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Fundamental holes and saturation points of a commutative semigroup and their applications to contingency tables

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Abstract: Does a given system of linear equations with nonnegative constraints have an integer solution? This problem appears in many areas, such as number theory, operations research, and statistics. To study a family of systems with no integer solution, we focus on a commutative semigroup generated by the columns of its defining matrix. In this paper we will study a commutative semigroup generated by a finite subset of $\mathbb{Z}^d$ and its saturation. We show the necessary and sufficient conditions for the given semigroup to have a finite number of elements in the difference between the semigroup and its saturation. Also we define fundamental holes and saturation points of a commutative semigroup. Then, we show the simultaneous finiteness of the difference between the semigroup and its saturation, the set of non-saturation points of the semigroup, and the set of generators for saturation points, which is a set of generator of a monoid. We apply our results to some three and four dimensional contingency tables.

Key words and phrases: Keywords: semigroup, monoid, saturation, Hilbert basis, linear integer feasibility problem, Frobenius problem, contingency tables, data security.

1. Introduction

Suppose we have the *semigroup* generated by the columns of a matrix $A \in \mathbb{Z}^{d\times n}$. To study the points in the semigroup finds its applications in several areas. For example, one can find an application in integer linear optimization. Let $\boldsymbol{b} \in \mathbb{Z}^d$. Suppose we have the following system of linear equations and inequalities:

$$A\boldsymbol{x} = \boldsymbol{b}, \;\; \boldsymbol{x} \geq 0. \tag{1.1}$$

In order to have an integral solution which satisfies the system in (1.1), a vector $\boldsymbol{b}$ must be in the semigroup generated by the columns of A. A *linear integer feasibility problem* is to ask whether the system in (1.1) has integral solution or not. If we fix a vector $\boldsymbol{b}$, then Aardal and collaborators have written fairly effective methods using the Lenstra-Lenstra-Lovász (LLL) procedure for testing integer feasibility [Aardal et al., 1998, 2002]. Another application can be found in number theory. When $d = 1$, the *Frobenius problem* asks to characterize the semigroup of A. In 2002, Barvinok and Woods introduced an algorithm to encode all points inside the semigroup generated by the columns of A into its generating function as a sum of short rational functions in polynomial time when $d = 1$ and n is fixed [Barvinok and Woods, 2003].

In statistics, one can find an application in *multi-way contingency tables*, or *tabular data*. The *3-dimensional integer planar transportation problem (3-DIPTP)* is an integer feasibility problem which asks whether there exists a three dimensional contingency table with the given 2-margins or not. Vlach in [Vlach, 1986] provides an excellent summary of attempts on the problem. 3-DIPTP has received recent attention in statistics, particularly for data security [Cox, 2002]. In order to publish the data to public, national statistical offices (NSOs) subject statistical data to a range of verification and "cleaning" processes. Data in a statistical database can come from multiple sources, at various times, and may have been modified through a variety of statistical procedures such as rounding. In 2000, Cox demonstrates that any of these factors can produce an infeasible table. Even though marginal totals satisfy obvious necessary conditions, for some instances, there does not exist any feasible solution with the marginals [Cox, 2000]. 3-DIPTP can be formulated as a linear integer feasibility problem by the following: Consider a 3-table $X = (x_{ijk})$ of size $(m,\ l,\ p)$, where m, q, and p are natural numbers. Let the integral matrices $M_1 = (a_{jk})$, $M_2 = (b_{ik})$, and $M_3 = (c_{ij})$ be 2-marginals of X, where M_1, M_2, and M_3 are integral matrices of type $q \times p$, $m \times p$, and $m \times q$, respectively. Then, a 3-table $X = (x_{ijk})$ of size $(m,\ q,\ p)$ with the given marginals satisfies the system of equations and inequalities:

$$\begin{array}{l}
\sum_{i=1}^{m} x_{ijk} = a_{jk}, \quad (j = 1, \ldots, q,\ k = 1, \ldots, p), \\
\sum_{j=1}^{q} x_{ijk} = b_{ik}, \quad (i = 1, \ldots, m,\ k = 1, \ldots, p), \\
\sum_{k=1}^{p} x_{ijk} = c_{ij}, \quad (i = 1, \ldots, m,\ j = 1, \ldots, q), \\
x_{ijk} \geq 0, \quad (i = 1, \ldots, m,\ j = 1, \ldots, q,\ k = 1, \ldots, p).
\end{array} \tag{1.2}$$

Then, 3-DIPTP is a linear integer feasibility problem with the system in (1.2).

In this paper we study the semigroup generated by the columns of $A \in \mathbb{Z}^{d\times n}$ and study the difference between the semigroup and its *saturation*. In Section 2 we will introduce notation of *saturation points* and in Section 3, we will show the necessary and sufficient condition for the finiteness of the difference between the semigroup and its saturation. In Section 4 we will show the simultaneous finiteness of the difference between the semigroup and its saturation, the set of non-saturation points of the semigroup, and the set of generators for saturation points. Section 5 shows some computational simulations on 3-DIPTP.

2. Notation and definitions

In this section we will remind the reader of some definitions and we will set appropriate notation. We follow the notation in Chapter 7 of [Miller and Sturmfels, 2004] and [Sturmfels, 1996]. Let $A \in \mathbb{Z}^{d\times n}$ and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ denote the columns of A. Let $\mathbb{N} = \mathbb{Z}_+ = \{0, 1, \ldots\}$. We assume that there exists $\boldsymbol{c} \in \mathbb{Q}^d$ such that $\boldsymbol{c} \cdot \boldsymbol{a}_i > 0$ for $i = 1, \ldots, n$, where $\cdot$ is the standard inner product.

Definition 1. *Let Q be the semigroup generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, let $K = \mathrm{cone}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ be the cone generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, and let L be the lattice generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Then the semigroup $Q_{\mathrm{sat}} = K \cap L$ is called the* saturation *of the semigroup Q. $Q \subset Q_{\mathrm{sat}}$ and we call Q* saturated *if $Q = Q_{\mathrm{sat}}$ (also this is called* normal*). $H = Q_{\mathrm{sat}} \setminus Q$ is the set of holes. $\boldsymbol{a} \in Q$ is called a saturation point if $\boldsymbol{a} + Q_{\mathrm{sat}} \subset Q$.*

We assume $L = \mathbb{Z}^d$ without loss of generality for our theoretical developments in Sections 3 and 4. The following is a list of some notations through this paper:

$$
\begin{aligned}
Q &= A\mathbb{N}^n = \{\lambda_1\boldsymbol{a}_1 + \cdots + \lambda_n\boldsymbol{a}_n : \lambda_1, \cdots, \lambda_n \in \mathbb{N}\}\\
K &= A\mathbb{R}^n_+ = \{\lambda_1\boldsymbol{a}_1 + \cdots + \lambda_n\boldsymbol{a}_n : \lambda_1, \cdots, \lambda_n \in \mathbb{R}_+\}\\
L &= A\mathbb{Z}^n = \{\lambda_1\boldsymbol{a}_1 + \cdots + \lambda_n\boldsymbol{a}_n : \lambda_1, \cdots, \lambda_n \in \mathbb{Z}\}\\
Q_{\mathrm{sat}} &= K \cap L = \text{saturation of } A \supset Q\\
H &= Q_{\mathrm{sat}} \setminus Q = \text{holes in } Q_{\mathrm{sat}}\\
S &= \{\boldsymbol{a} \in Q : \boldsymbol{a} + Q_{\mathrm{sat}} \subset Q\} = \text{saturation points of } Q\\
\bar{S} &= Q \setminus S = \text{non-saturation points of } Q
\end{aligned}
$$

Under the assumption above K and Q are *pointed* and S is non-empty by Problem

7.15 of [Miller and Sturmfels, 2004]. $Q_{\rm sat}$ is partitioned as

$$Q_{\rm sat} = H \cup \bar{S} \cup S = H \cup Q.$$

Equivalently

$$S \subset Q \subset Q_{\rm sat} \tag{2.1}$$

and the differences of these two inclusions are $\bar{S}$ and H, respectively.

If Q is saturated (equivalently $H = \emptyset$), then $0 \in S$, and by the monotonicity of S, $S = Q$. Therefore $S = Q = Q_{\rm sat}$ in (2.1). Similarly if $S = Q$, then $0 \in S$ and $Q_{\rm sat} \subset Q$, implying Q is saturated. From this consideration it follows that either $S = Q = Q_{\rm sat}$ or the two inclusions in (2.1) are simultaneously strict.

We now consider minimal points of S with respect to S, Q and $Q_{\rm sat}$. We call $\boldsymbol{a} \in S$ an S-minimal (a Q-minimal, a $Q_{\rm sat}$-minimal, resp.) if there exists no other $\boldsymbol{b} \in S$, $\boldsymbol{b} \neq \boldsymbol{a}$, such that $\boldsymbol{a} - \boldsymbol{b} \in S$ (Q, $Q_{\rm sat}$, resp.). More formally $\boldsymbol{a} \in S$ is

a) an S-minimal saturation point if $(\boldsymbol{a} + (-(S \cup \{0\}))) \cap S = \{\boldsymbol{a}\}$,

b) a Q-minimal saturation point if $(\boldsymbol{a} + (-Q)) \cap S = \{\boldsymbol{a}\}$,

c) a $Q_{\rm sat}$-minimal saturation point if $(\boldsymbol{a} + (-Q_{\rm sat})) \cap S = \{\boldsymbol{a}\}$.

Let $\min(S; S)$ denote the set of S-minimal saturation points, $\min(S; Q)$ the set of Q-minimal saturation points, and $\min(S; Q_{\rm sat})$ the set of $Q_{\rm sat}$-minimal saturation points. Because of the inclusion (2.1), it follows that

$$\min(S; Q_{\rm sat}) \subset \min(S; Q) \subset \min(S; S). \tag{2.2}$$

If $\boldsymbol{a} \in H$, then for any $\boldsymbol{b} \in Q$, either $\boldsymbol{a} - \boldsymbol{b} \notin Q_{\rm sat}$ or $\boldsymbol{a} - \boldsymbol{b} \in H$. This is because if $\boldsymbol{a} - \boldsymbol{b} \in Q_{\rm sat}$ and $\boldsymbol{a} - \boldsymbol{b} \notin H$, then $\boldsymbol{a} - \boldsymbol{b} \in Q$, and hence $\boldsymbol{a} = \boldsymbol{b} + (\boldsymbol{a} - \boldsymbol{b}) \in Q$, which contradicts $\boldsymbol{a} \in H$. This relation can be expressed as

$$Q_{\rm sat} \cap (H + (-Q)) = H.$$

This relation suggests the following definition.

Definition 2. *We call $\boldsymbol{a} \in Q_{\rm sat}$, $\boldsymbol{a} \neq 0$, a* fundamental hole *if*

$$Q_{\rm sat} \cap (\boldsymbol{a} + (-Q)) = \{\boldsymbol{a}\}.$$

Let H_0 be the set of fundamental holes.

Example 3. *Consider the one-dimensional example* $A = (3\ 5\ 7)$. $Q_{\mathrm{sat}} = \{0, 1, \ldots\}$, $Q = \{0, 3, 5, 6, 7, \ldots\}$, $-Q = \{0, -3, -5, -6, -7, \ldots\}$. $H = \{1, 2, 4\}$. *Among the 3 holes,* 1 *and* 2 *are fundamental. For example,* $2 \in H$ *is fundamental because*

$$\{0, 1, \ldots\} \cap \{2, -1, -3, -4, -5, \ldots\} = \{2\}.$$

On the other hand $4 \in H$ *is not fundamental because*

$$\{0, 1, \ldots\} \cap \{4, 1, -1, -2, -3, \ldots\} = \{4, 1\}.$$

If $0 \neq \boldsymbol{a} \in Q$, then $Q_{\mathrm{sat}} \cap (\boldsymbol{a} + (-Q)) \supset \{\boldsymbol{a}, 0\}$ and $\boldsymbol{a}$ is not a fundamental hole. This implies that a fundamental hole is a hole. For every non-fundamental hole $\boldsymbol{x}$, there exists $\boldsymbol{y} \in H$ such that $0 \neq \boldsymbol{x} - \boldsymbol{y} \in Q$. If $\boldsymbol{y}$ is not fundamental we can repeat this procedure. Since the procedure has to stop in finite number of steps, it follows that every non-fundamental hole $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{a}, \qquad \boldsymbol{y} \in H_0, \quad \boldsymbol{a} \in Q,\ \boldsymbol{a} \neq 0.$$

We will also focus on a *Hilbert basis* of a cone K and in the next section we will show a relation between the set of holes H and the *minimal Hilbert basis* of a pointed cone K.

Definition 4. *We call a finite integral vector subset* $B \subset K \cap \mathbb{Z}^d$ *a* Hilbert basis *of a cone* K *if any integral point in* K *can be written as a nonnegative integral linear combination of elements in* B. *If* B *is minimal in terms of inclusion then we call a* minimal Hilbert basis *of* K.

Note that there exists a Hilbert basis for any cone and also if a cone is pointed then there exists a unique minimal Hilbert basis ([Schrijver, 1986] for more details).

Example 5. *Let* A *be an integral matrix such that*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

The hole H *consists of only one element* $\{(1,2)^t\}$. $\bar{S} = \{(1,0)^t, (1,1)^t, (1,2)^t\}$. $\min(S; S) = \{(1,3)^t, (1,4)^t, (2,0)^t, (2,1)^t, (2,2)^t, (2,3)^t, (2,4)^t, (3,0)^t, (3,1)^t, (3,2)^t\}$. *Thus,* H, $\bar{S}$, *and* $\min(S; S)$ *are all finite.*

Example 6. *Let A be an integral matrix such that*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{pmatrix}.$$

The hole H consists of elements $\{(k,1) : k \in \mathbb{Z},\, k \geq 1\}$. $\bar{S} = \{(i,0)^t, : i \in \mathbb{Z},\, i \geq 1\}$, and $\min(S;S) = \{(k,j)^t : k \in \mathbb{Z},\, k \geq 1,\, 2 \leq j \leq 3\} \cup \{(1,4)^t\}$. Thus, H, $\bar{S}$, and $\min(S;S)$ are all infinite. However, $\min(S;Q) = \{(1,2)^t,\, (1,3)^t,\, (1,4)^t\}$ is finite.

3. Necessary and sufficient condition of finiteness of a set of holes

In this section we give a necessary and sufficient condition of finiteness of the set of holes H. Firstly we will show the necessary and sufficient condition in terms of the set of fundamental holes H_0. Then we generalize the statement, such that it is stated in terms of the minimal Hilbert basis of K.

First we show that H_0 is finite.

Proposition 7. *H_0 is finite.*

Proof. Every $\boldsymbol{a} \in Q_{\mathrm{sat}}$ can be written as

$$\boldsymbol{a} = c_1\boldsymbol{a}_1 + \cdots + c_n\boldsymbol{a}_n, \tag{3.1}$$

where c_i's are non-negative rational numbers. If $c_1 > 1$, then $\boldsymbol{a}$ can be written as

$$\boldsymbol{a} = \{(c_1 - \lfloor c_1 \rfloor)\boldsymbol{a}_1 + \cdots + c_n\boldsymbol{a}_n\} + \lfloor c_1 \rfloor \boldsymbol{a}_1 = \tilde{\boldsymbol{a}} + \lfloor c_1 \rfloor \boldsymbol{a}_1, \qquad \text{(say)}$$

and $\tilde{\boldsymbol{a}} = \boldsymbol{a} - \lfloor c_1 \rfloor \boldsymbol{a}_1$. Therefore

$$Q_{\mathrm{sat}} \cap (a + (-Q)) \supset \{\boldsymbol{a}, \tilde{\boldsymbol{a}}\}$$

and $\boldsymbol{a}$ is not a fundamental hole. In this argument we can replace c_1 with any c_i, $i \geq 2$. This shows each fundamental hole has an expression (3.1), where $0 \leq c_i \leq 1$, $i = 1, \ldots, n$. But then fundamental holes belong to a compact set. Since the lattice points in a compact set are finite, H_0 is finite. □

Let $H_0 = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_M\}$. Now for each $\boldsymbol{y}_h \in H_0$ and each $\boldsymbol{a}_i$ define $\bar{\lambda}_{hi}$ as follows. If there exists some $\lambda \in \mathbb{Z}$ such that $\boldsymbol{y}_h + \lambda \boldsymbol{a}_i \in Q$, let

$$\bar{\lambda}_{hi} = \min\{\lambda \in \mathbb{Z} \mid \boldsymbol{y}_h + \lambda \boldsymbol{a}_i \in Q\}. \tag{3.2}$$

Otherwise define $\bar{\lambda}_{hi} = \infty$. Then we have the following result:

Theorem 8. *H is finite if and only if $\bar{\lambda}_{hi} < \infty$ for all $h = 1, \ldots, M$ and all $i = 1, \ldots, n$.*

Proof. For one direction, assume that $\bar{\lambda}_{hi} = \infty$ for some h and i. Then $\boldsymbol{y}_h + \lambda \boldsymbol{a}_i$, $\lambda = 1, 2, \ldots$, all belong to Q_{sat} but do not belong to Q. Therefore they are holes. Hence H is infinite.

For the other direction, assume that $\bar{\lambda}_{hi} < \infty$ for all $h = 1, \ldots, M$ and all $i = 1, \ldots, n$. Each hole can be written as

$$\boldsymbol{x} = \boldsymbol{y}_h + \sum_{i=1}^{n} \lambda_i \boldsymbol{a}_i$$

for some h and $\lambda_i \in \mathbb{N}$, $i = 1, \ldots, n$. Now suppose that $\lambda_i \geq \bar{\lambda}_{hi}$ for some i. Then

$$\boldsymbol{y}_h + \lambda_i \boldsymbol{a}_i \in Q$$

and

$$\boldsymbol{x} = \boldsymbol{y}_h + \lambda_i \boldsymbol{a}_i + \sum_{j \neq i} \lambda_j \boldsymbol{a}_j \in Q,$$

which contradicts that $\boldsymbol{x}$ is a hole. Therefore if $\boldsymbol{x}$ is a hole, then $\lambda_i < \bar{\lambda}_{hi}$ for all i. Then

$$H \subset \{\boldsymbol{y}_h + \sum_{i=1}^{n} \lambda_{hi} \boldsymbol{a}_i \mid h = 1, \ldots, M,\ 0 \leq \lambda_{hi} < \bar{\lambda}_{hi}\}.$$

The right-hand side is finite. $\square$

There are several remarks to make. For each $1 \leq i \leq n$, let

$$\tilde{Q}_{(i)} = \{\sum_{j \neq i} \lambda_j \boldsymbol{a}_j \mid \lambda_j \in \mathbb{N},\ j \neq i\}$$

be the semigroup spanned by $\boldsymbol{a}_j, j \neq i$. Furthermore write

$$\bar{Q}_{(i)} = \mathbb{Z}\boldsymbol{a}_i + \tilde{Q}_{(i)}.$$

For each h and i, $\bar{\lambda}_{hi}$ is finite if and only if $\boldsymbol{y}_h \in \bar{Q}_{(i)}$. Since $\boldsymbol{y}_h$ is a hole, actually we only need to check

$$\boldsymbol{y}_h \in (-\mathbb{N}\boldsymbol{a}_i) + \tilde{Q}_{(i)}.$$

But $(-\mathbb{N}\boldsymbol{a}_i) + \tilde{Q}_{(i)}$ is another semigroup, where $\boldsymbol{a}_i$ in A is replaced by $-\boldsymbol{a}_i$. Therefore this problem is a standard membership problem in a semigroup.

Also we only need to check i such that $\boldsymbol{a}_i$ is an extreme ray. Assume, without loss of generality, that $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$, $k \leq n$, is the set of the extreme rays. The following corollary says that we only need to consider $i \leq k$.

Corollary 9. *H is finite if and only if $\bar{\lambda}_{hi} < \infty$ for all $h = 1, \ldots, M$ and all $i = 1, \ldots, k$.*

Proof. The first direction is the same as above.

For the converse direction, we show that if $\bar{\lambda}_{hi} < \infty$, $1 \leq h \leq M$, $1 \leq i \leq k$, then $\bar{\lambda}_{hi} < \infty$, $1 \leq h \leq M$, $k+1 \leq i \leq n$. Now any non-extreme ray $\boldsymbol{a}_i$, $i \geq k+1$, can be written as a non-negative rational combination of extreme rays:

$$\boldsymbol{a}_i = \sum_{j=1}^{k} q_{ij} \boldsymbol{a}_j, \qquad i \geq k+1. \tag{3.3}$$

Let $\bar{q}_i > 0$ denote the l.c.m. of the denominators of $q_{i1}, \ldots, q_{jk}$. Then multiplying both sides by $\bar{q}_i$, we have

$$\bar{q}_i \boldsymbol{a}_i = \sum_{j=1}^{k} (\bar{q}_i q_{ij}) \boldsymbol{a}_j, \quad \bar{q}_i q_{ij} \in \mathbb{N}.$$

Also note that there is at least one $q_{ij} > 0$, say q_{ij_0}. Consider $\bar{q}_i \boldsymbol{a}_i, 2\bar{q}_i \boldsymbol{a}_i, 3\bar{q}_i \boldsymbol{a}_i, \ldots$. Take $\lambda \in \mathbb{N}$ such that

$$\lambda \bar{q}_i q_{ij_0} \geq \bar{\lambda}_{hj_0}.$$

Then

$$\boldsymbol{y}_h + \lambda \bar{q}_i \boldsymbol{a}_i = \boldsymbol{y}_h + \lambda \bar{q}_i q_{ij_0} \boldsymbol{a}_{j_0} + \sum_{j \neq j_0}^{k} \lambda \bar{q}_i q_{ij} \boldsymbol{a}_j \in Q.$$

□

Another important point is that we want to state Theorem 8 in terms of Hilbert basis. Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_L\}$ denote the Hilbert basis of Q_{sat}. As above, if $\boldsymbol{b}_l + \lambda \boldsymbol{a}_i \in Q$ for some $\lambda \in \mathbb{Z}$ let

$$\bar{\mu}_{li} = \min\{\lambda \in \mathbb{Z} \mid \boldsymbol{b}_l + \lambda \boldsymbol{a}_i \in Q\}$$

and $\bar{\mu}_{li} = \infty$ otherwise. Then we have the following theorem.

Theorem 10. *H is finite if and only if $\bar{\mu}_{li} < \infty$ for all $l = 1, \ldots, L$ and all $i = 1, \ldots, n$.*

Proof. The first direction is the same as the above proofs.

For the converse direction, assume that $\bar{\mu}_{li} < \infty$ for all $l = 1, \dots, L$ and all $i = 1, \dots, n$. Let $\boldsymbol{y}_h$ be a fundamental hole. It can be written as non-negative integral combination of Hilbert basis

$$\boldsymbol{y}_h = \sum_{l=1}^{L} \alpha_{hl} \boldsymbol{b}_l.$$

Let

$$\lambda = \sum_{l=1}^{L} \alpha_{hl} \bar{\mu}_{li}.$$

Then

$$\begin{aligned} \boldsymbol{y}_h + \lambda \boldsymbol{a}_i &= \sum_{l=1}^{L} \alpha_{hl} \boldsymbol{b}_l + \big(\sum_{l=1}^{L} \alpha_{hl} \bar{\mu}_{li} \big) \boldsymbol{a}_i \\ &= \sum_{l=1}^{L} \alpha_{hl} (\boldsymbol{b}_l + \bar{\mu}_{li} \boldsymbol{a}_i) \in Q. \end{aligned}$$

This implies $\bar{\lambda}_{hi} < \infty$. □

Also it is clear that Corollary 9 can be applied to Hilbert basis.

Remark 11. *In summary, it seems that determining finiteness of H is straightforward. We obtain the Hilbert basis B of $Q_{\rm sat}$. For each $\boldsymbol{b} \in B$ and for each extreme $\boldsymbol{a}_i$, we check*

$$\boldsymbol{b} \in (-\mathbb{N}\boldsymbol{a}_i) + \tilde{Q}_{(i)}.$$

Example 12. *Let A be an integral matrix such that*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

Then B consists of 5 elements

$$B = \{\boldsymbol{b}_1 = (1,0)^t, \boldsymbol{b}_2 = (1,1)^t, \boldsymbol{b}_3 = (1,2)^t, \boldsymbol{b}_4 = (1,3)^t, \boldsymbol{b}_5 = (1,4)^t\}.$$

Then we can write $\boldsymbol{b}_3$ as the following:

$$\begin{aligned} (1,2)^t &= -(1,0)^t + 2 \cdot (1,1)^t \\ &= (1,0)^t - (1,1)^t + (1,3)^t \\ &= (1,1)^t - (1,3)^t + (1,4)^t \\ &= 2 \cdot (1,3)^t - (1,4)^t. \end{aligned}$$

Thus, in this case, we have $\bar{\mu}_{3i} = 1$ *for each* $i = 1, \dots, 4$ *and* $\bar{\mu}_{li} = 0$*, where* $l \neq 3$ *for each* $i = 1, \dots, 4$*. Thus by Theorem 10, the number of elements in* H *is finite. Note that* H *consists of only one elements* $\{\boldsymbol{b}_3 = (1, 2)^t\}$*.*

3. Simultaneous finiteness of holes, nonsaturation points, and minimal saturation points

In this section we will show the simultaneous finiteness of holes, nonsaturation points, and S-minimal saturation points. As in the previous section let $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_k\}$, $k \leq n$, be the set of the extreme rays. First, we will show the following lemmas.

Lemma 13. *Suppose that* Q *is not saturated.* $\boldsymbol{a} \in Q$ *is a saturation point if and only if* $\boldsymbol{a} + \boldsymbol{y} \in Q$ *for all essential holes* $\boldsymbol{y}$*.*

Proof. If $\boldsymbol{a} \in Q$ is a saturation point, then $\boldsymbol{a} + \boldsymbol{y} \in Q$ for all $\boldsymbol{y} \in Q_{\mathrm{sat}}$. In particular $\boldsymbol{a} + \boldsymbol{y} \in Q$ for all essential holes $\boldsymbol{y}$.

Now suppose that $\boldsymbol{a} \in Q$ is not a saturation point. Then there exists $\boldsymbol{y} \in Q_{\mathrm{sat}}$ such that $\boldsymbol{a} + \boldsymbol{y}$ is a hole. This $\boldsymbol{y}$ has to be a hole, because otherwise $\boldsymbol{a} + \boldsymbol{y} \in Q$. $\boldsymbol{y}$ can be written as $\boldsymbol{y} = \boldsymbol{y}_h + \boldsymbol{b}$ for some fundamental hole $\boldsymbol{y}_h$ and $\boldsymbol{b} \in Q$. Then $\boldsymbol{a} + \boldsymbol{y} = \boldsymbol{a} + \boldsymbol{y}_h + \boldsymbol{b}$. Then $\boldsymbol{a} + \boldsymbol{y}_h$ has to be hole. Therefore we have shown that if $\boldsymbol{a}$ is not a saturation point, then $\boldsymbol{a} + \boldsymbol{y}$ is a hole for some essential hole $\boldsymbol{y}$. □

Lemma 14. *Suppose that* Q *is not saturated. Consider any column* $\boldsymbol{a}_i$ *of* A*. There exists some* $n_i \in \mathbb{N}$ *such that* $n_i \boldsymbol{a}_i \in S$ *if and only if* $\bar{\lambda}_{hi} < \infty$ *in (3.2) for all* $h = 1, \dots, M$*.*

Proof. This follows easily from Lemma 13. If $n_i \boldsymbol{a}_i \in S$, $\bar{\lambda}_{hi} \leq n_i$. For the other direction take $n_i = \max_h \bar{\lambda}_{hi}$. □

Now we consider the following three conditions.

Condition 1 There exists a finite $M > 0$ such that every $\boldsymbol{a} \in Q$ with $\boldsymbol{c} \cdot \boldsymbol{a} > M$ belongs to S.

Condition 2 For each $\boldsymbol{a}_i$, there exists $n_i > 0$ such that $n_i \boldsymbol{a}_i \in S$.

Condition 3 For each extreme ray $\boldsymbol{a}_i$, $1 \leq i \leq k$, there exists $n_i > 0$ such that $n_i \boldsymbol{a}_i \in S$.

Proposition 15. *Condition* 1*, Condition* 2*, Condition* 3*, and the finiteness of* H *are equivalent.*

Proof. Suppose that Condition 1 holds. For each $\boldsymbol{a}_i$, let $n_i > M/(\boldsymbol{c} \cdot \boldsymbol{a}_i)$, then $\boldsymbol{c} \cdot (n_i \boldsymbol{a}_i) > M$ and $n_i \boldsymbol{a}_i \in S$. Therefore Condition 2 holds. Condition 3 trivially holds if Condition 2 holds.

On the other hand suppose that Condition 3 holds. Then each non-extreme $\boldsymbol{a}_i$, $k < i \leq n$, can be written as (3.3). As above let $\bar{q}_i > 0$ denote the l.c.m. of the denominators of $q_{i1}, \dots, q_{jk}$ and let $n_i = \bar{q}_i \times n_1 \times \cdots \times n_k$, then $n_i \boldsymbol{a}_i \in S$ and Condition 2 holds. Next suppose that Condition 2 holds. Then every non-saturation point $\boldsymbol{a} \in \bar{S}$ has an expression

$$\boldsymbol{a} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_n \boldsymbol{a}_n, \qquad 0 \leq \lambda_i < n_i, \forall i.$$

Therefore $\bar{S}$ is a subset of a compact set and hence finite. Choose M such that

$$M > \max_{\boldsymbol{a} \in \bar{S}} \boldsymbol{c} \cdot \boldsymbol{a}.$$

Then Condition 1 holds.

Finally we will show the equivalence between the finiteness of H and the other three conditions. Using Lemma 14, Condition 2 is equivalent to the condition in Theorem 8. Also Condition 3 is equivalent to the condition in Corollary 9. □

Now we prove the following theorem. In the theorem $\mathrm{cone}(S)$ denotes the set of finite nonnegative real combinations of elements of S.

Theorem 16. *The following statements are equivalent.*

1. $\min(S; S)$ *is finite.*
2. $\mathrm{cone}(S)$ *is a rational polyhedral cone.*
3. *There is some* $s \in S$ *on every extreme ray of* K*.*
4. H *is finite.*
5. $\bar{S}$ *is finite.*

Proof. 1. $\Longleftrightarrow$ 2. : $\min(S; S)$ is an integral generating set of the monoid $S \cup \{0\}$. We then apply Theorem 1.1 (b) of [Hemmecke and Weismantel, 2006] or Theorem 4 in [Jeroslow, 1978].

2. $\Longleftrightarrow$ 3. : If cone(S) is not polyhedral, there must be an extreme ray e of K not in cone(S), since K is polyhedral. Thus, $e \cap S = \emptyset$.

If cone(S) is polyhedral, then it is a rational polyhedron and has a finite integral generating set. Thus, by Theorem 8.8 in [Bertsimas and Weismantel, 2005], the polyhedron cone(S) contains all lattice points from its recession cone K and $(K \cap \mathbb{Z}^d) \setminus S$ is finite, which in this case can only happen if cone$(S) = K$. Thus, there is a point from S on each extreme ray of K.

3. $\Longleftrightarrow$ 4. : The statement 3. is equivalent to Condition 3. Thus, we can prove directly by Proposition 15.

4. $\Longleftrightarrow$ 5. : Suppose that H is finite. Then Condition 1 holds and as in the proof of Proposition 15 $\bar{S}$ is finite. For the opposite implication, suppose that H is infinite. Since Condition 2 does not hold, there exists some i such that $n\boldsymbol{a}_i \notin S$ for all $n \in \mathbb{N}$. Then $\{\boldsymbol{a}_i, 2\boldsymbol{a}_i, 3\boldsymbol{a}_i, \dots\} \subset \bar{S}$ and $\bar{S}$ is infinite. □

Now we consider the generators $\min(S; Q)$. We prove that $\min(S; Q)$ is always finite. Then by (2.2) $\min(S; Q_{\mathrm{sat}})$ is always finite as well.

Proposition 17. $\min(S; Q)$ *is finite.*

Proof. Note that Q is a finitely generated monoid. Consider the algebra, $k[Q] := k[t^{\boldsymbol{a}_1}, \cdots, t^{\boldsymbol{a}_n}]$, where k is any algebraic field. Then $k[Q]$ is a finitely generated k-algebra by Proposition 2.5 in [Bruns and Gubeladze, 2006] and therefore a Noetherian ring by a corollary of Hilbert's basis theorem (Corollary 1.3 in [Eisenbud, 1995]). Since $I_S :=< t^{\beta} : \beta \in S >$ is an ideal in $k[Q]$, we are done. □

Proposition 18.

$$\min(S; Q) \subset \min(S; Q_{\mathrm{sat}}) + (H_0 \cup \{0\}). \tag{3.1}$$

Proof. Let $\boldsymbol{a} \in \min(S; Q)$. We want to show that $\boldsymbol{a}$ can be written as $\boldsymbol{a} = \tilde{\boldsymbol{a}} + \boldsymbol{b}$, where $\tilde{\boldsymbol{a}} \in \min(S; Q_{\mathrm{sat}})$ and $\boldsymbol{b} \in H_0 \cup \{0\}$. If $\boldsymbol{a}$ itself belongs to $\min(S; Q_{\mathrm{sat}})$, then take $\boldsymbol{a} = \tilde{\boldsymbol{a}}$ and $\boldsymbol{b} = 0$. Otherwise, if $\boldsymbol{a} \notin \min(S; Q_{\mathrm{sat}})$, then by definition of Q_{sat}-minimality there exists $\boldsymbol{a}' \in S$ such that $0 \neq \boldsymbol{a} - \boldsymbol{a}' \in Q_{\mathrm{sat}}$. If $\boldsymbol{a}' \notin \min(S; Q_{\mathrm{sat}})$, then we can do the same operation to $\boldsymbol{a}'$. This operation has to stop in finite steps and we arrive at $\tilde{\boldsymbol{a}} \in \min(S; Q_{\mathrm{sat}})$ such that $\boldsymbol{b} = \boldsymbol{a} - \tilde{\boldsymbol{a}} \in Q_{\mathrm{sat}}$. If this

$\boldsymbol{b} \notin H_0$, then there exists $\boldsymbol{c} \in Q$, $\boldsymbol{c} \neq 0$, such that $\boldsymbol{b} - \boldsymbol{c} \in Q_{\text{sat}}$. Then

$$\boldsymbol{a} = \tilde{\boldsymbol{a}} + \boldsymbol{b} = \tilde{\boldsymbol{a}} + (\boldsymbol{b} - \boldsymbol{c}) + \boldsymbol{c},$$

where $\tilde{\boldsymbol{a}} \in S$, $\boldsymbol{b} - \boldsymbol{c} \in Q_{\text{sat}}$. By monotonicity of S, $\tilde{\boldsymbol{a}} + (\boldsymbol{b} - \boldsymbol{c}) \in S$. But this contradicts $\boldsymbol{a} \in \min(S; Q)$. $\square$

3. Applications to contingency tables

In this section we apply our theorem to some examples including $2 \times 2 \times 2 \times 2$ tables with 2-marginals (the complete graph with 4 nodes and with levels of 2 on each node, $K4$), and $2 \times 2 \times 2 \times 2$ tables with three 2-marginals and a 3-marginal ([12][13][14][234]). Also we apply our theorem to three dimensional contingency tables from [Vlach, 1986]. To compute minimal Hilbert bases of cones, we used `normaliz` [Bruns and Koch, 2001] and to compute each hyperplane representation and vertex representation we used `CDD` [Fukuda, 2005] and `lrs` [Avis, 2005]. Also we used `4ti2` [Hemmecke et al, 2005] to compute defining matrices.

$2 \times 2 \times 2 \times 2$ tables

$2 \times 2 \times 2 \times 2$ tables with 2-marginals

First, we would like to show some simulation result with $2 \times 2 \times 2 \times 2$ tables with 2-marginals, which can be seen as the complete graph with 4 nodes and with levels of 2 on each node, $K4$. The semigroup of $K4$ has 16 generators $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{16}$ in $\mathbb{Z}^{24}$ (without removing redundant rows) such that

```
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0
0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0
0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0
0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1
1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0
0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 0
```

```
0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1
1 1 0 0 0 0 0 1 1 0 0 0 0 0 0
0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0
0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0
0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1
1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0
0 0 1 1 0 0 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 0
0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
```

Remember that the columns of the given array are the generators of the semigroup. Note that all of these vectors are extreme rays of the cone (we verified via `cddlib`) [Fukuda, 2005]. The Hilbert basis of the cone generated by these 16 vectors contains 17 vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{17}$. The first 16 vectors are the same as $\boldsymbol{a}_i$, i.e. $\boldsymbol{b}_i = \boldsymbol{a}_i$, $i = 1, \ldots, 16$. The 17-th vector $\boldsymbol{b}_{17}$ is

$$\boldsymbol{b}_{17} = (1\ 1\ \ldots\ 1)^t$$

consisting of all 1's. Thus, $\boldsymbol{b}_{17} \notin Q$. Then we set the 16 systems of linear equations such that:

$$\begin{aligned} P_j: \quad & \boldsymbol{b}_1 x_1 + \boldsymbol{b}_2 x_2 + \cdots + \boldsymbol{b}_{16} x_{16} = \boldsymbol{b}_{17} \\ & x_j \in \mathbb{Z}_-,\ x_i \in \mathbb{Z}_+,\ \text{for } i \neq j, \end{aligned}$$

for $j = 1, 2, \cdots, 16$. We solved these systems via `lrs` and `LattE` [De Loera et al., 2003]. Then we have:

$$\begin{aligned}
\boldsymbol{b}_{17} &= -\boldsymbol{b}_1 + \boldsymbol{b}_2 + \boldsymbol{b}_3 + \boldsymbol{b}_5 + \boldsymbol{b}_9 + \boldsymbol{b}_{16}, \\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1 - \boldsymbol{b}_2 + \boldsymbol{b}_4 + \boldsymbol{b}_6 + \boldsymbol{b}_{10} + \boldsymbol{b}_{15}, \\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1 - \boldsymbol{b}_3 + \boldsymbol{b}_4 + \boldsymbol{b}_7 + \boldsymbol{b}_{11} + \boldsymbol{b}_{14}, \\
\boldsymbol{b}_{17} &= \boldsymbol{b}_2 + \boldsymbol{b}_3 - \boldsymbol{b}_4 + \boldsymbol{b}_8 + \boldsymbol{b}_{12} + \boldsymbol{b}_{13}, \\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1 - \boldsymbol{b}_5 + \boldsymbol{b}_6 + \boldsymbol{b}_7 + \boldsymbol{b}_{12} + \boldsymbol{b}_{13},
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{b}_{17} &= \boldsymbol{b}_2+\boldsymbol{b}_5-\boldsymbol{b}_6+\boldsymbol{b}_8+\boldsymbol{b}_{11}+\boldsymbol{b}_{14},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_3+\boldsymbol{b}_5-\boldsymbol{b}_7+\boldsymbol{b}_8+\boldsymbol{b}_{10}+\boldsymbol{b}_{15},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_4+\boldsymbol{b}_6+\boldsymbol{b}_7-\boldsymbol{b}_8+\boldsymbol{b}_9+\boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1+\boldsymbol{b}_8-\boldsymbol{b}_9+\boldsymbol{b}_{10}+\boldsymbol{b}_{11}+\boldsymbol{b}_{13},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_2+\boldsymbol{b}_7+\boldsymbol{b}_9-\boldsymbol{b}_{10}+\boldsymbol{b}_{12}+\boldsymbol{b}_{14},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_3+\boldsymbol{b}_6+\boldsymbol{b}_9-\boldsymbol{b}_{11}+\boldsymbol{b}_{12}+\boldsymbol{b}_{15},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_4+\boldsymbol{b}_5+\boldsymbol{b}_{10}+\boldsymbol{b}_{11}-\boldsymbol{b}_{12}+\boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_4+\boldsymbol{b}_5+\boldsymbol{b}_9-\boldsymbol{b}_{13}+\boldsymbol{b}_{14}+\boldsymbol{b}_{15},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_3+\boldsymbol{b}_6+\boldsymbol{b}_{10}+\boldsymbol{b}_{13}-\boldsymbol{b}_{14}+\boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_2+\boldsymbol{b}_7+\boldsymbol{b}_{11}+\boldsymbol{b}_{13}-\boldsymbol{b}_{15}+\boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1+\boldsymbol{b}_8+\boldsymbol{b}_{12}+\boldsymbol{b}_{14}+\boldsymbol{b}_{15}-\boldsymbol{b}_{16}.
\end{aligned}$$

Thus by Theorem 10, the number of elements in H is finite.

$2\times2\times2\times2$ tables with 2-marginals and a 3-marginal

Now we consider $2\times2\times2\times2$ tables with three 2-marginals and a 3-marginal as the simplicial complex on 4 nodes [12][13][14][234] and with levels of 2 on each node.

After removing redundant rows (we removed redundant rows using `cddlib`), $2\times2\times2\times2$ tables with 2-marginals and a 3-marginal has the 12×16 defining matrix. Thus the semigroup is generated by 16 vectors in $\mathbb{Z}^{12}$ such that:

```
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0
0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0
1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0
0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0
1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0
```

```
0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0
```

All of these vectors are extreme rays of the cone (we verified via `cddlib`). The Hilbert basis of the cone generated by these 16 vectors consists of these 16 vectors and two additional vectors

$$\boldsymbol{b}_{17} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0)^t, \qquad \boldsymbol{b}_{18} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1)^t.$$

Thus, $\boldsymbol{b}_{17}, \boldsymbol{b}_{18} \notin Q$. Then we set the system of linear equations such that:

$$\begin{gathered} \boldsymbol{b}_1 x_1 + \boldsymbol{b}_2 x_2 + \cdots + \boldsymbol{b}_{16} x_{16} = \boldsymbol{b}_{17} \\ x_1 \in \mathbb{Z}_-, \ x_i \in \mathbb{Z}_+, \ \text{for } i = 2, \cdots, 16. \end{gathered}$$

We solved the system via `lrs`, `CDD` and `LattE`. We noticed that this system has no real solution (infeasible). This means that

$$\boldsymbol{b}_{17} \notin (-\mathbb{N}\boldsymbol{a}_1) + \tilde{Q}_{(1)}.$$

Thus by Theorem 10, the number of elements in H is infinite.

Results on three dimensional tables

In this section we study the semigroups of defining matrices for some three dimensional tables. For $2 \times J \times K$ tables with 2-marginals for $J, K \in \mathbb{N}$, the semigroup of each defining matrix is saturated since $2 \times J \times K$ is the Lawrence lifting of $J \times K$, which is also unimodular (and Lawrence lifting of unimodular is unimodular [Sturmfels, 1996, Sturmfels, 1998]). Thus, an example in [Irving, 1994] does not have a table with integral entries as well as a table with real entries because this is unimodular.

For $3 \times 3 \times J$ tables with 2-marginals for $J \in \mathbb{N}$, we do not know that the semigroup of each defining matrix is saturated for all $J \in \mathbb{N}$. However for $3 \times 4 \times 6$ tables with 2-marginals, Vlach showed an example which has a table with real entries, but there does not exist a table with nonnegative integer entries [Vlach, 1986]. This example can be found in Figure 3.1. Also using this example, one can show that for $3 \times 4 \times 7$ tables and bigger tables have infinitely many holes. We take the example in Figure 3.1. Then we embed the table in a $3 \times 4 \times 7$ table. Then we put a single arbitrary positive integer at just one place of the seventh 3×4 slice. This positive integer is uniquely determined by 2-marginals of the seventh slice alone (Table 3.1). Thus for each choice of positive integer the beginning $3 \times 4 \times 6$ part remains to be a hole. Since the positive integer is arbitrary, $3 \times 4 \times 7$ table has infinite number of holes.

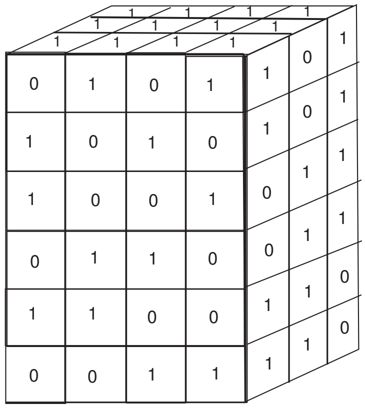

Figure 3.1: An example of $3 \times 4 \times 6$ table such that the given marginal condition is a hole of the semigroup.

					sum
	c	0	0	0	c
	0	0	0	0	0
	0	0	0	0	0
sum	c	0	0	0	c

Table 3.1: the 7-th 3×4 slice is uniquely determined by its row and its column sums. c is an arbitrary positive integer.

We can generalize this idea as follows. Let A_1 denote the integer matrix corresponding to problem of a smaller size. Suppose that A for a larger problem can be written as a partitioned matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_3 and A_4 are arbitrary. We consider the case that for A_1 there exists a hole. Now consider the semigroup associated with A_2. We assume that there exists infinite number of one-element fibers for the semigroup associated with A_2. This is usually the case, because the fibers on the extreme ray for A_2 is all one-element fibers, under the condition that A_2 does not contain multiple extreme rays.

Under these assumptions consider the equation

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where t_1 is a hole for A_1, t_2 is any of the one-element fibers for A_2 and t_3 is chosen to satisfy the equation. Then $(t_1, t_2, t_3)^t$ is a hole for each t_2. Therefore there exist infinite number of holes for the larger problem.

Example 19. *Let A_1 be an integral matrix such that*

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

and let $A_2 = (1)$. From Example 5, H consists of only one element $\{t_1 = (1,2)^t\}$ and with A_2 we can find a family of infinite number of one-element fibers, namely $F_a := \{a\}$, where a is an arbitrary positive integer. Let $t_2 = a$. Then we have a matrix A such that:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $(t_1, t_2)^t = (1, 2, a)$ is a hole for each $t_2 = a$. Thus, since a is an arbitrary positive integer, there exist infinitely many holes for the semigroup generated by the columns of the matrix A.

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