MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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(Communicated by Satoru IWATA)

METR 2007–11 February 2007

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Abstract

A tree metric is known to be representable as the sum of split metrics. This classical result can be derived geometrically via the *polyhedral split decomposition* of a tree metric. The present paper investigates the polyhedral split decomposition from the viewpoint of *discrete convex analysis*.

In discrete convex analysis, a *distance* such as a *tree metric* and *split metric* on a finite set is regarded as a *positively homogeneous M-convex function*. With the aid of the polyhedral split decomposition, the positively homogeneous M-convex function for a tree metric is decomposed into the sum of split functions. This paper shows that those split functions are positively homogeneous M-convex functions for split metrics. Then, we notice that the decomposition is in contrast with the general case of M-convex functions because the sum of M-convex functions is not necessarily an M-convex function. We study the decomposition by means of *polyhedral subdivisions* induced by M-convex functions. Those polyhedral subdivisions are known as *matroid subdivisions*.

This paper also deals with *quadratic M-convex functions*. The directional derivatives of an M-convex function can be defined at each point in the domain, and moreover, each of them is a positively homogeneous M-convex function as a function of directions. A discrete function on a finite set is called *split-decomposable* if its convex extension is represented as the sum of split functions and a linear function. For example, a tree metric is split-decomposable. This paper shows that each of the directional derivatives of a quadratic M-convex function is represented as the sum of a tree metric and a linear function. Thus, a quadratic M-convex function is a split-decomposable function at every point.

Keywords: positively homogeneous M-convex function, quadratic M-convex function, distance, tree metric, polyhedral split decomposition, matroid subdivision, split-decomposable.

1 Introduction

A tree metric is known to be representable as the sum of split metrics. This classical result can be derived geometrically via the *polyhedral split decomposition* of a tree metric [10]. The present paper investigates the polyhedral split decomposition from the viewpoint of *discrete convex analysis* [12].

In order to review the previous results, we begin by classifying a *distance*, *metric*, *tree metric* and *split metric* on a finite set X. A distance is defined as a function $d : X \times X \to \mathbb{R}$ such that $d(i,i) = 0$ for all $i \in X$ and $d(i,j) \leq d(i,k) + d(k,j)$ for all $i,j,k \in X$. A metric d is a symmetric and nonnegative distance, that is, a distance d such that $d(i,j) = d(j,i) \geq 0$ for all $i,j \in X$. A metric d is called a tree metric if there exits a tree with nonnegative edge lengths such that $d(i,j)$ is equal to the length of the path in the tree between the vertices indexed by i and j for all $i,j \in X$. An X-*splits* is a partition of X into two non-empty sets, i.e., a pair $\{A,B\}$ of A and B such that $\emptyset \neq A \subseteq X$,

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$\emptyset \neq B \subseteq X$, $A \cap B = \emptyset$ and $A \cup B = X$. The split metric $\xi_{\{A,B\}} : X \times X \to \{0,1\}$ associated with an X-split $\{A, B\}$ is defined by

$$\xi_{\{A,B\}}(i,j) = \begin{cases} 0 & \text{if } \ i,j \in A \text{ or } i,j \in B, \\ 1 & \text{otherwise,} \end{cases}$$

for all $i, j \in X$. Let X be the set of the vertices in a tree. Then, deleting an edge from the tree induces an X-split. As a consequence of this fact, we obtain the classical result that a tree metric can be represented as the sum of the split metrics corresponding to the edges of its designating tree. This classical result can be derived geometrically as a special case of the polyhedral split decomposition.

The polyhedral split decomposition of polyhedral convex functions is introduced by Hirai [5, 6] in order to extend the results of Bandelt and Dress [1]. A polyhedral convex function is a function whose epigraph is a convex polyhedron. By polyhedral split decomposition, a polyhedral convex function f on $\mathbb{R}^n$ can be decomposed as

$$f(x) = \sum_{\substack{(a,r) \in \mathbb{R}^n \times \mathbb{R}, \\ 0 < c^f_{a,r} < +\infty}} c^f_{a,r} |\langle a, x\rangle - r| + f'(x) \qquad (x \in \mathbb{R}^n),$$

where $c^f_{a,r}$ is given by $c^f_{a,r} = \sup\{t \geq 0 \mid f(x) - t|\langle a, x\rangle - r| \text{ is convex in } x\}$ and f', called the residue of f, is a polyhedral convex function such that $c^{f'}_{a,r} \in \{0, +\infty\}$ for any $(a, r) \in \mathbb{R}^n \times \mathbb{R}$. A function $l_{a,r}(x) := |\langle a, x\rangle - r|$ of x is called a *split function*.

The polyhedral split decomposition is extended to discrete functions with the aid of their homogeneous convex extensions. A *discrete function* is a function $g : K \to \mathbb{R}$ defined on a finite set K of points/vectors in $\mathbb{R}^n$. The *homogeneous convex extension* of g is defined by

$$\overline{g}(x) = \sup\left\{\langle p, x\rangle \mid p \in \mathbb{R}^n, \langle p, y\rangle \leq g(y) \ (y \in K)\right\} \qquad (x \in \mathbb{R}^n).$$

By definition, $\overline{g}$ is a positively homogeneous polyhedral convex function with $\operatorname{dom} g = \operatorname{cone} K$. Then, from the polyhedral split decomposition of $\overline{g}$, we obtain the *discrete split decomposition* of g as

$$g(x) = \sum_{\substack{(a,r) \in \mathbb{R}^n \times \mathbb{R}, \\ 0 < c^{\overline{g}}_{a,r} < +\infty}} c^{\overline{g}}_{a,r} l^K_{a,r}(x) + g'(x) \qquad (x \in K),$$

where $l^K_{a,r}$ denotes the restriction of a split function $l_{a,r}$ on K and $c^{\overline{g'}}_{a,r} \in \{0, +\infty\}$ for any $(a, r) \in \mathbb{R}^n \times \mathbb{R}$.

The polyhedral split decomposition is applied to distances as follows. Let $X = \{1, 2, \ldots, n\}$. For $A \subseteq X$, we denote by χ_A the characteristic vector of A defined by $\chi_A(i) = 1$ if $i \in A$ and $\chi_A(i) = 0$ if $i \notin A$. In particular, we write χ_i instead of $\chi_{\{i\}}$ for each $i \in X$. A distance $d : X \times X \to \mathbb{R}$ can be considered as a discrete function d on the point set $\Omega = \{\chi_i - \chi_j \mid i, j \in X\}$ by the correspondence: $d(\chi_i - \chi_j) \leftarrow d(i, j)$. The convex extension of d on Ω is as follows:

$$\overline{d}(x) = \sup\left\{\langle p, x\rangle \mid p \in \mathbb{R}^n, p(i) - p(j) \leq d(\chi_i - \chi_j) \ (i, j \in X)\right\} \qquad (x \in \mathbb{R}^n). \tag{1.1}$$

This $\overline{d}$ is a polyhedral convex function. Hence, the polyhedral split decomposition is applicable to $\overline{d}$. In [10], we show that, for a metric d, the polyhedral split decomposition of $\overline{d}$ is given by

$$\overline{d}(x) = \sum_{\{A,B\} \in \Sigma_b(d)} c^{\overline{d}}_{\chi_A - \chi_B, 0} \frac{|\langle \chi_A - \chi_B, x\rangle|}{2} + \overline{d'}(x) \qquad (x \in \mathbb{R}^n),$$

where $\Sigma_b(d)$ is defined by $\Sigma_b(d) = \{\{A, B\} \mid \{A, B\} : X\text{-split}, c^{\overline{d}}_{\chi_A-\chi_B,0} > 0\}$. Moreover, we reveal that $c^{\overline{d}}_{\chi_A-\chi_B,0}$ for an X-split $\{A, B\}$ coincides with the Buneman index [2] for the X-split and that the residue $\overline{d'}$ of $\overline{d}$ vanishes if and only if d is a tree metric.

On the other hand, in *discrete convex analysis* established by Murota [12], the notion of *M-convex functions* is introduced as a generalization of *valuated matroids* of Dress and Wenzel [4]. The class of *positively homogeneous M-convex functions* is the most fundamental one of M-convex functions and closely related to distances on a finite set. Indeed, it is known that $\overline{d}$ is a positively homogeneous M-convex function and every positively homogeneous M-convex function can be represented as (1.1) for some distance.

In this paper, we show that the split functions in the polyhedral split decomposition of $\overline{d}$ are positively homogeneous M-convex functions for split metrics. Moreover, even if d is not a tree metric, the residue of $\overline{d}$ is a positively homogeneous M-convex function. Hence, the polyhedral split decomposition of $\overline{d}$ can be interpreted as a decomposition of a positively homogeneous M-convex function into the sum of positively homogeneous M-convex functions, which is in contrast with the general case of M-convex functions because the sum of M-convex functions is not necessarily an M-convex function. We study the decomposition by means of *polyhedral subdivisions* induced by M-convex functions. Those polyhedral subdivisions are known as *matroid subdivisions* [11, 12, 14].

The notion of *split-decomposability* is suggested by Hirai [5, 6] in association with the polyhedral split decomposition. A discrete function $g : K \to \mathbb{R}$ is split-decomposable if its convex extension $\overline{g}$ can be decomposed into the sum of split functions and a linear function. The class of split-decomposable functions depends only on K [5, 6]. In fact, for an origin-symmetric points set K, such as Ω, the class is determined by the matroid associated with K [10]. In the case of Ω, a split-decomposable function coincides with a function such that it is represented as the sum of a tree metric and a linear function. Hence, the path metric $d : \{i, j, k, l\} \to \mathbb{R}$ on a square, i.e., $d(i, j) = d(j, k) = d(k, l) = d(l, i) = 1$ and $d(i, k) = d(j, l) = 2$ is not split-decomposable on Ω despite that d can be represented as the sum of the split metrics $\xi_{\{i,j\},\{k,l\}}$ and $\xi_{\{i,l\},\{j,k\}}$.

This paper also deals with *quadratic M-convex functions*. It is known that there is a one-to-one correspondence between tree metrics and quadratic M-convex functions on $\mathbb{Z}^n$ [8]. For an M-convex function on $\mathbb{Z}^n$, a *directional derivative* can be defined at each point in the domain, where the directional derivative is considered as a function of the directions along the vectors in Ω. In addition, the directional derivatives are positively homogeneous M-convex functions. We show that every directional derivative of a quadratic M-convex function can be represented as the sum of a tree metric and a linear function. Thus, a quadratic M-convex function is a split-decomposable function at every point.

The present paper is organized as follows. Section 2 contains definitions, notation and some fundamental lemmas. In Section 3, we describe a relation between distances and positively homogeneous M-convex functions and introduce some basic terms in discrete convex analysis. In Section 4, we introduce the polyhedral split decomposition of polyhedral convex functions and the discrete split decomposition of discrete functions. In Section 5, we apply the discrete split decomposition to a distance which is regarded as a discrete function on Ω. As a result, we obtain the polyhedral split decomposition of a positively homogeneous M-convex function which is the convex extension of the distance. In Section 6, we show that a quadratic M-convex function is split-decomposable at every point.

2 Preliminaries

Let $\mathbb{R}, \mathbb{R}_+, \mathbb{R}_{++}$ be the set of real numbers, nonnegative real numbers, and positive real numbers, respectively. We denote by $\mathbb{R}^n$ the n dimensional Euclidean space with the standard inner product $\langle \cdot, \cdot \rangle$. For a set $S \subseteq \mathbb{R}^n$, we denote by $\mathrm{conv}\, S$ and $\mathrm{cone}\, S$ the convex hull and the conical hull, respectively, i.e.,

$$\operatorname{conv} S = \Big\{ \sum_{t \in T} \lambda_t t \mid T \subseteq S : \text{a finite set}, \lambda \in \mathbb{R}_+^T, \sum_{t \in T} \lambda_t = 1 \Big\},$$

$$\operatorname{cone} S = \Big\{ \sum_{t \in T} \lambda_t t \mid T \subseteq S : \text{a finite set}, \lambda \in \mathbb{R}_+^T \Big\}.$$

For a set $S \subseteq \mathbb{R}^n$, let $\operatorname{ri} S$ denote the relative interior of S and let $\operatorname{int} S$ denote the interior of S.

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define $\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$, which is the *effective domain* of f, and $\operatorname{epi} f = \{(x, \beta) \in \mathbb{R}^n \times \mathbb{R} \mid \beta \geq f(x)\}$, which is the *epigraph* of f. The *subdifferential* of a function f at point $x \in \operatorname{dom} f$ is defined to be the set

$$\partial f(x) = \{p \in \mathbb{R}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \quad (\forall y \in \mathbb{R}^n)\}.$$

The *directional derivative* of f at point $x \in \operatorname{dom} f$ in a direction $d \in \mathbb{R}^n$ is defined by

$$f'(x; d) = \lim_{t \searrow 0} \frac{f(x + td) - f(x)}{t}.$$

The *indicator function* of a set $S \subseteq \mathbb{R}^n$ is the function $\delta_S : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\delta_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{if } x \notin S. \end{cases}$$

The *conjugate* of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is the function $f^\bullet : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^\bullet(p) = \sup_{x \in \mathbb{R}^n} \{\langle p, x \rangle - f(x)\} \quad (p \in \mathbb{R}^n). \tag{2.1}$$

For a function f and a vector $p \in \mathbb{R}^n$, $f[-p]$ denotes the function defined by

$$f[-p](x) = f(x) - \langle p, x \rangle \quad (x \in \mathbb{R}^n).$$

A function f is said to be *positively homogeneous* if $f(\lambda x) = \lambda f(x)$ holds for $\lambda \geq 0$ and $x \in \mathbb{R}^n$. If f is positively homogeneous, then $f^\bullet = \delta_{\partial f(0)}$ holds and hence $f = (\delta_{\partial f(0)})^\bullet$ is the support function of the set $\partial f(0)$.

The addition of support functions is equivalent to the Minkowski sum of sets.

Lemma 2.1. *Let $\delta_{B_1}^\bullet$ and $\delta_{B_2}^\bullet$ be the support functions of sets B_1 and B_2, respectively. Then we have*

$$\delta_{B_1}^\bullet + \delta_{B_2}^\bullet = \delta_{B_1 + B_2}^\bullet,$$

where $B_1 + B_2$ denotes the Minkowski sum of B_1 and B_2.

For $x, y \in \mathbb{R}^n$, let $[x, y]$ denote the closed line segment between x and y. We refer to an $(n-1)$ dimensional affine subspace of $\mathbb{R}^n$ as a hyperplane. In particular, for $(a, r) \in \mathbb{R}^n \times \mathbb{R}$, we define a hyperplane $H_{a,r} = \{x \in \mathbb{R}^n \mid \langle a, x \rangle = r\}$, closed half spaces $H_{a,r}^- = \{x \in \mathbb{R}^n \mid \langle a, x \rangle \leq r\}$ and $H_{a,r}^+ = \{x \in \mathbb{R}^n \mid \langle a, x \rangle \geq r\}$, and open half spaces $H_{a,r}^{--} = \{x \in \mathbb{R}^n \mid \langle a, x \rangle < r\}$ and $H_{a,r}^{++} = \{x \in \mathbb{R}^n \mid \langle a, x \rangle > r\}$. A set $P \subseteq \mathbb{R}^n$ is said to be a *polyhedron* if P can be represented as an intersection of finitely many closed half spaces.

A convex function f is said to be *polyhedral* if its epigraph $\operatorname{epi} f$ is a polyhedron. A polyhedral convex function f is represented as

$$f(x) = \max_{i \in I} \{\langle p_i, x \rangle - q_i\} + \sum_{j \in J} \delta_{H_{a_j, b_j}^-}(x) \quad (x \in \mathbb{R}^n), \tag{2.2}$$

where $\{(p_i, q_i) \mid i \in I\}$ and $\{(a_j, b_j) \mid j \in J\}$ are finite subsets of $\mathbb{R}^n \times \mathbb{R}$. The conjugate function $f^\bullet$ of a polyhedral function f is also polyhedral and $f^{\bullet\bullet} = f$ holds. We give a fundamental property of polyhedral convex functions in the following lemma.

Lemma 2.2. *The subdifferential of a polyhedral convex function f in* (2.2) *is given by*

$$\partial f(x) = \mathrm{conv}\{p_i \mid i \in I, f(x) = \langle p_i, x\rangle - q_i\} + \mathrm{cone}\{a_j \mid j \in J, x \in H_{a_j,b_j}\} \qquad (x \in \mathrm{dom} f).$$

A *polyhedral complex* $\mathcal{C}$ is a finite collection of polyhedra such that

(1) if $P \in \mathcal{C}$, all the faces of P are also in $\mathcal{C}$, and

(2) the nonempty intersection $P \cap Q$ of two polyhedra $P, Q \in \mathcal{C}$ is a face of P and Q.

The dimension of $\mathcal{C}$, denoted by $\dim \mathcal{C}$, is the largest dimension of a polyhedron in $\mathcal{C}$. The underlying set of $\mathcal{C}$ is the set $|\mathcal{C}| = \bigcup_{P \in \mathcal{C}} P$. A *polyhedral subdivision* of a polyhedron P is a polyhedral complex $\mathcal{C}$ with $|\mathcal{C}| = P$. A polyhedral subdivision is *pure* if its inclusionwise maximal elements are of the same dimension.

For a polyhedral convex function f, lower faces of $\mathrm{epi} f$ are bijectively projected on $\mathrm{dom} f$, and determine a polyhedral subdivision of $\mathrm{dom} f$, which is denoted by $\mathcal{T}(f)$. A polyhedral subdivision constructed in this way is said to be *regular*.

Lemma 2.3. *For a polyhedral convex function f, the polyhedral subdivision $\mathcal{T}(f)$ is given by*

$$\mathcal{T}(f) = \{F \subseteq \mathbb{R}^n \mid F = \mathrm{argmin}\, f[-p] \text{ for some } p \in \mathbb{R}^n\}.$$

For two polyhedral subdivisions $\mathcal{C}_1$ and $\mathcal{C}_2$, the *common refinement* $\mathcal{C}_1 \wedge \mathcal{C}_2$ is defined by $\mathcal{C}_1 \wedge \mathcal{C}_2 = \{F \cap G \mid F \in \mathcal{C}_1, G \in \mathcal{C}_2, F \cap G \neq \emptyset\}$. Note that $\mathcal{C}_1 \wedge \mathcal{C}_2$ is a polyhedral subdivision of $|\mathcal{C}_1| \cap |\mathcal{C}_2|$. In particular, for a finite set of hyperplanes $\mathcal{H}$, we define the polyhedral subdivision $\mathcal{A}(\mathcal{H})$ of $\mathbb{R}^n$ as

$$\mathcal{A}(\mathcal{H}) = \bigwedge_{H \in \mathcal{H}} \{H, H^+, H^-\}.$$

Namely, $\mathcal{A}(\mathcal{H})$ is the partition of $\mathbb{R}^n$ by hyperplanes in $\mathcal{H}$.

The polyhedral subdivision by the sum of two polyhedral convex functions amounts to the common refinement of the polyhedral subdivisions by the two polyhedral convex functions.

Lemma 2.4. *For two polyhedral convex functions f, g with $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$, we have*

$$\mathcal{T}(f + g) = \mathcal{T}(f) \wedge \mathcal{T}(g).$$

Let K be a finite set of points in $\mathbb{R}^n$. For a discrete function $f : K \to \mathbb{R}$, the *homogeneous convex extension* of f is defined by

$$\overline{f}(x) = \inf\Big\{\sum_{y \in K} \lambda_y f(y) \mid \sum_{y \in K} \lambda_y y = x, \lambda_y \geq 0 \ (y \in K)\Big\} + \delta_{\mathrm{cone} K}(x) \qquad (x \in \mathbb{R}^n). \tag{2.3}$$

By definition, $\overline{f}$ is a positively homogeneous polyhedral convex function with $\mathrm{dom} f = \mathrm{cone} K$. By linear programming duality, $\overline{f}$ is also expressed as

$$\overline{f}(x) = \sup\big\{\langle p, x\rangle \mid p \in \mathbb{R}^n, \langle p, y\rangle \leq f(y) \ (y \in K)\big\} \qquad (x \in \mathbb{R}^n).$$

Hence $\overline{f}$ is the support function of the polyhedron

$$Q(f) = \{p \in \mathbb{R}^n \mid \langle p, y\rangle \leq f(y) \ (y \in K)\},$$

and $Q(f) = \partial \overline{f}(0)$. The polyhedral subdivision $\mathcal{T}(\overline{f})$ of $\mathrm{cone} K$ is the intersection of the normal fan of $Q(f)$ with $\mathrm{cone} K$.

For a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote the restriction of f to K by f^K. A function $f : K \to \mathbb{R}$ is said to be *convex-extensible* if it satisfies $\overline{f}^K = f$. The set of convex-extensible functions is recognized as a fundamental class in discrete convex analysis [12].

3 Positively homogeneous M-convex functions

In this section, we describe a relation between the homogeneous convex extensions of a distance and the positively homogeneous M-convex function.

3.1 The convex extension of a distance

Let $X = \{1, 2, \ldots, n\}$, and let a finite set Ω defined by

$$\Omega = \{\chi_i - \chi_j \mid i, j \in X\}.$$

A distance γ on X is regarded as a discrete function defined on the set Ω by the correspondence:

$$\gamma(\chi_i - \chi_j) \leftarrow \gamma(i, j) \quad (i, j \in X).$$

The homogeneous convex extension of γ is defined by

$$\begin{aligned} \overline{\gamma}(x) &= \inf\{\sum_{i,j\in X} \lambda_{ij}\gamma(\chi_i - \chi_j) \mid \sum_{i,j\in X} \lambda_{ij}(\chi_i - \chi_j) = x,\ \lambda_{ij} \geq 0\ (i, j \in X)\} + \delta_{\mathrm{cone}\,\Omega}(x) \\ &= \sup\{\langle p, x\rangle \mid p \in \mathbb{R}^n, \langle p, \chi_i - \chi_j\rangle \leq \gamma(\chi_i - \chi_j)\ (i, j \in X)\} \quad (x \in \mathbb{R}^n). \end{aligned} \tag{3.1}$$

The effective domain of $\overline{\gamma}$ is $\mathrm{cone}\,\Omega = \{x \in \mathbb{R}^n \mid \sum_{i\in X} x(i) = 0\}$. From (3.1), $\overline{\gamma}$ is the support function of the polyhedron

$$Q(\gamma) = \{p \in \mathbb{R}^n \mid \langle p, \chi_i - \chi_j\rangle \leq \gamma(\chi_i - \chi_j)\ (i, j \in X)\} \tag{3.2}$$

The convex-extensibility is equivalent to satisfying the triangle inequality, i.e., a distance is a convex-extensible function on Ω.

Lemma 3.1. *A discrete function $f : \Omega \to \mathbb{R}$ with $f(0) = 0$ is convex-extensible if and only if f satisfies $f(\chi_i - \chi_j) \leq f(\chi_i - \chi_k) + f(\chi_k - \chi_j)$ for all $i, j, k \in X$.*

3.2 M-convex functions

For $x \in \mathbb{R}^n$, we define $\mathrm{supp}^+ x = \{i \mid x(i) > 0, i \in X\}$ and $\mathrm{supp}^- x = \{i \mid x(i) < 0, i \in X\}$.

Definition 3.2 (M-convex function). *A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is said to be M-convex if it satisfies the following exchange property:*

> (**M-EXC**[$\mathbb{Z}$]) *For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that*
>
> $$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

Definition 3.3 (polyhedral M-convex function). *A polyhedral convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is said to be M-convex if it satisfies the following exchange property:*

> (**M-EXC**[$\mathbb{R}$]) *For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exist $v \in \mathrm{supp}^-(x - y)$ and a positive number $\alpha_0 \in \mathbb{R}_{++}$ such that*
>
> $$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v))$$
>
> *for all $\alpha \in [0, \alpha_0]$.*

A polyhedral M-convex function that is positively homogeneous is called a *positively homogeneous M-convex function.* The effective domain of a polyhedral M-convex function is an M-convex polyhedron, which is defined as follows.

Definition 3.4 (M-convex polyhedron). *A nonempty polyhedron $B \subseteq \mathbb{R}^n$ is defined to be an M-convex polyhedron if it satisfies the following:*

> (**B-EXC**[$\mathbb{R}$]) *For $x, y \in B$ and $u \in \mathrm{supp}^+(x-y)$, there exist $v \in \mathrm{supp}^-(x-y)$ and a positive number $\alpha_0 \in \mathbb{R}_{++}$ such that $x - \alpha(\chi_u - \chi_v) \in B$ and $y + \alpha(\chi_u - \chi_v) \in B$ for all $\alpha \in [0, \alpha_0]$.*

A cone that is an M-convex polyhedron is called an M-convex cone.

Remark 3.5. As the property M-EXC[$\mathbb{Z}$] in Definition 3.2 reflects an exchange axiom of matroids, M-convex function is originally introduced as a generalization of *valuated matroids* for functions on integer lattice points. The property M-EXC[$\mathbb{R}$] in Definition 3.3 is devised to propagate the concept of M-convexity for functions in real variables through an appropriate adaptation of M-EXC[$\mathbb{Z}$]. Polyhedral M-convex functions are further extension of M-convexity M-EXC[$\mathbb{R}$] to polyhedral functions.

Recall that a positively homogeneous convex function is the support function of some convex set. In particular, a positively homogeneous M-convex function is the support function of some L-convex polyhedron [12, Theorem 8.4], which is defined as follows.

For two vectors $p, q \in \mathbb{R}^n$, $p \vee q$ and $p \wedge q$ are, respectively, the vectors of componentwise maxima and minima of p and q; i.e.,

$$(p \vee q)(i) = \max(p(i), q(i)), \quad (p \wedge q)(i) = \min(p(i), q(i)) \quad (1 \leq i \leq n).$$

Definition 3.6 (L-convex polyhedron). *A nonempty polyhedron $D \subseteq \mathbb{R}^n$ is defined to be an L-convex polyhedron if it satisfies*

> (**SBS**[$\mathbb{R}$]) $p, q \in D \Rightarrow p \vee q, p \wedge q \in D$,
> (**TBS**[$\mathbb{R}$]) $p \in D \Rightarrow p + \alpha \mathbf{1} \in D \quad (\forall \alpha \in \mathbb{R})$.

We associate a distance $\gamma : X \times X \to \mathbb{R}$ with the two-way directed complete graph K_n such that the length of an edge (i, j) is $\gamma(i, j)$ for all $i, j \in X$, where we distinguish an edge (i, j) from its opposite edge (j, i). If the graph K_n has no negative cycle, where a negative cycle means a directed cycle of negative length, we say that γ has no negative cycle.

It is known that $Q(\gamma)$ is the empty set for a distance γ having negative cycles. Moreover, there is a one-to-one correspondence between L-convex polyhedra and distances having no negative cycle [12, §5.2]. Every L-convex function is represented as (3.2) for a distance γ having no negative cycle. Therefore, $\overline{\gamma}$ is a positively homogeneous M-convex function.

We close this section by discussing the sum of M-convex functions. By Definitions 3.2 and 3.3, we see that, in general, the sum of M-convex functions is not necessarily an M-convex function, i.e., it does not necessarily satisfy (M-EXC[$\mathbb{Z}$]) or (M-EXC[$\mathbb{R}$]). This paper introduces a case that the sum of M-convex functions is also an M-convex function. In order to resolve the case, we apply the following theorem; see also [12, Theorem 6.63].

Theorem 3.7 (Murota and Shioura [13, Theorem 5.2]). *For a polyhedral convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, the following two conditions* (1) *and* (2) *are equivalent.*

(1) *f is a polyhedral M-convex function.*

(2) $\mathrm{argmin}\, f[-p]$ *is an M-convex polyhedron for every $p \in \mathbb{R}^n$ with* $\inf f[-p] > -\infty$.

Theorem 3.7 is one of the characterizations of polyhedral M-convex functions. According to Lemma 2.3 and Theorem 3.7, a polyhedral M-convex function induces a polyhedral subdivision such that each polyhedron in the polyhedral subdivision is an M-convex polyhedron. Such a polyhedral subdivision is called a *matroid subdivision*. Matroid subdivisions are studied in tropical geometry [14], surgery on Grassmannian [11], and discrete convex analysis [12]; see also [9]. Because the sum of polyhedral M-convex functions is obviously a polyhedral convex function, the M-convexity of the sum depends on whether the polyhedral subdivision induced by the sum is a matroid subdivision or not. This paper gives an example that the common refinement of matroid subdivisions is also a matroid subdivision.

4 Polyhedral and discrete split decomposition

This section introduces the polyhedral split decomposition of polyhedral convex functions and the discrete split decomposition of discrete functions. The discrete split decomposition of a discrete function is summarized as the polyhedral split decomposition of the convex extension of the discrete function.

4.1 Polyhedral split decomposition

We briefly explain the polyhedral split decomposition of polyhedral convex functions. The proofs of the propositions in this section can be found in [5, §2], [10, §7].

Definition 4.1 (split function). *For a hyperplane $H = H_{a,b}$ with $\|a\| = 1$, the split function $l_H : \mathbb{R}^n \to \mathbb{R}$ associated with H is defined by*

$$l_H(x) = |\langle a, x\rangle - b|/2 \quad (x \in \mathbb{R}^n).$$

By Lemma 2.2, the polyhedral subdivision induced by split function is given as follows.

Proposition 4.2. *Let l_H be the split function associated with a hyperplane $H = H_{a,b}$ with $\|a\| = 1$. The subdifferential of l_H is given by*

$$\partial l_H(x) = \begin{cases} \{a/2\} & \text{if } x \in H^{++}, \\ [-a/2, a/2] & \text{if } x \in H, \\ \{-a/2\} & \text{if } x \in H^{--}, \end{cases}$$

and polyhedral subdivisions $\mathcal{T}(l_H)$ and $\mathcal{T}(l_H^\bullet)$ are given by

$$\mathcal{T}(l_H) = \{H, H^+, H^-\},$$
$$\mathcal{T}(l_H^\bullet) = \{\{a/2\}, \{-a/2\}, [-a/2, a/2]\}.$$

For a polyhedral convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a hyperplane H, we define the *quotient* $c_H(f)$ of f by l_H as

$$c_H(f) = \sup\{t \in \mathbb{R}_+ \mid f - tl_H \text{ is convex}\},$$

and the set of hyperplanes $\mathcal{H}(f)$ as

$$\mathcal{H}(f) = \{H \mid 0 < c_H(f) < +\infty\}.$$

The basic idea for the polyhedral split decomposition of a polyhedral convex function f is to subtract split functions associated with hyperplanes in $\mathcal{H}(f)$ from f successively. In fact, if $\dim \operatorname{dom} f = n$, this idea directly applies to f because of the following proposition. Note that, for $H_1, H_2 \in \mathcal{H}(f)$ with $H_1 \cap \operatorname{int} \operatorname{dom} f \neq \emptyset$ and $H_2 \cap \operatorname{int} \operatorname{dom} f \neq \emptyset$, we have $H_1 = H_2$ if and only if $H_1 \cap \operatorname{dom} f = H_2 \cap \operatorname{dom} f$.

Proposition 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral convex function. Then, for $H, H' \in \mathcal{H}(f)$ and $t \in [0, c_H(f)]$, we have*

$$H = H' \Rightarrow c_{H'}(f - tl_H) = c_{H'}(f) - t,$$
$$H \cap \operatorname{dom} f \neq H' \cap \operatorname{dom} f \Rightarrow c_{H'}(f - tl_H) = c_{H'}(f).$$

If $\operatorname{dom} f$ is not full-dimensional, there exist infinitely many hyperplanes having the same intersection with $\operatorname{dom} f$. Moreover, $c_{H'}(f - tl_H)$ may not be equal to $c_{H'}(f)$ for $H, H' \in \mathcal{H}(f)$ and $t \in [0, c_H(f)]$ despite that $H \neq H'$. Hence, in the case of $\dim \operatorname{dom} f \neq n$, we must restrict $\mathcal{H}(f)$ to the set such that there are no hyperplanes having the same intersection with $\operatorname{dom} f$ in the set. Technically speaking, we define the equivalence relation $\sim$ by letting $H \sim H'$ if $H \cap \operatorname{dom} f = H' \cap \operatorname{dom} f$. Since a collection of representatives from the equivalence classes has the desirable property, we decompose f by using the representatives. Note that representatives is finite because $\mathcal{T}(f)$ is finite. If we fix representatives, denoted by $\mathcal{H}^\diamond(f)$, of $\mathcal{H}(f)/\sim$, the polyhedral split decomposition of f is uniquely defined as in the next theorem; see also [6, Thorem 2.2].

Theorem 4.4 ([10, Theorem 7.8]). *A polyhedral convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is uniquely decomposable as*

$$f = \sum_{H \in \mathcal{H}^\diamond(f)} c_H(f) l_H + f', \tag{4.1}$$

where $f' : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a polyhedral convex function with $c_{H'}(f') \in \{0, +\infty\}$ for any hyperplane H'.

Incidentally, the quotient $c_H(f)$ of f by l_H is written explicitly as in the next proposition.

Proposition 4.5. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral convex function, and let H be a hyperplane in $\mathbb{R}^n$. Then we have*

$$c_H(f) = \frac{1}{2} \inf \left\{ \frac{f(x) - f(w)}{l_H(x)} + \frac{f(y) - f(w)}{l_H(y)} \,\middle|\, \begin{array}{c} x \in \operatorname{dom} f \cap H^{++}, \\ y \in \operatorname{dom} f \cap H^{--}, \\ \{w\} = [x, y] \cap H \end{array} \right\}.$$

We conclude this section by the following lemma, which is used in Section 5.2.

Lemma 4.6. *For a polyhedral convex function $f : \mathbb{R}^n \to \mathbb{R}$, the polyhedral subdivision $\mathcal{T}(f)$ is decomposed as*

$$\mathcal{T}(f) = \mathcal{A}(\mathcal{H}(f)) \wedge \mathcal{T}(f'). \tag{4.2}$$

Proof. By Proposition 4.2, we have $\mathcal{T}(\alpha l_H) = \{H, H^+, H^-\}$ for any $\alpha \in \mathbb{R}_{++}$. Hence, by Lemma 2.4, we have (4.2) from the decomposition (4.1). □

4.2 Discrete split decomposition

We briefly describe the discrete split decomposition of discrete functions defined on a finite set K of points of $\mathbb{R}^n$. The proofs of the propositions in this section can be found in [5, §3], [6, §3], [10, §8].

The discrete split decomposition is based on the next proposition [6, Theorem 3.2] and Theorem 4.4. Recall that $\overline{f}$ is defined by (2.3).

Proposition 4.7. *For* $f : K \to \mathbb{R}$, $H \in \mathcal{H}(\overline{f})$, *and* $t \in [0, c_H(\overline{f})]$, *we have*

$$\overline{f} = tl_H + \overline{f - tl_H^K}.$$

We mention that the essential contents of the following theorem is derived as Theorem 3.2 in [6].

Theorem 4.8 ([10, Theorem 8.7]). *Let* $f : K \to \mathbb{R}$ *be a discrete function. Then,* f *is uniquely decomposable as*

$$f = \sum_{H \in \mathcal{H}^\diamond(\overline{f})} c_H(\overline{f}) l_H^K + f',$$

where $f' : K \to \mathbb{R}$ *satisfies* $c_{H'}(\overline{f'}) \in \{0, +\infty\}$ *for any linear hyperplane* H'. *Furthermore, we have*

$$\overline{f} = \sum_{H \in \mathcal{H}^\diamond(\overline{f})} c_H(\overline{f}) l_H + \overline{f'}.$$

If, in addition, f *is convex-extensible, then* f' *is also convex-extensible.*

We here describe a relation between K and $\mathcal{H}(\overline{f})$. Let f be a convex-extensible discrete function on K. Note that since $\mathcal{T}(\overline{f})$ is the intersection of the normal fan of $Q(f)$ with $\operatorname{cone} K$, each hyperplane $H \in \mathcal{H}(\overline{f})$ is linear, i.e., $H = H_{a,0}$ for some $a \in \mathbb{R}^n$. From the regularity of the subdivision $\mathcal{T}(\overline{f})$ induced by $\overline{f}$, we notice that possible hyperplanes appearing in $\mathcal{H}(\overline{f})$ is limited by the point set K. Then, we make the next definition.

Definition 4.9 (K-admissible). *A set of linear hyperplanes* $\mathcal{H}$ *is* K*-admissible if* $\mathcal{H}$ *satisfies*

(A1) $H \cap \operatorname{ri} \operatorname{cone} K \neq \emptyset$ *for each* $H \in \mathcal{H}$, *and*

(A2) $\operatorname{cone}(F \cap K) = F \cap \operatorname{cone} K$ *for each* $F \in \mathcal{A}(\mathcal{H})$.

Note that K-admissibility is determined solely by K. A justification for Definition 4.9 follows from the next lemma.

Lemma 4.10. *For* $f : K \to \mathbb{R}$, *the set of hyperplanes* $\mathcal{H}(\overline{f})$ *is* K*-admissible.*

Note that if a set of linear hyperplanes $\mathcal{H}$ is K-admissible, then any subset of $\mathcal{H}$ is also K-admissible. So we define the set of linear hyperplanes $\mathcal{H}_K$ as

$$\mathcal{H}_K = \{H \mid H : \text{a linear hyperplane}, \{H\} \text{ is } K\text{-admissible}\}.$$

By Lemma 4.10, $\mathcal{H}(f) \subseteq \mathcal{H}_K$ holds for any $f : K \to \mathbb{R}$. In the case of $\dim \operatorname{dom} \overline{f} \neq n$, we restrict $\mathcal{H}_K$ to representatives, denoted by $\mathcal{H}_K^\diamond$, of $\mathcal{H}_K/\sim$, so that we have $\mathcal{H}^\diamond(\overline{f}) = \mathcal{H}(\overline{f}) \cap \mathcal{H}_K^\diamond \subseteq \mathcal{H}_K^\diamond$.

The next theorem implies that the discrete split decomposition can be carried out without explicit construction of convex extensions; the quotient $c_H(\overline{f})$ can be calculated without the construction.

Theorem 4.11 (Hirai [6, Theorem 3.4]). *For a discrete function* $f : K \to \mathbb{R}$ *and a hyperplane* $H \in \mathcal{H}_K$, *let* $\tilde{c}_H(f)$ *be defined by*

$$\tilde{c}_H(f) = \frac{1}{2} \inf \left\{ \frac{f(x) - \overline{f^{K \cap H}}(w)}{l_H(x)} + \frac{f(y) - \overline{f^{K \cap H}}(w)}{l_H(y)} \;\middle|\; \begin{array}{c} x \in K \cap H^{++}, \\ y \in K \cap H^{--}, \\ \{w\} = [x, y] \cap H \end{array} \right\}.$$

Then we have

$$c_H(\overline{f}) = \max\{0, \tilde{c}_H(f)\}.$$

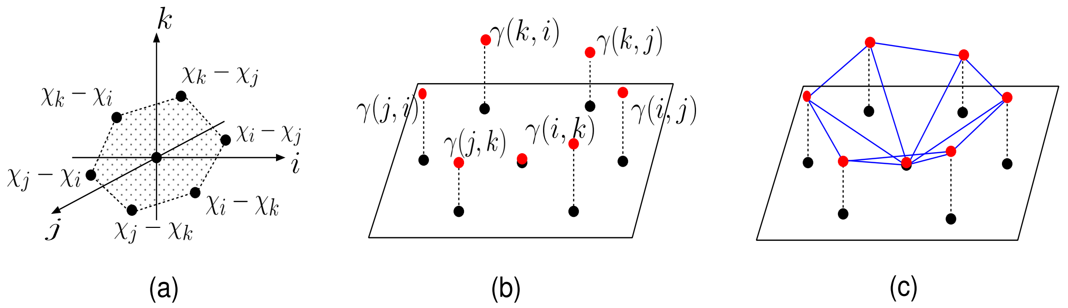

Figure 1: The homogeneous convex extension of a metric γ on $X = \{i, j, k\}$.

We close this section by introducing the notion of *split-decomposability*. A function $f \in \mathbb{R}^K$ is said to be *split-decomposable* if $f - \sum_{H \in \mathcal{H}^\diamond(f)} c_H(\overline{f}) l_H^K$ is the restriction of a linear function. We explain the relation between split-decomposable functions on K and K-admissible sets of hyperplanes.

Lemma 4.12. *Let $f : K \to \mathbb{R}$ be a discrete function. Then we have*

$$\overline{cf + (\langle q, \cdot \rangle)^K} = c\overline{f} + \langle q, \cdot \rangle + \delta_{\operatorname{cone} K} \quad (c \in \mathbb{R}_+, q \in \mathbb{R}^n).$$

By Lemma 4.12, the quotient of $\overline{cf + (\langle q, \cdot \rangle)^K}$ by a split function depends only on $c\overline{f}$. Hence, the discrete split decomposition of $cf + (\langle q, \cdot \rangle)^K$ is determined by $c\overline{f}$.

Proposition 4.13. *For $\mathcal{H} \subseteq \mathcal{H}_K^\diamond$ and $\alpha \in \mathbb{R}_{++}^{\mathcal{H}}$, let $f = \sum_{H \in \mathcal{H}} \alpha_H l_H^K$. Then the following conditions* (a)*,* (b) *and* (c) *are equivalent.*

(a) $\overline{f} = \sum_{H \in \mathcal{H}} \alpha_H l_H + \delta_{\operatorname{cone} K}$.

(b) $\mathcal{H} = \mathcal{H}^\diamond(\overline{f})$ *and* $\alpha_H = c_H(\overline{f})$ *for* $H \in \mathcal{H}$.

(c) $\mathcal{H}$ *is K-admissible.*

By Lemma 4.12 and Proposition 4.13, every split-decomposable function is constructed from a K-admissible set of hyperplanes, i.e., the sum of a positive combination of the restrictions of the split functions associated with the hyperplanes and the restriction of a linear function. Thus, split-decomposable functions are also determined by K through the K-admissible sets of hyperplanes since the K-admissibility depends on K.

5 The polyhedral split decomposition of a positively homogeneous M-convex function

5.1 The discrete split decomposition of a distance

In this subsection, we apply the discrete split decomposition to $\gamma : \Omega \to \mathbb{R}$. Recall that $\Omega = \{\chi_i - \chi_j \mid i, j \in X\}$. Figure 1 (c) illustrates the homogeneous convex extension of a metric γ on $X = \{i, j, k\}$. Since X on a linear space as in Figure 1 (a), we can project $\{(\chi_i - \chi_j, \gamma(i, j)) \mid i, j \in X\}$ to 3-dimensional space as shown in Figure 1 (b).

Although $\overline{\gamma}$ is a polyhedral convex function, its effective domain $\operatorname{dom}\overline{\gamma}$ is not fully dimensional. Then, we define representatives $\mathcal{H}_\Omega^\diamond$ as mentioned in Section 4.2. For an X-split $\{A, B\}$, we denote $H_{(\chi_A - \chi_B)/2\sqrt{|X|}, 0}$ by $H_{\{A,B\}}$. Let $\mathbf{1}$ be the all-one vector. Hereafter, coefficients for scaling vectors to unit ones are omitted for simplicity. We obtain $\mathcal{H}_\Omega := \{H \mid H : \text{a linear hyperplane}, \{H\} : \Omega\text{-admissible}\}$ as in the next proposition.

Proposition 5.1. $\mathcal{H}_\Omega = \{H_{\alpha(\chi_A-\chi_B)+\beta\mathbf{1},0} \mid \{A,B\} : \text{an } X\text{-split},\ \alpha,\beta\in\mathbb{R}\}$.

Since it is immediate that

$$H_{\{A,B\}} \cap \operatorname{dom}\overline{\gamma} = H_{\alpha(\chi_A-\chi_B)+\beta\mathbf{1},0} \cap \operatorname{dom}\overline{\gamma}$$

for all $\alpha,\beta\in\mathbb{R}$, we define the hyperplanes:

$$\mathcal{H}_\Omega^\diamond := \{H_{\{A,B\}} \mid \{A,B\} : \text{an } X\text{-split}\}.$$

Proposition 5.2. *For each $H\in\mathcal{H}_\Omega$, there exists a hyperplane $H^\diamond$ such that*

$$H \cap \operatorname{dom}\overline{\gamma} = H^\diamond \cap \operatorname{dom}\overline{\gamma}$$

in $\mathcal{H}_\Omega^\diamond$. Moreover, for all $H,H'\in\mathcal{H}_\Omega^\diamond$, $H\cap\operatorname{dom}\overline{\gamma} = H'\cap\operatorname{dom}\overline{\gamma}$ if and only if $H=H'$.

By Proposition 5.2, $\mathcal{H}_\Omega^\diamond$ constitutes representatives of $\mathcal{H}_\Omega/\!\sim$. Hence, Theorem 4.8 can be applied to γ and γ is decomposed uniquely with hyperplanes in $\mathcal{H}(\overline{\gamma})\cap\mathcal{H}_\Omega^\diamond$. We apply Theorem 4.11 to $\overline{\gamma}$, and then $\tilde{c}_{H_{\{A,B\}}}(\gamma)$ is equal to the minimum of

$$\frac{\gamma(\chi_i-\chi_k) - \overline{\gamma^{\Omega\cap H_{\{A,B\}}}}(w)}{2l_{H_{\{A,B\}}}(\chi_i-\chi_k)} + \frac{\gamma(\chi_l-\chi_j) - \overline{\gamma^{\Omega\cap H_{\{A,B\}}}}(w)}{2l_{H_{\{A,B\}}}(\chi_l-\chi_j)},$$

where $i,j\in A, k,l\in B$, and $\{w\} = H_{\{A,B\}}\cap[\chi_i-\chi_k,\chi_l-\chi_j]$. Hence, we have

$$\begin{aligned}\tilde{c}_{H_{\{A,B\}}}(\gamma) = \frac{\sqrt{|X|}}{2}\min_{i,j\in A,k,l\in B}\Bigg\{&\gamma(\chi_i-\chi_k)+\gamma(\chi_l-\chi_j) - 2\overline{\gamma^{\Omega\cap H_{\{A,B\}}}}\Big(\frac{\chi_i-\chi_k+\chi_l-\chi_j}{2}\Big),\\ &\gamma(\chi_i-\chi_l)+\gamma(\chi_k-\chi_j) - 2\overline{\gamma^{\Omega\cap H_{\{A,B\}}}}\Big(\frac{\chi_i-\chi_l+\chi_k-\chi_j}{2}\Big)\Bigg\}.\end{aligned}$$

Since γ satisfies the triangle inequality, we obtain

$$\overline{\gamma^{\Omega\cap H_{\{A,B\}}}}\Big(\frac{\chi_i-\chi_k+\chi_l-\chi_j}{2}\Big) = \frac{1}{2}(\gamma(\chi_i-\chi_j)+\gamma(\chi_l-\chi_k))$$

and

$$\overline{\gamma^{\Omega\cap H_{\{A,B\}}}}\Big(\frac{\chi_i-\chi_l+\chi_k-\chi_j}{2}\Big) = \frac{1}{2}(\gamma(\chi_i-\chi_j)+\gamma(\chi_k-\chi_l)).$$

Thus, we have

$$\begin{aligned}\tilde{c}_{H_{\{A,B\}}}(\gamma) =&\frac{\sqrt{|X|}}{2}\min_{i,j\in A,k,l\in B}\Big\{\gamma(\chi_i-\chi_k)+\gamma(\chi_l-\chi_j)-\gamma(\chi_i-\chi_j)-\gamma(\chi_l-\chi_k),\\ &\qquad\gamma(\chi_i-\chi_l)+\gamma(\chi_k-\chi_j)-\gamma(\chi_i-\chi_j)+\gamma(\chi_k-\chi_l)\Big\}\\ =&\frac{\sqrt{|X|}}{2}\min_{i,j\in A,k,l\in B}\Big\{\gamma(i,k)+\gamma(l,j)-\gamma(i,j)-\gamma(l,k),\gamma(i,l)+\gamma(k,j)-\gamma(i,j)-\gamma(k,l)\Big\},\end{aligned} \tag{5.1}$$

and so, we have $c_{H_{\{A,B\}}}(\overline{\gamma}) = \max\{0,\tilde{c}_{H_{\{A,B\}}}(\gamma)\}$. As a result, the next theorem is obtained.

Theorem 5.3. *Let $\gamma : X \times X \to \mathbb{R}$ be a distance. Then γ can be decomposed as*

$$\gamma = \sum_{\sigma \in \Sigma(\gamma)} c_{H_\sigma}(\overline{\gamma}) l^{\Omega}_{H_\sigma} + \gamma',$$

where $\Sigma(\gamma)$ is defined by

$$\Sigma(\gamma) = \{\sigma \mid \sigma : \text{an } X\text{-split}, c_{H_\sigma}(\overline{\gamma}) > 0\}$$

and $\gamma' : X \times X \to \mathbb{R}$ is a distance with $c_{H_{\sigma'}}(\overline{\gamma'}) = 0$ for any X-split σ'. Furthermore, we have

$$\overline{\gamma} = \sum_{\sigma \in \Sigma(\gamma)} c_{H_\sigma}(\overline{\gamma}) l_{H_\sigma} + \overline{\gamma'}. \tag{5.2}$$

We here define the compatibility of X-splits.

Definition 5.4 (compatible). *Let $\{A, B\}$ and $\{C, D\}$ be X-splits. Two X-splits $\{A, B\}$ and $\{C, D\}$ are compatible if at least one of the sets $A \cap C, A \cap D, B \cap C$ and $B \cap D$ is the empty set.*

A collection of X-splits is called *pairwise compatible* if any two X-splits in the collection are compatible. For a subset $\mathcal{H}$ of $\mathcal{H}^{\diamond}_{\Omega}$, we denote $\Sigma_{\mathcal{H}} = \{\{A, B\} \mid \{A, B\} : \text{an } X\text{-split}, H_{\{A,B\}} \in \mathcal{H}\}$. We can translate the Ω-admissibility of $\mathcal{H}$ into the pairwise compatibility of $\Sigma_{\mathcal{H}}$.

Proposition 5.5 ([10, Proposition 9.8]). *A set of hyperplanes $\mathcal{H} \subseteq \mathcal{H}^{\diamond}_{\Omega}$ is Ω-admissible if and only if $\Sigma_{\mathcal{H}}$ is pairwise compatible.*

Hence, we obtain the following proposition.

Proposition 5.6. *A metric γ is a tree metric if and only if γ is decomposed as*

$$\gamma = \sum_{\sigma \in \Sigma(\gamma)} c_{H_\sigma}(\overline{\gamma}) l^{\Omega}_{H_\sigma}.$$

By Proposition 5.6, a split-decomposable function on Ω corresponds to the sum of a tree metric and a linear function.

We introduce the *Buneman index* for the convenience. For a metric $\gamma : X \times X \to \mathbb{R}$ and an X-split $\{A, B\}$, the Buneman index is defined by

$$b^{\gamma}_{\{A,B\}} = \frac{1}{2} \min_{i,j \in A, k,l \in B} \left\{ \min \left\{ \begin{matrix} \gamma(i,k) + \gamma(j,l), \\ \gamma(i,l) + \gamma(j,k) \end{matrix} \right\} - \gamma(i,j) - \gamma(k,l) \right\}.$$

If γ is a metric, we obtain from (5.1)

$$\begin{aligned} \tilde{c}_{H_{\{A,B\}}}(\gamma) &= \frac{\sqrt{|X|}}{2} \min_{i,j \in A, k,l \in B} \left\{ \min \left\{ \begin{matrix} \gamma(i,k) + \gamma(j,l), \\ \gamma(i,l) + \gamma(j,k) \end{matrix} \right\} - \gamma(i,j) - \gamma(k,l) \right\} \\ &= \sqrt{|X|}\, b^{\gamma}_{\{A,B\}}. \end{aligned}$$

Therefore, we have $c_{H_{\{A,B\}}}(\overline{\gamma}) = \max\{0, \tilde{c}_{H_{\{A,B\}}}(\gamma)\} = \sqrt{|X|} \max\{0, b^{\gamma}_{\{A,B\}}\}$. As a result of the discrete split decomposition of metrics on Ω, the next theorem is obtained.

Theorem 5.7. *Let $\gamma : X \times X \to \mathbb{R}$ be a metric. Then γ can be decomposed as*

$$\gamma = \sum_{\sigma \in \Sigma_b(\gamma)} \sqrt{|X|}\, b^{\gamma}_{\sigma} l^{\Omega}_{H_\sigma} + \gamma', \tag{5.3}$$

where $\Sigma_b(\gamma)$ is defined by

$$\Sigma_b(\gamma) = \{\sigma \mid \sigma : \text{an } X\text{-split}, b^{\gamma}_{\sigma} > 0\}$$

and $\gamma' : X \times X \to \mathbb{R}$ is a metric with $b^{\gamma'}_{\sigma'} \leq 0$ for any X-split σ'.

Figure 2 illustrates the polyhedral split decomposition of a metric on X with $|X| = 3$. It is known that every 3-point metric is representable as the sum of split metrics, i.e., $\gamma' = 0$ in the decomposition (5.3).

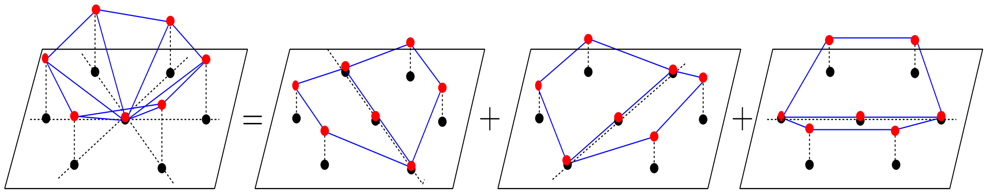

Figure 2: The polyhedral split decomposition of a metric on $X = \{i, j, k\}$.

5.2 M-convexity of split functions

In this subsection, we describe the split functions in the polyhedral split decomposition of a distance are also positively homogeneous M-convex functions and unravel why the sum of the split functions is an M-convex function as the main result of this paper.

Theorem 5.8. *Let $l_{H_{\{A,B\}}} : \mathbb{R}^n \to \mathbb{R}$ be the split function associated with an X-split $\{A, B\}$. Then $\phi := l_{H_{\{A,B\}}} + \delta_{\mathrm{cone}\,\Omega}$ is a positively homogeneous M-convex function from $\mathbb{R}^n$ into $\mathbb{R}$.*

Theorem 5.8 implies that ϕ is the support function of some L-convex polyhedron. Then we prove Theorem 5.8 by taking the conjugate of ϕ and showing that it is the indicator function of an L-convex polyhedron, which is represented as (3.2) for a (scalar multiple) *split metric*. Hence, $\phi = l_{H_{\{A,B\}}} + \delta_{\mathrm{cone}\,\Omega}$ turn out to be the homogeneous convex extension of the (scalar multiple) split metric $\xi_{\{A,B\}}$. Note that Theorem 5.8 can be obtained as a consequence of the fact that ϕ^{Ω} is a (scalar multiple) split metric and Proposition 4.13.

Proof of Theorem 5.8. For simplicity, we redefine $\phi := \sqrt{|X|}\, l_{H_{\{A,B\}}} + \delta_{\mathrm{cone}\,\Omega}$. Obviously, the effective domain of ϕ is $\mathrm{cone}\,\Omega$. By the definition (2.1), the conjugate of ϕ is defined as follows:

$$\begin{aligned}
\phi^{\bullet}(p) &= \sup\{\langle p, x\rangle - \phi(x) \mid x \in \mathbb{R}^n\} \\
&= \sup\{\langle p, x\rangle - |\langle(\chi_A - \chi_B)/2, x\rangle| \mid x \in \mathrm{cone}\,\Omega\} \\
&= \max\{\max\{\langle p - (\chi_A - \chi_B)/2, x\rangle \mid x \in \mathrm{cone}\,\Omega, x(A) \geq x(B)\}, \\
&\qquad \max\{\langle p + (\chi_A - \chi_B)/2, x\rangle \mid x \in \mathrm{cone}\,\Omega, x(A) \leq x(B)\}\} \quad (p \in \mathbb{R}^n).
\end{aligned}$$

The maximum; $\max\{\langle p - (\chi_A - \chi_B)/2, x\rangle \mid x \in \mathrm{cone}\,\Omega, x(A) \geq x(B)\}$ is zero if $p - (\chi_A - \chi_B)/2 = \alpha\mathbf{1} + \beta(-\chi_A + \chi_B)$ for some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$, otherwise $+\infty$. Hence, we have

$$p(i) = \begin{cases} \alpha - \beta + \frac{1}{2} & (i \in A), \\ \alpha + \beta - \frac{1}{2} & (i \in B), \end{cases}$$

and so

$$p(i) - p(j) = \begin{cases} 0 & (i, j \in A \text{ or } i, j \in B), \\ -2\beta + 1 & (i \in A, j \in B). \end{cases}$$

Similarly, the maximum; $\max\{\langle p + (\chi_A - \chi_B)/2, x\rangle \mid x \in \mathrm{cone}\,\Omega, x(A) \le x(B)\}$ is zero if $p + (\chi_A - \chi_B)/2 = \xi\mathbf{1} + \eta(\chi_A - \chi_B)$ for some $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}_+$, otherwise $+\infty$. Hence, we have

$$p(i) = \begin{cases} \xi + \eta - \frac{1}{2} & (i \in A), \\ \xi - \eta + \frac{1}{2} & (i \in B), \end{cases}$$

and so

$$p(i) - p(j) = \begin{cases} 0 & (i, j \in A \text{ or } i, j \in B), \\ 2\eta - 1 & (i \in A, j \in B). \end{cases}$$

Since β and η must satisfy $-2\beta + 1 = 2\eta - 1, \beta \ge 0$ and $\eta \ge 0$, we have $0 \le \beta \le 1, 0 \le \eta \le 1$. Therefore, the effective domain of $\phi^\bullet$ is the polyhedron $Q(\phi)$ defined by

$$Q(\phi) = \left\{ p \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} p(i) - p(j) \le 0 & (i, j \in A \text{ or } i, j \in B), \\ p(i) - p(j) \le 1 & (i \in A, j \in B \text{ or } i \in B, j \in A) \end{array} \right\}.$$

Obviously, $Q(\phi)$ is the L-convex polyhedron defined by the split metric $\xi_{\{A,B\}}$ and $\phi^\bullet(p)$ takes zero for $p \in Q(\phi)$ and $+\infty$ for $p \notin Q(\phi)$. Therefore, ϕ is the support function of the L-convex polyhedron $Q(\phi)$, that is, ϕ is a positively homogeneous M-convex function. □

Note that a residue $\overline{\gamma'}$ appearing in (5.2) is also a positively homogeneous M-convex function because γ' is a distance. As a consequence of Theorem 5.8, $\overline{\gamma}$ is decomposed as the sum of positively homogeneous M-convex functions, which is a case that the sum of M-convex functions is also an M-convex function. As mentioned in Section 3, we give a description based on Theorem 3.7 for this case.

By Theorem 3.7, it is sufficient for showing the M-convexity of $\overline{\gamma}$ that each cone in $\mathcal{T}(\overline{\gamma})$ is an M-convex cone by Lemma 2.3. We begin by proving the next proposition, which, combined with Theorem 3.7, provides an alternative proof of Theorem 5.8 since $\mathrm{cone}\,\Omega$ is obviously M-convex cone.

Proposition 5.9. *For an M-convex cone D, we define $Y = \{\chi_i - \chi_j \in D \mid i, j \in X\}$. Let $\{A, B\}$ be an X-split. Then the following (1), (2) and (3) hold.*

(1) $\mathrm{cone}\,(H_{\{A,B\}} \cap Y) = H_{\{A,B\}} \cap \mathrm{cone}\,Y$ *if and only if* $H_{\{A,B\}} \cap \mathrm{cone}\,Y$ *is an M-convex cone.*

(2) $\mathrm{cone}\,(H^+_{\{A,B\}} \cap Y) = H^+_{\{A,B\}} \cap \mathrm{cone}\,Y$ *if and only if* $H^+_{\{A,B\}} \cap \mathrm{cone}\,Y$ *is an M-convex cone.*

(3) $\mathrm{cone}\,(H^-_{\{A,B\}} \cap Y) = H^-_{\{A,B\}} \cap \mathrm{cone}\,Y$ *if and only if* $H^-_{\{A,B\}} \cap \mathrm{cone}\,Y$ *is an M-convex cone.*

Proof. We show (1). Every M-convex cone B_0 can be represented as

$$B_0 = \{ \sum_{(i,j)\in F} c_{ij}(\chi_i - \chi_j) \mid c_{ij} \ge 0 \ ((i, j) \in F)\}$$

for some $F \subseteq X \times X$, where we may assume that F is transitive, i.e., $(i, j) \in F$ and $(j, k) \in F$ imply $(i, k) \in F$. Hence, $\mathrm{cone}\,(H_{\{A,B\}} \cap Y)$ is obviously an M-convex cone, from which the necessity is immediate because $H_{\{A,B\}} \cap \mathrm{cone}\,Y = \mathrm{cone}\,(H_{\{A,B\}} \cap Y)$.

We show the sufficiency. By hypothesis, $H_{\{A,B\}} \cap \mathrm{cone}Y$ is an M-convex cone. Thereby $H_{\{A,B\}} \cap \mathrm{cone}Y$ is represented as

$$H_{\{A,B\}} \cap \mathrm{cone}Y = \{ \sum_{(i,j)\in F'} c_{ij}(\chi_i - \chi_j) \mid c_{ij} \geq 0 \ ((i,j) \in F')\}$$

for some $F' \subseteq X \times X$. By definition, Y is transitive, i.e., $\chi_i - \chi_j \in Y$ and $\chi_j - \chi_k \in Y$ imply $\chi_i - \chi_k \in Y$. It follows that $\chi_i - \chi_j \in \mathrm{cone}Y$ is equivalent to $\chi_i - \chi_j \in Y$. Hence, we have $\chi_i - \chi_j \in H_{\{A,B\}} \cap Y$ for any $(i,j) \in F'$, and thus $\mathrm{cone}(H_{\{A,B\}} \cap Y) = H_{\{A,B\}} \cap \mathrm{cone}Y$.

The assertions (2) and (3) are shown similarly. □

By Lemma 4.6, $\mathcal{T}(\overline{\gamma})$ is the common refinement of $\mathcal{A}(\mathcal{H}^\diamond(\overline{\gamma}))$ and $\mathcal{T}(\overline{\gamma'})$. We interpret $\mathcal{T}(\overline{\gamma})$ as the result of the successive refinements of $\mathcal{T}(\overline{\gamma'})$ by hyperplanes in $\mathcal{H}^\diamond(\overline{\gamma})$. Note on $\mathcal{T}(\overline{\gamma'})$ as the initial state that each cone in $\mathcal{T}(\overline{\gamma'})$ is an M-convex cone since γ' is a distance. A hyperplane in $\mathcal{H}^\diamond(\overline{\gamma})$ satisfies for each cone in $\mathcal{T}(\overline{\gamma'})$ the conditions in the left-hand sides of (1), (2), and (3) in Proposition 5.9. Thus, the common refinement of $\mathcal{T}(\overline{\gamma'})$ and the hyperplane consists of M-convex cones. We repeat such a refinement process until we obtain $\mathcal{T}(\overline{\gamma})$. At each step, a refinement of $\mathcal{T}(\overline{\gamma'})$ is composed of M-convex cones because of Proposition 5.9 and the Ω-admissibility of $\mathcal{H}^\diamond(\overline{\gamma})$ as in Lemma 4.10. Therefore, we obtain the following proposition.

Proposition 5.10. *Let $\mathcal{H} \subseteq \mathcal{H}^\diamond(\overline{\gamma})$. Each polyhedron in the common refinement of $\mathcal{T}(\overline{\gamma'})$ and $\mathcal{A}(\mathcal{H})$ is an M-convex cone. Therefore, each polyhedron in the polyhedral subdivision $\mathcal{T}(\overline{\gamma})$ is an M-convex cone.*

It follows from Theorem 3.7 and Proposition 5.10 that $\overline{\gamma}$ is a positively homogeneous M-convex function.

Remark 5.11. The conjugate structure is revealed as follows. By Lemma 2.1 and Theorem 5.8, the decomposition of $\overline{\gamma}$ implies that $Q(\gamma)$ is the Minkowski sum of L-convex polyhedra. We know that $Q(\gamma)$ is also an L-convex polyhedron. Note that, in general, the Minkowski sum of L-convex polyhedra is not necessarily an L-convex polyhedron. The L-convex polyhedron corresponding to a split function is a line segment $(+\{\alpha\mathbf{1} \mid \alpha \in \mathbb{R}\})$ by Proposition 4.2. Since the Minkowski sum of line segments is a *zonotope*, the polyhedral split decomposition of $\overline{\gamma}$ provides a decomposition of $Q(\gamma)$ into the Minkowski sum of a zonotope and some polyhedron, which is the dual operation of the polyhedral split decomposition. The zonotope is closely related to a *tight span* in T-theory [3]; see also [8].

6 Tree metrics and quadratic M-convex functions

In this section, we focus on the property of M-convex functions as described in the next theorem; see also [12, Theorem 6.61].

Theorem 6.1 (Murota and Shioura [13, Theorem 4.15]). *For an M-convex function $f : \mathbb{Z}^n \to \mathbb{R}$ and $x \in \mathrm{dom} f$, define $\gamma_{f,x}(u,v) = f(x + \chi_u - \chi_v) - f(x)$ $(u, v \in X)$. Then $\gamma_{f,x}$ is a distance.*

By Theorem 6.1, $\gamma_{f,x}$ can be regraded as a discrete function on Ω for each $x \in \mathrm{dom} f$ by the correspondence:

$$\gamma_{f,x}(\chi_i - \chi_j) \leftarrow \gamma_{f,x}(i,j) \ (i,j \in X).$$

In fact, the discrete split decomposition is applicable to $\gamma_{f,x}$. We are particularly interested in the case that f is a quadratic M-convex function on $\mathbb{Z}^n \cap \mathrm{cone}\Omega$, i.e., f can be represented as $f(x) = \frac{1}{2}x^\top Ax$ for all $x \in \mathbb{Z}^n \cap \mathrm{cone}\Omega$ with some coefficient matrix A. The reason is that tree metrics and quadratic M-convex functions are closely related. For a tree metric $d : X \times X \to \mathbb{R}_+$, an $n \times n$ square matrix $D = (d_{ij})$ is defined as $d_{ij} = d(i,j)$ for all $i, j \in X$ and called a tree metric matrix.

Theorem 6.2 (Hirai and Murota [8, Theorem 3.1]). *A quadratic form $f(x)$ defined on $\mathbb{Z}^n \cap \mathrm{cone}\,\Omega$ is M-convex if and only if there exists a tree metric $d : X \times X \to \mathbb{R}_+$ such that*

$$f(x) = -\frac{1}{2}x^\top Dx \quad (x \in \mathbb{Z}^n \cap \mathrm{cone}\,\Omega),$$

where D is a tree metric matrix for d.

Let f be a quadratic M-convex function on $\mathbb{Z}^n \cap \mathrm{cone}\,\Omega$, and let $f(x) = -\frac{1}{2}x^\top Dx$ for some tree metric matrix D. Then $\gamma_{f,x}$ for f and $x \in \mathbb{Z}^n \cap \mathrm{cone}\,\Omega$ is as follows:

$$\begin{aligned}
\gamma_{f,x}(u,v) &= f(x + \chi_u - \chi_v) - f(x) \\
&= -\frac{1}{2}(x + \chi_u - \chi_v)^\top D(x + \chi_u - \chi_v) + \frac{1}{2}x^\top Dx \\
&= -x^\top D\chi_u + x^\top D\chi_v - \frac{1}{2}\chi_u^\top D\chi_u + \chi_v^\top D\chi_u + \frac{1}{2}\chi_v^\top D\chi_v \\
&= \langle -x^\top D, \chi_u - \chi_v\rangle + d(u,v) \quad (u, v \in X).
\end{aligned}$$

Therefore, by the correspondence:

$$d(\chi_i - \chi_j) \leftarrow d(i,j) \quad (i, j \in X),$$

$\gamma_{f,x}$ can be regarded as a discrete function on Ω as follows:

$$\gamma_{f,x}(\cdot) = d(\cdot) + (\langle -x^\top D, \cdot\rangle)^\Omega.$$

By Lemma 4.12, we have $\overline{\gamma_{f,x}} = \overline{d} + \langle -x^\top D, \cdot\rangle + \delta_{\mathrm{cone}\,\Omega}$. Furthermore, $\tilde{c}_{H_{A,B}}(\gamma_{f,x})$ for $H_{\{A,B\}}$ is represented as follows:

$$\begin{aligned}
\tilde{c}_{H_{\{A,B\}}}(\gamma_{f,x}) &= \frac{\sqrt{|X|}}{2} \min_{i,j \in A, k,l \in B} \Big\{ \gamma_{f,x}(\chi_i - \chi_k) + \gamma_{f,x}(\chi_l - \chi_j) - 2\overline{\gamma_{f,x}^{\Omega \cap H_{\{A,B\}}}}\Big(\frac{\chi_i - \chi_k + \chi_l - \chi_j}{2}\Big), \\
&\qquad \gamma_{f,x}(\chi_i - \chi_l) + \gamma_{f,x}(\chi_k - \chi_j) - 2\overline{\gamma_{f,x}^{\Omega \cap H_{\{A,B\}}}}\Big(\frac{\chi_i - \chi_l + \chi_k - \chi_j}{2}\Big)\Big\} \\
&= \frac{\sqrt{|X|}}{2} \min_{i,j \in A, k,l \in B} \Big\{ d(i,k) + d(l,j) - d(i,j) - d(l,k), d(i,l) + d(k,j) - d(i,j) - d(k,l) \Big\} \\
&= \sqrt{|X|}\, b^d_{\{A,B\}}
\end{aligned}$$

because, by direct calculation, we have

$$\overline{\gamma_{f,x}^{\Omega \cap H_{\{A,B\}}}}\Big(\frac{\chi_i - \chi_k + \chi_l - \chi_j}{2}\Big) = \frac{1}{2}(d(\chi_i - \chi_j) + \langle -x^\top D, \chi_i - \chi_j\rangle + d(\chi_l - \chi_k) + \langle -x^\top D, \chi_l - \chi_k\rangle)$$

and

$$\overline{\gamma^{\Omega \cap H_{\{A,B\}}}}\Big(\frac{\chi_i - \chi_l + \chi_k - \chi_j}{2}\Big) = \frac{1}{2}(d(\chi_i - \chi_j) + \langle -x^\top D, \chi_i - \chi_j\rangle + d(\chi_k - \chi_l) + \langle -x^\top D, \chi_k - \chi_l\rangle).$$

Since d is a tree metric, we have

$$\gamma_{f,x}(\cdot) = \sum_{\sigma \in \Sigma_b(d)} \sqrt{|X|}\, b^d_\sigma l^\Omega_{H_\sigma}(\cdot) + (\langle -x^\top D, \cdot\rangle)^\Omega$$

by Theorem 5.7 and Proposition 5.6.

For an M-convex function $f : \mathbb{Z}^n \to \mathbb{R}$ and $x \in \mathrm{dom}\, f$, the convex extension of $\gamma_{f,x}(\cdot)$ coincides with the directional derivative $f'(x; \cdot)$ of the convex extension of f, where $f'(x; \cdot)$ is considered as a function of directions. Then, we regard $\gamma_{f,x}(\cdot)$ as the directional derivative of f at a point x. Hence, for a quadratic M-convex function f on $\mathbb{Z}^n \cap \mathrm{cone}\,\Omega$, we conclude that the directional derivative $\gamma_{f,x}(\cdot)$ for each $x \in \mathbb{Z}^n \cap \mathrm{cone}\,\Omega$ is split-decomposable.

Acknowledgments

The author is grateful to Satoru Iwata for his useful comments. The author also thanks Kazuo Mutora for his helpful suggestions. The present work is supported by the 21st Century COE Program on Information Science and Technology Strategic Core.

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