MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

Deductive Inference for the Interiors and Exteriors of Horn Theories

Kazuhisa MAKINO and Hirotaka ONO

METR 2007–06 February 2007

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/edu/course/mi/index_e.shtml

Deductive Inference for the Interiors and Exteriors of Horn Theories

Kazuhisa MAKINO† and Hirotaka ONO‡

† Department of Mathematical Informatics,
University of Tokyo, Tokyo, 113-8656, Japan.
makino@mist.i.u-tokyo.ac.jp

‡ Department of Computer Science and Communication Engineering,
Kyushu University, Fukuoka, 819-0395, Japan.
ono@csce.kyushu-u.ac.jp

February 2007

Abstract

In this paper, we investigate the deductive inference for interiors and exteriors of Horn knowledge bases, where the interiors and exteriors were introduced by Makino and Ibaraki [11] to study stability properties of knowledge bases. We present a linear time algorithm for the deduction for the interiors and show that it is co-NP-complete for the deduction for the exteriors. Under model-based representation, we show that both the deduction problems are intractable. As for Horn envelopes of the exteriors, we show that it is linearly solvable under model-based representation, while it is co-NP-complete under formula-based representation. We also discuss polynomially solvable cases for all the intractable problems.

1 Introduction

Knowledge-based systems are commonly used to store the sentences as our knowledge for the purpose of having automated reasoning such as deduction applied to them (see e.g., [1]). Deductive inference is a fundamental mode of reasoning, and usually abstracted as follows: Given the knowledge base KB, assumed to capture our knowledge about the domain in question, and a query χ that is assumed to capture the situation at hand, decide whether KB implies χ, denoted by $KB \models \chi$, which can be understood as the question: "Is χ consistent with the current state of knowledge ?"

In this paper, we consider the interiors and exteriors of knowledge base. Formally, for a given positive integer α, the α-interior of KB, denoted by

$\sigma_{-\alpha}(KB)$, is a knowledge that consists of the models (or assignments) v satisfying that the α-neighbors of v are all models of KB, and the α-exterior of KB, denoted by $\sigma_\alpha(KB)$, is a knowledge that consists of the models v satisfying that at least one of the α-neighbors of v is a model of KB [11]. Intuitively, the interior consists of the models v that *strongly* satisfy KB, since all neighbors of v are models of KB, while the exterior consists of the models v that *weakly* satisfy KB, since at least one of the α-neighbors of v is a model of KB. Here we note that v might not satisfy KB, even if we say that it weakly satisfies KB. As mentioned in [11], the interiors and exteriors of knowledge base merit study in their own right, since they shed light on the structure of knowledge base. Moreover, let us consider the situation in which knowledge base KB is *not perfect* in the sense that some sentences in KB are wrong and/or some are missing in KB. In this case, we may make use of the interiors and/or exteriors to be on safe side.

Main problems considered. In this paper, we study the deductive Inference for the interiors and exteriors of propositional Horn theories, where Horn theories are ubiquitous in Computer Science, cf. [13], and are of particular relevance in Artificial Intelligence and Databases. It is known that important reasoning problems like deductive inference and satisfiability checking, which are intractable for arbitrary propositional theories, are solvable in linear time for Horn theories (cf. [3]).

More precisely, we address the following problems:

- Given a Horn theory Σ, a clause c, and nonnegative integer α, we consider the problems of deciding if deductive queries hold for the α-interior and exterior of Σ, i.e., $\sigma_{-\alpha}(\Sigma) \models c$ and $\sigma_\alpha(\Sigma) \models c$. It is well-known [3] that a deductive query for a Horn theory can be answered in linear time. Note that it is intractable to construct the interior and exterior for a Horn theory [11, 12], and hence a direct method (i.e., first construct the interior (or exterior) and then check a deductive query) is not possible efficiently.

- We contrast traditional formula-based (syntactic) with model-based (semantic) representation of Horn theories. The latter form of representation has been proposed as an alternative form of representing and accessing a logical knowledge base, cf. [2, 4, 5, 7, 8, 6, 9, 10]. In model-based reasoning, Σ is represented by a subset of its models $\mathcal{M}$, which are commonly called *characteristic models*. As shown by Kautz *et al.* [7], the deductive inference can be done in polynomial time, given its characteristic models.

- Finally, we consider Horn approximations for the exteriors of Horn theories. Note that the interiors of Horn theories are Horn, while the exteriors might not be Horn. We deal with the least upper bounds, called the *Horn envelopes* [15], for the exteriors of Horn theories.

Main results. We investigate the problems mentioned above from an algorithmical viewpoint, and find answers to all of them. Our main results can be summarized as follows (see Figure 1).

- We present a linear time algorithm for the deduction for the interiors of a given Horn theory, and show that it is co-NP-complete for the deduction for the exteriors. Thus, the positive result for ordinary deduction for Horn theories extends to the interiors, but does not to the exteriors. We also show that the deduction for the exteriors is possible in polynomial time, if α is bounded by a constant or if $|N(c)|$ is bounded by a logarithm of the input size, where $N(c)$ corresponds to the set of negative literals in c.

- Under model-based representation, we show that the consistency problem for the interiors of Horn theories is co-NP-hard. This implies that the deduction for the interiors is NP-hard, where it is currently open if the problem belong to NP. As for the exteriors, we show that the deduction is co-NP-complete. We also show that the deduction for the interiors is possible in polynomial time if α is bounded by a constant, and so is for the exteriors, if α or $|P(c)|$ is bounded by a constant, or if $|N(c)|$ is bounded by a logarithm of the input size, where $P(c)$ corresponds to the set of positive literals in c.

- As for Horn envelopes of the exteriors of Horn theories, we show that it is linearly solvable under model-based representation, while it is co-NP-complete under formula-based representation. The former contracts to the negative result for the exteriors. We also present a polynomial algorithm for formula-based representation, if α is bounded by a constant or if $|N(c)|$ is bounded by a logarithm of the input size.

The rest of the paper is organized as follows. In the next section, we review the basic concepts and fix notations. Sections 3 and 4 investigate the deductive inference for the interiors and exteriors of Horn theories. Section 5 considers the deductive inference for the envelopes of the exteriors of Horn theories.

2 Preliminaries

Horn Theories

We assume a standard propositional language with atoms $At = \{x_1, x_2, \ldots, x_n\}$, where each x_i takes either value 1 (true) or 0 (false). A *literal* is either an atom x_i or its negation, which we denote by $\overline{x}_i$. The opposite of a literal ℓ is denoted by $\overline{\ell}$, and the opposite of a set of literals L by $\overline{L} = \{\overline{\ell} \mid \ell \in A\}$. Furthermore, $Lit = At \cup \overline{At}$ denotes the set of all literals.

	Interiors	Exteriors	Envelopes of Exteriors
Formula-Based Representation	P	co-NP-complete*	co-NP-complete*
Model-Based Representation	NP-hard†	co-NP-complete‡	P

$\star$: It is polynomially solvable, if $\alpha = O(1)$ or $|N(c)| = O(\log \|\Sigma\|)$.

$\dagger$: It is polynomially solvable, if $\alpha = O(1)$.

$\ddagger$: It is polynomially solvable, if $\alpha = O(1)$, $|P(c)| = O(1)$, or $|N(c)| = O(\log n|char(\Sigma)|)$.

Figure 1: Complexity of the deduction problems for the interiors and exteriors of Horn theories

A *clause* is a disjunction $c = \bigvee_{i\in P(c)} x_i \vee \bigvee_{i\in N(c)} \overline{x}_i$ of literals, where $P(c)$ and $N(c)$ are the sets of indices whose corresponding variables occur positively and negatively in c and $P(c) \cap N(c) = \emptyset$. Dually, a *term* is conjunction $t = \bigwedge_{i\in P(t)} x_i \wedge \bigwedge_{i\in N(t)} \overline{x}_i$ of literals, where $P(t)$ and $N(t)$ are similarly defined. We also view clauses and terms as sets of literals. A *conjunctive normal form* (*CNF*) is a conjunction of clauses. A clause c is *Horn*, if $|P(c)| \leq 1$. A *theory* Σ is any set of formulas; it is *Horn*, if it is a set of Horn clauses. As usual, we identify Σ with $\varphi = \bigwedge_{c\in\Sigma} c$, and write $c \in \varphi$ etc. It is known [3] that the deductive query for a Horn theory, i.e., deciding if $\Sigma \models c$ for a clause c is possible in linear time.

We recall that Horn theories have a well-known semantic characterization. A *model* is a vector $v \in \{0,1\}^n$, whose i-th component is denoted by v_i. For a model v, let $ON(v) = \{i \mid v_i = 1\}$ and $OFF(v) = \{i \mid v_i = 0\}$. The value of a formula φ on a model v, denoted $\varphi(v)$, is inductively defined as usual; satisfaction of φ in v, i.e., $\varphi(v) = 1$, will be denoted by $v \models \varphi$. The set of models of a formula φ (resp., theory Σ), denoted by $mod(\varphi)$ (resp., $mod(\Sigma)$), and logical consequence $\varphi \models \psi$ (resp., $\Sigma \models \psi$) are defined as usual. For two models v and w, we denote by $v \leq w$ the usual componentwise ordering, i.e., $v_i \leq w_i$ for all $i = 1, 2, \ldots, n$, where $0 \leq 1$; $v < w$ means $v \neq w$ and $v \leq w$. Denote by $v \bigwedge w$ componentwise AND of models $v, w \in \{0,1\}^n$, and by $Cl_\wedge(\mathcal{M})$ the closure of $\mathcal{M} \subseteq \{0,1\}^n$ under $\bigwedge$. Then, a theory Σ is Horn representable if and only if $mod(\Sigma) = Cl_\wedge(mod(\Sigma))$ (see [2, 9]) for proofs).

Example 1 Consider $\mathcal{M}_1 = \{(0101), (1001), (1000)\}$ and $\mathcal{M}_2 = \{(0101),$

(1001), (1000), (0001), (0000)}. Then, for $v = (0101)$, $w = (1000)$, we have $w, v \in \mathcal{M}_1$, while $v \bigwedge w = (0000) \notin \mathcal{M}_1$; hence $\mathcal{M}_1$ is not the set of models of a Horn theory. On the other hand, $Cl_\wedge(\mathcal{M}_2) = \mathcal{M}_2$, thus $\mathcal{M}_2 = mod(\Sigma)$ for some Horn theory Σ.

As discussed by Kautz *et al.* [7], a Horn theory Σ is semantically represented by its characteristic models, where $v \in mod(\Sigma)$ is called *characteristic* (or *extreme* [2]), if $v \notin Cl_\wedge(mod(\Sigma) \setminus \{v\})$. The set of all such models, the *characteristic set of* Σ, is denoted by $char(\Sigma)$. Note that $char(\Sigma)$ is unique. E.g., $(0101) \in char(\Sigma_2)$, while $(0000) \notin char(\Sigma_2)$; we have $char(\Sigma_2) = \mathcal{M}_1$.

It is known [7] that the deductive query for a Horn theory Σ from the characteristic set $char(\Sigma)$ is possible in linear time, i.e., $O(n|char(\Sigma)|)$ time.

Interior and Exterior of Theories

For a model $v \in \{0,1\}^n$ and a nonnegative integer α, its *α-neighborhood* is defined by

$$\mathcal{N}_\alpha(v) = \{w \in \{0,1\}^n \mid \parallel w - v \parallel \leq \alpha\},$$

where $\parallel v \parallel$ denotes $\sum_{i=1}^{n} |v_i|$. For a theory Σ and a nonnegative integer α, the α-interior and α-exterior of Σ, denoted by $\sigma_{-\alpha}(\Sigma)$ and $\sigma_\alpha(\Sigma)$ respectively, are theories defined by

$$mod(\sigma_{-\alpha}(\Sigma)) = \{v \in \{0,1\}^n \mid \mathcal{N}_\alpha(v) \subseteq mod(\Sigma)\} \quad (1)$$

$$mod(\sigma_{\alpha}(\Sigma)) = \{v \in \{0,1\}^n \mid \mathcal{N}_\alpha(v) \cap mod(\Sigma) \neq \emptyset\}. \quad (2)$$

By definition, $\sigma_0(\Sigma) = \sigma$, $\sigma_\alpha(\Sigma) \models \sigma_\beta(\Sigma)$ for integers α and β with $\alpha < \beta$, and $\sigma_\alpha(\Sigma_1) \models \sigma_\alpha(\Sigma_2)$ holds for any integer α, if two theories Σ_1 and Σ_2 satisfy $\Sigma_1 \models \Sigma_2$.

Example 2 Let us consider a Horn theory $\Sigma = \{\overline{x}_1 \vee x_3, \overline{x}_2 \vee x_3, \overline{x}_2 \vee x_4\}$ of 4 variables, where $mod(\Sigma)$ is given by

$$mod(\Sigma) = \{(1111), (1011), (1010), (0111), (0011), (0010), (0001), (0000)\}$$

(See Figure 2). Then we have $\sigma_\alpha(\Sigma) = \{\emptyset\}$ for $\alpha \leq -2$, $\{\overline{x}_1, \overline{x}_2, x_3, x_4\}$ for $\alpha = -1$, Σ for $\alpha = 0$, $\{\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee x_4\}$ for $\alpha = 1$, and $\emptyset$ for $\alpha \geq 2$. For example, (0011) is the unique model of $mod(\sigma_{-1}(\Sigma))$, since $\mathcal{N}_1(0011) \subseteq mod(\Sigma)$ and $\mathcal{N}_1(v) \not\subseteq mod(\Sigma)$ holds for all the other models v. For the 1-exterior, we can see that all models v with $(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee x_4)(v) = 1$ satisfy $\mathcal{N}_1(v) \cap mod(\Sigma) \neq \emptyset$, and no other such model exists. For example, (0101) is a model of $\sigma_1(\Sigma)$, since $(0111) \in \mathcal{N}_1(0101) \cap mod(\Sigma)$. On the other hand, (1100) is not a model of $\sigma_1(\Sigma)$, since $\mathcal{N}_1(1100) \cap mod(\Sigma) = \emptyset$. Notice that $\sigma_{-1}(\Sigma)$ is Horn, while $\sigma_1(\Sigma)$ is not.

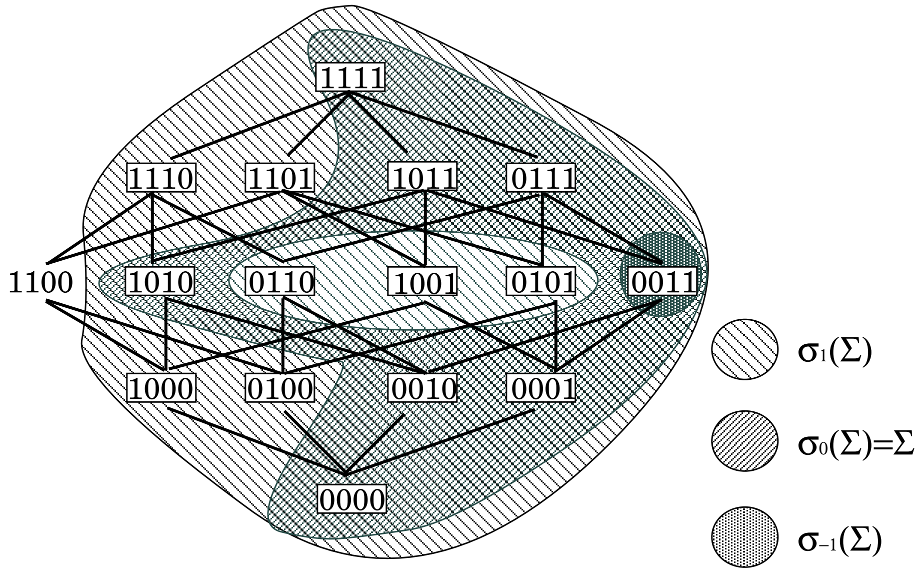

Figure 2: A Horn theory and its interiors and exteriors

Makino and Ibaraki [11] introduced the interiors and exteriors to analyze stability of Boolean functions, and studied their basic properties and complexity issues on them (see also [12]). For example, it is known [11] that, for a theory Σ and nonnegative integers α and β, $\sigma_{-\alpha}(\sigma_{-\beta}(\Sigma)) = \sigma_{-\alpha-\beta}(\Sigma)$, $\sigma_{\alpha}(\sigma_{\beta}(\Sigma)) = \sigma_{\alpha+\beta}(\Sigma)$, and

$$\sigma_{\alpha}(\sigma_{-\beta}(\Sigma)) \models \sigma_{\alpha-\beta}(\Sigma) \models \sigma_{-\beta}(\sigma_{\alpha}(\Sigma)). \tag{3}$$

For a nonnegative integer α and two theories Σ_1 and Σ_2, we have

$$\sigma_{-\alpha}(\Sigma_1 \cup \Sigma_2) = \sigma_{-\alpha}(\Sigma_1) \cup \sigma_{-\alpha}(\Sigma_2) \tag{4}$$

$$\sigma_{\alpha}(\Sigma_1 \cup \Sigma_2) \models \sigma_{\alpha}(\Sigma_1) \cup \sigma_{\alpha}(\Sigma_2), \tag{5}$$

where $\sigma_{\alpha}(\Sigma_1 \cup \Sigma_2) \neq \sigma_{\alpha}(\Sigma_1) \cup \sigma_{\alpha}(\Sigma_2)$ holds in general.

As demonstrated in Example 2, it is not difficult to see that the interiors of any Horn theory are Horn, which is, for example, proved by (4) and Lemma 1, while the exteriors might be not Horn.

3 Deductive Inference from Horn Theories

In this section, we investigate the deductive inference for the interiors and exteriors of a given Horn theory.

3.1 Interiors

Let us first consider the deduction for the α-interiors of a Horn theory: Given a Horn theory Σ, a clause c, and a positive integer α, decide if $\sigma_{-\alpha}(\Sigma) \models c$ holds. We show that the problem is solvable in linear time after showing a series of lemmas.

The following lemma is a basic properties of the interiors.

Lemma 1 *Let c be a clause. Then for a nonnegative integer α, we have $\sigma_{-\alpha}(c) = \bigvee_{\substack{S\subseteq c:\\ |S|=\alpha+1}} (\bigwedge_{\ell \in S} \ell) = \bigwedge_{\substack{S\subseteq c:\\ |S|=|c|-\alpha}} (\bigvee_{\ell \in S} \ell)$.*

This, together with (4), implies that for a CNF φ and a nonnegative integer α, we have

$$\sigma_{-\alpha}(\varphi) \;=\; \bigwedge_{c\in\varphi}\Big(\bigvee_{\substack{S\subseteq c:\\ |S|=\alpha+1}} (\bigwedge_{\ell\in S} \ell)\Big) \;=\; \bigwedge_{c\in\varphi}\Big(\bigwedge_{\substack{S\subseteq c:\\ |S|=|c|-\alpha}} (\bigvee_{\ell\in S} \ell)\Big),$$

where we regard c as a set of literals.

Lemma 2 *Let Σ be a Horn theory, and let c be a clause. For a nonnegative integer α, if there exists a clause $d \in \Sigma$ such that $|N(d) \setminus N(c)| \leq \alpha - 1$ or ($|N(d) \setminus N(c)| = \alpha$ and $P(d) \subseteq P(c)$), then we have $\sigma_{-\alpha}(\Sigma) \models c$.*

Proof. If Σ has a clause d that satisfies $|N(d) \setminus N(c)| \leq \alpha - 1$, then $|(N(d) \setminus N(c)) \cup P(d)| \leq \alpha$ holds. Thus by Lemma 1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i \in N(c)\cap N(d)} \overline{x}_i \models c$. Therefore, by (4), $\sigma_{-\alpha}(\Sigma) \models c$ holds.

On the other hand, if Σ has a clause d such that $|N(d) \setminus N(c)| = \alpha$ and $P(d) \subseteq P(c)$, then by Lemma 1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i\in P(c)} x_i \vee \bigvee_{i \in N(c)\cap N(d)} \overline{x}_i \models c$. Therefore, by (4), $\sigma_{-\alpha}(\Sigma) \models c$ holds. $\Box$

Lemma 3 *Let Σ be a Horn theory, and let c be a clause. For a nonnegative integer α, if* (i) *$|N(d) \setminus N(c)| \geq \alpha$ holds for all $d \in \Sigma$ and* (ii) *$\emptyset \neq P(d) \subseteq N(c)$ holds for all $d \in \Sigma$ with $|N(d) \setminus N(c)| = \alpha$, then we have $\sigma_{-\alpha}(\Sigma) \not\models c$.*

Proof. Let v be the unique minimal model that does not satisfy c, i.e., $v_i = 1$ if $\overline{x}_i \in c$ and 0, otherwise. We show that $v \models \sigma_{-\alpha}(\Sigma)$, which implies $\sigma_{-\alpha}(\Sigma) \not\models c$.

Let d be a clause in Σ with $|N(d) \setminus N(c)| \geq \alpha + 1$, and let t be a term obtained by conjuncting arbitrary $\alpha + 1$ literals in $N(d) \setminus N(c)$. Then we have $t(v) = 1$ and $t \models \sigma_{-\alpha}(d)$ by Lemma 1. On the other hand, for a clause d in Σ with $|N(d) \setminus N(c)| = \alpha$, let t be a term obtained by conjuncting all literals in $(N(d) \setminus N(c)) \cup P(d)$. Then we have $|t| = \alpha + 1$ and $t \models \sigma_{-\alpha}(d)$ by Lemma 1. Moreover, it holds that $t(v) = 1$ by $P(d) \subseteq N(c)$. Therefore, by (4), we have $v \models \sigma_{-\alpha}(\Sigma)$. $\Box$

By Lemmas 2 and 3, we can easily answer the deductive queries, if Σ satisfies certain conditions mentioned in them. In the remaining case, we have the following lemma.

Lemma 4 *For a Horn theory Σ that satisfies none of the conditions in Lemmas 2 and 3, let d be a clause in Σ such that $|N(d) \setminus N(c)| = \alpha$, and $P(d) = P(d) \setminus (P(c) \cup N(c)) = \{j\}$. Then $\sigma_{-\alpha}(\Sigma) \models c \vee x_j$ holds.*

Proof. By Lemma 1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i \in N(c) \cap N(d)} \overline{x}_i \vee x_j \models c \vee x_j$. This implies $\sigma_{-\alpha}(\Sigma) \models c \vee x_j$ by (4). □

From this lemma, we have only to check a deductive query $\sigma_{-\alpha}(\Sigma) \models c \vee \overline{x}_j$, instead of $\sigma_{-\alpha}(\Sigma) \models c$. Since $|c| < |c \vee \overline{x}_j| \leq n$, we can answer the deduction by checking the conditions in Lemmas 2 and 3 at most n times.

Algorithm DEDUCTION-INTERIOR-FROM-HORN-THEORY

Input: A Horn theory Σ, a clause c and a nonnegative integer α.

Output: Yes, if $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, No.

Step 0. Let $N := N(c)$ and $P := P(c)$.

Step 1. /* Check the condition in Lemma 2. */
If there exists a clause $d \in \Sigma$ such that $|N(d) \setminus N| \leq \alpha - 1$ or ($|N(d) \setminus N| = \alpha$ and $P(d) \subseteq P$, **then** output Yes and halt.

Step 2. /* Check the condition in Lemma 3. */
If $P(d) \subseteq N$ holds for all $d \in \Sigma$ with $|N(d) \setminus N| = \alpha$, **then** output No and halt.

Step 3. /* Update N by Lemma 4. */
For a clause d in Σ such that $|N(d) \setminus N| = \alpha$ and $P(d) = P(d) \setminus (P \cup N) = \{j\}$, update $N := N \cup \{j\}$ and return to Step 1. □

We can see that a straightforward implementation of the algorithm requires $O(n(\| \Sigma \| + |c|))$ time, where $\| \Sigma \|$ denotes the length of Σ, i.e., $\| \Sigma \| = \sum_{d \in \Sigma} |d|$. However, it is not difficult to see that we have a linear time algorithm for the problem, if $N(d) \setminus N$ for $d \in \Sigma$ is maintained by using the proper data structure.

Theorem 1 *Given a Horn theory Σ, a clause c and a nonnegative integer α, a deductive query $\sigma_{-\alpha}(\Sigma) \models c$ can be answered in linear time, i.e., $O(\| \Sigma \| + |c|)$ time.*

3.2 Exteriors

Let us next consider the deduction for the α-exteriors of a Horn theory. In contract to the interior case, we have the following negative result.

Theorem 2 *Given a Horn theory Σ, a clause c and a positive integer α, it is* co-NP*-complete to decide whether a deductive query $\sigma_\alpha(\Sigma) \models c$ holds, even if $P(c) = \emptyset$.*

Proof. By definition, $\sigma = \alpha(\Sigma) \not\models c$ if and only if there exists a model v of Σ such that some model in $\mathcal{N}_\alpha(v)$ does not satisfy c. The latter is equivalent to the condition that there exists a model v of Σ such that $|ON(v) \cap P(c)| + |OFF(v) \cap N(c)| \leq \alpha$, which can be checked in polynomial time. Thus the problem is in co-NP.

We then show the hardness by reducing a well-known NP-complete problem Independent Set to the complement of our problem. Independent Set is the problem of deciding if a given graph $G = (V, E)$ has an independent set $W \subseteq V$ such that $|W| \geq k$ for a given integer k. Here we call a subset $W \subseteq V$ is an *independent set* of G if $|W \cap e| \leq 1$ for all edges $e \in E$. For a problem instance $G = (V = \{1, 2, \ldots, n\}, E)$ and k of Independent Set, let us define a Horn theory Σ_G over $At = \{x_1, x_2, \ldots, x_n\}$ by

$$\Sigma_G = \{(\bar{x}_i \vee \bar{x}_j) \mid \{i, j\} \in E\}.$$

Let $c = \bigvee_{i=1}^n \overline{x}_i$ and $\alpha = n - k$. Note that $(11 \cdots 1)$ is the unique model that does not satisfy c. Thus $\sigma_\alpha(\Sigma) \not\models c$ if and only if $\sigma_\alpha(\Sigma)(11 \cdots 1) = 1$. Since W is an independent set of G if and only if Σ_G contains a model w defined by $ON(w) = W$, $\sigma_\alpha(\Sigma_G)(11 \cdots 1) = 1$ is equivalent to the condition that G has an independent set of size at least $k\,(= n - \alpha)$. This completes the proof. □

We remark that this result can also be derived from the ones in [11].

However, by using the next lemma, a deductive query can be answered in polynomial time, if α or $N(c)$ is small.

Lemma 5 *Let Σ_1 and Σ_2 be theories. For a nonnegative integer α, Then $\sigma_\alpha(\Sigma_1) \models \Sigma_2$ if and only if $\Sigma_1 \models \sigma_{-\alpha}(\Sigma_2)$*

Proof. For the if part, if $\Sigma_1 \models \sigma_{-\alpha}(\Sigma_2)$ holds, then, we have $\sigma_\alpha(\Sigma_1) \models \sigma_\alpha(\sigma_{-\alpha}(\Sigma_2)) \models \Sigma_2$ by (3). On the other hand, if $\sigma_\alpha(\Sigma_1) \models \Sigma_2$, then we have $\Sigma_1 \models \sigma_{-\alpha}(\sigma_\alpha(\Sigma_1)) \models \sigma_{-\alpha}(\Sigma_2)$ by (3). □

From Lemma 5, the deductive query for the α-interior of a theory Σ, i.e., $\sigma_\alpha(\Sigma) \models c$ for a given clause c is equivalent to the condition that $\Sigma \models \sigma_{-\alpha}(c)$. Since we have $\sigma_{-\alpha}(c) = \bigwedge_{\substack{S \subseteq c: \\ |S| = |c| - \alpha}} \left(\bigvee_{\ell \in S} \ell\right)$ by Lemma 1, the deductive query for the α-interior can be done by checking $\binom{|c|}{\alpha}$ deductions for Σ. More precisely, we have the following lemma.

Lemma 6 *Let Σ be a Horn theory, let c be a clause, and α be a nonnegative integer. Then $\sigma_\alpha(\Sigma) \models c$ holds if and only if, for each subset S of $N(c)$ such that $|S| \geq |N(c)| - \alpha$, at least $(\alpha - |N(c)| + |S| + 1)$ j's in $P(c)$ satisfy $\Sigma \models \bigvee_{i \in S} \overline{x}_i \vee x_j$.*

Proof. From Lemmas 1 and 5, we have $\sigma_\alpha(\Sigma) \models c$ if and only if $\Sigma \models \bigwedge_{\substack{S \subseteq c: \\ |S| = |c| - \alpha}} \left(\bigvee_{\ell \in S} \ell\right)$. It is known that for a Horn theory Σ and clause d, $\Sigma \models d$ if and only if $\Sigma \models \bigvee_{i \in N(d)} \overline{x}_i \vee x_j$ holds for some $j \in P(d)$ (i.e., All the prime implicates of Horn theory are Horn). This proves the lemma. □

This lemma implies that the deductive query can be answered by checking the number of j's in $P(c)$ that satisfy $\Sigma \models \bigvee_{i \in S} \overline{x}_i \vee x_j$ for each S. Since we can check this condition in linear time and there are $\sum_{p=0}^{\alpha} \binom{|N(c)|}{p}$ such S's, we have the following result, which complements Theorem 2 that the problem is intractable, even if $P(c) = \emptyset$.

Theorem 3 *Let Σ be a Horn theory, let c be a clause, and let α be a nonnegative integer. Then a deductive query $\sigma_\alpha(\Sigma) \models c$ can be answered in $O\left(\sum_{p=0}^{\alpha} \binom{|N(c)|}{p} \parallel \Sigma \parallel + |P(c)|\right)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$ or $|N(c)| = O(\log \parallel \Sigma \parallel)$.*

4 Deductive Inference from Characteristic Sets

In this section, we consider the case when Horn knowledge bases can be represented by characteristic sets. Different from formula-based representation, the deductions for interiors and exteriors are both intractable, unless P=NP.

4.1 Interiors

We first present an algorithm to solve the deduction problem for the interiors of Horn theories. This algorithm requires exponential time in general, but it is polynomial when α is small.

Let Σ be a Horn theory given by its characteristic set $char(\Sigma)$, and let c be a clause. Then for a nonnegative integer α, we have

$$\sigma_{-\alpha}(\Sigma) \models c \text{ if and only if } \sigma_{-\alpha}(\Sigma) \wedge \overline{c} \equiv 0. \tag{6}$$

Let v^* be a unique minimal model such that $c(v^*) = 0$ (i.e., $\overline{c}(v^*) = 1$). By the definition of interiors, v^* is a model of $\sigma_{-\alpha}(\Sigma)$ if and only if all v's in $\mathcal{N}_\alpha(v^*)$ are models of Σ. Therefore, for each model v in $\mathcal{N}_\alpha(v^*)$, we check if $v \in mod(\Sigma)$, which is equivalent to

$$\bigwedge_{\substack{w \in char(\Sigma) \\ w \geq v}} w \;=\; v. \tag{7}$$

If (7) holds for all models v in $\mathcal{N}_\alpha(v^*)$, then we can immediately conclude by (6) that $\sigma_{-\alpha}(\Sigma) \not\models c$. On the other hand, if there exists a model v in $\mathcal{N}_\alpha(v^*)$ such that (7) does not hold, let $J = ON(\bigwedge_{\substack{w \in char(\Sigma) \\ w \geq v}} w) \setminus ON(v)$. By definition, we have $J \neq \emptyset$, and we can see that $\sigma_{-\alpha}(\Sigma) \models \bigvee_{i \in N(c)} \overline{x}_i \vee x_j$ for $j \in J$. Thus, if J contains an index in $P(c)$, then we can conclude that $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, we check the condition $\sigma_{-\alpha}(\Sigma) \models c \vee \bigvee_{j \in J} \overline{x}_j$, instead of $\sigma_{-\alpha}(\Sigma) \models c$. Since a new clause $d = c \vee \bigvee_{j \in J} \overline{x}_j$ is longer than c, after at most n iterations, we can answer the deductive query. Formally, our algorithm can be described as follows.

Algorithm DEDUCTION-INTERIOR-FROM-CHARSET

Input: The characteristic set $char(\Sigma)$ of a Horn theory Σ, a clause c and a nonnegative integer α.

Output: Yes, if $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, No.

Step 0. Let $N := N(c)$ and let $d := c$.

Step 1. Let v^* be the unique minimal model such that $d(v^*) = 0$.

Step 2. **For** each v in $\mathcal{N}_\alpha(v^*)$ **do**

If (7) does not hold,

then let $J = ON(\bigwedge_{\substack{w \in char(\Sigma) \\ w \geq v}} w) \setminus ON(v)$.

If $J \cap P(c) \neq \emptyset$, **then** output yes and halt.

Let $N := N \cup J$ and $d := \bigvee_{i \in N(d)} \overline{x}_i \vee \bigvee_{i \in P(d)} x_i$.

Go to Step 1.

end{for}

Step 3. Output No and halt. □

Theorem 4 *Given the characteristic model $char(\Sigma)$ of a Horn theory Σ, a clause c and a nonnegative integer α, a deductive query $\sigma_{-\alpha}(\Sigma) \models c$ can be answered in $O(n^{\alpha+2}|char(\Sigma)|)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$.*

Proof. Since we can see algorithm DEDUCTION-INTERIOR-FROM-CHARSET correctly answers a deductive query from the discussion before the description, we only estimate the running time of the algorithm.

Clearly, Steps 0, 1 and 3 require $O(n)$ time. Step 2 requires $O(n^{\alpha+1} \cdot |char(\Sigma)|)$ time, since (7) can be checked in $O(n|char(\Sigma)|)$ time. Since we have at most n iterations between Steps 1 and 2, the algorithm requires $O(n^{\alpha+2}|char(\Sigma)|)$ time. □

However, in general, the problem is intractable, which contracts to the formula-model representation.

Theorem 5 *Given the characteristic set* $char(\Sigma)$ *of a Horn theory* Σ *and a positive integer* α*, it is* co-NP*-hard to decide whether* $\sigma_{-\alpha}(\Sigma)$ *is consistent, i.e.,* $mod(\sigma_{-\alpha}(\Sigma)) \neq \emptyset$*.*

Proof. We show the co-NP-hardness by reducing INDEPENDENT SET to our problem. Given a problem instance $G = (V = \{1, 2, \ldots, n\}, E)$ and k of INDEPENDENT SET, let us define a Horn theory Σ_G over $At = \{x_1, x_2, \ldots, x_n\}$ by

$$char(\Sigma_G) = \{v^{(i,j)}, v^{(i,j,l)} \mid \{i, j\} \in E, l \in V \setminus \{i, j\}\},$$

where $v^{(i,j)}$ and $v^{(i,j,l)}$ are respectively the vectors defined by $OFF(v^{(i,j)}) = \{i, j\}$ and $OFF(v^{(i,j,l)}) = \{i, j, l\}$. Let $\alpha = n - k$. Note that Σ_G is a negative theory, and hence $\sigma_{-\alpha}(\Sigma_G)$ is consistent if and only if $(00 \cdots 0)$ is a model of $\sigma_{-\alpha}(\Sigma_G)$. Moreover, the latter condition is equivalent to the one that G has no independent set of size at least k $(= n - \alpha)$. This completes the proof. □

It is easy to see that the consistency problem for the interiors of Horn theories is in Σ_2^P, but it is open whether the problem is Σ_2^P-complete.

This result immediately implies the following corollary.

Corollary 1 *Given the characteristic set* $char(\Sigma)$ *of a Horn theory* Σ*, a clause* c *and a positive integer* α*, it is* NP*-hard to decide whether a deductive query* $\sigma_{-\alpha}(\Sigma) \models c$ *holds, even if* $c = \emptyset$*.*

Different from the other hardness results, the hardness is not sensitive to the size of c.

4.2 Exteriors

Let us consider the exteriors. Similarly to the formula-based representation, we have the following negative result.

Theorem 6 *Given the characteristic set* $char(\Sigma)$ *of a Horn theory* Σ*, a clause* c *and a positive integer* α*, it is* co-NP*-complete to decide if a deductive query* $\sigma_{\alpha}(\Sigma) \models c$ *holds.*

Proof.

From Lemmas 1 and 5, $\sigma_{\alpha}(\Sigma) \not\models c$ if and only if there exists a subclause d of c such that $|d| = |c| - \alpha$ and $\Sigma \not\models d$. This d is a witness that the problem belongs to co-NP.

We then show the hardness by a reduction from VERTEX COVER which is known to be NP-hard. VERTEX COVER is the problem to decide if a given graph $G = (V, E)$ has a vertex cover U such that $|U| \leq k$ for a given integer k $(< n)$. Here $U \subseteq V$ is called *vertex cover* if $U \cap e \neq \emptyset$ holds for all $e \in E$. For this problem instance, we construct our problem instance. For each $e \in E$, let $W_e = \{e_1, e_2, \ldots, e_{|V|}\}$, and let $W = \bigcup_{e \in E} W_e$. Let $m^{(v)}$, $v \in V$, be a model over $V \cup W$ such that

$$ON(m^{(v)}) = (V \setminus \{v\}) \cup \bigcup_{v \notin e} W_e,$$

and let $char(\Sigma)$ be the characteristic set for some Horn theory Σ defined by $char(\Sigma) = \{m^{(v)} \mid v \in V\}$. We define c and α by

$$c = \bigvee_{i \in V} \overline{x}_i \vee \bigvee_{i \in W} x_i \quad \text{and} \quad \alpha = k,$$

respectively. For this instance, we show that $\sigma_\alpha(\Sigma) \not\models c$ if and only if the corresponding G has a vertex cover U of size at most $k\,(= \alpha)$.

For the if part, let U be such a vertex cover of G. For this U, we consider model $m^{(U)} \stackrel{\text{def}}{=} \bigwedge_{v \in U} m^{(v)}$, which is a model of Σ by the intersection property of a Horn theory. Note that $m^{(U)}$ does not satisfy a clause $d = \bigvee_{i \in V \setminus U} \overline{x}_i \vee \bigvee_{i \in W} x_i$. Since d is a subclause of c of length at least $|c| - \alpha$, $m^{(U)}$ is not a model of $\sigma_{-\alpha}(c)$ by Lemma 1. This completes the if part by Lemma 5.

For the only-if part, let us assume that $\sigma_\alpha(\Sigma) \not\models c$. Then by Lemmas 1 and 5, there exists a subclause d of c such that $|d| = |c| - \alpha$ and $\Sigma \not\models d$. This implies that $\Sigma \wedge \overline{d}$ contains a model m. By $\alpha < n$, for each $e \in E$, there exist an index j in W_e such that $m_j = 0$. Since any model m' in Σ satisfy either $m'_i = 0$ or $m'_i = 1$ for all $i \in W_e$, we have $m_i = 0$ for all $i \in W$. This means that $V \setminus ON(m)$ is a vertex cover of G, and since $|V \setminus ON(m)| \leq k$, we have the only-if part. □

By using Lemma 6, we can see that the problem can be solved in polynomial time, if α or $|N(c)|$ is small. Namely, for each subset S of $N(c)$ such that $|S| \geq |N(c)| - \alpha$, let v^S denotes the model such that $ON(v^S) = S$. Then $w^S = \bigwedge_{\substack{w \in char(\Sigma): \\ w \geq v^S}} w$ is the unique minimal model of Σ such that $ON(w^S) \supseteq S$, and hence it follows from Lemma 6 that it is enough to check if $|ON(w^s) \cap P(c)| \geq \alpha - |N(c)| + |S| + 1$. Clearly, this can be done in in $O\left(\sum_{p=0}^{\alpha} \binom{|N(c)|}{p} n |char(\Sigma)|\right)$ time.

Moreover, if $|P(c)|$ is small, then the problem also become tractable, which contrasts with Theorem 2.

Lemma 7 *Let Σ be a Horn theory, let c be a clause, and α be a nonnegative integer. Then $\sigma_\alpha(\Sigma) \models c$ holds if and only if each $S \subseteq P(c)$ such that $|S| \geq |P(c)| - \alpha$ satisfies*

$$|OFF(w) \cap N(c)| \geq \alpha - |P(c)| + |S| + 1 \tag{8}$$

for all models w of Σ such that $OFF(w) \cap P(c) = S$.

Note that (8) is monotone in the sense that, if a model w satisfies (8), then all models v with $v < w$ also satisfy it. Thus it is sufficient to check if (8) holds for all *maximal* models w of Σ such that $OFF(w) \cap P(c) = S$. Since such maximal models w can be obtained from $w^{(i)}$ ($i \in S$) with $i \in OFF(w^{(i)}) \cap P(c) \subseteq S$ by their intersection $w = \bigwedge_{i \in S} w^{(i)}$, we can answer the deduction problem in $O\left(n \sum_{p=|P(c)|-\alpha}^{|P(c)|} \binom{|P(c)|}{p} |char(\Sigma)|^p\right)$ time.

Theorem 7 *Given the characteristic set $char(\Sigma)$ of a Horn theory, a clause c, and a nonnegative integer α, a deductive query $\sigma_\alpha(\Sigma) \models c$ can be answered in $O\Big(n \min\{\sum_{p=0}^{\alpha} \binom{|N(c)|}{p} |char(\Sigma)|, \sum_{p=|P(c)|-\alpha}^{|P(c)|} \binom{|P(c)|}{p} | \cdot char(\Sigma)|^p\}\Big)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$, $|P(c)| = O(1)$, or $|N(c)| = O(\log n |char(\Sigma)|)$.*

5 Deductive Inference for Envelopes of the Exteriors of Horn Theories

We have considered the deduction for the interiors and exteriors of Horn theories. As mentioned before, the exteriors of Horn theories are also Horn, while this does not hold for the interiors. This means that the interiors of Horn theories might lose beneficial properties of Horn theories. One of the ways to overcome such a hurdle is *Horn Approximation*, that is, approximating a theory by a Horn theory [15]. There are several methods to do that, but one of the most natural ones is to approximate a theory by its *Horn envelope*. Namely, for a theory Σ, its *Horn envelope* is the Horn theory Σ_e such that $mod(\Sigma_e) = Cl_\wedge(mod(\Sigma))$. Since Horn theories are closed under intersection, Horn envelope is the least Horn upper bound for Σ, i.e., $char(\Sigma_e) \supseteq char(\Sigma)$ and and there exists no Horn theory Σ^* such that $char(\Sigma_e) \supsetneq char(\Sigma^*) \supseteq char(\Sigma)$. In this section, we consider the deduction for Horn envelopes of interiors of Horn theories, i.e., $\sigma_\alpha(\Sigma)_e \models c$.

5.1 Model-Based Representations

Let us first consider the case in which knowledge bases are represented by characteristic sets.

Proposition 1 Let Σ be a Horn theory, and let α be a nonnegative integer. Then we have

$$mod(\sigma_\alpha(\Sigma)_e) \quad = \quad Cl_\wedge(\bigcup_{v \in char(\Sigma)} \mathcal{N}_\alpha(v)) \tag{9}$$

Proof. By definition,

$$mod(\sigma_\alpha(\Sigma)_e) = Cl_\wedge(mod(\sigma_\alpha(\Sigma))) \supseteq Cl_\wedge(\bigcup_{v \in char(\Sigma)} \mathcal{N}_\alpha(v))$$

holds. We then show the converse direction. Let v^* be a model of Horn envelope of the α-exterior, i.e., $v^* \in mod(\sigma_\alpha(\Sigma)_e)$. Then v^* can be represented by $v^* = \bigwedge_{w \in W} w$ for some $W \subseteq mod(\sigma_\alpha(\Sigma))$. Assume that $w \in W$ is contained in $\mathcal{N}_\alpha(u)$ for some model u of Σ. Since such a u can be represented by $u = \bigwedge_{z \in S_w} z$ for some $S_w \subseteq char(\Sigma)$, w belongs to $Cl_\wedge(\bigcup_{v \in S_w} \mathcal{N}_\alpha(v))$. This, together with $v^* = \bigwedge_{w \in W} w$, implies that v^* also belongs to $Cl_\wedge(\bigcup_{v \in char(\Sigma)} \mathcal{N}_\alpha(v))$. □

For a clause c, let v^* be the unique minimal model such that $c(v^*) = 0$. We recall that, for a Horn theory Φ,

$$\Phi \models c \text{ if and only if } c(\bigwedge_{\substack{v \in char(\Phi) \\ v \geq v^*}} v) = 1. \tag{10}$$

Therefore, Proposition 1 immediately implies an algorithm for the deduction for $\sigma_\alpha(\Sigma)_e$ from $char(\Sigma)$, since $char(\sigma_\alpha(\Sigma)_e) \subseteq \bigcup_{v \in char(\Sigma)} \mathcal{N}_\alpha(v)$ holds. However, for a general α, $\bigcup_{v \in char(\Sigma)} \mathcal{N}_\alpha(v)$ is exponentially larger than $char(\Sigma)$, and hence this direct method is not efficient. The following lemma helps developing a polynomial time algorithm.

Lemma 8 *Let Σ be a Horn theory, let c be a clause, and let α be a non-negative integer. Then $\sigma_\alpha(\Sigma)_e \models c$ holds if and only if the following two conditions are satisfied.*

(i) *$|OFF(v) \cap N(c)| \geq \alpha$ holds for all $v \in char(\Sigma)$.*

(ii) *If $S = \{v \in char(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\} \neq \emptyset$, $P(c)$ is not covered with $OFF(v)$ for models v in S, i.e., $P(c) \not\subseteq \bigcup_{\substack{v \in char(\Sigma) \\ |OFF(v) \cap N(c)| = \alpha}} OFF(v)$.*

Proof. To show the if part, let us first assume that (i) and (ii) in the lemma holds. Let v be a model in $char(\Sigma)$ such that $|OFF(v) \cap N(c)| > \alpha$. Then all models w in $\mathcal{N}_\alpha(v)$ satisfy $OFF(w) \cap N(c) \neq \emptyset$. Therefore, if all the models v in $char(\Sigma)$ satisfy $|OFF(v) \cap N(c)| > \alpha$, then by Proposition 1, we have $OFF(w) \cap N(c) \neq \emptyset$ for any model w of $\sigma_\alpha(\Sigma)_e$. This implies $\sigma_\alpha(\Sigma)_e \models c$. Therefore, let us consider the case when $S = \{v \in char(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\}$ is not empty. Let v^* be the unique minimal model such that $c(v^*) = 0$. Then by Proposition 1, we have

$$\begin{aligned} &\{v \in char(\sigma_\alpha(\Sigma)_e) \mid v \geq v^*\} \\ &\qquad \subseteq \{w \mid ON(w) = ON(v) \cup N(c) \text{ for some } v \in S\}. \end{aligned} \tag{11}$$

Since $P(c)$ is not covered with $OFF(v)$ for models v in S, this, together with (10) implies $\sigma_\alpha(\Sigma)_e \models c$.

Let us next show the only-if part. Assume that (i) is satisfied, but (2) is not. Then (10) and (11) imply $\sigma_\alpha(\Sigma)_e \not\models c$. On the other hand, if (1) is not satisfied, i.e., there exists a $v \in char(\Sigma)$ such that $|OFF(v) \cap N(c)| < \alpha$, let $w^{(i)}$, $i \in P(c)$, be a model in $\mathcal{N}_\alpha(v)$ such that $ON(w^{(i)}) \supseteq N(c)$ and $OFF(w^{(i)}) \supseteq \{i\}$, and let $w^* = \bigwedge_{i \in P(c)} w^{(i)}$. Then we have $c(w^*) = 0$ and $w^* \in mod(\sigma_\alpha(\Sigma)_e)$ by Proposition 1. This implies $\sigma_\alpha(\Sigma)_e \not\models c$. □

The lemma immediately implies the following theorem.

Theorem 8 *Given the characteristic set $char(\Sigma)$ of a Horn theory Σ, a clause c, and a nonnegative integer α, a deductive query $\sigma_\alpha(\Sigma)_e \models c$ can be answered in linear time.*

We remark that this contrasts with Corollary 1. Namely, if we are given the characteristic set $char(\Sigma)$ of a Horn theory Σ, then $\sigma_\alpha(\Sigma)_e \models c$ is polynomially solvable, while it is co-NP-complete to decide if $\sigma_\alpha(\Sigma) \models c$.

5.2 Formula-Based Representation

Recall that a *negative* theory (i.e., a theory consisting of clauses with no positive literal) is Horn and the exteriors of negative theory are also negative, and hence Horn. This means that, for a negative theory Σ, we have $\sigma_\alpha(\Sigma)_e = \sigma_\alpha(\Sigma)$. Therefore, we can again make use of the reduction in the proof of Theorem 2, since the reduction uses negative theories.

Theorem 9 *Given the characteristic set $char(\Sigma)$ of a Horn theory Σ, a clause c, and a nonnegative integer α, it is* co-NP*-complete to decide whether $\sigma_\alpha(\Sigma)_e \models c$ holds, even if $P(c) = \emptyset$.*

Proof. Since the hardness is proved similarly to Theorem 2, we show that the problem belongs to co-NP.

Let v be a model of $\sigma_\alpha(\Sigma)_e$. Then v can be represented by $v = \bigwedge_{w \in W} w$ for some $W \subseteq char(\sigma_\alpha(\Sigma))$. Since we have $char(\sigma_\alpha(\Sigma)) \subseteq \bigcup_{w \in char(\Sigma)} \mathcal{N}_\alpha(w)$ by Proposition 1,

$$v = \bigwedge_{w \in char(\Sigma)} \Big(\bigwedge_{u \in S_w} u \Big) \tag{12}$$

holds for some $S_w \subseteq \mathcal{N}_\alpha(w)$. We claim that there exists such a representation that $|S_w| \leq n$ holds for all w's in (12). Let $w^* = \bigwedge_{u \in S_w} u$, and let $I = ON(w^*) \cap OFF(w)$ and $J = OFF(w^*) \cap ON(w)$. Then we have $w^* = \bigwedge_{j \in J} \Big(w - e^{(j)} + \sum_{i \in I} e^{(i)} \Big)$, where $e^{(i)}$ denotes the ith unit model. Since $w - e^{(j)} + \sum_{i \in I} e^{(i)} \in \mathcal{N}_\alpha(w)$ for all $j \in J$, the claim is proved.

Note that $\sigma_\alpha(\Sigma)_e \not\models c$ if and only if there exists a model v of $\sigma_\alpha(\Sigma)_e$ such that $c(v) = 0$. Since any model v of $\sigma_\alpha(\Sigma)_e$ can be represented by $v = \bigwedge_{w \in char(\Sigma)} \Big(\bigwedge_{u \in S_w} u \Big)$ for some $S_w \subseteq \mathcal{N}_\alpha(w)$ with $|S_w| \leq n$ by our claim, the problem is in co-NP. □

However, if α or $N(c)$ is small, the problem becomes tractable.

Algorithm DEDUCTION-ENVELOPE-EXTERIOR-FROM-HORN-THEORY

Input: A Horn theory Σ, a clause c and a nonnegative integer α.

Output: Yes, if $\sigma_\alpha(\Sigma)_e \models c$; Otherwise, No.

Step 1. /* Check if there exists a model v of Σ such that $|OFF(v) \cap N(c)| < \alpha$. */

For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha + 1$ **do**

Check if the theory obtained from Σ by assigning $x_i = 1$ for $i \in N$ is satisfiable.

If so, **then** output No and halt.

end{for}

Step 2. /* Check if there exists a set $S = \{v \in mod(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\}$ such that $\bigcup_{v \in S} OFF(v) \supseteq P(c)$. */

Let $J := \emptyset$.

For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha$ **do**

Compute a unique minimal satisfiable model v for the theory obtained from Σ by assigning $x_i = 1$ for $i \in N$ is satisfiable.

Update $J := J \cup \{j \in P(c) \mid v_j = 0\}$.

end{for}

If $J = P(c)$, **then** output NO and halt.

Step 3. Output Yes and halt. □

The algorithm above is based on a necessary and sufficient condition for $\sigma_\alpha(\Sigma)_e \models c$, which is obtained from Lemma 8 by replacing all $char(\Sigma)$'s with $mod(\Sigma)$'s. It is not difficult to see that such a condition holds from the proof of Lemma 8.

Theorem 10 *Given a Horn theory Σ, a clause c, and a nonnegative integer α, a deductive query $\sigma_\alpha(\Sigma)_e \models c$ can be answered in $O\Big(\big(\binom{|N(c)|}{\alpha-1} + \binom{|N(c)|}{\alpha}\big) \| \Sigma \| + |P(c)|\Big)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$ or $|N(c)| = O(\log \| \Sigma \|)$.*

Proof. The correctness of the algorithm follows from the discussion after its description. For the time complexity, it is known [3] that the satisfiability problem, together with computing a unique minimal model for a Horn theory, is possible in linear time. Since the number of the iterations of for-loops in Steps 2 and 3 are bounded by $\binom{|N(c)|}{\alpha-1}$ and $\binom{|N(c)|}{\alpha}$, respectively, the algorithm requires $O\Big(\big(\binom{|N(c)|}{\alpha-1} + \binom{|N(c)|}{\alpha}\big) \| \Sigma \| + |P(c)|\Big)$ time. □

References

[1] R. J. Brachman and H. J. Levesque. *Knowledge Representation and Reasoning*. Elsevier, 2004.

[2] R. Dechter and J. Pearl. Structure identification in relational data. *Artificial Intelligence*, 58 (1992) 237–270.

[3] W. Dowling and J. H. Galliear. Linear-time algorithms for testing the satisfiability of propositional Horn theories. *Journal of Logic Programming* 3 (1983) 267–284.

[4] T. Eiter, T. Ibaraki, and K. Makino. Computing intersections of Horn theories for reasoning with models. *Artificial Intelligence*, 110 (1999) 57-101.

[5] T. Eiter and K. Makino. On computing all abductive explanations. In *Proceedings AAAI-2002* (2002) 62-67.

[6] D. Kavvadias, C. Papadimitriou, and M. Sideri. On Horn Envelopes and Hypergraph Transversals. In W. Ng, editor, *Proceedings 4th International Symposium on Algorithms and Computation (ISAAC-93)*, LNCS 762, pages 399–405, Hong Kong, December 1993. Springer.

[7] H. Kautz, M. Kearns, and B. Selman. Reasoning with characteristic models. In *Proceedings AAAI-93* (1993) 34–39.

[8] H. Kautz, M. Kearns, and B. Selman. Horn approximations of empirical data. *Artificial Intelligence*, 74 (1995) 129–245.

[9] R. Khardon and D. Roth. Reasoning with models. *Artificial Intelligence*, 87 (1996) 187–213.

[10] R. Khardon and D. Roth. Defaults and relevance in model-based reasoning. *Artificial Intelligence*, 97 (1997) 169–193.

[11] K. Makino and T. Ibaraki, Interior and exterior functions of Boolean functions, *Discrete Applied Mathematics,* 69 (1996) 209–231.

[12] K. Makino, H. Ono and T. Ibaraki, Interior and exterior functions of positive Boolean functions, *Discrete Applied Mathematics,* 130 (2003) 417–436.

[13] J. Makowsky. Why Horn formulas matter for computer science: Initial structures and generic examples. *Journal of Computer and System Sciences*, 34 (1987) 266–292.

[14] J. McKinsey. The decision problem for some classes of sentences without quantifiers. *Journal of Symbolic Logic*, 8 (1943) 61–76.

[15] B. Selman and H. Kautz, Knowledge compilation and theory approximation, *Journal of ACM*, 43-2 (1996) 193–224.