime factorizations are given by (2). Let introduce the following index set:*

$$\mathbb{N}_z := \{k : \tilde{N}(z_k) = 0,\ z_k \in \mathbb{C}_+\}, \tag{13}$$

or $\mathbb{N}_z$ is the set of all common non-minimum phase zeros of $P(s)$ with counting multiplicities, and define the inner-outer factorization

$$\begin{bmatrix} W_v \\ W_y N \\ -1 \end{bmatrix} = \Lambda_{\mathrm{i}}\Lambda_{\mathrm{o}}.$$

Then, under Assumption 1 we have

$$E_{\mathrm{c}}^* = E_{\mathrm{cm}} + E_{\mathrm{cn}}, \tag{14}$$

where

$$\begin{aligned} E_{\mathrm{cm}} &= 2\sum_{k=1}^{n_p} p_k + \frac{1}{\pi}\int_0^\infty \log\left(1 + \|W_v(\mathrm{j}\omega)\|^2 + \|W_y(\mathrm{j}\omega)P(\mathrm{j}\omega)\|^2\right) \mathrm{d}\omega, \\ E_{\mathrm{cn}} &= \sum_{k,\ell\in\mathbb{N}_z} \frac{4\,\mathrm{Re}\, z_k\, \mathrm{Re}\, z_\ell}{\bar{a}_k a_\ell(\bar{z}_k + z_\ell)}\, \bar{\alpha}_k \alpha_\ell, \end{aligned}$$

with

$$\begin{aligned} a_k &:= \begin{cases} 1 & ;\ \#\mathbb{N}_z = 1 \\ \displaystyle\prod_{\ell\in\mathbb{N}_z,\ell\neq k} \frac{z_\ell - z_k}{z_\ell + \bar{z}_k}, & ;\ \#\mathbb{N}_z \geq 2 \end{cases}, \\ \alpha_k &:= 1 - \Lambda_{\mathrm{o}}(z_k)\prod_{i=1}^{n_p} \frac{z_k + \bar{p}_i}{z_k - p_i}. \end{aligned}$$

Proof The proof of E_{cm} can be accomplished by following that of [5, Th. 3]. We may prove E_{cn} by performing standard partial fraction expansion technique used in [6]. Expression of E_{cn} shares some similarities with that of [9, Th. 2] or [11, Prop. 3.1]. ■

Theorem 1, which is valid for both SISO and SIMO systems, shows that the best regulation performance depends not only on the plant unstable poles and non-minimum phase zeros but also on its gain and a relation between unstable poles and zeros. We may obtain an explicit formula for $\Lambda_{\rm o}(s)$ if $W_y = 0$, since $\Lambda_{\rm o}(s)$ does not depend the plant $P(s)$ (see Sec. 6). In general, since $\Lambda_{\rm o}$ is stable and minimum phase, its absolute value at z_k can be obtained from Lemma 2, that is

$$|\Lambda_{\rm o}(z_k)| = \exp\left\{\frac{2}{\pi}\int_0^\infty {\rm Re}\left(\frac{1+{\rm j}\omega z_k}{z_k+{\rm j}\omega}\right)\frac{\log|\Lambda_{\rm o}({\rm j}\omega)|}{1+\omega^2}{\rm d}\omega\right\},$$

where

$$|\Lambda_{\rm o}({\rm j}\omega)| = \sqrt{1+\|W_v({\rm j}\omega)\|^2+\|W_y({\rm j}\omega)P({\rm j}\omega)\|^2}.$$

For the SIMO case, condition z_k ($k \in \mathbb{N}_z$) means that only the common non-minimum phase zeros give effects. Hence, if P has no common non-minimum phase zero then $E_{\rm cn} = 0$. Particularly, if $W_v = 0$ and $W_y = 0$, which imply $\Lambda_{\rm o} = 1$, then we can recover the existing results on energy regulation problem obtained in [9] and [11].

Now we provide a simple example to confirm the validity of the result in Theorem 1.

Example 1 *Consider an SISO plant described by*

$$P(s) = \frac{s-3}{(s-\frac{1}{4})(s-p)}.$$

Clearly, $P(s)$ has one non-minimum phase zero at $s = 3$ and possibly two unstable poles at $s = \frac{1}{4}$ and $s = p$. We compute the optimal regulation performance $E_{\rm c}^*$ obtained by Theorem 1 (circled-line) and numerically calculated by MATLAB *toolbox (starred-line) for p from -1 to 5. Here we take $W_v(s) = 1/(s+1)$ and $W_y(s) = 1$. Fig. 2 shows that two computations match rather well. Particularly, when p closes to 3, the performance will be unbounded since it happens almost unstable pole-zero cancellation.*

4 Discrete-time Case

In this section, we formulate and solve the optimal regulation problem for discrete-time systems. Recall the coprime factorizations of $P(z)$ in (2), it is possible to set

$$M(z) = B(z) := \prod_{k=1}^{n_\lambda} \frac{1+\bar{\lambda}_k}{1+\lambda_k}\,\frac{z-\lambda_k}{\bar{\lambda}_k z-1} \tag{15}$$

without loss of generality, where λ_k ($k = 1,\ldots,n_\lambda$) are the unstable poles of $P(z)$. It is important to note that

$$B(-1) = 1, \quad B(\infty) = \prod_{k=1}^{n_\lambda}\frac{1+\bar{\lambda}_k}{1+\lambda_k}\,\frac{1}{\bar{\lambda}_k}.$$

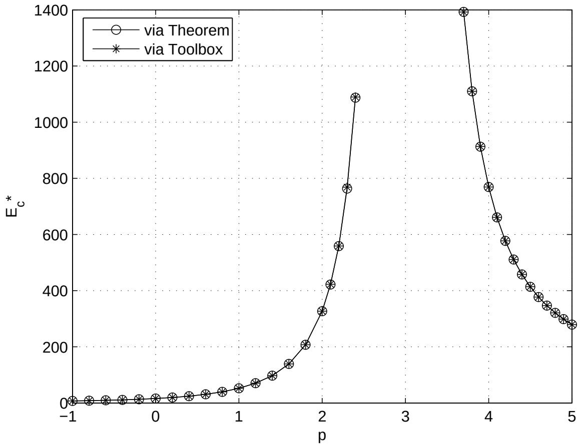

Figure 2: $E_{\rm c}^*$ with respect to pole location p.

Also note that the plant $P(z)$ and the weighting function $W_v(z)$ are not necessary to be strictly proper to regulate a discrete-time system.

Theorem 2 (Discrete-time) *Suppose that $P(z)$ has unstable poles $\lambda_k (k = 1, \ldots, n_\lambda)$ and its coprime factorizations are given by (2). Let define*

$$\breve{N}(z) = z\tilde{N}(z),$$

introduce the following index set:

$$\mathbb{N}_\eta := \{k : \breve{N}(\eta_k) = 0, \eta_k \in \bar{\mathbb{D}}^{\rm c}\}, \tag{16}$$

or $\mathbb{N}_\eta$ is the set of all common non-minimum phase zeros of $P(z)$ with counting multiplicities except one zero at infinity [1]*, and define the inner-outer factorization*

$$\begin{bmatrix} W_v \\ W_y N \\ -1 \end{bmatrix} = \Lambda_{\rm i}\Lambda_{\rm o}.$$

Then, we have

$$E_{\rm d}^* = E_{\rm dm} + E_{\rm dn}, \tag{17}$$

where

$$E_{\rm dm} = \exp\left\{\frac{1}{\pi}\int_0^\pi \log(1 + \|W_v({\rm e}^{{\rm j}\theta})\|^2 + \|W_y({\rm e}^{{\rm j}\theta})P({\rm e}^{{\rm j}\theta})\|^2){\rm d}\theta\right\}\prod_{k=1}^{n_\lambda}|\lambda_k|^2 - 1,$$

$$E_{\rm dn} = \sum_{k,\ell\in\mathbb{N}_\eta} \frac{(\bar{\eta}_k + 1)(\eta_\ell + 1)(|\eta_k|^2 - 1)(|\eta_\ell|^2 - 1)}{(\eta_k + 1)(\bar{\eta}_\ell + 1)\bar{b}_k b_\ell(\bar{\eta}_k\eta_\ell - 1)}\,\bar{\beta}_k\beta_\ell,$$

[1] The number of zeros at infinity in $\mathbb{N}_\eta$ is equal to the relative degree of $P(z)$ minus one.

with

$$b_k \ := \ \begin{cases} 1 & ; \#\mathbb{N}_\eta = 1 \\ \displaystyle\prod_{\ell \in \mathbb{N}_\eta, \ell \neq k} \frac{1+\bar{\eta}_k}{1+\eta_k} \frac{\eta_\ell - \eta_k}{\bar{\eta}_k \eta_\ell - 1}, & ; \#\mathbb{N}_\eta \geq 2 \end{cases},$$

$$\beta_k \ := \ \Lambda_{\rm o}(\infty) \prod_{i=1}^{n_\lambda} \bar{\lambda}_i - \Lambda_{\rm o}(\eta_k) \prod_{i=1}^{n_\lambda} \frac{\bar{\lambda}_i \eta_k - 1}{\eta_k - \lambda_i}.$$

Proof The proof is completely different from the continuous-time case. See Appendix A. ■

Theorem 2 reveals that there are at least two fundamental properties which differ from its continuous-time counterpart, Theorem 1. First, the contribution of unstable poles is given in product way rather than in summation. Second, the effect of plant gain is expressed in exponential way rather than in plain manner. These facts suggest us that in discrete-time case, unstable poles and plant gain contribute more detrimental effects than those in continuous-time case.

The remarks below Theorem 1 for the continuous-time case are also valid for the discrete-time case. Particulary, the exact expressions for $|\Lambda_{\rm o}(\infty)|$ and $|\Lambda_{\rm o}(\eta_k)|$ can be derived from Lemma 1. Again, if $W_v = 0$ and $W_y = 0$, which imply $\Lambda_{\rm o} = 1$, then we can recover the existing result on energy regulation problem derived in [8] and on LTI networked control in [11].

The following illustrative example confirms the validity of the result in Theorem 2.

Example 2 *We consider an SISO plant given by*

$$P(z) = \frac{4z^2 - 9}{z(3z+4)(z-\lambda)},$$

which has non-minimum phase zeros at $z = 3/2$ and $z = -3/2$, and unstable poles at $z = -4/3$ and possibly at $z = \lambda$. Fig. 3 plots Theorem 2 based computation (circled-line) and MATLAB *toolbox-based computation (starred-line) for λ from -3 to 3, where we set $W_v(z) = 1$ and $W_y(z) = 1$. The figure clearly shows that the expression in Theorem 2 is correct and that $E_{\rm d}^*$ becomes larger when λ approaches to one of the non-minimum phase zeros.*

5 Unified Results

In this section we reformulate and solve the optimal regulation problem in terms of the delta operator [12] in order to link the continuous-time and discrete-time results derived in the previous sections.

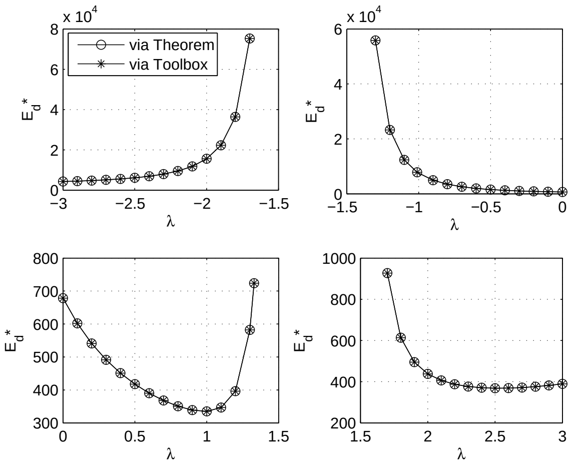

Figure 3: E_{d}^{*} with respect to pole location λ.

5.1 Delta Transforms

The delta operator δ is defined by

$$\delta x(k) = \frac{x(k+1) - x(k)}{T},$$

for any sequence $x(k)$, $k = 1, 2, \ldots$, with $T > 0$ is the sampling time. By taking the $\mathcal{Z}$-transform of above equation we obtain $\delta \hat{x}(z) = \frac{z-1}{T} \hat{x}(z)$. Later, the variable δ is used as the delta operator variable and is analogous to the Laplace variable s for continuous-time systems and the $\mathcal{Z}$-transform variable z for discrete-time systems. We then obtain the following relationship:

$$\delta = \frac{z-1}{T} \Leftrightarrow z = T\delta + 1. \tag{18}$$

For any sequence $x(k)$ we define its delta transform by

$$\mathcal{D}\{x(k)\} = \hat{x}_T(\delta) := T \sum_{k=0}^{\infty} x(k)(T\delta + 1)^{-k},$$

or equivalently, $\hat{x}_T(\delta) = T\hat{x}(z)|_{z=T\delta+1}$. The inner product equipped in $\mathcal{L}_2$ is then defined as

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} f^{\mathrm{H}} \left(\frac{\mathrm{e}^{\mathrm{j}\omega T} - 1}{T} \right) g \left(\frac{\mathrm{e}^{\mathrm{j}\omega T} - 1}{T} \right) \mathrm{d}\omega.$$

Let $F(z)$ be given and define $G(\delta) := F(T\delta + 1)$. Then by setting $\theta = \omega T$, it is easy to verify that the following $\mathcal{H}_2$ norms relation holds,

$$\|G(\delta)\|_2^2 = \|F(z)\|_2^2 / T. \tag{19}$$

For $T > 0$, we denote $\mathbb{D}_T := \{\delta \in \mathbb{C} : |T\delta + 1| < 1\}$ and $\bar{\mathbb{D}}_T^{\mathrm{c}} := \{\delta \in \mathbb{C} : |T\delta + 1| > 1\}$.

5.2 Problem Reformulation and Its Solution

Consider the coprime factorizations of $P(\delta)$ given in (2), in which without lost of generality we may set $M = H$, where

$$H(\delta) := B(T\delta + 1) = \prod_{k=1}^{n_\lambda} \frac{1+\bar{\lambda}_k}{1+\lambda_k} \frac{(T\delta+1)-\lambda_i}{\bar{\lambda}_i(T\delta+1)-1}, \tag{20}$$

with λ_k are the unstable poles of $P(z)$. It is easy to show that H is inner in δ-domain, i.e., $H(\frac{\mathrm{e}^{-\mathrm{j}\omega T}-1}{T})H(\frac{\mathrm{e}^{\mathrm{j}\omega T}-1}{T}) = 1$ and $H(\infty) = B(\infty)$. We remark that H possesses non-minimum phase zeros $\rho_k \in \bar{\mathbb{D}}_T^{\mathrm{c}}$ at $\rho_k = (\lambda_k - 1)/T\,(k = 1, 2, \ldots, n_\rho)$, in which they also act as the unstable poles of $P(\delta)$. Note that $n_\rho = n_\lambda$.

We consider the following performance index

$$E_\delta := T\sum_{k=0}^{\infty}(\|z(k)\|^2 + |u(k)|^2), \tag{21}$$

where $z(k)$ is given by

$$z(k) = \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix} = \begin{pmatrix} \mathcal{D}^{-1}\{W_v(\delta)\hat{v}(\delta)\} \\ \mathcal{D}^{-1}\{W_y(\delta)\hat{y}(\delta)\} \end{pmatrix}.$$

As a disturbance signal, we consider impulse function in the form of

$$d(k) = \begin{cases} \frac{1}{T}, & \text{for } k = 0 \\ 0, & \text{for } k \neq 0 \end{cases}, \tag{22}$$

where its δ-transform is $\hat{d}_T(\delta) = 1$.

Now we are ready to provide the delta-time solution. First we reformulate Lemmas 1 and 3 in δ-domain as follow, respectively.

Lemma 4 *Let h is analytic in $\bar{\mathbb{D}}_T^{\mathrm{c}}$ and σ_k $(k = 1, \ldots, n_\sigma)$ be the zeros of h in $\bar{\mathbb{D}}_T^{\mathrm{c}}$, counting their multiplicities. If $\delta \in \bar{\mathbb{D}}_T^{\mathrm{c}}$ and $h(\delta) \neq 0$, then*

$$\begin{aligned}\log|h(\delta)| &= \frac{T}{\pi}\int_0^{\pi/T} \mathrm{Re}\left(\frac{(T\delta+1)\mathrm{e}^{\mathrm{j}\omega T}+1}{(T\delta+1)\mathrm{e}^{\mathrm{j}\omega T}-1}\right)\log\left|h\left(\frac{\mathrm{e}^{\mathrm{j}\omega T}-1}{T}\right)\right|\mathrm{d}\omega \\ &\quad - \sum_{k=1}^{n_\sigma}\log\left|\frac{T\delta\bar{\sigma}_k+\bar{\sigma}_k+\delta}{\delta-\sigma_k}\right|.\end{aligned}$$

Lemma 5 *If $h \in \mathbb{R}\mathcal{H}_\infty$, then*

$$\frac{1}{\pi}\int_0^{\pi/T}\mathrm{Re}\, h\left(\frac{\mathrm{e}^{\mathrm{j}\omega T}-1}{T}\right)\mathrm{d}\omega = \frac{h(\infty)}{T}.$$

Theorem 3 (Delta-time) *Suppose that $P(\delta)$ has unstable poles ρ_k ($k = 1,\dots,n_\rho$) and its coprime factorizations are given by (2). Let define*

$$\breve{N}(\delta) = \delta\tilde{N}(\delta),$$

and introduce the following index set:

$$\mathbb{N}_\zeta := \{k : \breve{N}(\zeta_k) = 0, \zeta_k \in \bar{\mathbb{D}}_T^{\mathrm{c}}\}. \tag{23}$$

Let denote $\eta_k = T\zeta_k + 1$, $\lambda_k = T\rho_k + 1$, and define the inner-outer factorization

$$\begin{bmatrix} W_v \\ W_y N \\ -1 \end{bmatrix} = \Lambda_{\mathrm{i}}\Lambda_{\mathrm{o}}.$$

Then, we have

$$E_\delta^* = E_{\delta\mathrm{m}} + E_{\delta\mathrm{n}}, \tag{24}$$

where

$$\begin{aligned} E_{\delta\mathrm{m}} &= \frac{1}{T}\left(|\Lambda_{\mathrm{o}}(\infty)|^2 \prod_{k=1}^{n_\rho} |T\rho_k + 1|^2 - 1\right), \\ E_{\delta\mathrm{n}} &= \frac{1}{T}\sum_{k,\ell\in\mathbb{N}_\zeta} \frac{(\bar{\eta}_k + 1)(\eta_\ell + 1)(|\eta_k|^2 - 1)(|\eta_\ell|^2 - 1)}{(\eta_k + 1)(\bar{\eta}_\ell + 1)\bar{g}_k g_\ell(\bar{\eta}_k\eta_\ell - 1)}\,\bar{\gamma}_k\gamma_\ell, \end{aligned}$$

with

$$\begin{aligned} |\Lambda_{\mathrm{o}}(\infty)|^2 &= \exp\left\{\frac{T}{\pi}\int_0^{\pi/T} \log(1 + \|W_v(\frac{\mathrm{e}^{\mathrm{j}\omega T} - 1}{T})\|^2 + \right. \\ &\qquad \left. \|W_y(\frac{\mathrm{e}^{\mathrm{j}\omega T} - 1}{T})P(\frac{\mathrm{e}^{\mathrm{j}\omega T} - 1}{T})\|^2)\,\mathrm{d}\omega \right\}, \\ g_k &:= \begin{cases} 1 & ;\ \#\mathbb{N}_\zeta = 1 \\ \displaystyle\prod_{\ell\in\mathbb{N}_\zeta,\ell\neq k} \frac{1 + \bar{\eta}_k}{1 + \eta_k}\frac{\eta_\ell - \eta_k}{\bar{\eta}_k\eta_\ell - 1}, & ;\ \#\mathbb{N}_\zeta \geq 2 \end{cases}, \\ \gamma_k &:= \Lambda_{\mathrm{o}}(\infty)\prod_{i=1}^{n_\lambda}\bar{\lambda}_i - \Lambda_{\mathrm{o}}(\zeta_k)\prod_{i=1}^{n_\lambda}\frac{\bar{\lambda}_i\eta_k - 1}{\eta_k - \lambda_i}. \end{aligned}$$

Proof See Appendix B. ■

We here remark that the optimal solution in δ-domain shares many similarities with its predecessor, i.e., Theorem 2. Except the existence of sampling time factor $1/T$, all parameters are parallel with the relationship between z and δ, that is, $z = T\delta + 1$.

5.3 Continuity Properties

In this subsection we show the continuity properties of the δ-domain solution, i.e., we will demonstrate that E_δ^* converges to $E_{\rm c}^*$ when the sampling time T tends to zero. To this end we consider a continuous-time plant $P(s)$ which has common non-minimum phase zeros z_k and unstable poles p_k. Under the zero-order hold operations we obtain the corresponding discrete-time plant $P(z)$ which has those of η_k and λ_k, and also the delta-time plant $P(\delta)$ which has those of ζ_k and ρ_k.

5.3.1 Convergence of $E_{\delta{\rm m}}$

It is easy to verify that $E_{\delta{\rm m}}$ can be written as $E_{\delta{\rm m}} = E_H + E_R$, where

$$\begin{aligned} E_H &:= \frac{1}{T}\left(\prod_{k=1}^{n_\rho} |T\rho_k + 1|^2 - 1\right), \\ E_R &:= \frac{|\Lambda_{\rm o}(\infty)|^2 - 1}{T} \prod_{k=1}^{n_\rho} |T\rho_k + 1|^2. \end{aligned}$$

Since

$$E_H \approx 2\sum_{k=1}^{n_\rho} \rho_k = 2\sum_{k=1}^{n_p} \frac{{\rm e}^{p_k T} - 1}{T}$$

by spectral mapping theorem, then we get $\lim_{T\to 0} E_H = 2\sum_{k=1}^{n_p} p_k$. Next, since $\frac{{\rm e}^{{\rm j}\omega T}-1}{T}$ tends to ${\rm j}\omega$ and $|T\rho_k + 1|^2$ tends to 1 as T tends to zero, we have

$$\lim_{T\to 0} E_R = \frac{1}{\pi}\int_0^\infty \log\ \left(1 + \|W_v({\rm j}\omega)\|^2 + \|W_y({\rm j}\omega)P({\rm j}\omega)\|^2\right) {\rm d}\omega.$$

These two facts then show that

$$\lim_{T\to 0} E_{\delta{\rm m}} = E_{\rm cm}. \tag{25}$$

5.3.2 Convergence of $E_{\delta{\rm n}}$

We show the convergence of $E_{\delta{\rm n}}$ part by part. First, we denote

$$E_\eta := \frac{1}{T}\sum_{k,\ell\in\mathbb{N}_\zeta} \frac{(\bar\eta_k + 1)(\eta_\ell + 1)(|\eta_k|^2 - 1)(|\eta_\ell|^2 - 1)}{(\eta_k + 1)(\bar\eta_\ell + 1)(\bar\eta_k\eta_\ell - 1)}.$$

Then by noting that $\eta_k = {\rm e}^{z_k T}$,

$$\lim_{T\to 0} E_\eta = \sum_{k,\ell\in\mathbb{N}_z} \lim_{T\to 0} \frac{({\rm e}^{2T\,{\rm Re}\,z_k} - 1)({\rm e}^{2T\,{\rm Re}\,z_\ell} - 1)}{T({\rm e}^{(\bar z_k + z_\ell)T} - 1)} = \sum_{k,\ell\in\mathbb{N}_z} \frac{4\,{\rm Re}\,z_k\,{\rm Re}\,z_\ell}{\bar z_k + z_\ell}.$$

Next we deal with g_k for $\#\mathbb{N}_\zeta \geq 2$. Immediately we obtain

$$\lim_{T\to 0} g_k = \prod_{\ell\in\mathbb{N}_z,\ell\neq k} \lim_{T\to 0} \frac{\mathrm{e}^{z_\ell T} - \mathrm{e}^{z_k T}}{\mathrm{e}^{(\bar{z}_k + z_\ell)T} - 1} = \prod_{\ell\in\mathbb{N}_z,\ell\neq k} \frac{z_\ell - z_k}{z_\ell + \bar{z}_k},$$

which shows that $\lim_{T\to 0} g_k = a_k$. To inspect the convergence of γ_k, we know that $\lim_{T\to 0} |\Lambda_{\mathrm{o}}(\infty)|^2 = 1$. Thus, $\lim_{T\to 0} \Lambda_{\mathrm{o}}(\infty) = 1$. We also know that

$$\lim_{T\to 0} \prod_{i=1}^{n_\lambda} \bar{\lambda}_i = \prod_{i=1}^{n_p} \lim_{T\to 0} \mathrm{e}^{\bar{p}_i T} = 1,$$

$$\lim_{T\to 0} \prod_{i=1}^{n_\lambda} \frac{\bar{\lambda}_i \eta_k - 1}{\eta_k - \lambda_i} = \prod_{i=1}^{n_p} \lim_{T\to 0} \frac{\mathrm{e}^{(\bar{p}_i + z_k)T} - 1}{\mathrm{e}^{z_k T} - \mathrm{e}^{\bar{p}_i T}} = \prod_{i=1}^{n_p} \frac{z_k + \bar{p}_i}{z_k - p_i}.$$

Now we only need to show the convergence of $\Lambda_{\mathrm{o}}(\zeta_k)$. From Lemma 4 we have

$$|\Lambda_{\mathrm{o}}(\zeta_k)| = \exp\left\{ \frac{T}{\pi} \int_0^{\pi/T} \mathrm{Re}\left(\frac{(T\zeta_k + 1)\mathrm{e}^{\mathrm{j}\omega T} + 1}{(T\zeta_k + 1)\mathrm{e}^{\mathrm{j}\omega T} - 1} \right) \log \left| \Lambda_{\mathrm{o}}(\frac{\mathrm{e}^{\mathrm{j}\omega T} - 1}{T}) \right| \mathrm{d}\omega \right\}.$$

Since

$$\lim_{T\to 0} T\,\mathrm{Re}\left[\frac{(T\zeta_k + 1)\mathrm{e}^{\mathrm{j}\omega T} + 1}{(T\zeta_k + 1)\mathrm{e}^{\mathrm{j}\omega T} - 1} \right] = 2\,\mathrm{Re}\left[\frac{1}{z_k + \mathrm{j}\omega} \right] = \frac{2}{1+\omega^2}\,\mathrm{Re}\left[\frac{1 + \mathrm{j}\omega z_k}{z_k + \mathrm{j}\omega} \right]$$

implies $\lim_{T\to 0} \Lambda_{\mathrm{o}}(\zeta_k) = \Lambda_{\mathrm{o}}(z_k)$, then we achieve $\lim_{T\to 0} \gamma_k = \alpha_k$. Communicating all the above facts yields

$$\lim_{T\to 0} E_{\delta\mathrm{n}} = E_{\mathrm{cn}}. \tag{26}$$

Therefore, (25) and (26) conclude

$$\lim_{T\to 0} E_\delta^* = E_{\mathrm{c}}^*,$$

i.e., the δ-domain solution converges to the corresponding s-domain solution as sampling time T approaches to zero.

Now we provide an example to demonstrate the continuity properties of the optimal delta solution.

Example 3 *Reconsider the SISO continuous-time plant given in Example 1. Implementation of zero-order hold operation yields the corresponding delta-time plant*

$$P(\delta) = \frac{c(\delta - \zeta)}{(\delta - \rho_1)(\delta - \rho_2)},$$

which has also one non-minimum phase zero at $\delta = \zeta$ *and possibly two unstable poles at* $\delta = \rho_1$ *and* $\delta = \rho_2$. *Under the corresponding weighting functions* $W_v(\delta)$ *and* $W_y(\delta)$, *Fig. 4 shows the computation of* E_{c}^* *of Theorem 1 (solid line) and that of* E_δ^* *of Theorem 3 for sampling time* $T = 0.10, 0.05$ *seconds (dashed and dash-dotted lines, respectively). This confirms our result that* E_δ^* *converges to* E_{c}^* *as* T *approaches zero.*

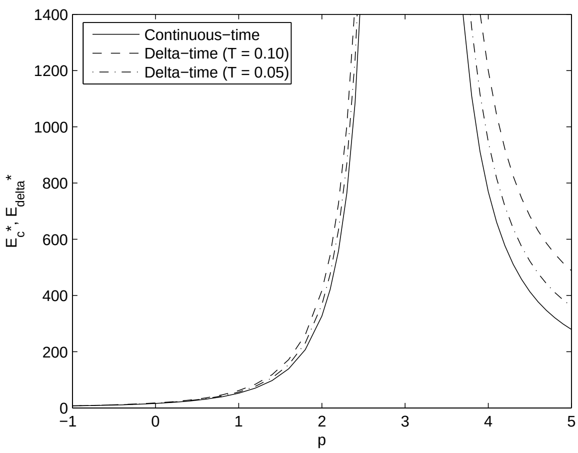

Figure 4: The convergence of the delta-time solution to its continuous-time counterpart.

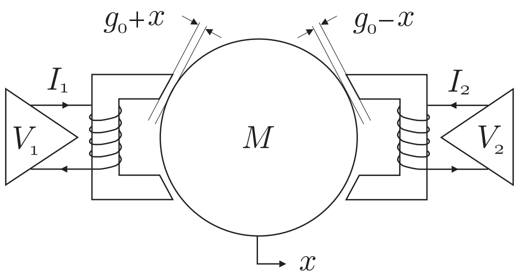

Figure 5: Active magnetic bearing.

6 Application to Magnetic Bearing System

6.1 Problem Setting

In this section we implement our results to study the performance limitations in a magnetic bearing system investigated in [10]. We consider a simple active magnetic bearing (AMB) depicted in Fig. 5. AMBs suspend the levitated object (generally, a rotor) of mass M by forces of two opposing magnetic attractions which are supplied by power switching amplifiers of voltages V_1, V_2 and currents I_1, I_2. AMBs use actively controlled electromagnetic forces to control the position of the rotor or other ferromagnetic body in air which has nominal air gap g_0.

A dimensionless non-linear model for the AMB system of Fig. 5 can be found in [10]. If we assume that the state variable can be forced to track some constant trajectory Φ_0 by appropriate choice of control input u, then a linearizing model may be realized as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ v \\ \phi \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \Phi_0 \\ \mu\Phi_0 & 0 & -\mu \end{bmatrix}\begin{bmatrix} x \\ v \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} -\Phi_0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ v \\ \phi \end{bmatrix},\end{aligned}$$

where x and v respectively denote the normalized position of the rotor and its derivative, and ϕ, u, and y are respectively normalized differences of the fluxes (Φ_1, Φ_2), input voltages (V_1, V_2), and output currents (I_1, I_2) of left and right magnetics, which are given by

$$\phi := \frac{\Phi_1 - \Phi_2}{A_{\mathrm{g}} B_{\mathrm{sat}}},\ u := \frac{V_1 - V_2}{V_0},\ y := \frac{I_1 - I_2}{I_{\mathrm{sat}}}$$

with appropriate constants A_{g}, B_{sat}, V_0, and I_{sat}.

We define by P_{c} the transfer function from the control input to the current sensor, i.e., from u to y, by P_{p} the transfer function from the control input to the position sensor, i.e., from u to x, and by P_{cp} the transfer function from the control input to the both of current and position sensors, i.e., from u to x and y. In other words, we define

$$P_{\mathrm{c}}(s) = \frac{s^2 - \Phi_0^2}{s^3 + \mu s^2 - \mu\Phi_0^2}, \tag{27}$$

$$P_{\mathrm{p}}(s) = \frac{\Phi_0}{s^3 + \mu s^2 - \mu\Phi_0^2}, \tag{28}$$

$$P_{\mathrm{cp}}(s) = \begin{bmatrix} P_{\mathrm{c}}(s) \\ P_{\mathrm{p}}(s) \end{bmatrix} \tag{29}$$

It is clear that $P_{\mathrm{c}}(s)$ has one non-minimum phase zero at Φ_0 and one unstable pole p lying between 0 and Φ_0, depending on the value of Φ_0 and μ. It is known that physically reasonable value of $\sigma := \Phi_0/\mu$ ranges between about 0.3 and 3 and that, the unstable pole p ranges from about $0.6\Phi_0$ and $0.9\Phi_0$ as used in [10]. Further, $P_{\mathrm{p}}(s)$ has no non-minimum phase zero but one unstable pole at p. While, $P_{\mathrm{cp}}(s)$ has no common non-minimum phase zero but one unstable pole, also at p.

We here examine the regulation performance limitation of the AMB as measured by the $\mathcal{H}_2$ norm. In other words, we minimize the following performance measure

$$E = \|\hat{z}_1\|_2^2 + \|\hat{z}_2\|_2^2 + \|\hat{u}\|_2^2. \tag{30}$$

To facilitate our analysis, we define by $E^*_{\mathrm{c,c}}$, $E^*_{\mathrm{c,p}}$, $E^*_{\mathrm{c,cp}}$, the optimal performances with respect to $P_{\mathrm{c}}(s)$, $P_{\mathrm{p}}(s)$, and $P_{\mathrm{cp}}(s)$, respectively. We examine the following three cases:

$$
\begin{array}{lll}
(1) & W_{y,\mathrm{c}}(s) = \begin{bmatrix} \rho_{\mathrm{c}} \\ \frac{\rho_{\mathrm{p}}\Phi_0}{(s+\Phi_0)^2} \end{bmatrix}, & \text{for } P(s) = P_{\mathrm{c}}(s), \\
(2) & W_{y,\mathrm{p}}(s) = \begin{bmatrix} \frac{\rho_{\mathrm{c}}(s+\Phi_0)^2}{\Phi_0} \\ \rho_{\mathrm{p}} \end{bmatrix}, & \text{for } P(s) = P_{\mathrm{p}}(s), \\
(3) & W_{y,\mathrm{cp}}(s) = \begin{bmatrix} \rho_{\mathrm{c}} & 0 \\ 0 & \rho_{\mathrm{p}} \end{bmatrix}, & \text{for } P(s) = P_{\mathrm{cp}}(s).
\end{array}
$$

Note that the definitions $W_y(s)$ above assure that the performance indexes for the three cases are completely same. For all the cases we choose the weighting function W_v as

$$W_v(s) = \frac{w}{1+\tau s},$$

where τ is the bandwidth. The measure E in (30) then can be written as

$$E = \|\hat{z}_1\|_2^2 + \|\rho_{\mathrm{c}}\hat{z}_{2\mathrm{c}}\|_2^2 + \|\rho_{\mathrm{p}}\hat{z}_{2\mathrm{p}}\|_2^2 + \|\hat{u}\|_2^2, \tag{31}$$

where $\hat{z}_{2\mathrm{c}}$ and $\hat{z}_{2\mathrm{p}}$ are the current and position sensor outputs, respectively.

6.2 Continuous-time Case

We first note that the first terms of (14), E_{cm}, are the same for all the three cases. Since $P_{\mathrm{p}}(s)$ has no non-minimum phase zero and $P_{\mathrm{cp}}(s)$ has no common one (note that only common non-minimum phase zero gives limitation), we can see that $E_{\mathrm{cn}} = 0$ and hence the optimal performances of these two cases are equal. On the other hand, $P_{\mathrm{c}}(s)$ has one non-minimum phase zero at Φ_0, and hence E_{cn} is always positive. This observation implies that the following relations generally hold:

$$E^*_{\mathrm{c,c}} > E^*_{\mathrm{c,p}} = E^*_{\mathrm{c,cp}},$$

where

$$E^*_{\mathrm{c,c}} - E^*_{\mathrm{c,p}} = 2\Phi_0 \left[1 - \Lambda_{\mathrm{o}}(\Phi_0)\frac{\Phi_0 + p}{\Phi_0 - p}\right]^2.$$

This can be confirmed by the following further investigation. First we consider a case where $\rho_{\mathrm{c}} = \rho_{\mathrm{p}} = 0$, i.e., $W_y(s) = 0$. For this special case, clearly we obtain

$$\Lambda_{\mathrm{o}}(s) = \frac{\sqrt{1+w^2} + \tau s}{1+\tau s}.$$

Then, the closed-form expression of the optimal performances then can be expressed as

$$
\begin{aligned}
E^*_{\mathrm{c,p}} = E^*_{\mathrm{c,cp}} &= 2p + \frac{1}{\pi}\int_0^\infty \log\left(1 + \frac{w^2}{1+\omega^2\tau^2}\right) \mathrm{d}\omega \\
&= 2p + \frac{\sqrt{1+w^2} - 1}{|\tau|}
\end{aligned}
$$

and

$$E^*_{\mathrm{c,c}} - E^*_{\mathrm{c,p}} = 2\Phi_0 \left(1 - \frac{\sqrt{1+w^2}+\tau\Phi_0}{1+\tau\Phi_0}\,\frac{\Phi_0+p}{\Phi_0-p}\right)^2 > 0.$$

We now compute the optimal performances with the following physical parameters [10]: $\Phi_0 = 0.288$, $\mu = 0.582$, from which we get $p = 0.242$. For the weighting function $W_v(s)$, we take $w = 1$ and $\tau = 1$. The computation results give $E^*_{\mathrm{c,c}} = 117.7013$ and $E^*_{\mathrm{c,p}} = E^*_{\mathrm{c,cp}} = 0.8983$.

Next, we consider a case where $\rho_{\mathrm{c}} = \rho_{\mathrm{p}} = 1$. Note that in computation $E^*_{\mathrm{c,c}}$, $\Lambda_{\mathrm{o}}(s)$ is determined from the inner-outer factorization

$$\begin{bmatrix} W_v(s) \\ W_{y,\mathrm{c}}(s)N_{\mathrm{c}}(s) \\ -1 \end{bmatrix} = \Lambda_{\mathrm{i}}(s)\Lambda_{\mathrm{o}}(s),$$

where $N_{\mathrm{c}}(s)$ is the coprime factor of $P_{\mathrm{c}}(s)$, i.e., $P_{\mathrm{c}}(s) = N_{\mathrm{c}}(s)M_{\mathrm{c}}^{-1}(s)$. The computation results provide $E^*_{\mathrm{c,c}} = 716.5626$ and $E^*_{\mathrm{c,p}} = E^*_{\mathrm{c,cp}} = 1.5821$.

6.3 Discrete-time Case

We here discuss the discrete-time case. We assume that the corresponding discrete-time transfer functions $P_{\mathrm{c}}(z)$, $P_{\mathrm{p}}(z)$, $P_{\mathrm{cp}}(z)$ are obtained from the zero-order hold operations of $P_{\mathrm{c}}(s)$, $P_{\mathrm{p}}(z)$, $P_{\mathrm{cp}}(s)$, respectively. By these operations, we know that $P_{\mathrm{c}}(z)$ has two non-minimum phase zeros at 1.3351 and at infinity. $P_{\mathrm{p}}(z)$ has also two non-minimum phase zeros at -3.2498 and at infinity, while $P_{\mathrm{cp}}(z)$ has no common non-minimum phase zero except at infinity. All the plants have one unstable pole at 1.2738. The optimal performance then can be computed based on Theorem 2, and we can show the following relations hold:

$$E^*_{\mathrm{d,c}} > E^*_{\mathrm{d,p}} > E^*_{\mathrm{d,cp}}.$$

For example, for $\rho_{\mathrm{c}} = \rho_{\mathrm{p}} = 0$ with the sampling time $T = 1$ second, we have $E^*_{\mathrm{d,c}} = 150.3615$, $E^*_{\mathrm{d,p}} = 1.7506$, and $E^*_{\mathrm{d,cp}} = 1.3383$, which implies that using multiple sensors has an advantage for the discrete-time system.

7 Conclusion

In this paper, we have examined $\mathcal{H}_2$ output regulation problems for SIMO LTI feedback control systems. We have derived analytical closed-form expressions of the optimal regulation performance for unstable/non-minimum phase continuous-time and discrete-time systems. In general, our results show that the best achievable output regulation performance depends upon plant unstable poles, plant non-minimum phase zeros, plant gain, and a certain outer factors. We clarified several special cases where the formulae do not include the outer factors, which directly characterize the performance limitations by unstable poles, non-minimum phase zeros, and gain of the plant.

We have then applied those results to a magnetic bearing system. For the continuous-time system, we showed that using a position sensor rather than a current sensor or multiple sensors significantly improves the achievable sensitivity performance, while in the discrete-time case, implementing multiple sensors gives an advantage to using a position sensor only.

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A Proof of Theorem 2

From Bezout identity (3) we have $B^{-1}Y\tilde{N} = -B^{-1} + \tilde{X}$. This fact then enables us to write (11) as

$$\begin{aligned} E_{\mathrm{d}}^{*} &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W_v(B^{-1} + B^{-1}Y\tilde{N} - Q\tilde{N}) \\ W_y(X\tilde{N} - NQ\tilde{N}) \\ B^{-1}Y\tilde{N} - Q\tilde{N} \end{bmatrix} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W_v(\tilde{X} - Q\tilde{N}) \\ W_y(N\tilde{X} - NQ\tilde{N}) \\ -B^{-1} + \tilde{X} - Q\tilde{N} \end{bmatrix} \right\|_2^2 . \end{aligned}$$

For any $Q \in \mathbb{R}\mathcal{H}_\infty$ such that $(B^{-1}(\infty) - \tilde{X} + Q\tilde{N}) \in \mathcal{H}_2$ and together with the fact that $(B^{-1}(\infty) - B^{-1}) \in \mathcal{H}_2^{\perp}$ then

$$E_{\mathrm{d}}^{*} = E_B + E_Q,$$

where

$$
\begin{aligned}
E_B &:= \left\| \begin{bmatrix} 0 \\ 0 \\ B^{-1}(\infty) - B^{-1} \end{bmatrix} \right\|_2^2, \\
E_Q &:= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W_v(\tilde{X} - Q\tilde{N}) \\ W_y(N\tilde{X} - NQ\tilde{N}) \\ B^{-1}(\infty) - \tilde{X} + Q\tilde{N} \end{bmatrix} \right\|_2^2.
\end{aligned}
$$

Since $B(z)$ is inner then

$$
E_B = \|B^{-1}(\infty)B(z) - 1\|_2^2 = \left\| \prod_{k=1}^{n_\lambda} \frac{\bar{\lambda}_k z - |\lambda_k|^2}{\bar{\lambda}_k z - 1} - 1 \right\|_2^2.
$$

Let define

$$
E_{B,n} = \left\| \prod_{k=1}^{n} \frac{\bar{\lambda}_k z - |\lambda_k|^2}{\bar{\lambda}_k z - 1} - 1 \right\|_2^2, \tag{32}
$$

and claim that

$$
E_{B,n} = \prod_{k=1}^{n} |\lambda_k|^2 - 1.
$$

We rely on the mathematical induction to prove our claim. For $n = 1$, it is true that

$$
E_{B,1} = \left\| \frac{1 - |\lambda_1|^2}{\lambda_1 z - 1} \right\|_2^2 = |\lambda_1|^2 - 1.
$$

Further, let define

$$
\xi(z) = \frac{\bar{\lambda}_n z - 1}{\bar{\lambda}_n z - |\lambda_n|^2} \frac{\bar{\lambda}_n - |\lambda_n|^2}{\bar{\lambda}_n - 1}.
$$

It is easy to show that $\xi(z)$ is an inner function. Then by pre-multiplying $\xi(z)$ to (32), we may obtain

$$
\begin{aligned}
E_{B,n} &= \left\| \frac{\bar{\lambda}_n - |\lambda_n|^2}{\bar{\lambda}_n - 1} \left[\prod_{k=1}^{n-1} \frac{\bar{\lambda}_k z - |\lambda_k|^2}{\bar{\lambda}_k z - 1} - \frac{\bar{\lambda}_n z - 1}{\bar{\lambda}_n z - |\lambda_n|^2} \right] \right\|_2^2 \\
&= |\lambda_n|^2 \left\| \left[\prod_{k=1}^{n-1} \frac{\bar{\lambda}_k z - |\lambda_k|^2}{\bar{\lambda}_k z - 1} - 1 \right] - \left[\frac{\bar{\lambda}_n z - 1}{\bar{\lambda}_n z - |\lambda_n|^2} - 1 \right] \right\|_2^2 \\
&= |\lambda_n|^2 \left(E_{B,n-1} + \left\| \frac{\bar{\lambda}_n z - 1}{\bar{\lambda}_n z - |\lambda_n|^2} - 1 \right\|_2^2 \right).
\end{aligned}
$$

A direct calculation shows that

$$
\left\| \frac{\bar{\lambda}_n z - 1}{\bar{\lambda}_n z - |\lambda_n|^2} - 1 \right\|_2^2 = \frac{1}{|\lambda_n|^2} \left\| \frac{|\lambda_n|^2 - 1}{z - \lambda_n} \right\|_2^2 = \frac{|\lambda_n|^2 - 1}{|\lambda_n|^2}.
$$

Hence, we may write $E_{B,n}$ as a recursive expression

$$E_{B,n} = |\lambda_n|^2 E_{B,n-1} + |\lambda_n|^2 - 1.$$

Suppose it is true that

$$E_{B,n-1} = \prod_{k=1}^{n-1} |\lambda_k|^2 - 1.$$

Then we get

$$E_{B,n} = |\lambda_n|^2 \left(\prod_{k=1}^{n-1} |\lambda_k|^2 - 1 \right) + |\lambda_n|^2 - 1 = \prod_{k=1}^{n} |\lambda_k|^2 - 1.$$

This proves our claim and show that

$$E_B = \prod_{k=1}^{n_\lambda} |\lambda_k|^2 - 1. \tag{33}$$

Next, we may write

$$E_Q = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W\tilde{X} \\ B^{-1}(\infty) - \tilde{X} \end{bmatrix} - \begin{bmatrix} W \\ -1 \end{bmatrix} Q\tilde{N} \right\|_2^2,$$

where

$$W := \begin{bmatrix} W_v \\ W_y N \end{bmatrix}.$$

Perform the inner-outer factorization such that

$$\begin{bmatrix} W \\ -1 \end{bmatrix} = \Lambda_{\mathrm{i}} \Lambda_{\mathrm{o}},$$

where the inner factor Λ_{i} is stable factor and the outer factor Λ_{o} represents the minimum phase part. Note that Λ_{i} is a column vector transfer function, Λ_{o} is a scalar transfer function, and

$$\Lambda_{\mathrm{o}}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\theta}) \Lambda_{\mathrm{o}}(\mathrm{e}^{\mathrm{j}\theta}) = W^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\theta}) W(\mathrm{e}^{\mathrm{j}\theta}) + 1.$$

Let

$$\Gamma(z) = \begin{bmatrix} \Lambda_{\mathrm{i}}^{\sim}(z) \\ I - \Lambda_{\mathrm{i}}(z) \Lambda_{\mathrm{i}}^{\sim}(z) \end{bmatrix},$$

i.e., $\Gamma(z)$ is an inner function. By pre-multiplying Γ we obtain

$$\begin{aligned} E_Q &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \Gamma \left\{ \begin{bmatrix} W\tilde{X} \\ B^{-1}(\infty) - \tilde{X} \end{bmatrix} - \begin{bmatrix} W \\ -1 \end{bmatrix} Q\tilde{N} \right\} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| C_1 - \Lambda_{\mathrm{o}} Q\tilde{N} \right\|_2^2 + \|C_2\|_2^2, \end{aligned}$$

where

$$\begin{array}{rcl} C_1 & := & \Lambda_\mathrm{o}\tilde{X} - \Lambda_\mathrm{o}^{-\mathrm{H}} B^{-1}(\infty), \\ C_2 & := & \left[\begin{array}{c} W(\Lambda_\mathrm{o}^\mathrm{H}\Lambda_\mathrm{o})^{-1} \\ 1-(\Lambda_\mathrm{o}^\mathrm{H}\Lambda_\mathrm{o})^{-1} \end{array} \right] B^{-1}(\infty). \end{array}$$

Further,

$$\begin{array}{rcl} E_Q & = & \inf\limits_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \Lambda_\mathrm{o}\tilde{X} - \Lambda_\mathrm{o}(\infty)B^{-1}(\infty) - \Lambda_\mathrm{o}Q\tilde{N} \right\|_2^2 \\ & & + \left|B^{-1}(\infty)\right|^2 \left\|\Lambda_\mathrm{o}^{-\mathrm{H}} - \Lambda_\mathrm{o}(\infty)\right\|_2^2 + \|C_2\|_2^2. \end{array}$$

Let denote

$$\begin{array}{rcl} E_{Q_1} & := & \inf\limits_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \Lambda_\mathrm{o}\tilde{X} - \Lambda_\mathrm{o}(\infty)B^{-1}(\infty) - \Lambda_\mathrm{o}Q\tilde{N} \right\|_2^2, \\ E_{Q_2} & := & \left|B^{-1}(\infty)\right|^2 \left\|\Lambda_\mathrm{o}^{-\mathrm{H}} - \Lambda_\mathrm{o}(\infty)\right\|_2^2, \\ E_{Q_3} & := & \|C_2\|_2^2. \end{array}$$

We start with the calculation of E_{Q_1}, i.e.,

$$E_{Q_1} = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| z\left[\Lambda_\mathrm{o}(\infty)B^{-1}(\infty) - \Lambda_\mathrm{o}\tilde{X}\right] + \Lambda_\mathrm{o}Q\breve{N} \right\|_2^2,$$

where $\breve{N}(z) = z\tilde{N}(z)$, which indicate that we decrease by one the relative degree of $P(z)$ since we only consider the biproper controllers. Since $\breve{N}(\eta_k) = 0$ for all $k \in \mathbb{N}_\eta$ and since $\breve{N}(z)$ is left-invertible, it can be factorized as

$$\breve{N}(z) = n(z)b(z),$$

where $n(z)$ is left-invertible in $\mathbb{R}\mathcal{H}_\infty$ and $b(z)$ is defined by

$$b(z) = \prod_{k\in\mathbb{N}_\eta} \frac{1+\bar{\eta}_k}{1+\eta_k}\frac{z-\eta_k}{\bar{\eta}_k z - 1}.$$

Therefore we obtain

$$E_{Q_1} = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \frac{z[\Lambda_\mathrm{o}(\infty)B^{-1}(\infty) - \Lambda_\mathrm{o}\tilde{X}]}{b} + \Lambda_\mathrm{o}Qn \right\|_2^2.$$

By following the standard partial fraction expansion procedure performed in [4, 9, 14], yields

$$\begin{array}{rcl} E_{Q_1} & = & \left\| \sum\limits_{k\in\mathbb{N}_\eta} \frac{1+\eta_k}{1+\bar{\eta}_k}\left[\frac{\bar{\eta}_k z-1}{z-\eta_k} - \bar{\eta}_k\right] \frac{z[\Lambda_\mathrm{o}(\infty)B^{-1}(\infty) - \Lambda_\mathrm{o}(\eta_k)\tilde{X}(\eta_k)]}{b_k} \right\|_2^2 \\ & & + \inf\limits_{Q\in\mathbb{R}\mathcal{H}_\infty} \|V + \Lambda_\mathrm{o}Qn\|_2^2, \end{array}$$

for some $V \in \mathbb{R}\mathcal{H}_\infty$. Here,

$$b_k := \prod_{\ell \in \mathbb{N}_\eta, \ell \neq k} \frac{1+\bar{\eta}_k}{1+\eta_k} \frac{\eta_\ell - \eta_k}{\bar{\eta}_k \eta_\ell - 1}.$$

Since $n(z)$ is left invertible we may select a Q such that

$$\inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \|V + \Lambda_{\mathrm{o}} Q n\|_2^2 = 0.$$

By fact that

$$\frac{1+\eta_k}{1+\bar{\eta}_k} \left[\frac{\bar{\eta}_k z - 1}{z - \eta_k} - \bar{\eta}_k \right] = \frac{\eta_k + 1}{\bar{\eta}_k + 1} \frac{|\eta_k|^2 - 1}{z - \eta_k},$$

we get

$$E_{Q_1} = \left\| \sum_{k \in \mathbb{N}_\eta} \frac{(\eta_k + 1)(|\eta_k|^2 - 1)}{(\bar{\eta}_k + 1) b_k} \frac{[\Lambda_{\mathrm{o}}(\infty) B^{-1}(\infty) - \Lambda_{\mathrm{o}}(\eta_k) \tilde{X}(\eta_k)]}{z - \eta_k} \right\|_2^2.$$

Since $\tilde{X}(z) = B^{-1}(z) Y(z) \tilde{N}(z) + B^{-1}(z)$, then $\tilde{X}(\eta_k) = B^{-1}(\eta_k)$ for all $k \in \mathbb{N}_\eta$. Further, by fact that

$$\left\| \frac{1}{z - \eta_k} \right\|_2^2 = \frac{1}{|\eta_k|^2 - 1},$$

we obtain

$$\begin{aligned} E_{Q_1} \;\; = \;\; & \sum_{k,\ell \in \mathbb{N}_\eta} \frac{(\bar{\eta}_k + 1)(\eta_\ell + 1)(|\eta_k|^2 - 1)(|\eta_\ell|^2 - 1)}{(\eta_k + 1)(\bar{\eta}_\ell + 1) \bar{b}_k b_\ell (\bar{\eta}_k \eta_\ell - 1)} \times \\ & [\Lambda_{\mathrm{o}}(\infty) B^{-1}(\infty) - \Lambda_{\mathrm{o}}(\eta_k) B^{-1}(\eta_k)]^{\mathrm{H}} \times \\ & [\Lambda_{\mathrm{o}}(\infty) B^{-1}(\infty) - \Lambda_{\mathrm{o}}(\eta_\ell) B^{-1}(\eta_\ell)]. \end{aligned}$$

Hence, we show that $E_{Q_1} = E_{\mathrm{dn}}$ by denoting

$$\beta_k := \Lambda_{\mathrm{o}}(\infty) B^{-1}(\infty) - \Lambda_{\mathrm{o}}(\eta_k) B^{-1}(\eta_k).$$

Next, by direct calculation we get

$$E_{Q_2} = \frac{|\Lambda_{\mathrm{o}}(\infty)|^2}{|B(\infty)|^2} + \frac{1}{2\pi |B(\infty)|^2} \int_{-\pi}^{\pi} \left(|\Lambda_{\mathrm{o}}^{-1}(\mathrm{e}^{\mathrm{j}\theta})|^2 - 2\,\mathrm{Re}\{\Lambda_{\mathrm{o}}^{-1}(\mathrm{e}^{\mathrm{j}\theta}) \Lambda_{\mathrm{o}}(\infty)\} \right) \mathrm{d}\theta,$$

and similarly,

$$E_{Q_3} = \frac{1}{|B(\infty)|^2} - \frac{1}{2\pi |B(\infty)|^2} \int_{-\pi}^{\pi} |\Lambda_{\mathrm{o}}^{-1}(\mathrm{e}^{\mathrm{j}\theta})|^2 \mathrm{d}\theta.$$

Therefore,

$$E_{Q_2} + E_{Q_3} = \frac{|\Lambda_\mathrm{o}(\infty)|^2 + 1}{|B(\infty)|^2} - \frac{2\Lambda_\mathrm{o}(\infty)}{\pi|B(\infty)|^2}\int_{-\pi}^{\pi} \mathrm{Re}\,\Lambda_\mathrm{o}^{-1}(\mathrm{e}^{\mathrm{j}\theta})\,\mathrm{d}\theta.$$

Since Λ_o is an outer factor, then Λ_o^{-1} is in $\mathbb{R}\mathcal{H}_\infty$. Invoking Lemma 3 yields

$$E_{Q_2} + E_{Q_3} = \frac{|\Lambda_\mathrm{o}(\infty)|^2 - 1}{|B(\infty)|^2}. \tag{34}$$

Since

$$|B(\infty)|^2 = \prod_{k=1}^{n_\lambda} \frac{1}{|\lambda_k|^2},$$

then (33) together with (34) produce

$$E_B + E_{Q_2} + E_{Q_3} = |\Lambda_\mathrm{o}(\infty)|^2 \prod_{k=1}^{n_\lambda} |\lambda_k|^2 - 1.$$

We then show that $E_B + E_{Q_2} + E_{Q_3} = E_\mathrm{dm}$ by application of Poisson-Jensen formula in Lemma 1, by fact that Λ_o is a stable and minimum phase function, i.e.,

$$|\Lambda_\mathrm{o}(\infty)|^2 = \exp\left\{\frac{1}{\pi}\int_0^{\pi} \log\left(1 + \|W(\mathrm{e}^{\mathrm{j}\theta})\|^2\right)\mathrm{d}\theta\right\}.$$

This completes the proof of Theorem 2.

B Proof of Theorem 3

To prove of Theorem 3 we follow the proof of Theorem 2 in rather straightforward manner. We readily have

$$\begin{aligned} E_\delta^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W_v(H^{-1} + H^{-1}Y\tilde{N} - Q\tilde{N}) \\ W_y(X\tilde{N} - NQ\tilde{N}) \\ H^{-1}Y\tilde{N} - Q\tilde{N} \end{bmatrix} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W_v(\tilde{X} - Q\tilde{N}) \\ W_y(N\tilde{X} - NQ\tilde{N}) \\ -H^{-1} + \tilde{X} - Q\tilde{N} \end{bmatrix} \right\|_2^2 . \end{aligned}$$

Based on decomposition of two orthogonal subspaces $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$, we get

$$E_\delta^* = E_H + E_R,$$

where

$$
\begin{aligned}
E_H &:= \left\| \begin{bmatrix} 0 \\ 0 \\ H^{-1}(\infty) - H^{-1} \end{bmatrix} \right\|_2^2, \\
E_R &:= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} W_v(\tilde{X} - Q\tilde{N}) \\ W_y(N\tilde{X} - NQ\tilde{N}) \\ H^{-1}(\infty) - \tilde{X} + Q\tilde{N} \end{bmatrix} \right\|_2^2.
\end{aligned}
$$

From (19) we obtain

$$
E_H = \left\| H^{-1}(\infty) - H^{-1}(\delta) \right\|_2^2 = \frac{1}{T} \left\| B^{-1}(\infty) - B^{-1}(z) \right\|_2^2.
$$

Hence, by (33)

$$
E_H = \frac{1}{T} \left(\prod_{k=1}^{n_\rho} |T\rho_k + 1|^2 - 1 \right), \tag{35}
$$

since $\lambda_k = T\rho_k + 1$ for $k = 1, 2, \ldots, n_\rho$. Further we may write

$$
\begin{aligned}
E_R = & \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \Lambda_\mathrm{o}\tilde{X} - \Lambda_\mathrm{o}(\infty) H^{-1}(\infty) - \Lambda_\mathrm{o} Q \tilde{N} \right\|_2^2 \\
& + |H^{-1}(\infty)|^2 \left\| \Lambda_\mathrm{o}^{-\mathrm{H}} - \Lambda_\mathrm{o}(\infty) \right\|_2^2 + \|D\|_2^2,
\end{aligned}
$$

where

$$
D := \begin{bmatrix} W(\Lambda_\mathrm{o}^\mathrm{H}\Lambda_\mathrm{o})^{-1} \\ 1 - (\Lambda_\mathrm{o}^\mathrm{H}\Lambda_\mathrm{o})^{-1} \end{bmatrix} H^{-1}(\infty).
$$

Let denote

$$
\begin{aligned}
E_{R_1} &:= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \Lambda_\mathrm{o}\tilde{X} - \Lambda_\mathrm{o}(\infty) H^{-1}(\infty) - \Lambda_\mathrm{o} Q \tilde{N} \right\|_2^2, \\
E_{R_2} &:= \left| H^{-1}(\infty) \right|^2 \left\| \Lambda_\mathrm{o}^{-\mathrm{H}} - \Lambda_\mathrm{o}(\infty) \right\|_2^2, \\
E_{R_3} &:= \|D\|_2^2.
\end{aligned}
$$

After implementing the standard partial fraction expansion over E_{R_1} and selecting the optimal free parameter Q we obtain

$$
E_{R_1} = \left\| \sum_{k \in \mathbb{N}_\zeta} \frac{(\eta_k + 1)(|\eta_k|^2 - 1)}{(\bar{\eta}_k + 1) g_k} \, \frac{[\Lambda_\mathrm{o}(\infty) H^{-1}(\infty) - \Lambda_\mathrm{o}(\zeta_k)\tilde{X}(\zeta_k)]}{(T\delta + 1) - \eta_k} \right\|_2^2,
$$

where $\eta_k = T\zeta_k + 1$. By fact that

$$
\left\| \frac{1}{(T\delta + 1) - \eta_k} \right\|_2^2 = \frac{1}{T} \left\| \frac{1}{z - \eta_k} \right\|_2^2 = \frac{1}{T} \frac{1}{|\eta_k|^2 - 1},
$$

then

$$\begin{aligned} E_{R_1} &= \frac{1}{T}\sum_{k,\ell\in\mathbb{N}_\zeta} \frac{(\bar{\eta}_k+1)(\eta_\ell+1)(|\eta_k|^2-1)(|\eta_\ell|^2-1)}{(\eta_k+1)(\bar{\eta}_\ell+1)\bar{g}_k g_\ell(\bar{\eta}_k\eta_\ell-1)} \times \\ &\quad [\Lambda_{\rm o}(\infty)H^{-1}(\infty)-\Lambda_{\rm o}(\zeta_k)H^{-1}(\zeta_k)]^{\rm H} \times \\ &\quad [\Lambda_{\rm o}(\infty)H^{-1}(\infty)-\Lambda_{\rm o}(\zeta_\ell)H^{-1}(\zeta_\ell)]. \end{aligned}$$

We show that $E_{R_1} = E_{\delta 2}$ by denoting

$$\begin{aligned} \gamma_k &:= \Lambda_{\rm o}(\infty)H^{-1}(\infty)-\Lambda_{\rm o}(\zeta_k)H^{-1}(\zeta_k) \\ &= \Lambda_{\rm o}(\infty)B^{-1}(\infty)-\Lambda_{\rm o}(\zeta_k)B^{-1}(\eta_k). \end{aligned}$$

Further, application of Lemma 5 yields

$$E_{R_2}+E_{R_3} = \frac{|\Lambda_{\rm o}(\infty)|^2-1}{T|H(\infty)|^2}. \tag{36}$$

Then, (35) together with (36) produce

$$E_H+E_{R_2}+E_{R_3} = \frac{1}{T}\left(|\Lambda_{\rm o}(\infty)|^2\prod_{k=1}^{n_\rho}|T\rho_k+1|^2-1\right).$$

We then show that $E_H+E_{R_2}+E_{R_3} = E_{\delta{\rm m}}$. The explicit expression of $|\Lambda_{\rm o}(\infty)|^2$ can be found by Lemma 4. This completes the proof of Theorem 3.