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The Independent Even Factor Problem

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Abstract

Cunningham and Geelen (1997) introduced the notion of independent path-matchings, which generalize both matchings and matroid intersection. Path-matchings are yet generalized to even factors in digraphs by Cunningham and Geelen (2001). Pap (2005) gave a combinatorial algorithm to find a maximum even factor in odd-cycle-symmetric digraphs, in which each arc in any odd dicycle has the reverse arc.

The even factor problem, however, does not contain the matroid intersection problem. Cunningham and Geelen (2001) proposed the notion of basic even factors, which generalize both of even factors and matroid intersection, and showed a polynomial reduction of the basic even factor problem to the matroid intersection problem, which applies a maximum even factor algorithm in each oracle call for independence test.

This paper deals with the independent even factor problem, which is a variant of the basic even factor problem. For odd-cycle-symmetric digraphs, a min-max formula is established as a common generalization of the Tutte-Berge formula for matchings and the min-max formula of Edmonds (1970) for matroid intersection. We devise a combinatorial efficient algorithm to find a maximum independent even factor in an odd-cycle-symmetric digraph accompanied with general matroids, which commonly extends two of the alternating-path type algorithms, the even factor algorithm and the matroid intersection algorithm. This algorithm gives a proof of the min-max formula, and contains a new operation on matroids, which corresponds to shrinking factor-critical components in the matching algorithm of Edmonds (1965). The running time of the algorithm is $\mathrm{O}(n^4 Q)$, where n is the number of vertices and Q is the time for an independence test. The algorithm also gives a common generalization of the Edmonds-Gallai decomposition for matchings and the principal partition for matroid intersection.

1 Introduction

One of the most fundamental problems in combinatorial optimization is the matching problem. Tutte [20] characterized graphs that have a perfect matching. Then, Berge [1] showed that Tutte's characterization implies a min-max formula, now called the Tutte-Berge formula. Edmonds [5] gave a combinatorial algorithm to find a maximum matching. This algorithm implies a structural result, which is also found independently by Gallai [9], and now called the Edmonds-Gallai decomposition.

The matroid intersection problem is also of importance. Edmonds [6] presented a min-max formula for that problem. Polynomial-time algorithms to find a maximum common independent set are known, such as those of Edmonds [7] and Lawler [13].

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As a common generalization of the matching and matroid intersection problems, Cunningham and Geelen [3] introduced the notion of *independent path-matchings*. They showed a min-max formula, a totally dual integral polyhedral description, and polynomial-time solvability of the independent path-matching problem, which generalize those in the matching and matroid intersection problems. The independent path-matching problem with free matroids is called the *path-matching problem*. For the path-matching problem, Frank and Szegő [8] simplified the min-max formula of Cunningham and Geelen [3] and provided its combinatorial proof. Spille and Szegő [18] showed that the path-matching problem has a property which generalizes the Edmonds-Gallai structure for matchings.

The proof of polynomial-time solvability in Cunningham and Geelen [3] relies on the ellipsoid method. A combinatorial polynomial-time algorithm to find an optimal path-matching had been of interest. Spille and Weismantel [19] proposed to generalize Edmonds' matching algorithm [5] to path-matchings.

For the sake of a combinatorial approach to path-matchings, Cunningham and Geelen [4] introduced the *even factor problem* in digraphs, which is yet a generalization of the path-matching problem. While finding a maximum even factor is NP-hard in general digraphs, they showed that the problem is solvable in polynomial time in *weakly symmetric* digraphs with an algebraic approach. A digraph is said to be weakly symmetric if every arc in any dicycle has the reverse arc. In addition, they presented a min-max formula and generalized the Edmonds-Gallai decomposition to the even factor problem in weakly symmetric digraphs, followed by simplifications of Pap and Szegő [15]. These properties are known to hold in a broader class of digraphs, called *odd-cycle-symmetric*. A digraph is odd-cycle-symmetric if each arc in any odd dicycle has the reverse arc. Pap [14] devised a combinatorial algorithm to find a maximum even factor in odd-cycle-symmetric digraphs, which provides an algorithmic proof of the min-max formula and the Edmonds-Gallai type structure. This algorithm has some properties similar to Edmonds' matching algorithm [5].

In fact, as mentioned above, the even factor problem leaves matroids out of consideration, or in other words, it only deals with free matroids. For a common generalization of matchings and matroid intersection, Cunningham and Geelen [4] proposed the notion of *basic even factors*. They showed a polynomial reduction of the basic even factor problem in weakly symmetric digraphs to the matroid intersection problem. In the resulting algorithm, one needs to compute the cardinality of a maximum even factor in a certain digraph for each independence test.

In this paper, we deal with the *independent even factor problem*, which is a variant of the basic even factor problem. Extending the approach of Cunningham and Geelen [4], we obtain a min-max formula for the independent even factor problem in odd-cycle-symmetric digraphs, which commonly extends the Tutte-Berge formula for matchings and the min-max formula for matroid intersection of Edmonds [6]. For the independent even factor problem in odd-cycle-symmetric digraphs accompanied with general matroids, we devise a combinatorial algorithm to find a maximum independent even factor. The algorithm commonly extends the even factor algorithm [14] and the matroid intersection algorithms [7, 13], in both of which the notion of *alternating-paths* plays an essential part. The algorithm runs in $\mathrm{O}(n^4Q)$ time, where n is the cardinality of the vertex set of the digraph and Q is the time needed to test if a given set is independent. By this algorithm, we obtain a different proof of the min-max formula.

The algorithm also gives a common extension of the Edmonds-Gallai decomposition for matchings and the principal partition for matroid intersection (cf. [11, 12]). Unlike the existing theorems on the Edmonds-Gallai type structure for path-matchings or even factors, we treat the two matroids equally to achieve a unified decomposition principle including the principal partition for matroid intersection.

The maximum independent even factor algorithm includes a new operation on matroids, called *shrinking*. For a matroid $\mathbf{M}$ and its independent set I, eliminate I and create a new element i to obtain a new matroid. We call this procedure *shrinking* I. Note that the resulting matroid differs from the one derived by contraction or deletion. This procedure corresponds to shrinking a factor-critical component in the matching algorithm of Edmonds [5] and the even factor algorithm of Pap [14].

Recently, Harvey [10] gave a new algebraic characterization of basic path-matchings, which achieved a maximum basic path-matching algorithm. His approach needs the matroids to be linearly represented over the same field. The algorithm runs in $\mathrm{O}(n^{\omega})$ time, where n is the number of vertices and ω is the exponent for matrix multiplication, bounded by $\omega < 2.38$ (due to Coppersmith and Winograd [2]). One can easily extend this approach to obtain an $\mathrm{O}(n^{\omega})$ algorithm for independent even factor problems in which the matroids are linearly represented over the same field. Our algorithm, however, does not need the assumption, that is, it can deal with general matroids.

Before closing this section, let us give some notations and definitions used in the following sections.

A digraph G with the vertex set V and the arc set A is denoted by $G = (V, A)$. For vertices u and v, let (u, v) denote an arc which starts in u and ends in v. For an arc $a = (u, v)$, let $\partial^+ a$ (resp. $\partial^- a$) denote the initial (resp. terminal) vertex of a, that is, $\partial^+ a = u$ and $\partial^- a = v$. For a vertex v, define $\delta^+ v = \{a \mid a \in A,\ \partial^+ a = v\}$, and $\delta^- v = \{a \mid a \in A,\ \partial^- a = v\}$. For an arc set B, define $\partial^+ B = \{v \mid v \in V,\ \exists a \in B,\ v = \partial^+ a\}$, and $\partial^- B = \{v \mid v \in V,\ \exists a \in B,\ v = \partial^- a\}$. For a vertex set U, denote $\delta^+ U = \{a \mid a \in A,\ \partial^+ a \in U,\ \partial^- a \notin U\}$, and $\delta^- U = \{a \mid a \in A,\ \partial^- a \in U,\ \partial^+ a \notin U\}$.

We denote the reverse arc of a by $\bar{a}$, i.e., $\bar{a} = (v, u)$ for $a = (u, v)$. An arc $a \in A$ is called *symmetric* if $\bar{a} \in A$. A digraph is said to be *symmetric* if every arc is symmetric.

For a subgraph H of G, the vertex set and the arc set of H are denoted by $V(H)$ and $A(H)$, respectively. We say H is *odd* (resp. *even*) if $|V(H)|$ is odd (resp. even). A strongly connected component C with $\delta^- V(C) = \emptyset$ (resp. $\delta^+ V(C) = \emptyset$) is called a *source-component* (resp. *sink-component*). For two vertex sets X^+ and X^-, let $G[X^+, X^-]$ denote the subgraph whose vertex set is $X^+ \cup X^-$ and arc set is $A[X^+, X^-] = \{a \mid a \in A,\ \partial^+ a \in X^+,\ \partial^- a \in X^-\}$. For a vertex set X, we abbreviate $G[X, X]$ and $A[X, X]$ as $G[X]$ and $A[X]$, respectively. We denote by $\mathrm{odd}^+(X)$ (resp. $\mathrm{odd}^-(X)$) the number of odd source-components (resp. odd sink-components) in $G[X]$.

For $v_0, v_1, \ldots, v_k \in V$ and $a_1, \ldots, a_k \in A$, a sequence $W = (v_0, a_1, v_1, a_2, v_2, \ldots, v_{k-1}, a_k, v_k)$ is called a *walk* if $a_i = (v_{i-1}, v_i)$ for all $i = 1, \ldots, k$. We denote the vertex set $\{v_0, \ldots, v_k\} = V(W)$ and the arc set $\{a_1, \ldots, a_k\} = A(W)$. For a walk $W = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$, the length k of W is denoted by $|W|$. A walk W is said to be *odd* (resp. *even*) if $|W|$ is odd (resp. even). A walk $W = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$ is called a *path* if $v_0, v_1, \ldots, v_k$ are pairwise distinct, and called a *cycle* if $v_0, v_1, \ldots, v_{k-1}$ are pairwise distinct and $v_0 = v_k$. For a walk $W = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$, let $\bar{W}$ denote the reverse walk of W (if exists), that is, $\bar{W} = (v_k, \bar{a}_k, v_{k-1}, \ldots, v_1, \bar{a}_1, v_0)$. Recall that a digraph is odd-cycle-symmetric if any odd cycle has the reverse cycle.

For two vertex sets X^+ and X^-, we call (X^+, X^-) a *stable pair* if there is neither an arc a with $\partial^+ a \in X^+ \setminus X^-$ and $\partial^- a \in X^-$, nor $\partial^+ a \in X^+$ and $\partial^- a \in X^- \setminus X^+$. This definition generalizes that of Cunningham and Geelen [3], who first introduced the notion of stable pairs for the independent path-matching problem.

The outline of this paper is as follows. Section 2 introduces even factors and matroid intersection, and describes some important results which are to be extended in independent even factors. Section 3 provides the operation of shrinking on matroids. Section 4 introduces independent even factors and proves a min-max formula structurally. Section 5 presents a new algorithm of ours. Finally, Section 6 reveals the Edmonds-Gallai type structure for independent even factors.

2 Even Factors and Matroid Intersection

2.1 Even Factors

Let $G = (V, A)$ be a digraph. An *even factor* is the arc set of a vertex-disjoint collection of some paths and even cycles. For a digraph G, let $\nu(G)$ denote the cardinality of a maximum even factor in G.

One approach to computing $\nu(G)$ is extending the notion of the Tutte matrix in the matching problem [20]. Let $\{x_{uv} \mid (u, v) \in A\}$ be the indeterminates such that $x_{uv} = -x_{vu}$ for $(u, v), (v, u) \in A$. The *Tutte matrix* $T = (t_{uv})$ is defined by

$$t_{uv} = \begin{cases} x_{uv} & ((u, v) \in A), \\ 0 & (\text{otherwise}), \end{cases}$$

where rows and columns are both indexed by V. Cunningham and Geelen [4] asserted that

$$\nu(G) = \operatorname{rank} T \tag{2.1}$$

if G is weakly symmetric. Based on this fact, the algorithm of Cunningham and Geelen [4] figures out $\nu(G)$ by computing $\operatorname{rank} T$ for a weakly symmetric digraph G.

In fact, the proof of (2.1) in Cunningham and Geelen [4] is adaptable even for odd-cycle-symmetric digraphs. That is, (2.1) and their algorithm are valid for odd-cycle-symmetric digraphs.

Theorem 2.1. *Let G be an odd-cycle-symmetric digraph and let T be the Tutte matrix of G. Then $\nu(G) = \operatorname{rank} T$ holds.*

Another approach to $\nu(G)$, proposed by Pap [14], is extending the matching algorithm of Edmonds [5]. Pap's algorithm finds a maximum even factor in an odd-cycle-symmetric digraph and brings about the following min-max formula.

Theorem 2.2. *For an odd-cycle-symmetric digraph $G = (V, A)$, it holds that*

$$\nu(G) = \min_{(X^+, X^-)} \left\{ |V \setminus X^+| + |V \setminus X^-| + |X^+ \cap X^-| - \operatorname{odd}^+(X^+ \cap X^-) \right\},$$

where (X^+, X^-) runs over all stable pairs.

This formula modestly extends that of Cunningham and Geelen [4, Theorem 1.3], in which the minimum is taken over all (X^+, X^-) such that $G[X^+, X^-]$ is symmetric. One easily sees that such (X^+, X^-) forms a stable pair.

2.2 Matroid Intersection

Let V be a finite set and its subset family $\mathcal{I} \subseteq 2^V$ satisfy the following (I0)–(I2):

(I0) $\emptyset \in \mathcal{I}$,

(I1) $I \subseteq J \in \mathcal{I} \Rightarrow I \in \mathcal{I}$,

(I2) $I, J \in \mathcal{I},\ |I| < |J| \Rightarrow \exists j \in J \setminus I,\ I \cup \{j\} \in \mathcal{I}$.

A set function $\rho : 2^V \to \mathbb{Z}_+$ is defined by

$$\rho(X) = \max\{|J| \mid J \subseteq X, J \in \mathcal{I}\} \quad (X \subseteq V).$$

For V, $\mathcal{I}$, and ρ defined above, we say $\mathbf{M} = (V, \mathcal{I}, \rho)$ forms a *matroid*. We call V the *ground set*, $\mathcal{I}$ the *independent set family*, $I \in \mathcal{I}$ an *independent set*, and ρ the *rank function*. A subset of V is said to be *dependent* if it is not an independent set.

Given a matroid $(V, \mathcal{I}, \rho)$, a subset $B \subseteq V$ is called a *base* if B is an inclusion-wise maximal independent set, and a subset $C \subseteq V$ is called a *circuit* if C is an inclusion-wise minimal dependent set.

The *closure function* $\mathrm{cl} : 2^V \to 2^V$ is defined by

$$\mathrm{cl}(X) = \{j \mid \rho(X \cup \{j\}) = \rho(X)\} \quad (X \subseteq V).$$

For any $I \in \mathcal{I}$ and $j \in \mathrm{cl}(I) \setminus I$, the union $I \cup \{j\}$ contains a unique circuit, called the *fundamental circuit* denoted by $C(I \mid j)$.

Let us be given two matroids on the same ground set, say $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$. The *matroid intersection problem* is a problem to find a maximum common independent set $I \in \mathcal{I}^+ \cap \mathcal{I}^-$.

A min-max formula for the matroid intersection problem is given by Edmonds [6].

Theorem 2.3 (Edmonds [6]). *For two matroids $\mathbf{M}_1 = (V, \mathcal{I}_1, \rho_1)$ and $\mathbf{M}_2 = (V, \mathcal{I}_2, \rho_2)$, it holds that*

$$\max\{|I| \mid I \subseteq V,\ I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{X \subseteq V}\{\rho_1(V \setminus X) + \rho_2(X)\}.$$

There are some combinatorial polynomial-time algorithms to find a maximum common independent set, such as Lawler's [13] and Edmonds' [7]. We remark that the notion of *alternating-paths* plays an essential part in those algorithms, as is the case with the matching algorithm of Edmonds [5] and the even factor algorithm of Pap [14].

3 Shrinking of Matroids

This section is devoted to the notion of *shrinking* of matroids. Note that it is different from the well-known notion of contraction of matroids.

3.1 Definition

Let $\mathbf{M} = (V, \mathcal{I}, \rho)$ be a matroid and $W \subseteq V$ be an independent set in $\mathbf{M}$. Then, consider $\tilde{V} = V \setminus W \cup \{w\}$, where w is a new element. Define a subset family $\tilde{\mathcal{I}}$ of $\tilde{V}$ and a set function $\tilde{\rho} : 2^{\tilde{V}} \to \mathbb{Z}_+$ by

$$\begin{aligned}\tilde{\mathcal{I}} = \{I \mid I \subseteq V \setminus W,\ \exists J \subseteq W, |J| = |W| - 1, I \cup J \in \mathcal{I}\}\\ \cup \{I \cup \{w\} \mid I \subseteq V \setminus W,\ I \cup W \in \mathcal{I}\},\\ \tilde{\rho}(X) = \max\{|J| \mid J \subseteq X, J \in \tilde{\mathcal{I}}\} \quad (X \subseteq \tilde{V}).\end{aligned}$$

Lemma 3.1. *The tuple $(\tilde{V}, \tilde{\mathcal{I}}, \tilde{\rho})$ forms a matroid.*

Proof. It is easily seen that $\tilde{\mathcal{I}}$ satisfies (I0) and (I1). We prove (I2) by showing that for any $\tilde{I}, \tilde{J} \in \tilde{\mathcal{I}}$ with $|\tilde{I}| < |\tilde{J}|$, there exists an element v such that

$$v \in \tilde{J} \setminus \tilde{I}, \quad \tilde{I} \cup \{v\} \in \tilde{\mathcal{I}}. \tag{3.1}$$

Suppose $w \in \tilde{I}$, i.e., $\tilde{I} = \tilde{I}' \cup \{w\}$ with $\tilde{I}' \subseteq V \setminus W$. As $\tilde{I}, \tilde{J} \in \tilde{\mathcal{I}}$, it follows that $I = \tilde{I}' \cup W \in \mathcal{I}$, and there exists $J \in \mathcal{I}$ with $|J| = |\tilde{J}| + |W| - 1$ and $J \setminus \tilde{J} \subseteq W$. Then, $|I| < |J|$ holds, which implies that there exists $v \in J \setminus I$ such that $I \cup \{v\} \in \mathcal{I}$. Hence, it holds that $\tilde{I} \cup \{v\} \in \mathcal{I}$ and $v \in \tilde{J} \setminus \tilde{I}$. Thus, v satisfies (3.1).

Suppose $w \notin \tilde{I}$. Then, there exists $W_I \subseteq W$ such that $|W_I| = |W| - 1$ and $I = \tilde{I} \cup W_I \in \mathcal{I}$. Define J and v as mentioned above. If $v \notin W$, we have that v satisfies (3.1). So, we may assume $v \in W$, which implies $W = W_I \cup \{v\}$ and $\tilde{I} \cup \{w\} \in \tilde{\mathcal{I}}$. If $w \in \tilde{J}$, then w satisfies (3.1). Otherwise, let $W_J = J \setminus \tilde{J}$. Note that $|W_J| = |W| - 1$ and $W_J \subseteq W$. Then, it holds that $I' = \tilde{I} \cup W_J \subseteq \tilde{I} \cup W$. Hence, by (I1) we have $I' \in \mathcal{I}$, and $|I'| = |\tilde{I}| < |J|$. Therefore, there exists an element $u \in J \setminus I' \subseteq \tilde{J} \setminus \tilde{I}$ such that $I' \cup \{u\} \in \mathcal{I}$, which implies (3.1) holds for u. $\square$

We denote the new matroid $(\tilde{V}, \tilde{\mathcal{I}}, \tilde{\rho})$ defined above by $\mathbf{M}_W$. The procedure to obtain $\mathbf{M}_W$ is called *shrinking* W, and we say that W is *shrunk into* w. Conversely, given an independent set I_W in $\mathbf{M}_W$, there exists an independent set $I \in \mathcal{I}$ such that $I_W \setminus \{w\} \subseteq I$ and $|I| = |I_W| + |W| - 1$. The procedure of obtaining I is said to be *expanding* W.

Lemma 3.1 can also be confirmed through combining well-known operations in matroid theory, induction through a bipartite graph and contraction.

Let $\mathbf{M} = (V, \mathcal{I}, \rho)$ be a matroid and $W \subseteq V$ be its independent set. Make a copy $\mathbf{M}' = (V', \mathcal{I}', \rho')$ of $\mathbf{M}$. We denote the copy of $v \in V$ (resp. $U \subseteq V$) by v' (resp. U'). Then, construct a bipartite graph $G = (V', V; A)$, where

$$A = \{(u', v) \mid u, v \in W\} \cup \{(v', v) \mid v \in V \setminus W\}.$$

Associated with $\mathcal{I}'$ and G, define a subset family $\mathcal{I}^\circ$ of V: a subset $I \subseteq V$ belongs to $\mathcal{I}^\circ$ if and only if there exists a matching $M \subseteq A$ in G such that $\partial^- M = I$ and $\partial^+ M \in \mathcal{I}'$. Perfect [16] observed that $\mathcal{I}^\circ$ satisfies (I0)–(I2). We denote the resulting matroid by $\mathbf{M}^\circ = (V, \mathcal{I}^\circ, \rho^\circ)$.

Next, choose an arbitrary subset S of W with $|S| = |W| - 1$ and contract S in $\mathbf{M}^\circ$ to obtain a new matroid $\tilde{\mathbf{M}}$. By denoting the unique element in $W \setminus S$ by w, one easily checks that $\tilde{\mathbf{M}}$ is the same as $\mathbf{M}_W$.

From this construction, we obtain an explicit description of $\tilde{\rho}$.

Theorem 3.2. *Let $\mathbf{M} = (V, \mathcal{I}, \rho)$ be a matroid, $W \subseteq V$ be an independent set in $\mathbf{M}$, and $\mathbf{M}_W = (\tilde{V}, \tilde{\mathcal{I}}, \tilde{\rho})$ be the matroid obtained by shrinking W. Then, for $X \subseteq \tilde{V}$, it holds that*

$$\tilde{\rho}(X) = \begin{cases} \rho(X \cup W) - |W| + 1 & (w \in X), \\ \min\{\rho(X),\ \rho(X \cup W) - |W| + 1\} & (w \notin X). \end{cases}$$

Proof. Given $\mathbf{M} = (V, \mathcal{I}, \rho)$ and $W \in \mathcal{I}$, define $G = (V', V; A)$ and $\mathbf{M}^\circ = (V, \mathcal{I}^\circ, \rho^\circ)$ as above and let S be a subset of W with $|S| = |W| - 1$. Then, for $X \subseteq \tilde{V}$, it holds that

$$\begin{aligned} \tilde{\rho}(X) &= \rho^\circ(X \cup S) - \rho^\circ(S) \\ &= \min_{Y \subseteq X \cup S} \{\rho(\Gamma(Y)) + |X \cup S \setminus Y|\} - |W| + 1, \end{aligned}$$

where $\Gamma(Y) = \{u \mid u \in V,\ \exists (u', v) \in A,\ v \in Y\}$ for $Y \subseteq V$.

Suppose $w \in X$. In this case, we have

$$\begin{aligned}
\tilde{\rho}(X) &= \min_{Y \subseteq X \cup S} \{\rho(\Gamma(Y)) + |X \cup S \setminus Y|\} - |W| + 1 \\
&= \min \left\{ \min_{Y \subseteq X \setminus \{w\}} \{\rho(Y) + |X \cup S \setminus Y|\}, \min_{\substack{Y \subseteq X \cup S, \\ Y \cap W \neq \emptyset}} \{\rho(Y \cup W) + |X \cup S \setminus Y|\} \right\} - |W| + 1 \\
&= \min \{\rho(X \setminus \{w\}) + |W|, \rho(X \cup W)\} - |W| + 1 \\
&= \rho(X \cup W) - |W| + 1,
\end{aligned}$$

where the third equality can be seen through putting $Y = X \setminus \{w\}, X \cup S$.

Suppose $w \notin X$. In this case,

$$\begin{aligned}
\tilde{\rho}(X) &= \min_{Y \subseteq X \cup S} \{\rho(\Gamma(Y)) + |X \cup S \setminus Y|\} - |W| + 1 \\
&= \min \left\{ \min_{Y \subseteq X} \{\rho(Y) + |X \cup S \setminus Y|\}, \min_{\substack{Y \subseteq X \cup S, \\ Y \cap S \neq \emptyset}} \{\rho(Y \cup W) + |X \cup S \setminus Y|\} \right\} - |W| + 1 \\
&= \min\{\rho(X) + |W| - 1, \rho(X \cup W)\} - |W| + 1 \\
&= \min\{\rho(X), \rho(X \cup W) - |W| + 1\},
\end{aligned}$$

where the third equality can be seen through putting $Y = X, X \cup S$. □

3.2 Independence Test

Here we describe how to test independence in matroids obtained by repeated shrinking. Suppose we have an independence oracle for a matroid $\mathbf{M} = (V, \mathcal{I}, \rho)$, and we obtained a matroid $\tilde{\mathbf{M}} = (\tilde{V}, \tilde{\mathcal{I}}, \tilde{\rho})$ by applying to $\mathbf{M}$ the procedure shrinking repeatedly. Let $w_1, \ldots, w_k$ be the newly created elements in $\tilde{V}$, that is, $\{w_1, \ldots, w_k\} = \tilde{V} \setminus V$, and denote the set of vertices shrunk into the pseudo-vertex w_i by $W_i \subseteq V$ $(i = 1, \ldots, k)$. Assume that we have an independent set $\tilde{I} \in \tilde{\mathcal{I}}$, and we know how to expand W_i $(i = 1, \ldots, k)$ to obtain an independent set $I \in \mathcal{I}$. We focus on testing whether $\tilde{I} \cup \{v\}$ is independent for all $v \in \tilde{V} \setminus \tilde{I}$. Moreover, if $\tilde{I} \cup \{v\} \notin \tilde{\mathcal{I}}$, we enumerate all $u \in \tilde{I}$ such that $(\tilde{I} \setminus \{u\}) \cup \{v\} \in \tilde{\mathcal{I}}$. These independence tests can be performed efficiently through constructing an auxiliary graph.

Construct an auxiliary bipartite graph $G^\natural = (V^\natural, A^\natural; S^1, S^2)$ as follows. The vertex set $V^\natural = V^1 \cup V^2$, where V^1 and V^2 are copies of V. The copy of $v \in V$ (resp. $W \subseteq V$) in V^i is denoted by v^i (resp. W^i), for $i = 1, 2$. The arc set $A^\natural$ consists of $A^\natural = A_V \cup A_M \cup J$, where

$$\begin{aligned}
A_V &= \left(\bigcup_{i=1}^{k} \left\{ (u^1, v^2) \mid u, v \in W_i \right\} \right) \cup \left\{ (v^1, v^2) \mid v \in V \setminus (W_1 \cup \cdots \cup W_k) \right\}, \\
A_M &= \{(v^2, v^1) \mid v \in I\}, \\
J &= \left\{ (u^2, v^2) \mid u \in \mathrm{cl}(I) \setminus I,\ v \in I,\ I \setminus \{v\} \cup \{u\} \in \mathcal{I} \right\}.
\end{aligned}$$

The two subsets $S^1 \subseteq V^1$ and $S^2 \subseteq V^2$ are defined by

$$\begin{aligned}
S^1 &= \{v^1 \mid v \in V \setminus I\}, \\
S^2 &= \{v^2 \mid v \in V \setminus I,\ I \cup \{v\} \in \mathcal{I}\}.
\end{aligned}$$

We refer to S^1 and S^2 as the set of *source-vertices* and *sink-vertices*, respectively, and refer to a path which starts from S^1 and ends in S^2 as a *source-sink path*.

Fix $v \in V \setminus I$. Assume v is not a pseudo-vertex. If there exists a source-sink path in $G^\natural$ which starts from v^1, then $\tilde{I} \cup \{v\} \in \tilde{\mathcal{I}}$. If not, one can find all elements $u \in \tilde{I}$ which satisfies $(\tilde{I} \setminus \{u\}) \cup \{v\} \in \tilde{\mathcal{I}}$ as follows: for a vertex $u \in \tilde{I} \setminus \{w_1, \ldots, w_k\}$, it holds that $(\tilde{I} \setminus \{u\}) \cup \{v\} \in \tilde{\mathcal{I}}$ if and only if there exists a path in $G^\natural$ which starts from v^1 and ends in u^2; for a pseudo-vertex $w_i \in \tilde{I}$, it holds that $(\tilde{I} \setminus \{w_i\}) \cup \{v\} \in \tilde{\mathcal{I}}$ if and only if there exists a path in $G^\natural$ which starts from v^1 and ends in w^2 with $w \in W_i$.

Assume v is a pseudo-vertex, say $v = w_1$, and let w_* be the unique vertex in $W_1 \setminus I$. In that case, if there exists a source-sink path in $G^\natural$ which starts from w_*, then $\tilde{I} \cup \{v\} \in \tilde{\mathcal{I}}$. If not, one can find all elements $u \in \tilde{I}$ which satisfies $(\tilde{I} \setminus \{u\}) \cup \{v\} \in \tilde{\mathcal{I}}$ as follows: for a vertex $u \in \tilde{I} \setminus \{w_1, \ldots, w_k\}$, it holds that $(\tilde{I} \setminus \{u\}) \cup \{v\} \in \tilde{\mathcal{I}}$ if and only if there exists a path in $G^\natural$ which starts from w_*^1 and ends in u^2; for a pseudo-vertex $w_i \in \tilde{I}$, it holds that $(\tilde{I} \setminus \{w_i\}) \cup \{v\} \in \tilde{\mathcal{I}}$ if and only if there exists a path in $G^\natural$ which starts from w_*^1 and ends in w^2 with $w \in W_i$.

4 The Independent Even Factor Problem

From now on, we deal with the *independent even factor problem*, which is a variant of the *basic even factor problem* [4].

Definition 4.1. Let $G = (V, A)$ be a digraph and $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$, $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$ be matroids. An even factor M in G is called an *independent even factor* in $(G, \mathbf{M}^+, \mathbf{M}^-)$ if $\partial^+ M \in \mathcal{I}^+$ and $\partial^- M \in \mathcal{I}^-$. If $\partial^+ M$ and $\partial^- M$ are bases of $\mathbf{M}^+$ and $\mathbf{M}^-$, respectively, M is called a *basic even factor*.

For a digraph $G = (V, A)$ and matroids $\mathbf{M}^+ = (V, \mathcal{I}^+)$, $\mathbf{M}^- = (V, \mathcal{I}^-)$, let $\nu(G, \mathbf{M}^+, \mathbf{M}^-)$ denote the cardinality of a maximum independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$. The independent even factor problem contains the even factor problem and the matroid intersection problem as special cases.

Even Factor Problem: If $\mathbf{M}^+$ and $\mathbf{M}^-$ are free, an arc set is an independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$ if and only if it is an even factor in G.

Matroid Intersection Problem: Let $\mathbf{M}_1 = (V, \mathcal{I}_1, \rho_1)$ and $\mathbf{M}_2 = (V, \mathcal{I}_2, \rho_2)$ be an instance of the matroid intersection problem. Then, construct an associated bipartite digraph $G = (V_1, V_2; A)$ as follows. The vertex set V_1 and V_2 are copies of V, respectively, and the arc set A is given by

$$A = \{(v_1, v_2) \mid v \in V\},$$

where $v_1 \in V_1$ (resp. $v_2 \in V_2$) denotes the copy of $v \in V$. On the vertex set $V_1 \cup V_2$, define matroids $\mathbf{M}^+ = (V_1 \cup V_2, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V_1 \cup V_2, \mathcal{I}^-, \rho^-)$ by the following rank functions $\rho^+, \rho^- : 2^{V_1 \cup V_2} \to \mathbb{Z}_+$:

$$\rho^+(X) = \rho_1(X \cap V_1),$$
$$\rho^-(X) = \rho_2(X \cap V_2).$$

Then, a subset $I \subseteq V$ belongs to $\mathcal{I}_1 \cap \mathcal{I}_2$ if and only if the arc set $\{(v_1, v_2) \mid v \in I\}$ is an independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$.

For any independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$ and any stable pair in G, the following inequality holds.

Lemma 4.2. *Let $G = (V, A)$ be a digraph and $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$, $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$ be matroids. Then, for any independent even factor M in $(G, \mathbf{M}^+, \mathbf{M}^-)$ and any stable pair (X^+, X^-) in G, it holds that*

$$|M| \leq \rho^+(V \setminus X^+) + \rho^-(V \setminus X^-) + \left|X^+ \cap X^-\right| - \mathrm{odd}^+(X^+ \cap X^-). \tag{4.1}$$

Proof. As (X^+, X^-) is a stable pair, it holds that $M = M_1 \cup M_2 \cup M_3$, where

$$M_1 = \left\{a \mid a \in M,\ \partial^+ a \in V \setminus X^+\right\}, \tag{4.2}$$

$$M_2 = \left\{a \mid a \in M,\ \partial^- a \in V \setminus X^-\right\}, \tag{4.3}$$

$$M_3 = \left\{a \mid a \in M,\ a \in A[X^+ \cap X^-]\right\}. \tag{4.4}$$

Note that M_1 and M_2 are not necessarily disjoint. We accomplish the proof by showing

$$|M_1| \leq \rho^+(V \setminus X^+), \qquad |M_2| \leq \rho^-(V \setminus X^-), \qquad |M_3| \leq |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-). \tag{4.5}$$

First, as M is an independent even factor, it follows that $\partial^+ M \in \mathcal{I}^+$, especially $\partial^+ M_1 \in \mathcal{I}^+$. By the definition of M_1, we have $\partial^+ M_1 \subseteq V \setminus X^+$, which implies $|M_1| \leq \rho^+(V \setminus X^+)$.

Next, a similar argument shows that $|M_2| \leq \rho^-(V \setminus X^-)$.

Finally, let us estimate $|M_3|$ by counting $|\partial^- M_3|$. For a source-component C in $G[X^+ \cap X^-]$, it holds that $|V(C) \cap \partial^- M_3| = |A(C) \cap M_3|$. Hence, if C is odd and $|V(C) \cap \partial^- M_3| = |C|$, then $A(C) \cap M_3$ would contain an odd cycle, which contradicts that M is an even factor in G. Thus, for every odd source-component C in $X^+ \cap X^-$, it holds that $|V(C) \cap \partial^- M_3| \leq |C| - 1$, which implies $|M_3| = |\partial^- M_3| \leq |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-)$. □

A stable pair (X^+, X^-) is called *minimizing* if (X^+, X^-) minimizes the right hand side of (4.1). In fact, for a maximum independent even factor M and a minimizing stable pair (X^+, X^-), (4.1) holds with equality if G is odd-cycle-symmetric.

Theorem 4.3. *For an odd-cycle-symmetric digraph $G = (V, A)$ and matroids $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$, $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$, it holds that*

$$\nu(G, \mathbf{M}^+, \mathbf{M}^-) = \min_{(X^+, X^-)} \left\{\rho^+(V \setminus X^+) + \rho^-(V \setminus X^-) + \left|X^+ \cap X^-\right| - \mathrm{odd}^+(X^+ \cap X^-)\right\}, \tag{4.6}$$

where (X^+, X^-) runs over all stable pairs.

One easily sees that Theorem 4.3 generalizes Theorems 2.2 and 2.3.

Theorem 4.3 can be proved by extending Cunningham and Geelen's approach [4]. Let $G = (V, A)$ be an odd-cycle-symmetric digraph, $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$ be matroids, and T be the Tutte matrix of G. Let us denote by $T[I^+, I^-]$ the submatrix of T whose row set is $I^+ \subseteq V$ and column set $I^- \subseteq V$. Note that the submatrix $T[I^+, I^-]$ is the Tutte matrix of the subgraph $G[I^+, I^-]$. By Theorem 2.1, we have that if a submatrix $T[I^+, I^-]$ such that $I^+ \in \mathcal{I}^+$ and $I^- \in \mathcal{I}^+$ is nonsingular, there exists an independent even factor M with $\partial^+ M = I^+$ and $\partial^- M = I^-$. As for such submatrices, there exists a min-max formula, which is a special case of Schrijver's one for linking systems [17, Theorem 3.6].

Theorem 4.4. *Let T be a matrix whose row set and column set are both indexed by V. and let $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$ be matroids. Then, it holds that*

$$\begin{aligned}\max\left\{|I^+| \mid I^+ \in \mathcal{I}^+,\ I^- \in \mathcal{I}^-,\ T[I^+, I^-] \textit{ is nonsingular}\right\} \\ = \min_{U^+, U^- \subseteq V} \left\{\rho^+(V \setminus U^+) + \rho^-(V \setminus U^-) + \operatorname{rank} T[U^+, U^-]\right\}.\end{aligned}$$

We remark here that Schrijver [17] proved the min-max formula using Theorem 2.3. Combining Theorems 2.2 and 4.4 procures us a proof of Theorem 4.3.

Proof of Theorem 4.3. Let T be the Tutte matrix of G. By Theorem 4.4, we have

$$\nu(G, \mathbf{M}^+, \mathbf{M}^-) = \min_{U^+, U^- \subseteq V} \left\{\rho^+(V \setminus U^+) + \rho^-(V \setminus U^-) + \operatorname{rank} T[U^+, U^-]\right\}.$$

Moreover, we know by Theorems 2.1 and 2.2 that

$$\begin{aligned}\operatorname{rank} T[U^+, U^-] &= \nu(G[U^+, U^-]) \\ &= \min_{(X^+, X^-)} \left\{|U^+ \setminus X^+| + |U^- \setminus X^-| + |X^+ \cap X^-| - \operatorname{odd}^+(X^+ \cap X^-)\right\},\end{aligned}$$

where (X^+, X^-) runs over all stable pairs in $G[U^+, U^-]$. Note that the second equality follows from a variant of Theorem 2.2, which can also be proved by Pap's even factor algorithm [14]. Therefore, we have that

$$\begin{aligned}\nu(G, \mathbf{M}^+, \mathbf{M}^-) &= \min_{U^+, U^- \subseteq V} \Big\{\rho^+(V \setminus U^+) + \rho^-(V \setminus U^-) \\ &\qquad + \min_{(X^+, X^-)} \left\{|U^+ \setminus X^+| + |U^- \setminus X^-| + |X^+ \cap X^-| - \operatorname{odd}^+(X^+ \cap X^-)\right\}\Big\} \\ &= \min_{(X^+, X^-)} \Big\{\min_{U^+, U^-} \left\{\rho^+(V \setminus U^+) + \rho^-(V \setminus U^-) + |U^+ \setminus X^+| + |U^- \setminus X^-|\right\} \\ &\qquad + |X^+ \cap X^-| - \operatorname{odd}^+(X^+ \cap X^-)\Big\},\end{aligned}$$

where U^+ and U^- satisfy that (X^+, X^-) is a stable pair in $G[U^+, U^-]$. The inner minimum in the right hand side is achieved when $U^+ = X^+$ and $U^- = X^-$. Hence, it follows that

$$\nu(G, \mathbf{M}^+, \mathbf{M}^-) = \min_{(X^+, X^-)} \left\{\rho^+(V \setminus X^+) + \rho^-(V \setminus X^-) + |X^+ \cap X^-| - \operatorname{odd}^+(X^+ \cap X^-)\right\},$$

where (X^+, X^-) runs over all stable pairs. □

5 An Alternating-Path Algorithm

In this section, we describe a combinatorial algorithm to find a maximum independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$, where G is odd-cycle-symmetric. This algorithm is a natural generalization of the maximum even factor algorithm [14] and the matroid intersection algorithms [7,13], in both of which the notion of *alternating-path* plays an essential part. The algorithm also finds a minimizing stable pair which certificates (4.6).

5.1 The Algorithm

Let $G = (V, A)$ be an odd-cycle-symmetric digraph, and $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$ be matroids, whose closure functions and fundamental circuits are denoted by $\mathrm{cl}^+(\cdot), \mathrm{cl}^-(\cdot)$ and $C^+(\cdot \mid \cdot)$, $C^-(\cdot \mid \cdot)$, respectively. Assume we have an independent even factor M in $(G, \mathbf{M}^+, \mathbf{M}^-)$. We show a procedure to find an independent even factor M' with $|M'| = |M| + 1$, or determine that M is a maximum independent even factor.

For an independent even factor M, construct an auxiliary graph $G_M = (V^*, A^*; S^+, S^-)$ as follows. The vertex set V^* is given by $V^* = V^+ \cup V^-$, where V^+ and V^- are two copies of V. For a vertex $v \in V$, we denote the copy of v in V^+ (resp. V^-) by v^+ (resp. v^-).

The arc set A^* is composed of

$$A^* = A^\circ \cup M^\circ \cup J^+ \cup J^-,$$

where

$$\begin{aligned}
A^\circ &= \left\{(u^+, v^-) \mid (u, v) \in A\right\} \qquad \text{(copy of } A\text{)},\\
M^\circ &= \left\{(v^-, u^+) \mid (u, v) \in M\right\} \quad \text{(reverse copy of } M\text{)},\\
J^+ &= \left\{(u^+, v^+) \mid u \in \partial^+ M,\ v \in \mathrm{cl}^+(\partial^+ M) \setminus \partial^+ M,\ \partial^+ M \setminus \{u\} \cup \{v\} \in \mathcal{I}^+\right\},\\
J^- &= \left\{(u^-, v^-) \mid u \in \mathrm{cl}^-(\partial^- M) \setminus \partial^- M,\ v \in \partial^- M,\ \partial^- M \setminus \{v\} \cup \{u\} \in \mathcal{I}^-\right\}.
\end{aligned}$$

We call the arcs in J^+ and J^- the *jumping arcs*.

The two subsets $S^+ \subseteq V^+$ and $S^- \subseteq V^-$ are defined by

$$\begin{aligned}
S^+ &= \{v^+ \mid v \in V \setminus \partial^+ M,\ \partial^+ M \cup \{v\} \in \mathcal{I}^+\},\\
S^- &= \{v^- \mid v \in V \setminus \partial^- M,\ \partial^- M \cup \{v\} \in \mathcal{I}^-\}.
\end{aligned}$$

We refer to S^+ and S^- as the set of *source-vertices* and *sink-vertices*, respectively.

A vertex $v \in V^*$ is called *source-reachable* if there exists a path which starts in S^+ and ends in v, and called *sink-reachable* if there exists a path which starts in v and ends in S^-. A path P is called a *source-sink path* if P starts in S^+ and ends in S^-. For a path P in G_M, we denote the master copy of $A(P) \cap (A^\circ \cup M^\circ)$ by $A^\circ(P)$ (that is, $A^\circ(P) \subseteq A$).

Suppose that G_M has source-sink paths. Let P be the shortest source-sink path and $A^\circ(P) = \{a_1, m_1, a_2, \ldots, a_k, m_k, a_{k+1}\}$, which is in order of appearance in P. Note that $a_1, \ldots, a_{k+1} \in A \setminus M$ and $m_1, \ldots, m_k \in M$. Then, we obtain an independent even factor M' with $|M'| = |M| + 1$ by putting $M' = M \triangle A^\circ(P) = (M \setminus \{m_1, \ldots, m_k\}) \cup \{a_1, \ldots, a_{k+1}\}$, if M' does not contain odd cycles. We refer to this procedure as $\mathsf{Augment}(M, P)$.

If there exist odd cycles in M', we avoid the creation of the cycles as follows. Denote $\{a_1, m_1, a_2, \ldots, a_i, m_i\} = A^\circ(P_i)$ for each $i \leq k$. Let j be the largest number such that $M \triangle A^\circ(P_j)$ does not contain odd cycles. If $j = k$, it follows that there exists a unique odd cycle in $M \triangle A^\circ(P)$, which contains a_{k+1}. Else, there exists a unique odd cycle in $M \triangle A^\circ(P_{j+1})$. Let us denote such an odd cycle by C, and note that $V(C) \in \mathcal{I}^+ \cap \mathcal{I}^-$. In those cases, take the symmetric difference of M and $A^\circ(P_j)$, say $N = M \triangle A^\circ(P_j) = (M \setminus \{m_1, \ldots, m_j\}) \cup \{a_1, \ldots, a_j\}$. Let us remark that N is an independent even factor with $|N| = |M|$ and $|N \cap A(C)| = |C| - 1$. Then, shrink $V(C)$ into a new vertex v_C to obtain a new digraph $\tilde{G} = (\tilde{V}, \tilde{A})$ and matroids $\tilde{\mathbf{M}}^+$, $\tilde{\mathbf{M}}^-$. In need of explicit representation of the shrunk odd cycle, we shall denote $\tilde{G}$, $\tilde{V}$, and $\tilde{A}$ by G_C, V_C, and A_C, respectively. Note that the arc set $\tilde{N} = N \cap \tilde{A}$ is an independent even factor in $(\tilde{G}, \tilde{\mathbf{M}}^+, \tilde{\mathbf{M}}^-)$ with $|\tilde{N}| = |N| - |C| + 1$. We call this procedure $\mathsf{Shrink}(M, P, C)$.

We show the validity of the procedure $\mathsf{Shrink}(M, P, C)$ by the propositions below. First, the following proposition assures the odd-cycle-symmetry of $\tilde{G}$.

Proposition 5.1 (Pap [14]). *Let $G = (V, A)$ be an odd-cycle-symmetric digraph. For an odd cycle C, it holds that $G_C = (V_C, A_C)$ is also odd-cycle-symmetric.* $\square$

By Lemma 3.1 and Proposition 5.1, it is ensured that the algorithm also runs in $(\tilde{G}, \tilde{\mathbf{M}}^+, \tilde{\mathbf{M}}^-)$.

Next, for an independent even factor $\tilde{M}$ in $(G_C, \mathbf{M}^+_{V(C)}, \mathbf{M}^-_{V(C)})$, we describe how to obtain an independent even factor M in $(G, \mathbf{M}^+, \mathbf{M}^-)$ with $|M| = |\tilde{M}| + |C| - 1$. We only discuss the case where $v_C \in \partial^+\tilde{M} \cap \partial^-\tilde{M}$. Similar arguments hold in the other cases.

Since $v_C \in \partial^+\tilde{M} \cap \partial^-\tilde{M}$, we have that $\partial^+ M = (\partial^+\tilde{M} \setminus \{v_C\}) \cup V(C) \in \mathcal{I}^+$ and $\partial^- M = (\partial^-\tilde{M} \setminus \{v_C\}) \cup V(C) \in \mathcal{I}^-$. In this case, it suffices to add $|C| - 1$ arcs in $A(C) \cup A(\bar{C})$ to $\tilde{M}$ so that the resulting arc set forms an even factor. Let us denote by a^+ (resp. a^-) $\in \tilde{A}$ the unique arc in $\delta^+ v_C \cap \tilde{M}$ (resp. $\delta^- v_C \cap \tilde{M}$). As C is odd and G is odd-cycle-symmetric, there exists an even path P from $\partial^- a^-$ to $\partial^+ a^+$ in C or $\bar{C}$. In addition, there exist cycles of length two that cover every vertex $v \in V(C)$ exposed by P exactly once. The union of $\tilde{M}$, $A(P)$ and the arcs in these cycles is an independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$ of size $|\tilde{M}| + |C| - 1$. We call this procedure $\mathsf{Expand}(\tilde{M}, C)$.

The procedure $\mathsf{Expand}(\tilde{M}, C)$ ensures that for every independent even factor $\tilde{M}$ in $(G_C, \mathbf{M}^+_{V(C)}, \mathbf{M}^-_{V(C)})$, there exists an independent even factor M in $(G, \mathbf{M}^+, \mathbf{M}^-)$ with $|M| = |\tilde{M}| + |C| - 1$. Hence, the following proposition holds.

Proposition 5.2. *Let $G = (V, A)$ be an odd-cycle-symmetric digraph, and $\mathbf{M}^+$, $\mathbf{M}^-$ be two matroids on V. For an odd cycle C with $V(C) \in \mathcal{I}^+ \cup \mathcal{I}^-$, it follows that*

$$\nu\left(G, \mathbf{M}^+, \mathbf{M}^-\right) \geq \nu\left(G_C, \mathbf{M}^+_{V(C)}, \mathbf{M}^-_{V(C)}\right) + |C| - 1.$$

We are now ready to describe the steps to find an independent even factor of cardinality larger than M by one.

Step 1: Construct the auxiliary graph G_M. If there exists a source-sink path in G_M, then go to Step 2. Otherwise, go to Step 4.

Step 2: Let P be the shortest source-sink path. If $M \triangle A^\circ(P)$ does not contain any odd cycle, apply $\mathsf{Augment}(M, P)$, and then go to Step 4. Otherwise, go to Step 3.

Step 3: Apply $\mathsf{Shrink}(M, P, C)$, and $(G, \mathbf{M}^+, \mathbf{M}^-) := (\tilde{G}, \tilde{\mathbf{M}}^+, \tilde{\mathbf{M}}^-)$, $M := \tilde{N}$. Then, go to Step 1.

Step 4: Apply $\mathsf{Expand}(M, C)$ while there exists a pseudo-vertex v_C.

In the case that no source-sink path is found in Step 1, the algorithm determines that M is a maximum independent even factor. We certify this by showing that there exists a stable pair (X^+, X^-) such that M and (X^+, X^-) certificate (4.6).

Algorithmic Proof of Theorem 4.3. Let $(G, \mathbf{M}^+, \mathbf{M}^-)$ be the tuple in which no source-sink path was found in Step 1. Let R be the set of the source-reachable vertices in G_M, $X^+ = \{v \mid v^+ \in R\}$, and $X^- = \{v \mid v^- \notin R\}$. Then, we can easily see that (X^+, X^-) is a stable pair, $M_1 \cap M_2 = \emptyset$, and $M_3 = \emptyset$. Moreover, $A[X^+ \cap X^-] = \emptyset$, which implies that $|X^+ \cap X^-| = \mathrm{odd}^+(X^+ \cap X^-)$.

Pick up $v \in (V \setminus X^+) \setminus \partial^+ M$. As v^+ is not source-reachable, v^+ is not a source-vertex, that is, $\partial^+ M \cup \{v\} \notin \mathcal{I}^+$. Then, consider the fundamental circuit $C^+(\partial^+ M \mid v)$. If $C^+(\partial^+ M \mid v)$ and X^+ intersect at some vertex u, there would be a jumping arc from u^+ to v^+, which contradicts that v^+ is not source-reachable. Hence, we have that $C^+(\partial^+ M \mid v) \subseteq V \setminus X^+$ for all $v \in (V \setminus X^+) \setminus \partial^+ M$, which implies that $|M_1| = \rho^+(V \setminus X^+)$. A similar argument shows that $|M_2| = \rho^-(V^- \setminus X^-)$. Therefore, it holds that $|M| = \rho^+(V \setminus X^+) + \rho^-(V \setminus X^-) + |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-)$.

The argument above just assures that (4.6) holds in $(G, \mathbf{M}^+, \mathbf{M}^-)$, which may be achieved as a result of applying $\mathsf{Shrink}(\cdot,\cdot,\cdot)$ repeatedly. We show below that (4.6) holds before the repeated $\mathsf{Shrink}(\cdot,\cdot,\cdot)$.

Suppose that $(G, \mathbf{M}^+, \mathbf{M}^-)$ is the tuple obtained by applying $\mathsf{Shrink}(\hat{M}, P, C)$ to $(\hat{G}, \hat{\mathbf{M}}^+, \hat{\mathbf{M}}^-)$, where $\hat{G} = (\hat{V}, \hat{A})$, $\hat{\mathbf{M}}^+ = (\hat{V}, \hat{\mathcal{I}}^+, \hat{\rho}^+)$, and $\hat{\mathbf{M}}^- = (\hat{V}, \hat{\mathcal{I}}^-, \hat{\rho}^-)$. Associated with X^+ and X^- mentioned above, define $\hat{X}^+, \hat{X}^- \subseteq \hat{V}$ as the inverse-image of X^+, X^-, respectively, that is,

$$\hat{X}^+ = \begin{cases} X^+ & (v_C \notin X^+), \\ X^+ \cup V(C) & (v_C \in X^+), \end{cases}$$

and so is $\hat{X}^-$. Now we prove that $(\hat{X}^+, \hat{X}^-)$ is a stable pair which certificates (4.6).

Firstly, one can easily see that $(\hat{X}^+, \hat{X}^-)$ forms a stable pair in $\hat{G}$, which follows from the fact that (X^+, X^-) is a stable pair in G.

Secondly, we estimate the value $\hat{\rho}^+(\hat{V} \setminus \hat{X}^+) + \hat{\rho}^-(\hat{V} \setminus \hat{X}^-) + |\hat{X}^+ \cap \hat{X}^-| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-)$.

One can easily see that v_C is a source-vertex in G_M, which implies that $v_C \in X^+$. Then, by Theorem 3.2, we have

$$\hat{\rho}^+(\hat{V} \setminus \hat{X}^+) = \rho(V \setminus X^+),$$

since $\hat{\rho}^+(\hat{V} \setminus \hat{X}^+) \leq \hat{\rho}^+\left((\hat{V} \setminus \hat{X}^+) \cup V(C)\right) - |C| + 1$ follows that we can extend M with $|C| - 1$ arcs in $A(C) \cup A(\bar{C})$ to obtain an independent even factor in $(\hat{G}, \hat{\mathbf{M}}^+, \hat{\mathbf{M}}^-)$.

Assume $v_C \in X^+ \cap X^-$. Then, we have $\hat{\rho}^-(\hat{V} \setminus \hat{X}^-) = \rho^-(V \setminus X^-)$ from an argument similar to the above one. Moreover, the equation $|\hat{X}^+ \cap \hat{X}^-| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) = |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-) + |C| - 1$ follows the equations: $|\hat{X}^+ \cap \hat{X}^-| = |X^+ \cap X^-| + |C| - 1$; $\mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) = \mathrm{odd}^+(X^+ \cap X^-)$.

Assume $v_C \in X^+ \setminus X^-$. In this case, Theorem 3.2 implies that $\hat{\rho}^-(\hat{V} \setminus \hat{X}^-) = \rho^-(V \setminus X^-) + |C| - 1$, and it is obvious that $|\hat{X}^+ \cap \hat{X}^-| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) = |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-)$.

Therefore, in any case we have

$$\begin{aligned} &\hat{\rho}^+(\hat{V} \setminus \hat{X}^+) + \hat{\rho}^-(\hat{V} \setminus \hat{X}^-) + |\hat{X}^+ \cap \hat{X}^-| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) \\ &\quad = \rho^+(V \setminus X^+) + \rho^-(V \setminus X^-) + |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-) + |C| - 1. \end{aligned} \tag{5.1}$$

Hence, combining Proposition 5.2 and (5.1), we have that (4.6) holds for $(\hat{G}, \hat{\mathbf{M}}^+, \hat{\mathbf{M}}^-)$. □

5.2 Complexity

We discuss the complexity of the algorithm described in § 5.1. Let $(G, \mathbf{M}^+, \mathbf{M}^-)$, where $G = (V, A)$, $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$, be the original instance. Recall that $n = |V|$ and Q is the time needed to test if a given set $U \subseteq V$ is independent in $\mathbf{M}^+$ or $\mathbf{M}^-$.

The most time-consuming part of the algorithm is the construction of the auxiliary graphs $G_M = (V', A'; S^+, S^-)$. We can identify the source-vertices S^+, sink-vertices S^-, and jumping arcs J^+, J^- by searching paths in the auxiliary graphs $G^\natural$ described in § 3.2. The construction of the auxiliary graph $G^\natural$ takes $\mathrm{O}(n^2 Q)$ time. In the procedure $\mathsf{Shrink}(\cdot,\cdot,\cdot)$, the number of vertices decreases at least by two, which implies that we have to construct $\mathrm{O}(n)$ auxiliary graphs G_M between augmentations. Since there are at most n augmentations, the total time complexity of the algorithm is $\mathrm{O}(n^4 Q)$.

We remark that the auxiliary graph $G^\natural$ helps us finding which vertices in $V(C)$ one should choose to be in $\partial^+ M$ and $\partial^- M$ in the procedure $\mathsf{Expand}(M, C)$.

6 An Edmonds-Gallai Type Structure

In this section, we show that independent even factors have a structure which generalizes both of the Edmonds-Gallai decomposition for matchings or even factors, and the principal partition for matroid intersection.

Theorem 6.1. *Let $G = (V, A)$ be an odd-cycle-symmetric digraph and $\mathbf{M}^+ = (V, \mathcal{I}^+, \rho^+), \mathbf{M}^- = (V, \mathcal{I}^-, \rho^-)$ be matroids. Define*

$$\begin{aligned}
V_D^+ &= \{v \mid v \in V,\ \exists \text{maximum independent even factor } M,\ v \notin \mathrm{cl}^+(\partial^+ M)\},\\
V_A^+ &= \{v \mid v \in V,\ \exists \text{maximum independent even factor } M,\ v^- \text{ is source-reachable in } G_M\},\\
V_D^- &= \{v \mid v \in V,\ \exists \text{maximum independent even factor } M,\ v \notin \mathrm{cl}^-(\partial^- M)\},\\
V_A^- &= \{v \mid v \in V,\ \exists \text{maximum independent even factor } M,\ v^+ \text{ is sink-reachable in } G_M\}.
\end{aligned}$$

Then, the following (i)–(iv) hold.

(i) The pair $(V_D^+, V \setminus V_A^+)$ is a minimizing stable pair (i.e., minimizes the right hand side of (4.1) *among all stable pairs). Similarly, $(V \setminus V_A^-, V_D^-)$ is a stable pair which minimizes*

$$\rho^+(V \setminus Y^+) + \rho^-(V \setminus Y^-) + \left|Y^+ \cap Y^-\right| - \mathrm{odd}^-(Y^+ \cap Y^-), \tag{6.1}$$

where (Y^+, Y^-) runs over all stable pairs.

(ii) Every strongly connected component C in $G[V_D^+ \cap (V \setminus V_A^+)]$ or in $G[V_D^- \cap (V \setminus V_A^-)]$ is an odd component with $V(C) \in \mathcal{I}^+ \cap \mathcal{I}^-$.

(iii) If M is a maximum independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$, then the following (a)–(c) hold.

(a) For every strongly connected component C in $G[V_D^+ \cap (V \setminus V_A^+)]$ or in $G[V_D^- \cap (V \setminus V_A^-)]$, $|M \cap A(C)| = |C| - 1$.

(b) $|(V \setminus V_D^+) \cap \partial^+ M| = \rho^+(V \setminus V_D^+)$, $|V_A^+ \cap \partial^- M| = \rho^-(V_A^+)$, $|V_A^- \cap \partial^+ M| = \rho^+(V_A^-)$, and $|(V \setminus V_D^-) \cap \partial^- M| = \rho^-(V \setminus V_D^-)$.

(c) For a vertex $v \in V_A^+ \cap \partial^- M$, there exists a vertex $u \in V_D^+$ with $(u, v) \in M$, and for a vertex $v \in V_A^- \cap \partial^+ M$, there exists a vertex $u \in V_D^-$ with $(v, u) \in M$.

(iv) Any minimizing stable pair (X^+, X^-) satisfies that $V_D^+ \subseteq X^+$ and $V_D^- \subseteq X^-$. Similarly, any stable pair (Y^+, Y^-) that minimizes (6.1) *satisfies that $V_D^+ \subseteq Y^+$ and $V_D^- \subseteq Y^-$.*

Proof. Here we only show the statements on the stable pair $(V_D^+, V \setminus V_A^+)$. Those on $(V \setminus V_A^-, V_D^-)$ can be proved similarly by exchanging the roles of $\mathbf{M}^+$ and $\mathbf{M}^-$ in the maximum independent even factor algorithm.

Assertions (i) and (ii) follow from the fact that $(V_D^+, V \setminus V_A^+)$ is the minimizing stable pair which the algorithm finds.

For a maximum independent even factor M, define M_1, M_2, M_3 by (4.2)–(4.4), where $X^+ = V_D^+$ and $X^- = V \setminus V_A^+$. As M is a maximum independent even factor and (X^+, X^-) is a minimizing stable pair, Inequations (4.5) hold with equalities and $M_1 \cap M_2 = \emptyset$. One checks that these conditions induce (iii) (cf. the proof of Lemma 4.2).

We show (iv). Suppose there exists a minimizing stable pair (X^+, X^-) such that there is a vertex u with $u \in V_D^+ \setminus X^+$. As $u \in V_D^+$, there is a maximum independent even factor M with $u \notin \mathrm{cl}^+(\partial^+ M)$. Then, the stable pair (X^+, X^-) and M satisfy (4.5) with equality. The equation $|M_1| = \rho^+(V \setminus X^+)$, however, contradicts that $u \in V \setminus X^+$ and $u \notin \mathrm{cl}^+(\partial^+ M)$. Therefore, for any minimizing stable pair (X^+, X^-), it holds that $V_D^+ \subseteq X^+$. $\square$

Let us describe how Theorem 6.1 generalizes the Edmonds-Gallai decomposition for matchings and the principal partition for matroid intersection.

Edmonds-Gallai Decomposition for Matchings Let $G = (V, E)$ be an undirected graph with the vertex set V and the edge set E, and $\vec{G} = (V, A)$ be a symmetric digraph whose arc set is

$$A = \{(u, v), (v, u) \mid \exists\text{edge between } u \text{ and } v \text{ in } E\}.$$

Consider an instance of matching problem G and an instance of the independent even factor problem $(\vec{G}, \mathbf{M}^+, \mathbf{M}^-)$ where $\mathbf{M}^+$ and $\mathbf{M}^-$ are free matroids. Then, it follows that $V_D^+ = V_D^- = V_D$, $V_A^+ = V_A^- = V_A$, where

$$\begin{aligned}
V_D &= \{v \mid v \in V,\ \exists\text{maximum matching which does not cover } v\},\\
V_A &= \{v \mid v \in V \setminus V_D,\ v \text{ is adjacent to some vertex in } V_D\}.
\end{aligned}$$

By Theorem 6.1, the following statements on the undirected graph G hold.

- The cardinality of a maximum matching in G is equal to

$$\begin{aligned}
\frac{1}{2}\nu(\vec{G}, \mathbf{M}^+, \mathbf{M}^-) &= \frac{1}{2}\left(|V \setminus V_D^+| + |V_A^+| + |V_D^+ \cap (V \setminus V_A^+)| - \mathrm{odd}^+(V_D^+ \cap (V \setminus V_A^+))\right)\\
&= \frac{1}{2}\left(|V| + |V_A| - \mathrm{odd}(V_D)\right),
\end{aligned}$$

  where $\mathrm{odd}(X)$ denotes the number of odd component in $G[X]$ $(X \subseteq V)$.
- Every connected component C in $G[V_D]$ is odd and factor-critical.
- If M is a maximum matching, then it holds that
  - for every connected component C in $G[V_D]$, C has $(|C| - 1)/2$ edges in M,
  - every vertex in $V \setminus V_D$ is covered by M,
  - for a vertex $v \in V_A$, there exists a vertex $u \in V_D$ such that an edge in M connects u and v.
- For any $V'_D \subseteq V$ which minimizes $|V'_A| - \mathrm{odd}(V'_D)$, it follows that $V_D \subseteq V'_D$.

These statements are exactly the statements of the Edmonds-Gallai decomposition.

Principal Partition for Matroid Intersection Let $\mathbf{M}_1 = (V, \mathcal{I}_1, \rho_1)$ and $\mathbf{M}_2 = (V, \mathcal{I}_2, \rho_2)$ be an instance of the matroid intersection problem, and define

$$\begin{aligned}
V_D^1 &= \{v \mid v \in V, \exists\text{maximum common independent set } I,\ v \notin \mathrm{cl}_1(I)\},\\
V_D^2 &= \{v \mid v \in V, \exists\text{maximum common independent set } I,\ v \notin \mathrm{cl}_2(I)\}.
\end{aligned}$$

Recalling the instance of the independent even factor problem $(G, \mathbf{M}^+, \mathbf{M}^-)$ to which we reduced the matroid intersection instance in § 4, we have that

$$V_D^+ = V_A^+ = \bigcup_{v \in V_D^1} \{v_1, v_2\}, \quad V_D^- = V_A^- = \bigcup_{v \in V_D^2} \{v_1, v_2\}.$$

Applying Theorem 6.1 to $(G, \mathbf{M}^+, \mathbf{M}^-)$, we obtain the following statements.

- The cardinality of a maximum common independent set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is equal to

$$\begin{aligned}\nu(G, \mathbf{M}^+, \mathbf{M}^-) &= \rho_1(V \setminus V_D^1) + \rho_2(V_D^1) \\ &= \rho_1(V_D^2) + \rho_2(V \setminus V_D^2).\end{aligned}$$

- If I is a maximum common independent set, then it holds that $|(V \setminus V_D^1) \cap I| = \rho_1(V \setminus V_D^1)$, $|V_D^1 \cap I| = \rho_2(V_D^1)$, $|V_D^2 \cap I| = \rho_1(V_D^2)$, and $|(V \setminus V_D^2) \cap I| = \rho_2(V \setminus V_D^2)$.
- Any subset $X \subseteq V$ that minimizes $\rho_1(V \setminus X) + \rho_2(X)$ satisfies that $V_D^1 \subseteq X$ and $V_D^2 \subseteq V \setminus X$. That is, V_D^1 (resp. V_D^2) is the minimal (resp. maximal) subset minimizing the submodular function $\rho_1(V \setminus X) + \rho_2(X)$.

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