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# Operations on M-convex Functions on Jump Systems

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**Abstract**

A jump system is a set of integer points with an exchange property, which is a generalization of a matroid, a delta-matroid, and a base polyhedron of an integral polymatroid (or a submodular system). Recently, the concept of M-convex functions on constant-parity jump systems is introduced by Murota as a class of discrete convex functions that admit a local criterion for global minimality. M-convex functions on constant-parity jump systems generalize valuated matroids, valuated delta-matroids, and M-convex functions on base polyhedra.

This paper reveals that the class of M-convex functions on constant-parity jump systems is closed under a number of natural operations such as splitting, aggregation, convolution, composition, and transformation by networks. The present results generalize hitherto-known similar constructions for matroids, delta-matroids, valuated matroids, valuated delta-matroids, and M-convex functions on base polyhedra.

## 1 Introduction

A jump system [6] is a set of integer points with an exchange property (to be described later); see also [16], [17]. It is a generalization of a matroid [8], a delta-matroid [4], [7], [9], and a base polyhedron of an integral polymatroid (or a submodular system) [14].

Study of nonseparable nonlinear functions on matroidal structures was started with valuated matroids [10], [12], which have come to be accepted as discrete concave functions; see [19], [21]. This concept has been generalized to M-convex functions on base polyhedra [20], which play a central role in discrete convex analysis [22]. Valuated delta-matroids [11] afford another generalization of valuated matroids. As a common generalization of valuated delta-matroids and M-convex functions on base polyhedra, the concept of M-convex functions on constant-parity jump systems is introduced in [24]. To distinguish between M-convex functions on base polyhedra and those on constant-parity jump systems, we sometimes refer to the former as $\mathrm{M}^{\mathrm{B}}$-convex functions and the latter as $\mathrm{M}^{\mathrm{J}}$-convex functions. A separable convex function in the degree sequences of a graph is a typical example of $\mathrm{M}^{\mathrm{J}}$-convex functions. In all these generalizations global optimality is equivalent to local optimality defined in an appropriate manner. In addition, discrete duality theorems such as discrete separation

Table 1: Sum of discrete structures.

| | |
|---|---|
| Matroids | Rado (1942) [26] (see [27]) |
| | (explicitly by Edmonds (1968) [13]) |
| Base polyhedra | McDiarmid (1975) [18] |
| Delta-matroids | Bouchet (1989) [5] |
| (Constant-parity) Jump systems | Bouchet and Cunningham (1995) [6] |

Table 2: Convolution of discrete functions.

| | |
|---|---|
| Valuated matroids | Murota (1996) [20] (see also [21]) |
| $\mathrm{M}^{\mathrm{B}}$-convex functions | Murota (1996) [20] |
| Valuated delta-matroids | |
| $\mathrm{M}^{\mathrm{J}}$-convex functions | This paper |

and min-max formula hold for valuated matroids and $\mathrm{M}^{\mathrm{B}}$-convex functions, whereas they fail for valuated delta-matroids and $\mathrm{M}^{\mathrm{J}}$-convex functions.

A number of operations can be defined on matroidal structures and functions.

For example, union (or sum) can be defined for two matroids to yield another matroid. When translated in terms of incidence vectors, union can be understood as Minkowski sum, followed by truncation by the vector $\mathbf{1} = (1, 1, \ldots, 1)$. Sum can also be defined for delta-matroids, base polyhedra, and (constant-parity) jump systems (see Table 1).

Convolution (or infimum convolution) of functions is a quantitative extension of sum, and the first result of the present paper (Theorem 12) is that $\mathrm{M}^{\mathrm{J}}$-convex functions are closed under convolution. This generalizes the known facts that valuated matroids and $\mathrm{M}^{\mathrm{B}}$-convex functions are closed under convolution (see Table 2).

Aggregation is another fundamental operation. For instance, it is known that any polymatroid can be obtained as an aggregation of a matroid [14] and that any jump system can be obtained as an aggregation of a delta-matroid [16]. The second result of the present paper (Theorem 11) is that $\mathrm{M}^{\mathrm{J}}$-convex functions are closed under aggregation. It is mentioned that the first result on convolution can be derived from this. A kind of converse of aggregation operation is splitting, which divides variables into several copies and generates a new function on a higher dimensional space. We show that splitting of $\mathrm{M}^{\mathrm{J}}$-convex functions is again $\mathrm{M}^{\mathrm{J}}$-convex.

Transformation (or induction) by graphs or networks is one of the most general operations. The fundamental fact in this direction is that a matroid can be transformed to another matroid through matchings in a bipartite graph. This construction works also delta-matroids [4]. As for functions, valuated matroids are closed under transformation by bipartite graphs defined in an appropriate manner [20], [21], and $\mathrm{M}^{\mathrm{B}}$-convex functions are closed under transformation by networks [20]. The third result of the present paper (Theorem 14) is that this construction extends to $\mathrm{M}^{\mathrm{J}}$-convex functions, that is, transformation of $\mathrm{M}^{\mathrm{J}}$-convex functions by networks, to be defined precisely in

Section 6, preserves M$^{\rm J}$-convexity. Aggregation, convolution and splitting may be obtained as special cases of this construction, whereas our proof for the network transformation is based on the combination of aggregation, splitting, and other basic operations.

Here is a remark on the proof technique of the present paper. Our proofs consist of repeated applications of the defining exchange axiom of M$^{\rm J}$-convex functions. This is particularly true of the proof given in Section 7. For M$^{\rm B}$-convex functions, on the other hand, an alternative "geometric" or "polyhedral" approach is possible on the basis of the convex extension of the functions. To be specific, such "polyhedral" proofs are known for convolution and network transformation of M$^{\rm B}$-convex functions (see [14], [20], [23]). M$^{\rm J}$-convex functions, however, seem to deny such "polyhedral" approach, because jump systems can have "holes" within the convex hull, and accordingly, jump systems are not determined by their convex hulls. It is also noted that M$^{\rm J}$-convex functions are not necessarily extensible to ordinary convex functions, although they possess a number of nice properties that justify the name of "convex functions."

## 2 Definitions and Exchange Axioms

Let $V$ be a finite set. For $x = (x(v)), y = (y(v)) \in \mathbf{Z}^V$ define

$$\begin{aligned} x(V) &= \sum_{v \in V} x(v), \\ ||x||_1 &= \sum_{v \in V} |x(v)|, \end{aligned}$$

$$[x, y] = \{z \in \mathbf{Z}^V \mid \min(x(v), y(v)) \leq z(v) \leq \max(x(v), y(v)), \forall v \in V\}.$$

We denote by $\mathbf{0}$ the zero vector of an appropriate dimension. For $u \in V$ we denote by $\chi_u$ the *characteristic vector* of $u$, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \neq u$. A vector $s \in \mathbf{Z}^V$ is called an $(x, y)$*-increment* if $s = \chi_u$ or $s = -\chi_u$ for some $u \in V$ and $x + s \in [x, y]$. An $(x, y)$*-increment pair* will mean a pair of vectors $(s, t)$ such that $s$ is an $(x, y)$-increment and $t$ is an $(x + s, y)$-increment.

A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if it satisfies an exchange axiom, called the *2-step axiom*: for any $x, y \in J$ and for any $(x, y)$-increment $s$ with $x + s \notin J$, there exists an $(x+s, y)$-increment $t$ such that $x+s+t \in J$. A set $J \subseteq \mathbf{Z}^V$ is a *constant-sum system* if $x(V) = y(V)$ for any $x, y \in J$, and a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$.

For constant-parity jump systems, Geelen [15] introduced a stronger exchange axiom:

**(J-EXC)** For any $x, y \in J$ and for any $(x, y)$-increment $s$, there exists an $(x + s, y)$-increment $t$ such that $x + s + t \in J$ and $y - s - t \in J$.

This property characterizes a constant-parity jump system, a fact communicated to one of the authors by J. Geelen (see [24] for a proof).

**Theorem 1** ([15])**.** *A nonempty set $J$ is a constant-parity jump system if and only if it satisfies* (J-EXC)*.*

Next we turn to functions defined on integer points $J$. We call $f : J \to \mathbf{R}$ an *$M^J$-convex function* if it satisfies the following exchange axiom:

**($\mathrm{M^J}$-EXC)** For any $x, y \in J$ and for any $(x, y)$-increment $s$, there exists an $(x+s, y)$-increment $t$ such that $x + s + t \in J$, $y - s - t \in J$, and

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

It follows from ($\mathrm{M^J}$-EXC) that $J$ satisfies (J-EXC), and hence is a constant-parity jump system.

We adopt the convention that $f(x) = +\infty$ for $x \notin J$. For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ we denote the *effective domain* of $f$ by

$$\mathrm{dom} f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}.$$

Then, it can be seen that if $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies ($\mathrm{M^J}$-EXC) then its effective domain $J$ satisfies (J-EXC).

It is known that if $J$ satisfies (J-EXC), the exchange axiom ($\mathrm{M^J}$-EXC) is equivalent to a local exchange axiom:

**($\mathrm{M^J}$-EXC$_{\mathrm{loc}}$)** For any $x, y \in J$ with $||x - y||_1 = 4$ there exists an $(x, y)$-increment pair $(s, t)$ such that $x + s + t \in J$, $y - s - t \in J$, and

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

**Theorem 2** ([24]). *A function $f : J \to \mathbf{R}$ defined on a constant-parity jump system $J$ satisfies* ($\mathrm{M^J}$-EXC) *if and only if it satisfies* ($\mathrm{M^J}$-EXC$_{\mathrm{loc}}$).

In what follows, we refer to $\mathrm{M^J}$-convexity simply as M-convexity; in particular, when we talk about an M-convex function it is presumed that its effective domain is a constant-parity jump system.

The definition of an M-convex function is consistent with the previously considered special cases where (i) $J$ is a constant-sum jump system, and (ii) $J$ is a constant-parity jump system contained in $\{0, 1\}^V$. Case (i) is equivalent to $J$ being the set of integer points in the base polyhedron of an integral submodular system [14], and then M-convex function is the same as the $\mathrm{M^B}$-convex function investigated in [20], [22]. Case (ii) is equivalent to $J$ being an even delta-matroid [29], [30], and then $f$ is M-convex if and only if $-f$ is a valuated delta-matroid in the sense of [11].

For an M-convex function, it is known that global optimality (minimality) is guaranteed by local optimality in the neighborhood of $\ell_1$-distance two, which generalizes the optimality criterion in [1] for separable convex function minimization over a jump system. The efficient algorithm for the minimization problem of M-convex functions follows from the optimality criterion [24], [25].

**Theorem 3** ([24]). *Let $f : J \to \mathbf{R}$ be an M-convex function on a constant-parity jump system $J$, and let $x \in J$. Then $f(x) \leq f(y)$ for all $y \in J$ if and only if $f(x) \leq f(y)$ for all $y \in J$ with $||x - y||_1 \leq 2$.*

It is also known that global optimality (minimality) for constrained minimization on a hyperplane of a constant component sum is guaranteed by local optimality in the neighborhood of $\ell_1$-distance four.

**Theorem 4** ([24])**.** *Let $f : J \to \mathbf{R}$ be an M-convex function on a constant-parity jump system $J \subseteq \mathbf{Z}^V$, let $J_k = \{x \in J \mid x(V) = k\}$, and let $x \in J_k$. Then $f(x) \leq f(y)$ for all $y \in J_k$ if and only if $f(x) \leq f(y)$ for all $y \in J_k$ with $||x - y||_1 \leq 4$.*

This optimality criterion for M-convex functions helps us deepen our understanding of the result of Apollonio and Sebő [2], [3]. They provided a polynomial algorithm for the minconvex factor problem, which is, given an undirected graph possibly containing loops and parallel edges and a separable convex function on the degree sequences, to find a subgraph with a specified number of edges that minimizes the function. The key observation in [2], [3] is that global optimality is guaranteed by local optimality in the neighborhood of $\ell_1$-distance at most four in the space of degree sequences. Since a separable convex function on the degree sequences of a graph is an M-convex function, this result can be seen as a special case of Theorem 4.

## 3 Basic Operations

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function. We introduce some basic operations on $f$ that preserve M-convexity. Though too simple to be interesting in their own right, these operations are stated explicitly in view of their use in our proofs.

For subsets $U \subseteq V$ and $W \supseteq V$, we define the *coordinate inversion* $f_U^- : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ of $U$, the *restriction* $f_U : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$ to $U$, and the 0-*augmentation* $f^W : \mathbf{Z}^W \to \mathbf{R} \cup \{+\infty\}$ to $W$ by

$$\begin{aligned} f_U^-(y, z) &= f(-y, z) \quad (y \in \mathbf{Z}^U, z \in \mathbf{Z}^{V \setminus U}), \\ f_U(y) &= f(y, \mathbf{0}) \quad (y \in \mathbf{Z}^U, \mathbf{0} \in \mathbf{Z}^{V \setminus U}), \\ f^W(y, z) &= \begin{cases} f(y) & \text{if } z = \mathbf{0} \\ +\infty & \text{otherwise} \end{cases} \quad (y \in \mathbf{Z}^V, z \in \mathbf{Z}^{W \setminus V}), \end{aligned}$$

respectively. For a linear function $p : \mathbf{Z}^V \to \mathbf{R}$, we define $f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f[p](x) = f(x) + p(x).$$

It is obvious that they are M-convex.

We say that $\phi : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is a *separable convex function* if it is represented as

$$\phi(x) = \sum_{u \in V} \phi_u(x(u)),$$

where for each $u \in V$, $\phi_u : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ is a convex function, that is, for any integers $\xi < \eta$

$$\phi_u(\xi) + \phi_u(\eta) \geq \phi_u(\xi + 1) + \phi_u(\eta - 1).$$

Note that this condition is equivalent to the following: for any integer $\xi$

$$\phi_u(\xi - 1) + \phi_u(\xi + 1) \geq 2\phi_u(\xi).$$

For a separable convex function $\phi$, we define $f + \phi : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$(f + \phi)(x) = f(x) + \phi(x).$$

**Theorem 5.** *If $f$ is M-convex and $\phi$ is a separable convex function, then $f + \phi$ is M-convex.*

*Proof.* It suffices to show that for a one-dimensional convex function $\phi_u$ with a particular $u \in V$ the function $g(x) = f(x) + \phi_u(x(u))$ is M-convex. Suppose that $x = (x(v)) \in \mathbf{Z}^V,\ y = (y(v)) \in \mathbf{Z}^V$ and $s$ is an $(x, y)$-increment. By M-convexity of $f$, there exists an $(x + s, y)$-increment $t$ such that

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t),$$

and it holds that

$$\phi_u(x(u)) + \phi_u(y(u)) \geq \phi_u(x(u) + s(u) + t(u)) + \phi_u(y(u) - s(u) - t(u))$$

by convexity of $\phi_u$. Thus we have

$$g(x) + g(y) \geq g(x + s + t) + g(y - s - t),$$

which completes the proof. $\square$

# 4 Splitting

Splitting is an operation which generates a new function by dividing some variables. The objective of this section is to show that if a given function is M-convex, then the function obtained by splitting is also M-convex (Theorem 7 below). Although splitting is a simple operation, it plays an important role when we deal with transformation by networks in Section 6.

First we introduce an elementary operation, called elementary splitting, which divides one variable into two variables. Elementary splitting preserves M-convexity, from which we can show that splitting preserves M-convexity.

For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the *elementary splitting* of $f$ at $v \in V$ is a function $f' : \mathbf{Z}^{V'} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f'(x_0; x(v'), x(v'')) = f(x_0; x(v') + x(v'')),$$

where $V' = (V \setminus \{v\}) \cup \{v', v''\}$ and $x_0 \in \mathbf{Z}^{V \setminus \{v\}}$.

**Lemma 6.** *If $f$ is M-convex, then its elementary splitting $f'$ is M-convex.*

*Proof.* For a concise description, let $V = \{1, 2, \dots, n\}$ and $V' = \{1, 2, \dots, n-1, a, b\}$. We show that if $f$ is M-convex then its elementary splitting $f'$ at $n$ defined by

$$f'(x_0; x_a, x_b) = f(x_0; x_a + x_b)$$

is M-convex. For $u \in V'$ we denote by $\chi'_u$ the characteristic vector of $u$ in $V'$. It suffices to show that $f'$ satisfies (M$^{\mathrm{J}}$-EXC), that is, for any two vectors $x' = (x_0; x_a, x_b) \in \mathrm{dom} f'$, $y' = (y_0; y_a, y_b) \in \mathrm{dom} f'$, and for any $(x', y')$-increment $s'$, there exists an $(x' + s', y')$-increment $t'$ such that

$$f'(x') + f'(y') \geq f'(x' + s' + t') + f'(y' - s' - t').$$

We put $\xi = x_a + x_b$ and $\eta = y_a + y_b$. We also put $x = (x_0; \xi)$ and $y = (y_0; \eta)$.

**Case 1.** Suppose that $s' = \pm\chi'_k$ is an $(x', y')$-increment, where $1 \leq k \leq n-1$. We denote $\pm\chi_k$ by $s$. Since $f$ is M-convex and $s$ is an $(x, y)$-increment, there exists an $(x + s, y)$-increment $t$ such that

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

If $t = \pm\chi_l$ with $1 \leq l \leq n-1$, then $t' = \pm\chi'_l$ is an $(x' + s', y')$-increment and

$$f'(x') + f'(y') \geq f'(x' + s' + t') + f'(y' - s' - t').$$

Otherwise we have $l = n$. Without loss of generality, we may assume that $\xi < \eta$ and $t = \chi_n$. Since $\xi < \eta$ implies that at least one of $x_a < y_a$ and $x_b < y_b$ holds, at least one of $\chi'_a$ and $\chi'_b$, say $t'$, is an $(x' + s', y')$-increment and it holds that

$$f'(x') + f'(y') = f(x) + f(y) \geq f(x + s + t) + f(y - s - t) = f'(x' + s' + t') + f'(y' - s' - t').$$

**Case 2.** Suppose that $s' = \pm\chi'_a$ or $\pm\chi'_b$ is an $(x', y')$-increment. In this case, without loss of generality, we may assume that $s' = \chi'_b$ and $x_b < y_b$.

If $x_a > y_a$ then $t' = -\chi'_a$ is an $(x' + s', y')$-increment and

$$f'(x') + f'(y') = f(x) + f(y) = f'(x' + s' + t') + f'(y' - s' - t').$$

Suppose that $x_a \leq y_a$. Then we have $\xi < \eta$ and $\chi_n$ is an $(x, y)$-increment. Since $f$ is M-convex, by applying (M$^{\mathrm{J}}$-EXC) with $s = \chi_n$, there exists an $(x + s, y)$-increment $t$ such that

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

If $t = \pm\chi_k$ with $1 \leq k \leq n-1$, then $t' = \pm\chi'_k$ is an $(x' + s', y')$-increment and

$$f'(x') + f'(y') = f(x) + f(y) \geq f(x + s + t) + f(y - s - t) = f'(x' + s' + t') + f'(y' - s' - t').$$

Otherwise, we have $t = \chi_n$ and $\xi + 2 \leq \eta$. Thus at least one of $x_b + 2 \leq y_b$ and $x_a + 1 \leq y_a$ holds, and hence at least one of $\chi'_b$ and $\chi'_a$, say $t'$, is an $(x' + s', y')$-increment. We then have

$$f'(x') + f'(y') = f(x) + f(y) \geq f(x + s + t) + f(y - s - t) = f'(x' + s' + t') + f'(y' - s' - t').$$

This shows the existence of $t'$ in Case 2. $\square$

Suppose that we are given a finite set $V = \{v_1, v_2 \ldots, v_n\}$ and a family of nonempty disjoint sets $\{U_v \mid v \in V\}$ indexed by $v \in V$. Let $U = \bigcup_{v \in V} U_v$. For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we define the *splitting* of $f$ to $U$ as a function $f' : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$ given by

$$f'(\tilde{x}_{v_1}, \tilde{x}_{v_2}, \ldots, \tilde{x}_{v_n}) = f(\xi_{v_1}, \xi_{v_2}, \ldots, \xi_{v_n}),$$

where $\tilde{x}_v \in \mathbf{Z}^{U_v}$ and $\xi_v = \tilde{x}_v(U_v)$ for $v \in V$. We now have the following theorem.

**Theorem 7.** *If $f$ is M-convex then its splitting $f'$ is M-convex.*

*Proof.* We can obtain splitting $f'$ by applying elementary splittings $\sum_{v \in V}(|U_v| - 1)$ times. Hence, by Lemma 6, $f'$ is M-convex. □

Theorem 7 implies that if $\mathrm{dom} f$ is a constant-parity jump system, then $\mathrm{dom} f'$ is also a constant-parity jump system.

# 5 Aggregation and Convolution

Minkowski sum is a fundamental operation on matroid structures, and jump systems are closed under Minkowski sum. In this section, we deal with an operation for functions, called convolution, which is a quantitative extension of sum, and also a related operation, called aggregation. The objective of this section is to show that M-convexity is preserved under these operations. As with splitting, aggregation plays an important role when we deal with transformations by networks in Section 6.

For two jump systems $J_1 \subseteq \mathbf{Z}^V$ and $J_2 \subseteq \mathbf{Z}^V$, their *sum* $J_1 + J_2 \subseteq \mathbf{Z}^V$ is defined by

$$J_1 + J_2 = \{x_1 + x_2 \mid x_1 \in J_1, x_2 \in J_2\},$$

which is known to be a jump system.

**Theorem 8** ([6]). *The sum of two jump systems is a jump system.*

While this theorem is shown directly in [6], Kabadi and Sridhar [16] gave an alternative proof by showing that a related elementary operation preserves M-convexity. They showed that if $J \subseteq \mathbf{Z}^V$ is a jump system then its *elementary aggregation* $\tilde{J} \subseteq \mathbf{Z}^{\tilde{V}}$ at $v_1 \in V$ and $v_2 \in V$ defined by

$$\tilde{J} = \{(x_0, x(v_1) + x(v_2)) \mid (x_0, x(v_1), x(v_2)) \in J\}$$

is also a jump system, where $\tilde{V} = (V \setminus \{v_1, v_2\}) \cup \{v\}$ and $x_0 \in \mathbf{Z}^{V \setminus \{v_1, v_2\}}$. Theorem 8 can be derived from the following fact.

**Lemma 9** ([16]). *An elementary aggregation of a jump system is a jump system.*

Convolution is a quantitative extension of sum. For two functions $f_1 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we define their (infimum) *convolution* as a function $f_1 \Box f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty, -\infty\}$ given by

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x,\ x_1 \in \mathbf{Z}^V,\ x_2 \in \mathbf{Z}^V\}.$$

To show that convolution preserves M-convexity (Theorem 12 below), we introduce a quantitative extension of elementary aggregation.

For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the *elementary aggregation* of $f$ at $v_1 \in V$ and $v_2 \in V$ is a function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty, -\infty\}$ defined by

$$\tilde{f}(x_0; \xi) = \inf\{f(x_0; x(v_1), x(v_2)) \mid \xi = x(v_1) + x(v_2)\},$$

where $\tilde{V} = (V \setminus \{v_1, v_2\}) \cup \{v\}$ and $x_0 \in \mathbf{Z}^{V \setminus \{v_1, v_2\}}$. Then we can show that if $f$ is M-convex then $\tilde{f}$ is M-convex; the proof is given in Section 7.

**Lemma 10.** *If $f$ is M-convex then its elementary aggregation $\tilde{f}$ is M-convex, provided $\tilde{f} > -\infty$.*

A general aggregation is defined as the result of repeated applications of elementary aggregations. More formally, let $V$ be a finite set and $\pi$ be its partition $V = V_1 \cup V_2 \cup \cdots \cup V_n$ into disjoint subsets. For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we define the *aggregation* of $f$ with respect to $\pi$ as a function $\tilde{f} : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty, -\infty\}$ given by

$$\tilde{f}(\xi_1, \xi_2, \ldots, \xi_n) = \inf\left\{f(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbf{Z}^{V_i},\ x_i(V_i) = \xi_i\right\}.$$

Then we have the following theorem.

**Theorem 11.** *If $f$ is M-convex then its aggregation $\tilde{f}$ is M-convex, provided $\tilde{f} > -\infty$.*

*Proof.* By applying elementary aggregations $|V| - n$ times, we can obtain $\tilde{f}$, which is M-convex by Lemma 10. □

We are now ready to show that convolution preserves M-convexity.

**Theorem 12.** *If $f_1$ and $f_2$ are M-convex functions then their convolution $f_1 \Box f_2$ is M-convex, provided $f_1 \Box f_2 > -\infty$.*

*Proof.* First we make the direct sum $f : \mathbf{Z}^V \times \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ of $f_1$ and $f_2$ defined by

$$f(x_1, x_2) = f_1(x_1) + f_2(x_2),$$

where $x_1, x_2 \in \mathbf{Z}^V$. Then $f$ is M-convex because $f_1$ and $f_2$ are M-convex. Let $\pi$ be the partition consisting of pairs of the corresponding elements. Then the aggregation of $f$ coincides with $f_1 \Box f_2$. Hence, by Theorem 11, $f_1 \Box f_2$ is M-convex. □

Finally, we consider another operation, called composition. Let $f_1 : \mathbf{Z}^{S_1} \to \mathbf{R} \cup \{+\infty\}$ and $f_2 : \mathbf{Z}^{S_2} \to \mathbf{R} \cup \{+\infty\}$ be M-convex functions. Put $V_0 = S_1 \cap S_2$, $V_1 = S_1 \setminus V_0$, and $V_2 = S_2 \setminus V_0$. We define the *composition* of $f_1$ and $f_2$ to be a function $f : \mathbf{Z}^{V_1 \cup V_2} \to \mathbf{R} \cup \{+\infty, -\infty\}$ given by

$$f(x_1, x_2) = \inf\{f_1(x_1, y_1) + f_2(x_2, y_2) \mid y_1 = y_2 \in \mathbf{Z}^{V_0}\} \quad (x_1 \in \mathbf{Z}^{V_1}, x_2 \in \mathbf{Z}^{V_2}).$$

**Theorem 13.** *The composition of two M-convex functions is M-convex, provided it does not take the value $-\infty$.*

*Proof.* Consider M-convex functions $\tilde{f}_1$ and $\tilde{f}_2$ defined by

$$\tilde{f}_1(x_1, y_1, \mathbf{0}) = f_1(x_1, y_1) \quad (x_1 \in \mathbf{Z}^{V_1}, y_1 \in \mathbf{Z}^{V_0}, \mathbf{0} \in \mathbf{Z}^{V_2}),$$
$$\tilde{f}_2(\mathbf{0}, (-y_2), x_2) = f_2(x_2, y_2) \qquad (\mathbf{0} \in \mathbf{Z}^{V_1}, (-y_2) \in \mathbf{Z}^{V_0}, x_2 \in \mathbf{Z}^{V_2}).$$

Their convolution $\tilde{f}_1 \Box \tilde{f}_2$ is M-convex by Theorem 12, and the restriction of $\tilde{f}_1 \Box \tilde{f}_2$ to $V_1 \cup V_2$ coincides with the composition. $\Box$

Note that the composition of M-convex functions is a generalization of the *composition* of (constant-parity) jump systems. It is known that the composition of two jump systems is a jump system [6], and Theorem 13 generalizes this fact.

## 6 Transformation by Networks

In this section, we consider the transformation of an M-convex function through a network. We show that it preserves M-convexity on the basis of splitting, aggregation, and other basic operations discussed above.

Let $G = (V, A; S, T)$ be a directed graph with vertex set $V$, arc set $A$, entrance set $S$, and exit set $T$, where $S$ and $T$ are disjoint subsets of $V$. For each $a \in A$, the cost of integer-flow in $a$ is represented by a function $\phi_a : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$, which is assumed to be convex.

Given a function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ associated with the entrance set $S$ of the network, we define a function $\tilde{f} : \mathbf{Z}^T \to \mathbf{R} \cup \{+\infty, -\infty\}$ on the exit set $T$ by

$$\tilde{f}(y) = \inf_{\xi, x} \{ f(x) + \sum_{a \in A} \phi_a(\xi(a)) \mid \partial\xi = (x, -y, \mathbf{0}),$$
$$\xi \in \mathbf{Z}^A, (x, -y, \mathbf{0}) \in \mathbf{Z}^S \times \mathbf{Z}^T \times \mathbf{Z}^{V \setminus (S \cup T)} \} \quad (y \in \mathbf{Z}^T),$$

where $\partial\xi \in \mathbf{Z}^V$ is the vector given by

$$\partial\xi(v) = \sum_{a: a \text{ leaves } v} \xi(a) - \sum_{a: a \text{ enters } v} \xi(a) \quad (v \in V).$$

If such $(\xi, x)$ does not exist, we define $\tilde{f}(y) = +\infty$. We may think of $\tilde{f}(y)$ as the minimum cost to meet a demand specification $y$ at the exit, where the cost consists of two parts, the cost $f(x)$ of supply or production of $x$ at the entrance and the cost $\sum_{a \in A} \phi_a(\xi(a))$ of transportation through arcs; the sum of these is to be minimized over varying supply $x$ and flow $\xi$ subject to the flow conservation constraint $\partial\xi = (x, -y, \mathbf{0})$. We regard $\tilde{f}$ as a result of *transformation* (or *induction*) of $f$ by the network.

**Theorem 14.** *Assume that $f$ is M-convex and $\phi_a$ is convex for each $a \in A$. Then the function $\tilde{f}$ induced by a network $G = (V, A; S, T)$ is M-convex, provided $\tilde{f} > -\infty$.*

To prove this theorem, we first show that transformations by some simple bipartite networks preserve M-convexity. When $V = S \cup T$, we denote the graph $G$ simply by $G = (S, T; A)$. It is noted that some arcs are directed from $S$ to $T$ and the others are from $T$ to $S$.

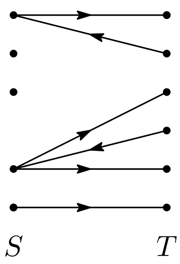

Figure 1: Splitting.

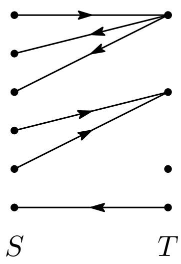

Figure 2: Aggregation.

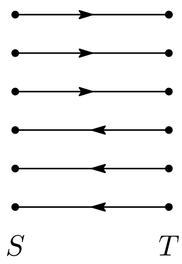

Figure 3: Addition.

**Lemma 15.** *Let $G = (S, T; A)$ be a bipartite network, where each vertex in $T$ has exactly one incident arc (see Fig. 1). If $f$ is M-convex and $\phi_a = 0$ for each $a \in A$, the function $\tilde{f}$ induced by $G$ is M-convex.*

*Proof.* We can obtain $\tilde{f}$ from $f$ by restriction and splitting. Hence, if $f$ is M-convex then $\tilde{f}$ is M-convex by Theorem 7. $\square$

**Lemma 16.** *Let $G = (S, T; A)$ be a bipartite network, where each vertex in $S$ has exactly one incident arc (see Fig. 2). If $f$ is M-convex and $\phi_a = 0$ for each $a \in A$, the function $\tilde{f}$ induced by $G$ is M-convex, provided $\tilde{f} > -\infty$.*

*Proof.* We can obtain $\tilde{f}$ from $f$ by aggregation and 0-augmentation. Hence, if $f$ is M-convex then $\tilde{f}$ is M-convex by Theorem 11. $\square$

**Lemma 17.** *Let $G = (S, T; A)$ be a bipartite network, as in Fig. 3, where $S = \{s_1, \ldots, s_n\}$, $T = \{t_1, \ldots, t_n\}$, and $A = \{a_1, \ldots, a_n\}$ with $a_i = (s_i, t_i)$ or $a_i = (t_i, s_i)$ for $i = 1, \ldots, n$. If $f$ is M-convex and $\phi_a$ is convex for each $a \in A$, the function $\tilde{f}$ induced by $G$ is M-convex.*

*Proof.* We may assume that

$$\begin{aligned} S^+ &= \{s_1, \ldots, s_m\}, & S^- &= \{s_{m+1}, \ldots, s_n\}, \\ T^+ &= \{t_1, \ldots, t_m\}, & T^- &= \{t_{m+1}, \ldots, t_n\}, \\ A^+ &= \{(s_i, t_i) \mid i = 1, \ldots, m\}, & A^- &= \{(t_i, s_i) \mid i = m+1, \ldots, n\}, \end{aligned}$$

and $A = A^+ \cup A^-$. Then, for $x = (x_i) \in \mathbf{Z}^n$, $\tilde{f}$ is expressed as

$$\tilde{f}(x) = f(x) + \sum_{i=1}^{m} \phi_a(x_i) + \sum_{i=m+1}^{n} \phi_a(-x_i),$$

and if $\phi_a(x)$ is convex then $\phi_a(-x)$ is convex for $a \in A^-$. Thus we can obtain $\tilde{f}$ by adding a separable convex function to $f$. Hence, if $f$ is M-convex then $\tilde{f}$ is M-convex by Theorem 5. $\square$

Using above lemmas, we see that transformation by bipartite networks preserves M-convexity.

**Theorem 18.** *Assume that $f$ is M-convex, $\phi_a$ is convex for each $a \in A$, and $G = (S, T; A)$ is a bipartite network. Then the function $\tilde{f}$ induced by $G$ is M-convex, provided $\tilde{f} > -\infty$.*

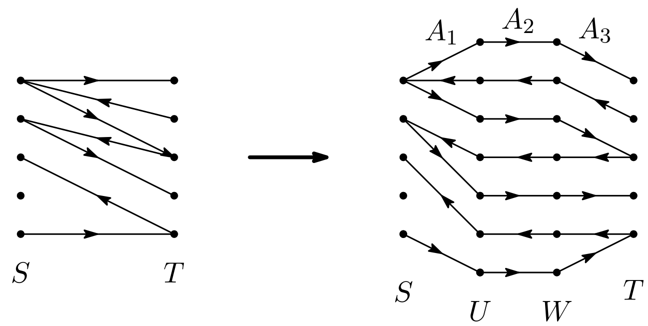

Figure 4: Transformation of a bipartite network.

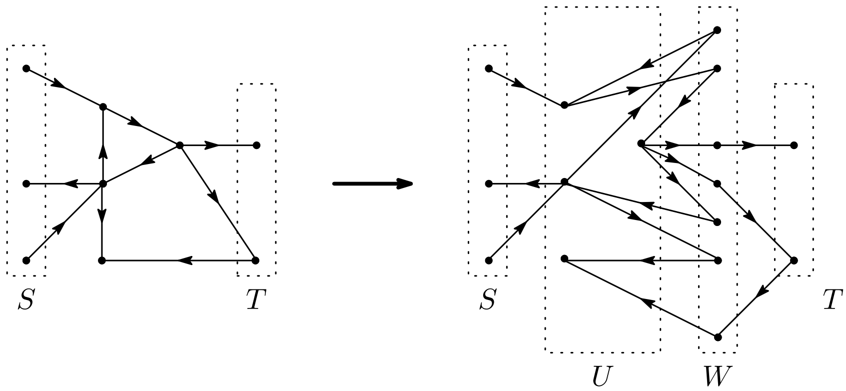

Figure 5: Transformation of a general network.

*Proof.* We construct a new network that represents the same transformation as the original network. The new network is obtained by subdividing each arc of $G$ into three arcs, as illustrated in Fig. 4. For each arc $a \in A$ we consider two new vertices $u_a$ and $w_a$; if $a$ is directed from $S$ to $T$, i.e., $a = (s, t)$ with $s \in S$ and $t \in T$, we will have three arcs $a_1 = (s, u_a)$, $a_2 = (u_a, w_a)$ and $a_3 = (w_a, t)$; and if $a = (t, s)$ with $t \in T$ and $s \in S$, we will have $a_3 = (t, w_a)$, $a_2 = (w_a, u_a)$ and $a_1 = (u_a, s)$. The cost $\phi_a$ is associated with arc $a_2$, whereas the arcs $a_1$ and $a_3$ are given 0 as the cost. Thus the new network consists of three bipartite graphs connected in series, $G_1 = (S, U; A_1)$, $G_2 = (U, W; A_2)$, and $G_3 = (W, T; A_3)$, where $U = \{u_a \mid a \in A\}$, $W = \{w_a \mid a \in A\}$, and $A_i = \{a_i \mid a \in A\}$ $(i = 1, 2, 3)$.

The given M-convex function $f$ on $S$ is transformed through $G_1$ to a function $f_1 : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$, which is M-convex by Lemma 15. Then $f_1$ is transformed through $G_2$ to a function $f_2 : \mathbf{Z}^W \to \mathbf{R} \cup \{+\infty\}$, which is M-convex by Lemma 17. Finally, $f_2$ is transformed through $G_3$ to a function $f_3 : \mathbf{Z}^T \to \mathbf{R} \cup \{+\infty, -\infty\}$, which is M-convex by Lemma 16. The resulting function $f_3$ coincides with the function $\tilde{f}$ induced from $f$ by $G$. □

We are now ready to show Theorem 14.

*Proof of Theorem 14.* We construct a new network that represents the same transformation as the original network. The new network is obtained by subdividing each arc of $G$ into some arcs, as illustrated in Fig. 5. We may assume, by subdividing arcs, that no arcs exist between the two vertices in $S \cup T$. Let $U = V \setminus (S \cup T)$, let $A_{UT}$ be the set of arcs connecting $U$ and $T$, and define $A_{SU}$ and $A_{UU}$ similarly. For each arc $a \in A_{UT}$, we consider a new vertex $w_a$; if $a$ is directed from $U$ to $T$, i.e., $a = (u, t)$ with $u \in U$ and $t \in T$, we will have two arcs $a_1 = (u, w_a), a_2 = (w_a, t)$; and if $a = (t, u)$ with $t \in T$ and $u \in U$, we will have $a_2 = (t, w_a), a_1 = (w_a, u)$. For each arc $a = (u_1, u_2) \in A_{UU}$ with $u_1, u_2 \in U$, we consider a new vertex $w_a$, and have two arcs $a_1 = (u_1, w_a), a_2 = (w_a, u_2)$. Thus the new network consists of three bipartite graphs connected in series, $G_1 = (S, U; A_1)$, $G_2 = (U, W; A_2)$, and $G_3 = (W, T; A_3)$, where $W = \{w_a \mid a \in A_{UT} \cup A_{UU}\}$, $A_1 = A_{SU}$, $A_2 = \{a_1 \mid a \in A_{UT}\} \cup \{a_1 \mid a \in A_{UU}\} \cup \{a_2 \mid a \in A_{UU}\}$, and $A_3 = \{a_2 \mid a \in A_{UT}\}$.

By Theorem 18, transformations by the networks $G_1, G_2$, and $G_3$ preserve M-convexity. Since the transformation by $G$ can be represented as a combination of the above three transformations, the function $\tilde{f}$ transformed from $f$ by $G$ is M-convex. $\square$

As we mentioned in Section 1, transformations by networks also preserve $\mathrm{M}^{\mathrm{B}}$-convexity. Two kinds of proofs for this fact are known (see [20], [21], [28]), one uses a dual variable and the other is a complicated algorithmic proof. We can see that our proof of Theorem 14 also works for $\mathrm{M}^{\mathrm{B}}$-convex functions, that is, by proving that splitting, aggregation, and other basic operations preserve $\mathrm{M}^{\mathrm{B}}$-convexity, we can show that transformations by networks preserve $\mathrm{M}^{\mathrm{B}}$-convexity.

# 7 Proof of Lemma 10 for Elementary Aggregation

In this section, we give a proof of Lemma 10. For a concise description, we denote $V = \{1, 2, \ldots, n-1, n\}$ and $\tilde{V} = \{1, 2, \ldots, n-2, a\}$. We show that if $f$ is M-convex then $\tilde{f}$ defined by

$$\tilde{f}(x_0; \xi) = \inf\{f(x_0; x_{n-1}, x_n) \mid \xi = x_{n-1} + x_n\}$$

is M-convex. For $u \in \tilde{V}$, we denote by $\tilde{\chi}_u$ the characteristic vector of $u$ in $\tilde{V}$.

We first deal with case where the effective domain of $f$ is bounded, whereas the general case is treated in Section 7.4.

## 7.1 Case of bounded effective domain

**Lemma 19.** *If $f$ is M-convex and* $\mathrm{dom} f$ *is bounded then its elementary aggregation $\tilde{f}$ is M-convex.*

*Proof.* Let $J$ and $\tilde{J}$ be the effective domains of $f$ and $\tilde{f}$, respectively. If $f$ is M-convex then $J$ is a constant-parity jump system, which implies by Lemma 9 that $\tilde{J}$ is also a constant-parity jump system. Hence, by Theorem 2, it is enough to show that $\tilde{f}$ satisfies ($\mathrm{M}^{\mathrm{J}}$-$\mathrm{EXC}_{\mathrm{loc}}$), that is, for any $\tilde{x} = (x_0; \xi), \tilde{y} = (y_0; \eta) \in \tilde{J}$ with $||\tilde{x} - \tilde{y}||_1 = ||x_0 - y_0||_1 + |\xi - \eta| = 4$, there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s, t)$ such that

$$\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x} + s + t) + \tilde{f}(\tilde{y} - s - t). \qquad (*)$$

Without loss of generality, we may assume that $\xi \geq \eta$. Take $x_{n-1}, x_n, y_{n-1}, y_n$ with the minimum value of $|x_{n-1} - y_{n-1}| + |x_n - y_n|$ such that

$$\tilde{f}(x_0; \xi) = f(x_0; x_{n-1}, x_n) \quad ((x_0; x_{n-1}, x_n) \in J,\ \xi = x_{n-1} + x_n),$$
$$\tilde{f}(y_0; \eta) = f(y_0; x_{n-1}, y_n) \quad ((y_0; y_{n-1}, y_n) \in J,\ \eta = y_{n-1} + y_n).$$

Note that such $x_{n-1}, x_n, y_{n-1}, y_n$ exist, because $J$ is finite and $(x_0;\xi), (y_0;\eta) \in \tilde{J}$.

If $x_{n-1} = y_{n-1}$ or $x_n = y_n$ then it is obvious by (M$^{\mathrm{J}}$-EXC) for $f$ that there exists an $(\tilde{x},\tilde{y})$-increment pair $(s,t)$ satisfying $(*)$. Since $x_{n-1} \geq y_{n-1}$ or $x_n \geq y_n$ holds by the assumption $\xi \geq \eta$, we may assume that $x_{n-1} > y_{n-1}$ and $x_n \neq y_n$.

**Case 1.** Suppose that $\xi \geq \eta + 2$. By (M$^{\mathrm{J}}$-EXC) for $f$ with $s = -\chi_{n-1}$, we have

$$f(x_0;x_{n-1},x_n) + f(y_0;y_{n-1},y_n) \geq \min \left\{ \begin{array}{c} f(x_0;x_{n-1}-1,x_n \pm 1) + f(y_0;y_{n-1}+1,y_n \mp 1), \\ f(x_0;x_{n-1}-2,x_n) + f(y_0;y_{n-1}+2,y_n), \\ f(x_0+t_0;x_{n-1}-1,x_n) + f(y_0-t_0;y_{n-1}+1,y_n) \end{array} \right\}, \quad (1)$$

where $t_0 \in \mathbf{Z}^{n-2}$ is an $(x_0,y_0)$-increment. Note that the signs in (1) are determined by the relations of components, and the second term exists only if $x_{n-1} - y_{n-1} \geq 2$. If the second term or the third term achieves the minimum, then $(s,t) = (-\tilde{\chi}_a, -\tilde{\chi}_a)$ or $(-\tilde{\chi}_a, \tilde{t})$, where $\tilde{t} = (t_0, 0) \in \mathbf{Z}^{\tilde{V}}$, is an $(\tilde{x},\tilde{y})$-increment pair satisfying $(*)$. Otherwise, we have

$$\begin{aligned} &f(x_0;x_{n-1},x_n) + f(y_0;y_{n-1},y_n) \geq f(x_0;x_{n-1}-1,x_n+1) + f(y_0;y_{n-1}+1,y_n-1), \\ \text{or } &f(x_0;x_{n-1},x_n) + f(y_0;y_{n-1},y_n) \geq f(x_0;x_{n-1}-1,x_n-1) + f(y_0;y_{n-1}+1,y_n+1). \end{aligned}$$

If $x_n > y_n$ then we have $f(x_0;x_{n-1},x_n) + f(y_0;y_{n-1},y_n) \geq f(x_0;x_{n-1}-1,x_n-1) + f(y_0;y_{n-1}+1,y_n+1)$, and so $\tilde{f}(x_0;\xi) + \tilde{f}(y_0;\eta) \geq \tilde{f}(x_0;\xi-2) + \tilde{f}(y_0;\eta+2)$. Thus $(s,t) = (-\tilde{\chi}_a, -\tilde{\chi}_a)$ is an $(\tilde{x},\tilde{y})$-increment pair satisfying $(*)$.

If $x_n < y_n$ then we have $f(x_0;x_{n-1},x_n) + f(y_0;y_{n-1},y_n) \geq f(x_0;x_{n-1}-1,x_n+1) + f(y_0;y_{n-1}+1,y_n-1)$. By the definition of $x_{n-1}, x_n, y_{n-1}, y_n$, we have $f(x_0;x_{n-1},x_n) = f(x_0;x_{n-1}-1,x_n+1)$ and $f(y_0;y_{n-1},y_n) = f(y_0;y_{n-1}+1,y_n-1)$. This contradicts the minimality of $|x_{n-1}-y_{n-1}| + |x_n - y_n|$.

**Case 2.** Suppose that $\xi = \eta$. It suffices to show that if $\tilde{y} = \mathbf{0}$ and $\tilde{x} = (1,1,1,1;0)$, $(2,1,1;0)$, $(3,1;0)$, $(2,2;0)$, or $(4;0)$, there exists an $(\tilde{x},\tilde{y})$-increment pair $(s,t)$ satisfying $(*)$. This is shown in Section 7.2.

**Case 3.** Suppose that $\xi = \eta + 1$. It suffices to show that if $\tilde{y} = \mathbf{0}$ and $\tilde{x} = (1,1,1;1)$, $(2,1;1)$, or $(3;1)$, there exists an $(\tilde{x},\tilde{y})$-increment pair $(s,t)$ satisfying $(*)$. This is shown in Section 7.3. □

## 7.2 Case 2 in the proof of Lemma 19

In this section, we deal with Case 2 in the proof of Lemma 19. First we show the essential case when $\tilde{x} = (1,1,1,1;0)$, whereas the other cases can be derived from this using the splitting technique discussed in Section 4.

### 7.2.1 The main lemma

Let $f : \mathbf{Z}^6 \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function with a bounded effective domain, and define

$$\tilde{f}(x_1,x_2,x_3,x_4;\xi) = \inf \{ f(x_1,x_2,x_3,x_4;x_5,x_6) \mid x_5 + x_6 = \xi \}.$$

We now show that if $\tilde{y} = \mathbf{0} \in \tilde{J}$ and $\tilde{x} = (1,1,1,1;0) \in \tilde{J}$ then there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s,t)$ satisfying $(*)$. We may assume that $\tilde{f}(0,0,0,0;0) = f(0,0,0,0;0,0)$ and $\tilde{f}(1,1,1,1;0) = f(1,1,1,1;k,-k)$ with $k > 0$. We denote $\mathbf{0} = (0,0,0,0;0,0)$, $\mathbf{1}_k = (1,1,1,1;k,-k)$, and $\chi_{1234} = \chi_1 + \chi_2 + \chi_3 + \chi_4$.

**Lemma 20.** *Suppose that* $\tilde{f}(0,0,0,0;0) = f(\mathbf{0})$ *and* $\tilde{f}(1,1,1,1;0) = f(\mathbf{1}_k)$ *with* $k > 0$. *Then we have*

$$f(\mathbf{0}) + f(\mathbf{1}_k) \geq \min \left\{ \begin{array}{c} \tilde{f}(1,1,0,0;0) + \tilde{f}(0,0,1,1;0), \\ \tilde{f}(1,0,1,0;0) + \tilde{f}(0,1,0,1;0), \\ \tilde{f}(1,0,0,1;0) + \tilde{f}(0,1,1,0;0) \end{array} \right\}. \tag{2}$$

*Proof.* First, by ($\mathrm{M}^{\mathrm{J}}$-EXC) for $f$ with $s = \chi_1$, we have

$$f(\mathbf{0}) + f(\mathbf{1}_k) \geq \min \left\{ \begin{array}{c} f(1,1,0,0;0,0) + f(0,0,1,1;k,-k), \\ f(1,0,1,0;0,0) + f(0,1,0,1;k,-k), \\ f(1,0,0,1;0,0) + f(0,1,1,0;k,-k), \\ f(1,0,0,0;1,0) + f(0,1,1,1;k-1,-k), \\ f(1,0,0,0;0,-1) + f(0,1,1,1;k,-k+1) \end{array} \right\}.$$

If one of the first three terms achieves the minimum, the desired inequality holds. Otherwise, we have

$$f(\mathbf{0}) + f(\mathbf{1}_k) \geq \min \left\{ \begin{array}{c} f(1,0,0,0;1,0) + f(0,1,1,1;k-1,-k), \\ f(1,0,0,0;0,-1) + f(0,1,1,1;k,-k+1) \end{array} \right\}. \tag{3}$$

We consider the following bipartite digraph $G = (U_G, V_G; A_G)$. The vertex sets $U_G$ and $V_G$ are defined by

$$\begin{aligned} U_G &= \{u_{(p,i)} \mid 1 \leq p \leq k,\ 1 \leq i \leq 4,\ f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) < +\infty\}, \\ V_G &= \{v_{(r,j)} \mid 1 \leq r \leq k,\ 1 \leq j \leq 4,\ f(\chi_j + r\chi_5 - (r-1)\chi_6) < +\infty\}. \end{aligned}$$

The arc set $A_G$ is defined as follows. For $u_{(p,i)} \in U_G$ an arc exists from $u_{(p,i)}$ to $v_{(r,j)}$ with $r \in \{1, \ldots, k\}$ and $j \in \{1,2,3,4\} \setminus \{i\}$ if there exists $q$ such that $0 \leq q \leq k$ and

$$f(\mathbf{0}) + f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \geq f(\chi_j + r\chi_5 - (r-1)\chi_6) + f(\chi_{1234} - \chi_i - \chi_j + q\chi_5 - q\chi_6).$$

Note that this inequality guarantees $v_{(r,j)} \in V_G$. Similarly, for $v_{(r,j)} \in V_G$ an arc exists from $v_{(r,j)}$ to $u_{(p,i)}$ with $p \in \{1, \ldots, k\}$ and $i \in \{1,2,3,4\} \setminus \{j\}$ if there exists $q$ such that $0 \leq q \leq k$ and

$$f(\mathbf{1}_k) + f(\chi_j + r\chi_5 - (r-1)\chi_6) \geq f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) + f(\chi_i + \chi_j + q\chi_5 - q\chi_6).$$

Note that this inequality guarantees $u_{(p,i)} \in U_G$.

Then the following lemma holds, which we prove in Section 7.2.2.

**Lemma 21.** *The out-degree of each vertex in* $G$ *is at least one.*

We mention here that $U_G \neq \emptyset$ and $V_G \neq \emptyset$. For, it follows from the inequality (3) that $u_{(k,1)} \in U_G$ or $v_{(1,1)} \in V_G$. Then Lemma 21 implies that $U_G \neq \emptyset$ and $V_G \neq \emptyset$.

By Lemma 21, $G$ has a directed cycle $C = (u_{(p_1,i_1)}, v_{(p_2,i_2)}, u_{(p_3,i_3)}, v_{(p_4,i_4)}, \dots, u_{(p_{2m-1},i_{2m-1})}, v_{(p_{2m},i_{2m})})$. This means, by the definition of $A_G$, that there exist $q_1, q_2, \dots, q_{2m}$ such that

$$
\begin{aligned}
f(\mathbf{0}) &+ f(\chi_{1234} - \chi_{i_1} + p_1\chi_5 - (p_1 - 1)\chi_6) \\
&\geq f(\chi_{i_2} + p_2\chi_5 - (p_2 - 1)\chi_6) + f(\chi_{1234} - \chi_{i_1} - \chi_{i_2} + q_1\chi_5 - q_1\chi_6), \\
f(\mathbf{1}_k) &+ f(\chi_{i_2} + p_2\chi_5 - (p_2 - 1)\chi_6) \\
&\geq f(\chi_{1234} - \chi_{i_3} + p_3\chi_5 - (p_3 - 1)\chi_6) + f(\chi_{i_3} + \chi_{i_2} + q_2\chi_5 - q_2\chi_6), \\
f(\mathbf{0}) &+ f(\chi_{1234} - \chi_{i_3} + p_3\chi_5 - (p_3 - 1)\chi_6) \\
&\geq f(\chi_{i_4} + p_4\chi_5 - (p_4 - 1)\chi_6) + f(\chi_{1234} - \chi_{i_3} - \chi_{i_4} + q_3\chi_5 - q_3\chi_6), \\
f(\mathbf{1}_k) &+ f(\chi_{i_4} + p_4\chi_5 - (p_4 - 1)\chi_6) \\
&\geq f(\chi_{1234} - \chi_{i_5} + p_5\chi_5 - (p_5 - 1)\chi_6) + f(\chi_{i_5} + \chi_{i_4} + q_4\chi_5 - q_4\chi_6), \\
&\dots \\
f(\mathbf{0}) &+ f(\chi_{1234} - \chi_{i_{2m-1}} + p_{2m-1}\chi_5 - (p_{2m-1} - 1)\chi_6) \\
&\geq f(\chi_{i_{2m}} + p_{2m}\chi_5 - (p_{2m} - 1)\chi_6) + f(\chi_{1234} - \chi_{i_{2m-1}} - \chi_{i_{2m}} + q_{2m-1}\chi_5 - q_{2m-1}\chi_6), \\
f(\mathbf{1}_k) &+ f(\chi_{i_{2m}} + p_{2m}\chi_5 - (p_{2m} - 1)\chi_6) \\
&\geq f(\chi_{1234} - \chi_{i_1} + p_1\chi_5 - (p_1 - 1)\chi_6) + f(\chi_{i_1} + \chi_{i_{2m}} + q_{2m}\chi_5 - q_{2m}\chi_6).
\end{aligned}
$$

By adding these inequalities, we obtain

$$
\begin{aligned}
m(f(\mathbf{0}) + f(\mathbf{1}_k)) \geq\ & f(\chi_{1234} - \chi_{i_1} - \chi_{i_2} + q_1\chi_5 - q_1\chi_6) \\
&+ f(\chi_{i_3} + \chi_{i_2} + q_2\chi_5 - q_2\chi_6) \\
&+ f(\chi_{1234} - \chi_{i_3} - \chi_{i_4} + q_3\chi_5 - q_3\chi_6) \\
&+ f(\chi_{i_5} + \chi_{i_4} + q_4\chi_5 - q_4\chi_6) \\
&+ \cdots \\
&+ f(\chi_{1234} - \chi_{i_{2m-1}} - \chi_{i_{2m}} + q_{2m-1}\chi_5 - q_{2m-1}\chi_6) \\
&+ f(\chi_{i_1} + \chi_{i_{2m}} + q_{2m}\chi_5 - q_{2m}\chi_6).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
m(f(\mathbf{0}) + f(\mathbf{1}_k)) \geq\ & \tilde{f}(\tilde{\chi}_{1234} - \tilde{\chi}_{i_1} - \tilde{\chi}_{i_2}) + \tilde{f}(\tilde{\chi}_{i_3} + \tilde{\chi}_{i_2}) \\
&+ \tilde{f}(\tilde{\chi}_{1234} - \tilde{\chi}_{i_3} - \tilde{\chi}_{i_4}) + \tilde{f}(\tilde{\chi}_{i_5} + \tilde{\chi}_{i_4}) \\
&+ \cdots \\
&+ \tilde{f}(\tilde{\chi}_{1234} - \tilde{\chi}_{i_{2m-1}} - \tilde{\chi}_{i_{2m}}) + \tilde{f}(\tilde{\chi}_{i_1} + \tilde{\chi}_{i_{2m}}),
\end{aligned}
$$

where $\tilde{\chi}_{1234} = \tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 + \tilde{\chi}_4$.

Here we note that

$$\begin{aligned} m\tilde{\chi}_{1234} &= (\tilde{\chi}_{1234} - \tilde{\chi}_{i_1} - \tilde{\chi}_{i_2}) + (\tilde{\chi}_{i_3} + \tilde{\chi}_{i_2}) \\ &\quad + (\tilde{\chi}_{1234} - \tilde{\chi}_{i_3} - \tilde{\chi}_{i_4}) + (\tilde{\chi}_{i_5} + \tilde{\chi}_{i_4}) \\ &\quad + \cdots \\ &\quad + (\tilde{\chi}_{1234} - \tilde{\chi}_{i_{2m-1}} - \tilde{\chi}_{i_{2m}}) + (\tilde{\chi}_{i_1} + \tilde{\chi}_{i_{2m}}). \end{aligned}$$

Then the desired inequality (2) follows from Lemma 22 below. □

**Lemma 22.** *If*

$$m\tilde{\chi}_{1234} = \sum_{1 \le i < j \le 4} m_{ij}(\tilde{\chi}_i + \tilde{\chi}_j) \tag{4}$$

*and*

$$m(f(\mathbf{0}) + f(\mathbf{1}_k)) \ge \sum_{1 \le i < j \le 4} m_{ij}\tilde{f}(\tilde{\chi}_i + \tilde{\chi}_j)$$

*for some nonnegative integers $m_{ij}$ and $m$, then*

$$f(\mathbf{0}) + f(\mathbf{1}_k) \ge \min\{\tilde{f}(\tilde{\chi}_1 + \tilde{\chi}_2) + \tilde{f}(\tilde{\chi}_3 + \tilde{\chi}_4),\ \tilde{f}(\tilde{\chi}_1 + \tilde{\chi}_3) + \tilde{f}(\tilde{\chi}_2 + \tilde{\chi}_4),\ \tilde{f}(\tilde{\chi}_1 + \tilde{\chi}_4) + \tilde{f}(\tilde{\chi}_2 + \tilde{\chi}_3)\}.$$

*Proof.* On the right-hand side of (4), the sum of the coefficients of $\tilde{\chi}_1$ and $\tilde{\chi}_2$ is $2m_{12} + m_{13} + m_{14} + m_{23} + m_{24}$. Meanwhile, that of $\tilde{\chi}_3$ and $\tilde{\chi}_4$ is $2m_{34} + m_{13} + m_{14} + m_{23} + m_{24}$. Hence $m_{12} = m_{34}$. We can show $m_{13} = m_{24}$, $m_{14} = m_{23}$ in the same way. Thus we have

$$m(f(\mathbf{0}) + f(\mathbf{1}_k)) \ge m_{12}(\tilde{f}(\tilde{\chi}_1 + \tilde{\chi}_2) + \tilde{f}(\tilde{\chi}_3 + \tilde{\chi}_4)) + m_{13}(\tilde{f}(\tilde{\chi}_1 + \tilde{\chi}_3) + \tilde{f}(\tilde{\chi}_2 + \tilde{\chi}_4)) + m_{14}(\tilde{f}(\tilde{\chi}_1 + \tilde{\chi}_4) + \tilde{f}(\tilde{\chi}_2 + \tilde{\chi}_3))$$

and

$$m_{12} + m_{13} + m_{14} = m,$$

which imply the desired inequality. □

### 7.2.2 Proof of Lemma 21

The out-degree of vertex $u_{(p,i)}$ is nonzero by Lemma 24 below, which relies on the following lemma.

**Lemma 23.** *For any integers $p \le q$ and for any $i \in \{1, 2, 3, 4\}$,* (A) *or* (B) *holds.*

(A) *There exists an integer $r$ such that $p \le r \le q + 1$ and*

$$\begin{aligned} &f((p+2)\chi_5 - p\chi_6) + f(\chi_{1234} - \chi_i + q\chi_5 - (q+1)\chi_6) \\ &\quad \ge \min \left\{ \begin{array}{l} f(\chi_{j_1} + (p+q+2-r)\chi_5 - (p+q+1-r)\chi_6) + f(\chi_{j_2} + \chi_{j_3} + r\chi_5 - r\chi_6), \\ f(\chi_{j_2} + (p+q+2-r)\chi_5 - (p+q+1-r)\chi_6) + f(\chi_{j_3} + \chi_{j_1} + r\chi_5 - r\chi_6), \\ f(\chi_{j_3} + (p+q+2-r)\chi_5 - (p+q+1-r)\chi_6) + f(\chi_{j_1} + \chi_{j_2} + r\chi_5 - r\chi_6) \end{array} \right\}, \end{aligned}$$

*where $\{j_1, j_2, j_3\} = \{1, 2, 3, 4\} \setminus \{i\}$.*

(B) *There exists an integer $r$ such that $p+1 \le r \le q+1$ and*

$$\begin{aligned}f((p+2)\chi_5 - p\chi_6) &+ f(\chi_{1234} - \chi_i + q\chi_5 - (q+1)\chi_6) \\ &\ge f(r\chi_5 - r\chi_6) + f(\chi_{1234} - \chi_i + (p+q+2-r)\chi_5 - (p+q+1-r)\chi_6).\end{aligned}$$

*Proof.* We show by induction on $q-p$.

If $q-p=0$ then, by ($\mathrm{M^J}$-EXC) with $s=-\chi_6$, we have

$$\begin{aligned}&f((p+2)\chi_5 - p\chi_6) + f(\chi_{1234} - \chi_i + q\chi_5 - (q+1)\chi_6) \\ &\ge \min\left\{\begin{array}{c} f(\chi_{j_1} + (p+2)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_{j_1} - \chi_i + q\chi_5 - q\chi_6), \\ f(\chi_{j_2} + (p+2)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_{j_2} - \chi_i + q\chi_5 - q\chi_6), \\ f(\chi_{j_3} + (p+2)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_{j_3} - \chi_i + q\chi_5 - q\chi_6), \\ f((p+1)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_i + (q+1)\chi_5 - q\chi_6) \end{array}\right\},\end{aligned}$$

where $\{j_1, j_2, j_3\} = \{1,2,3,4\} \setminus \{i\}$. If one of the first three terms achieves the minimum, then (A) holds with $r=q$; otherwise (B) holds with $r=p+1$.

If $q-p=1$ then, by ($\mathrm{M^J}$-EXC) with $s=-\chi_5$, we have

$$\begin{aligned}&f((p+2)\chi_5 - p\chi_6) + f(\chi_{1234} - \chi_i + q\chi_5 - (q+1)\chi_6) \\ &\ge \min\left\{\begin{array}{c} f(\chi_{j_1} + (p+1)\chi_5 - p\chi_6) + f(\chi_{1234} - \chi_{j_1} - \chi_i + (q+1)\chi_5 - (q+1)\chi_6), \\ f(\chi_{j_2} + (p+1)\chi_5 - p\chi_6) + f(\chi_{1234} - \chi_{j_2} - \chi_i + (q+1)\chi_5 - (q+1)\chi_6), \\ f(\chi_{j_3} + (p+1)\chi_5 - p\chi_6) + f(\chi_{1234} - \chi_{j_3} - \chi_i + (q+1)\chi_5 - (q+1)\chi_6), \\ f((p+1)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_i + (q+1)\chi_5 - q\chi_6) \end{array}\right\},\end{aligned}$$

where $\{j_1, j_2, j_3\} = \{1,2,3,4\} \setminus \{i\}$. If one of the first three terms achieves the minimum, then (A) holds with $r=q+1$; otherwise (B) holds with $r=p+1$.

Suppose that $q-p \ge 2$. By ($\mathrm{M^J}$-EXC) with $s=-\chi_6$, we have

$$\begin{aligned}&f((p+2)\chi_5 - p\chi_6) + f(\chi_{1234} - \chi_i + q\chi_5 - (q+1)\chi_6) \\ &\ge \min\left\{\begin{array}{c} f(\chi_{j_1} + (p+2)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_{j_1} - \chi_i + q\chi_5 - q\chi_6), \\ f(\chi_{j_2} + (p+2)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_{j_2} - \chi_i + q\chi_5 - q\chi_6), \\ f(\chi_{j_3} + (p+2)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_{j_3} - \chi_i + q\chi_5 - q\chi_6), \\ f((p+3)\chi_5 - (p+1)\chi_6) + f(\chi_{1234} - \chi_i + (q-1)\chi_5 - q\chi_6), \\ f((p+2)\chi_5 - (p+2)\chi_6) + f(\chi_{1234} - \chi_i + q\chi_5 - (q-1)\chi_6) \end{array}\right\},\end{aligned}$$

where $\{j_1, j_2, j_3\} = \{1,2,3,4\} \setminus \{i\}$. Note that the fourth term exists only if $q-p \ge 3$. If one of the first three terms achieves the minimum, then (A) holds with $r=q$, and if the last term achieves the minimum, then (B) holds with $r=p+2 \le q$. To the fourth term, the induction applies and yields (A) with $p+1 \le r \le q$ or (B) with $p+2 \le r \le q$. □

**Lemma 24.** *For any integer $1 \le p \le k$ and for any $i \in \{1,2,3,4\}$, there exist integers $q$ and $r$ such that $0 \le q \le k-1$, $1 \le r \le k$, and*

$$f(\mathbf{0}) + f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \ge \min\left\{\begin{array}{c} f(\chi_{j_1} + r\chi_5 - (r-1)\chi_6) + f(\chi_{j_2} + \chi_{j_3} + q\chi_5 - q\chi_6), \\ f(\chi_{j_2} + r\chi_5 - (r-1)\chi_6) + f(\chi_{j_3} + \chi_{j_1} + q\chi_5 - q\chi_6), \\ f(\chi_{j_3} + r\chi_5 - (r-1)\chi_6) + f(\chi_{j_1} + \chi_{j_2} + q\chi_5 - q\chi_6) \end{array}\right\},$$

*where $\{j_1, j_2, j_3\} = \{1,2,3,4\} \setminus \{i\}$.*

*Proof.* It suffices to consider $p$ which minimizes $f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6)$. Let $p$ be the minimum minimizer. By (M$^{\mathrm{J}}$-EXC) with $s = \chi_5$, we have

$$f(\mathbf{0})+f(\chi_{1234}-\chi_i+p\chi_5-(p-1)\chi_6) \geq \min \left\{ \begin{array}{c} f(\chi_{j_1}+\chi_5)+f(\chi_{j_2}+\chi_{j_3}+(p-1)\chi_5-(p-1)\chi_6), \\ f(\chi_{j_2}+\chi_5)+f(\chi_{j_3}+\chi_{j_1}+(p-1)\chi_5-(p-1)\chi_6), \\ f(\chi_{j_3}+\chi_5)+f(\chi_{j_1}+\chi_{j_2}+(p-1)\chi_5-(p-1)\chi_6), \\ f(\chi_5-\chi_6)+f(\chi_{1234}-\chi_i+(p-1)\chi_5-(p-2)\chi_6), \\ f(2\chi_5)+f(\chi_{1234}-\chi_i+(p-2)\chi_5-(p-1)\chi_6) \end{array} \right\}.$$

Note that the last two terms exist only if $p \geq 2$.

If one of the first three terms achieves the minimum, the claim holds with $q = p - 1$ and $r = 1$.

To consider the fourth term, suppose that $p \geq 2$ and

$$f(\mathbf{0}) + f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \geq f(\chi_5 - \chi_6) + f(\chi_{1234} - \chi_i + (p-1)\chi_5 - (p-2)\chi_6).$$

Then, since $f(\mathbf{0}) = \tilde{f}(0,0,0,0;0) \leq f(\chi_5 - \chi_6)$, we have

$$f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \geq f(\chi_{1234} - \chi_i + (p-1)\chi_5 - (p-2)\chi_6),$$

which contradicts the definition of $p$.

To consider the fifth term, suppose that $p \geq 2$ and

$$f(\mathbf{0}) + f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \geq f(2\chi_5) + f(\chi_{1234} - \chi_i + (p-2)\chi_5 - (p-1)\chi_6).$$

By Lemma 23, at least one of (A) and (B) holds.

(A) There exists an integer $r'$ such that $0 \leq r' \leq p-1$ and

$$\begin{aligned} &f(2\chi_5) + f(\chi_{1234} - \chi_i + (p-2)\chi_5 - (p-1)\chi_6) \\ &\quad \geq \min \left\{ \begin{array}{c} f(\chi_{j_1} + (p-r')\chi_5 - (p-r'-1)\chi_6) + f(\chi_{j_2} + \chi_{j_3} + r'\chi_5 - r'\chi_6), \\ f(\chi_{j_2} + (p-r')\chi_5 - (p-r'-1)\chi_6) + f(\chi_{j_3} + \chi_{j_1} + r'\chi_5 - r'\chi_6), \\ f(\chi_{j_3} + (p-r')\chi_5 - (p-r'-1)\chi_6) + f(\chi_{j_1} + \chi_{j_2} + r'\chi_5 - r'\chi_6) \end{array} \right\}. \end{aligned}$$

(B) There exists an integer $r'$ such that $1 \leq r' \leq p-1$ and

$$\begin{aligned} &f(2\chi_5) + f(\chi_{1234} - \chi_i + (p-2)\chi_5 - (p-1)\chi_6) \\ &\quad \geq f(r'\chi_5 - r'\chi_6) + f(\chi_{1234} - \chi_i + (p-r')\chi_5 - (p-r'-1)\chi_6). \end{aligned}$$

In case of (A) we have

$$\begin{aligned} &f(\mathbf{0}) + f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \\ &\quad \geq \min \left\{ \begin{array}{c} f(\chi_{j_1} + (p-r')\chi_5 - (p-r'-1)\chi_6) + f(\chi_{j_2} + \chi_{j_3} + r'\chi_5 - r'\chi_6), \\ f(\chi_{j_2} + (p-r')\chi_5 - (p-r'-1)\chi_6) + f(\chi_{j_3} + \chi_{j_1} + r'\chi_5 - r'\chi_6), \\ f(\chi_{j_3} + (p-r')\chi_5 - (p-r'-1)\chi_6) + f(\chi_{j_1} + \chi_{j_2} + r'\chi_5 - r'\chi_6) \end{array} \right\}, \end{aligned}$$

which implies the desired claim with $q = r'$ and $r = p - r'$.

In case of (B) we have $1 \leq r' \leq p-1$ and

$$f(\mathbf{0}) + f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \geq f(r'\chi_5 - r'\chi_6) + f(\chi_{1234} - \chi_i + (p-r')\chi_5 - (p-r'-1)\chi_6).$$

Since $f(\mathbf{0}) = \tilde{f}(0,0,0,0;0) \leq f(r'\chi_5 - r'\chi_6)$, we have

$$f(\chi_{1234} - \chi_i + p\chi_5 - (p-1)\chi_6) \geq f(\chi_{1234} - \chi_i + (p-r')\chi_5 - (p-r'-1)\chi_6),$$

which contradicts the definition of $p$. □

In the same way as Lemma 24, we have the following lemma, which means that the out degree of vertex $v_{(r,j)}$ is nonzero.

**Lemma 25.** *For any integer $1 \leq r \leq k$ and for any $j \in \{1,2,3,4\}$, there exist integers $p$ and $q$ such that $1 \leq p \leq k$, $1 \leq q \leq k$, and*

$$f(\mathbf{1}_k) + f(\chi_j + r\chi_5 - (r-1)\chi_6) \geq \min \left\{ \begin{array}{l} f(\chi_{1234} - \chi_{i_1} + p\chi_5 - (p-1)\chi_6) + f(\chi_{i_1} + \chi_j + q\chi_5 - q\chi_6), \\ f(\chi_{1234} - \chi_{i_2} + p\chi_5 - (p-1)\chi_6) + f(\chi_{i_2} + \chi_j + q\chi_5 - q\chi_6), \\ f(\chi_{1234} - \chi_{i_3} + p\chi_5 - (p-1)\chi_6) + f(\chi_{i_3} + \chi_j + q\chi_5 - q\chi_6) \end{array} \right\},$$

*where $\{i_1, i_2, i_3\} = \{1,2,3,4\} \setminus \{j\}$.*

*Proof.* We consider the coordinate transformation from $(x_1, x_2, x_3, x_4; x_5, x_6)$ to $(1-x_1, 1-x_2, 1-x_3, 1-x_4; k+x_6, -k+x_5)$. Applying Lemma 24 in the new coordinate system, we see the following fact:

> For any integer $1 \leq p' \leq k$ and for any $j \in \{1,2,3,4\}$, there exist integers $q'$ and $r'$ such that $0 \leq q' \leq k-1$, $1 \leq r' \leq k$ and
>
> $$\begin{aligned} & f(\mathbf{1}_k) + f(\chi_j + (k-p'+1)\chi_5 - (k-p')\chi_6) \\ & \geq \min \left\{ \begin{array}{l} f(\chi_{1234} - \chi_{i_1} + (k-r'+1)\chi_5 - (k-r')\chi_6) + f(\chi_{i_1} + \chi_j + (k-q')\chi_5 - (k-q')\chi_6), \\ f(\chi_{1234} - \chi_{i_2} + (k-r'+1)\chi_5 - (k-r')\chi_6) + f(\chi_{i_2} + \chi_j + (k-q')\chi_5 - (k-q')\chi_6), \\ f(\chi_{1234} - \chi_{i_3} + (k-r'+1)\chi_5 - (k-r')\chi_6) + f(\chi_{i_3} + \chi_j + (k-q')\chi_5 - (k-q')\chi_6) \end{array} \right\}, \end{aligned}$$
>
> where $\{i_1, i_2, i_3\} = \{1,2,3,4\} \setminus \{j\}$.

By setting $p = k - r' + 1$, $r = k - p' + 1$, and $q = k - q'$, we obtain the claim. □

Lemma 21 immediately follows from Lemmas 24 and 25.

### 7.2.3 Other cases in Case 2

The other cases in Case 2 are treated here with the aid of the splitting technique.

**Lemma 26.** *If $\tilde{y} = \mathbf{0}$ and $\tilde{x} = (1,1,1,1;0)$, $(2,1,1;0)$, $(3,1;0)$, $(2,2;0)$, or $(4;0)$, then there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s,t)$ satisfying $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x} + s + t) + \tilde{f}(\tilde{y} - s - t)$.*

*Proof.* If $\tilde{x} = (1,1,1,1;0)$ then the claim follows from Lemma 20.

Suppose that $\tilde{x} = (2,1,1;0)$. In this case, we may assume $\tilde{f}(0,0,0;0) = f(0,0,0;0,0)$ and $\tilde{f}(2,1,1;0) = f(2,1,1;k,-k)$ with $k > 0$. We define $f'$ as $f'(x_1,x_2,x_3,x_4;x_5,x_6) = f(x_1 + x_2,x_3,x_4;x_5,x_6)$, and $\tilde{f}'$ as

$$\tilde{f}'(x_1,x_2,x_3,x_4;\xi) = \inf\left\{f'(x_1,x_2,x_3,x_4;x_5,x_6) \mid x_5 + x_6 = \xi\right\}.$$

Then $\tilde{f}'(x_1,x_2,x_3,x_4;0) = \tilde{f}(x_1 + x_2,x_3,x_4;0)$. Since $f'$ is a splitting of $f$, it is M-convex by Theorem 7. By Lemma 20, we have

$$\begin{aligned}
f(0,0,0;0,0) + f(2,1,1;k,-k) &= f'(0,0,0,0;0,0) + f'(1,1,1,1;k,-k) \\
&\geq \min\left\{\begin{matrix} \tilde{f}'(1,1,0,0;0) + \tilde{f}'(0,0,1,1;0), \\ \tilde{f}'(1,0,1,0;0) + \tilde{f}'(0,1,0,1;0), \\ \tilde{f}'(1,0,0,1;0) + \tilde{f}'(0,1,1,0;0) \end{matrix}\right\} \\
&= \min\left\{\begin{matrix} \tilde{f}(2,0,0;0) + \tilde{f}(0,1,1;0), \\ \tilde{f}(1,1,0;0) + \tilde{f}(1,0,1;0), \\ \tilde{f}(1,0,1;0) + \tilde{f}(1,1,0;0) \end{matrix}\right\},
\end{aligned}$$

which means that there exists an $(\tilde{x},\tilde{y})$-increment pair $(s,t)$ satisfying $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x}+s+t) + \tilde{f}(\tilde{y}-s-t)$.

Suppose that $\tilde{x} = (3,1;0)$. In this case, we may assume $\tilde{f}(0,0;0) = f(0,0;0,0)$ and $\tilde{f}(3,1;0) = f(3,1;k,-k)$ with $k > 0$. We define $f'$ as $f'(x_1,x_2,x_3,x_4;x_5,x_6) = f(x_1 + x_2 + x_3,x_4;x_5,x_6)$, and $\tilde{f}'$ as

$$\tilde{f}'(x_1,x_2,x_3,x_4;\xi) = \inf\left\{f'(x_1,x_2,x_3,x_4;x_5,x_6) \mid x_5 + x_6 = \xi\right\}.$$

Then $\tilde{f}'(x_1,x_2,x_3,x_4;0) = \tilde{f}(x_1 + x_2 + x_3,x_4;0)$. Since $f'$ is a splitting of $f$, it is M-convex by Theorem 7. By Lemma 20, we have

$$\begin{aligned}
f(0,0;0,0) + f(3,1;k,-k) &= f'(0,0,0,0;0,0) + f'(1,1,1,1;k,-k) \\
&\geq \min\left\{\begin{matrix} \tilde{f}'(1,1,0,0;0) + \tilde{f}'(0,0,1,1;0), \\ \tilde{f}'(1,0,1,0;0) + \tilde{f}'(0,1,0,1;0), \\ \tilde{f}'(1,0,0,1;0) + \tilde{f}'(0,1,1,0;0) \end{matrix}\right\} \\
&= \min\left\{\begin{matrix} \tilde{f}(2,0;0) + \tilde{f}(1,1;0), \\ \tilde{f}(2,0;0) + \tilde{f}(1,1;0), \\ \tilde{f}(1,1;0) + \tilde{f}(2,0;0) \end{matrix}\right\} \\
&= \tilde{f}(2,0;0) + \tilde{f}(1,1;0),
\end{aligned}$$

which means that there exists an $(\tilde{x},\tilde{y})$-increment pair $(s,t)$ satisfying $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x}+s+t) + \tilde{f}(\tilde{y}-s-t)$.

Suppose that $\tilde{x} = (2,2;0)$. In this case, we may assume $\tilde{f}(0,0;0) = f(0,0;0,0)$ and $\tilde{f}(2,2;0) = f(2,2;k,-k)$ with $k > 0$. We define $f'$ as $f'(x_1,x_2,x_3,x_4;x_5,x_6) = f(x_1 + x_2,x_3 + x_4;x_5,x_6)$, and $\tilde{f}'$ as

$$\tilde{f}'(x_1,x_2,x_3,x_4;\xi) = \inf\left\{f'(x_1,x_2,x_3,x_4;x_5,x_6) \mid x_5 + x_6 = \xi\right\}.$$

Then $\tilde{f}'(x_1, x_2, x_3, x_4; 0) = \tilde{f}(x_1 + x_2, x_3 + x_4; 0)$. Since $f'$ is a splitting of $f$, it is M-convex by Theorem 7. By Lemma 20, we have

$$\begin{aligned}
f(0,0;0,0) + f(2,2;k,-k) &= f'(0,0,0,0;0,0) + f'(1,1,1,1;k,-k) \\
&\geq \min \left\{ \begin{array}{l} \tilde{f}'(1,1,0,0;0) + \tilde{f}'(0,0,1,1;0), \\ \tilde{f}'(1,0,1,0;0) + \tilde{f}'(0,1,0,1;0), \\ \tilde{f}'(1,0,0,1;0) + \tilde{f}'(0,1,1,0;0) \end{array} \right\} \\
&= \min \left\{ \begin{array}{l} \tilde{f}(2,0;0) + \tilde{f}(0,2;0), \\ \tilde{f}(1,1;0) + \tilde{f}(1,1;0), \\ \tilde{f}(1,1;0) + \tilde{f}(1,1;0) \end{array} \right\},
\end{aligned}$$

which means that there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s,t)$ satisfying $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x} + s + t) + \tilde{f}(\tilde{y} - s - t)$.

Suppose that $\tilde{x} = (4; 0)$. In this case, we may assume $\tilde{f}(0;0) = f(0;0,0)$ and $\tilde{f}(4;0) = f(4;k,-k)$ with $k > 0$. We define $f'$ as $f'(x_1, x_2, x_3, x_4; x_5, x_6) = f(x_1 + x_2 + x_3 + x_4; x_5, x_6)$, and $\tilde{f}'$ as

$$\tilde{f}'(x_1, x_2, x_3, x_4; \xi) = \inf \left\{ f'(x_1, x_2, x_3, x_4; x_5, x_6) \mid x_5 + x_6 = \xi \right\}.$$

Then $\tilde{f}'(x_1, x_2, x_3, x_4; 0) = \tilde{f}(x_1 + x_2 + x_3 + x_4; 0)$. Since $f'$ is a splitting of $f$, it is M-convex by Theorem 7. By Lemma 20, we have

$$\begin{aligned}
f(0;0,0) + f(4;k,-k) &= f'(0,0,0,0;0,0) + f'(1,1,1,1;k,-k) \\
&\geq \min \left\{ \begin{array}{l} \tilde{f}'(1,1,0,0;0) + \tilde{f}'(0,0,1,1;0), \\ \tilde{f}'(1,0,1,0;0) + \tilde{f}'(0,1,0,1;0), \\ \tilde{f}'(1,0,0,1;0) + \tilde{f}'(0,1,1,0;0) \end{array} \right\} \\
&= \tilde{f}(2;0) + \tilde{f}(2;0),
\end{aligned}$$

which means that there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s,t)$ satisfying $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x} + s + t) + \tilde{f}(\tilde{y} - s - t)$. $\square$

### 7.3 Case 3 in the proof of Lemma 19

In this section, we deal with Case 3 in the proof of Lemma 19. First we focus on the case of $\tilde{x} = (1,1,1;1)$, whereas the other cases are treated later using the splitting technique discussed in Section 4.

Let $f : \mathbf{Z}^5 \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function, and put

$$\tilde{f}(x_1, x_2, x_3; \xi) = \inf \{ f(x_1, x_2, x_3; x_4, x_5) \mid x_4 + x_5 = \xi \}.$$

We now show that if $\tilde{y} = \mathbf{0} \in \tilde{J}$ and $\tilde{x} = (1,1,1;1) \in \tilde{J}$ then there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s,t)$ satisfying $(*)$. We may assume $\tilde{f}(0,0,0;0) = f(0,0,0;0,0)$ and $\tilde{f}(1,1,1;1) = f(1,1,1;k+1,-k)$ with $k > 0$.

**Lemma 27.** *Suppose that* $\tilde{f}(0,0,0;0) = f(0,0,0;0,0)$ *and* $\tilde{f}(1,1,1;1) = f(1,1,1;k+1,-k)$ *with* $k > 0$*. Then we have*

$$f(0,0,0;0,0) + f(1,1,1;k+1,-k) \geq \min \begin{Bmatrix} \tilde{f}(1,1,0;0) + \tilde{f}(0,0,1;1), \\ \tilde{f}(1,0,1;0) + \tilde{f}(0,1,0;1), \\ \tilde{f}(0,1,1;0) + \tilde{f}(1,0,0;1) \end{Bmatrix}.$$

*Proof.* We define $f' : \mathbf{Z}^6 \to \mathbf{R} \cup \{+\infty\}$ as

$$f'(x_1, x_2, x_3, x_4; x_5, x_6) = \begin{cases} f(x_1, x_2, x_3; x_5, x_6) & \text{if } x_4 = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Since $f$ is M-convex, $f'$ is also M-convex. By Lemma 24 applied to $f'$ with $i = 4$, we have the following fact:

For any integer $1 \leq p \leq k'$, there exist integers $q$ and $r$ such that $0 \leq q \leq k'-1$, $1 \leq r \leq k'$, and

$$f'(0,0,0,0;0,0) + f'(1,1,1,0;p,-(p-1)) \geq \min \begin{Bmatrix} f'(1,0,0,0;r,-(r-1)) + f'(0,1,1,0;q,-q), \\ f'(0,1,0,0;r,-(r-1)) + f'(1,0,1,0;q,-q), \\ f'(0,0,1,0;r,-(r-1)) + f'(1,1,0,0;q,-q) \end{Bmatrix}.$$

By taking $k' \geq k+1$ and $p = k+1$ in the above, we have

$$\begin{aligned} f(0,0,0;0,0) + f(1,1,1;k+1,-k) &= f'(0,0,0,0;0,0) + f'(1,1,1,0;k+1,-k) \\ &\geq \min \begin{Bmatrix} f'(1,0,0,0;r,-(r-1)) + f'(0,1,1,0;q,-q), \\ f'(0,1,0,0;r,-(r-1)) + f'(1,0,1,0;q,-q), \\ f'(0,0,1,0;r,-(r-1)) + f'(1,1,0,0;q,-q) \end{Bmatrix} \\ &= \min \begin{Bmatrix} f(1,0,0;r,-(r-1)) + f(0,1,1;q,-q), \\ f(0,1,0;r,-(r-1)) + f(1,0,1;q,-q), \\ f(0,0,1;r,-(r-1)) + f(1,1,0;q,-q) \end{Bmatrix} \\ &\geq \min \begin{Bmatrix} \tilde{f}(1,0,0;1) + \tilde{f}(0,1,1;0), \\ \tilde{f}(0,1,0;1) + \tilde{f}(1,0,1;0), \\ \tilde{f}(0,0,1;1) + \tilde{f}(1,1,0;0) \end{Bmatrix}, \end{aligned}$$

which implies the lemma. □

**Lemma 28.** *If* $\tilde{y} = \mathbf{0}$ *and* $\tilde{x} = (1,1,1;1)$, $(2,1;1)$, *or* $(3;1)$*, then there exists an* $(\tilde{x}, \tilde{y})$*-increment pair* $(s,t)$ *satisfying* $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x}+s+t) + \tilde{f}(\tilde{y}-s-t)$.

*Proof.* If $\tilde{x} = (1,1,1;1)$ then the claim follows from Lemma 27.

Suppose that $\tilde{x} = (2,1;1)$. In this case, we may assume $\tilde{f}(0,0;0) = f(0,0;0,0)$ and $\tilde{f}(2,1;1) = f(2,1;k+1,-k)$ with $k > 0$. We define $f'$ as $f'(x_1,x_2,x_3;x_4,x_5) = f(x_1+x_2,x_3;x_4,x_5)$, and $\tilde{f}'$ as

$$\tilde{f}'(x_1,x_2,x_3;\xi) = \inf \left\{ f'(x_1,x_2,x_3;x_4,x_5) \mid x_4 + x_5 = \xi \right\}.$$

Then $\tilde{f}'(x_1, x_2, x_3; \xi) = \tilde{f}(x_1 + x_2, x_3; \xi)$. Since $f'$ is a splitting of $f$, it is M-convex by Theorem 7. By Lemma 27, we have

$$\begin{aligned} f(0,0;0,0) + f(2,1;k+1,-k) &= f'(0,0,0;0,0) + f'(1,1,1;k+1,-k) \\ &\geq \min \left\{ \begin{matrix} \tilde{f}'(1,0,0;1) + \tilde{f}'(0,1,1;0), \\ \tilde{f}'(0,1,0;1) + \tilde{f}'(1,0,1;0), \\ \tilde{f}'(0,0,1;1) + \tilde{f}'(1,1,0;0) \end{matrix} \right\} \\ &= \min \left\{ \begin{matrix} \tilde{f}(1,0;1) + \tilde{f}(1,1;0), \\ \tilde{f}(1,0;1) + \tilde{f}(1,1;0), \\ \tilde{f}(0,1;1) + \tilde{f}(2,0;0) \end{matrix} \right\}, \end{aligned}$$

which means that there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s, t)$ satisfying $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x} + s + t) + \tilde{f}(\tilde{y} - s - t)$.

Suppose that $\tilde{x} = (3; 1)$. In this case, we may assume $\tilde{f}(0; 0) = f(0; 0, 0)$ and $\tilde{f}(3; 1) = f(3; k+1, -k)$. We define $f'$ as $f'(x_1, x_2, x_3; x_4, x_5) = f(x_1 + x_2 + x_3; x_4, x_5)$, and $\tilde{f}'$ as

$$\tilde{f}'(x_1, x_2, x_3; \xi) = \inf \left\{ f'(x_1, x_2, x_3; x_4, x_5) \mid x_4 + x_5 = \xi \right\}.$$

Then $\tilde{f}'(x_1, x_2, x_3; \xi) = \tilde{f}(x_1 + x_2 + x_3; \xi)$. Since $f'$ is a splitting of $f$, it is M-convex by Theorem 7. By Lemma 27, we have

$$\begin{aligned} f(0;0,0) + f(3;k+1,-k) &= f'(0,0,0;0,0) + f'(1,1,1;k+1,-k) \\ &\geq \min \left\{ \begin{matrix} \tilde{f}'(1,0,0;1) + \tilde{f}'(0,1,1;0), \\ \tilde{f}'(0,1,0;1) + \tilde{f}'(1,0,1;0), \\ \tilde{f}'(0,0,1;1) + \tilde{f}'(1,1,0;0) \end{matrix} \right\} \\ &= \tilde{f}(1;1) + \tilde{f}(2;0), \end{aligned}$$

which means that there exists an $(\tilde{x}, \tilde{y})$-increment pair $(s, t)$ satisfying $\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x} + s + t) + \tilde{f}(\tilde{y} - s - t)$. $\square$

### 7.4 Case of unbounded effective domain

We now deal with the general case of Lemma 10 without boundness assumption on the effective domain.

*Proof of Lemma 10.* For $R = 1, 2, \ldots$, we define $f^{(R)} : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ by

$$f^{(R)}(x) = \begin{cases} f(x) & \text{if } \max_{v \in V} |x(v)| \leq R \\ +\infty & \text{otherwise} \end{cases} \qquad (x \in \mathbf{Z}^n),$$

which is an M-convex function with a bounded effective domain, provided that $R$ is large enough for $\mathrm{dom} f^{(R)} \neq \emptyset$. For each $R$ an elementary aggregation $\tilde{f}^{(R)}$ of $f^{(R)}$ is M-convex by Lemma 19. Take $x, y \in \mathrm{dom}\tilde{f}$. There exists $R_0 = R_0(x, y)$, depending on $x$ and $y$, such that $x, y \in \mathrm{dom}\tilde{f}^{(R)}$ for every $R \geq R_0$. Since $\tilde{f}^{(R)}$ is M-convex, there exists an $(x, y)$-increment pair $(s_R, t_R)$ such that

$$\tilde{f}^{(R)}(x) + \tilde{f}^{(R)}(y) \geq \tilde{f}^{(R)}(x + s_R + t_R) + \tilde{f}^{(R)}(y - s_R - t_R).$$

Since the set of all $(x, y)$-increment pairs is finite, at least one $(x, y)$-increment pair appears infinitely many times in the sequence $(s_{R_0}, t_{R_0}), (s_{R_0+1}, t_{R_0+1}), \ldots$. More precisely, there exists an $(x, y)$-increment pair $(s, t)$ and an increasing subsequence $R_1 < R_2 < \cdots$ such that $(s_{R_i}, t_{R_i}) = (s, t)$ for $i = 1, 2, \ldots$. By letting $R \to \infty$ along this subsequence in the above inequality we obtain

$$\tilde{f}(x) + \tilde{f}(y) \geq \tilde{f}(x + s + t) + \tilde{f}(y - s - t).$$

Thus $\tilde{f}$ satisfies ($\mathrm{M}^{\mathrm{J}}$-$\mathrm{EXC}_{\mathrm{loc}}$). This completes the proof of Lemma 10. $\square$


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