MATHEMATICAL ENGINEERING TECHNICAL REPORTS

Reduced-Order Proper H_∞ Controllers for Descriptor Systems: Existence Conditions and LMI-Based Design Algorithms

Xin XIN, Shinji HARA, and Masahiro KANEDA

(Communicated by Kazuo MUROTA)

METR 2007–13 March 2007

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/edu/course/mi/index_e.shtml

Reduced-Order Proper H_∞ Controllers for Descriptor Systems: Existence Conditions and LMI-Based Design Algorithms

Xin XIN[†], Shinji HARA[‡], and Masahiro KANEDA[†]

[†] Department of Communication Engineering
Faculty of Computer Science and System Engineering
Okayama Prefectural University
{xxin,kaneda}@c.oka-pu.ac.jp
[‡] Department of Information Physics and Computing
Graduate School of Information Science and Technology
The University of Tokyo
{Shinji_Hara}@ipc.i.u-tokyo.ac.jp

March, 2007

Abstract

In this paper, we present a new approach to investigate the existence and design of reduced-order proper H_∞ controllers that provide the same level of performance as that of full-order controllers. By examining some special features of the LMI-based solvability conditions for the H_∞ control problem for descriptor systems, we obtain a refined bound on the order of H_∞ controllers, which is *independent of* (*invariant under the allowed transformations on*) a descriptor realization of the generalized plant. Moreover, we provide two LMI-based algorithms to design reduced-order controllers; and we demonstrate the validity of the theoretical results obtained in this paper via two numerical examples. This paper not only extends in a satisfying way the results on reduced-order H_∞ controllers for state-space systems to descriptor systems, but also provides insight into the mechanism by which the order of H_∞ controllers for descriptor systems can be reduced through a consideration of the unstable finite zeros or infinite zeros.

1 Introduction

In many practical applications, the descriptor (also known as implicit, singular, semistate) system description provides a natural mathematical representation of many practical systems because it is able to describe nondynamic constraints, and finite dynamic and impulsive behavior simultaneously (see [8] and references therein).

Glover *et al.* [5] provided a descriptor representation of all the solutions to the four-block general distance problem, which arises in *standard* H_∞ optimal control. To remove the assumptions required in [2] on the infinite and finite $j\omega$-axis zeros of the generalized plant in the state-space model, Hara *et al.* [6] and Copeland and Safonov [1] employed a descriptor system representation and demonstrated that it is useful for solving the *singular* H_∞ control problem.

Subsequent studies have dealt with the descriptor H_∞ control problem, which treats the generalized plant in descriptor form [10], [15], [16], [20]. One application is to tackle the mixed sensitivity problem for a physical plant with nonproper weights, because it is often desirable to choose some nonproper weights [9] and because relaxing the requirements of the state-space model of the generalized plant provides more freedom in choosing the weights.

Without making any extra assumptions about the direct feedforward matrices in a descriptor realization of the generalized plant, Masubuchi [11, 12] obtained better solvability conditions for the H_∞ control problem for descriptor systems in terms of LMIs and a rank constraint, he showed that there exists a proper H_∞ controller with an order not greater than $\operatorname{rank} E$ for a solvable descriptor H_∞ control problem, see (3) for the definition of E.

For practical applications, it is very important to design reduced-order controllers; and a great deal of research has been done on reduced-order H_∞ controllers for state-space systems. Studies employing an LMI-based approach have appeared on the existence and design of reduced-order controllers for the H_∞ control problem with infinite zeros [21], with real unstable transmission zeros [19], and with unstable invariant zeros [22].

This paper concerns the existence and design of reduced-order proper H_∞ controllers with an order *strictly less than* $\operatorname{rank} E$ for descriptor systems, a topic that has not received much attention. If we directly apply the approach to analyzing and designing reduced-order H_∞ controllers for state-space systems in [22] to the LMI-based results in [11, 12], we can obtain a bound on the order of H_∞ controllers. Although this bound includes the corresponding result for state-space systems in [22] as a special case, it is dependent on a descriptor realization of the generalized plant and can not reveal the existence of a reduced-order controller for a particular descriptor realization.

In this paper, we present a new approach to investigate the existence and design of reduced-order proper H_∞ controllers by examining some special features of the LMI-based solvability conditions for the H_∞ control problem for descriptor systems. We obtain a refined bound on the order of H_∞ controllers, which is expressed in terms of the original parameter matrices of a descriptor realization of the generalized plant. We show that a prominent feature of this bound is the invariance under the *allowed transformations*, which are proposed by Verghese *et al.* [18] and are widely used in analyzing descriptor systems, *on a descriptor realization of the generalized plant*. In this sense, *this bound is independent of a particular descriptor realization of the generalized plant*. Moreover, we provide two LMI-based algorithms to design reduced-order controllers; and we demonstrate the validity of the theoretical results obtained in this paper via two numerical examples.

This paper not only extends the results on reduced-order H_∞ controllers for state-space systems to descriptor systems, but also provides insight into the mechanism by which the order of a controller can be reduced through a consideration of the unstable finite zeros or infinite zeros, and reveals some special features of the descriptor H_∞ control problem.

Notation

1. $\mathbb{C}$: the open complex plane.
2. $\mathbb{R}^{m\times r}$: the set of all $m \times r$ constant real matrices.
3. I_n: identity matrix of size $n \times n$.
4. A^{T}: the transpose of matrix A.
5. $\mathbf{He}\,A$: $A + A^{\mathrm{T}}$ for the square matrix A.
6. $B^\perp$: full-row-rank matrix with the maximal number of rows satisfying $B^\perp B = 0$; that is, the rows of $B^\perp$ represent the basis of the left null space of B.
7. $X > 0$ ($X \geq 0$, $X < 0$): X is symmetric positive (semi-positive, negative) definite.

2 Preliminaries

2.1 Descriptor Systems, and Finite and Infinite Zeros of Matrix Pencils

Consider system Σ with the following descriptor realization:

$$\Sigma : \left\{ \begin{array}{rcl} E\dot{x} & = & Ax + Bw \\ z & = & Cx + Dw \end{array} \right. , \tag{1}$$

where $x \in \mathbb{R}^n$ is the descriptor variable, w is the input, and z is the output of the system. Assume that $E \in \mathbb{R}^{n\times n}$ and $-sE + A$ is regular; that is, $\det(-sE + A) \not\equiv 0$. The finite eigenvalues of the pencil $-sE + A$ (which are the roots of $\det(-sE + A) = 0$) are called the finite dynamic modes of Σ. The infinite eigenvalues of $-sE + A$ are defined to be the zero eigenvalues of $-sA + E$. The infinite eigenvalues corresponding to grade-one infinite generalized eigenvectors, v_i^1, that satisfy $Ev_i^1 = 0$ are called the nondynamic modes of Σ. The infinite eigenvalues corresponding to the grade-k ($k \geq 2$) infinite generalized eigenvectors, v_i^k, that satisfy $Ev_i^k = Av_i^{k-1}$ are called the impulsive modes of Σ. The system Σ is *admissible* if Σ has neither any impulsive modes nor any unstable finite dynamic modes.

For the pencil (regular or singular) $H(\lambda) = -\lambda K + L$, $\lambda_0 \in \mathbb{C}$ is a *finite zero* of $H(\lambda)$ if $\operatorname{rank} H(\lambda_0) <$ normal $\operatorname{rank} H(\lambda)$, where normal $\operatorname{rank} H(\lambda)$ is the rank of $H(\lambda)$ almost everywhere in $\lambda \in \mathbb{C}$. The zero structure of $H(\lambda)$ at infinity is defined as the zero structure of $H(\lambda^{-1})$ at $\lambda = 0$ Verghese *et al.* [17] concluded that a kth-order infinite elementary divisor of a pencil (in the terminology of the Kronecker pencil theory) corresponds to a $(k-1)$th-order zero at infinity.

We review the definitions of the *restricted system equivalence (RSE) transformations* [13] and the *allowed transformations* [18] in Appendix A. The latter ones, which include the former ones as special cases, preserve the transfer function matrix of the original system, and are used more often in analyzing descriptor systems.

2.2 LMI-Based H_∞ Control for Descriptor Systems

We recall the LMI-based solvability conditions for the H_∞ control problem for descriptor systems in [11, 12]. Consider a generalized plant, $G(s)$, described by

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}. \tag{2}$$

Its descriptor realization is

$$\begin{cases} E\dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \ \text{with } D_{22} = 0, \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$ is the descriptor variable vector, $w \in \mathbb{R}^{m_1}$ is the exogenous input vector, $u \in \mathbb{R}^{m_2}$ is the control input vector, $z \in \mathbb{R}^{p_1}$ is the controlled error vector, and $y \in \mathbb{R}^{p_2}$ is the observation output vector. Assume that $E \in \mathbb{R}^{n \times n}$ and $-sE + A$ is regular. Let $r = \operatorname{rank} E$.

Consider a controller, $C(s)$, given by

$$\begin{bmatrix} E_c \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix}, \tag{4}$$

where $x_c \in \mathbb{R}^{n_c}$ and $E_c \in \mathbb{R}^{n_c \times n_c}$.

For a given $\gamma > 0$, the (suboptimal) H_∞ control problem is to find a control law, $u(s) = C(s)y(s)$, such that the closed-loop system is admissible (stable and impulsive-free) and $||T_{zw}(s)||_\infty < \gamma$, where $T_{zw}(s)$ is the closed-loop transfer function matrix from w to z.

Lemma 1 *[11, 12]* For a given $\gamma > 0$, the H_∞ control problem for the generalized plant (3) is solvable if and only if there exist matrices $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times m_1}$, and $Z \in \mathbb{R}^{m_1\times n}$ such that

$$\begin{bmatrix} E^{\mathrm{T}}X & E^{\mathrm{T}} \\ E & EY^{\mathrm{T}} \end{bmatrix} \geq 0, \tag{5}$$

$$E^{\mathrm{T}}W = 0, \quad EZ^{\mathrm{T}} = 0, \tag{6}$$

$$L_B(Y, Z) < 0, \tag{7}$$

$$L_C(X, W) < 0, \tag{8}$$

where

$$L_B(Y,Z) = \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} AY^{\mathrm{T}} + YA^{\mathrm{T}} & YC_1^{\mathrm{T}} & B_1 + AZ^{\mathrm{T}} \\ C_1Y^{\mathrm{T}} & -\gamma I & D_{11} + C_1Z^{\mathrm{T}} \\ B_1^{\mathrm{T}} + ZA^{\mathrm{T}} & D_{11}^{\mathrm{T}} + ZC_1^{\mathrm{T}} & -\gamma I \end{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^{\perp\mathrm{T}}, \tag{9}$$

$$L_C(X,W) = \begin{bmatrix} C_2^{\mathrm{T}} \\ D_{21}^{\mathrm{T}} \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} A^{\mathrm{T}}X + X^{\mathrm{T}}A & X^{\mathrm{T}}B_1 + A^{\mathrm{T}}W & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}X + W^{\mathrm{T}}A & B_1^{\mathrm{T}}W + W^{\mathrm{T}}B_1 - \gamma I & D_{11}^{\mathrm{T}} \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} C_2^{\mathrm{T}} \\ D_{21}^{\mathrm{T}} \\ 0 \end{bmatrix}^{\perp\mathrm{T}}. \tag{10}$$

If (5)–(8) are satisfied, then an H_∞ controller exists in the state-space form with order, r_c, satisfying

$$r_c \leq r_0(X,Y) := \operatorname{rank} \begin{bmatrix} E^{\mathrm{T}}X & E^{\mathrm{T}} \\ E & EY^{\mathrm{T}} \end{bmatrix} - r. \tag{11}$$

$\diamondsuit$

Using a solution satisfying (5)–(8), Masubuchi [11, 12] provided a synthesis method for obtaining E_c, A_c, B_c, C_c, and D_c in (4) such that $\operatorname{rank} E_c = r_0(X,Y)$ and $-sE_c + A_c$ is impulsive-free by solving an LMI and, if necessary, adding a small perturbation to the solution. Owing to $r_0(X,Y) \leq \operatorname{rank} E = r$, there exists an H_∞ controller in the state-space form with order, r_c, satisfying $r_c \leq r$.

Since *finding a solution satisfying (5)–(8) and $r_0(X,Y) < r$, in general, is non-convex*, we will study the existence conditions and LMI-based design algorithms for reduced-order proper H_∞ controllers with order *strictly* less than r. We investigate how to reduce the order of an H_∞ controller, mainly by exploiting LMI (7), and dually, how to reduce the order by exploiting LMI (8).

3 A Bound on the Order of H_∞ Controllers for Descriptor Systems: A Direct Approach

Without separating the matrix variables EY^{T} and $E^{\mathrm{T}}X$, which determine the order of the H_∞ controller (see (11)), from other matrix variables in the LMIs of Lemma 1, we obtain the following proposition via a direct application of a previous result on the analysis and design of reduced-order H_∞ controllers for state-space systems in [22] (which is Lemma B1 in the Appendix B of this paper).

Propostion 1 For a given $\gamma > 0$, suppose that the H_∞ control problem for the generalized plant (3) is solvable. Then, there exists a proper H_∞ controller whose order, r_c, satisfies

$$r_c \le n_{b0} := \min\left\{ \min_{\mathrm{Re}\,[\lambda]\ge 0} \rho_0(\lambda),\ \rho_{0\infty} \right\} \le r, \tag{12}$$

where $\mathrm{Re}\,[\lambda]$ denotes the real part of $\lambda \in \mathbb{C}$, and

$$\rho_0(\lambda) := \mathrm{rank} \begin{bmatrix} -\lambda E + AE^+E & B_2 \\ C_1E^+E & D_{12} \end{bmatrix} - \mathrm{rank} \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}, \tag{13}$$

$$\rho_{0\infty} := \mathrm{rank} \begin{bmatrix} E & B_2 \\ 0 & D_{12} \end{bmatrix} - \mathrm{rank} \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}, \tag{14}$$

where E^+ is the Moore-Penrose pseudoinverse of E. $\diamondsuit$

Proof: See Appendix C. ■

For the special case $E = I_n$, Proposition 1 reduces to the one in [22] for state-space systems. However, n_{b0} in Proposition 1, is *dependent* on a descriptor realization of a given transfer matrix, $G(s)$. To see this, first, for nonsingular matrices T and P, since in general $(TEP)^+ \ne P^{-1}E^+T^{-1}$ (because both TEE^+T^{-1} and $P^{-1}E^+EP$ being symmetrical is, in general, not true), n_{b0} in (12) is, in general, variant under a RSE transformation on (3).

Second, we illustrate that the values of n_{b0} are different for two descriptor realizations of $G(s)$ which can be transformed to each other via some allowed transformations. Consider a proper $G(s)$, for which a state-space realization is given in (3) with $E = I_n$. By introducing an additional descriptor variable in two different ways, we obtain the following two descriptor realizations for $G(s)$:

$$\begin{bmatrix} \dot{x} \\ 0 \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ 0 & I_{m_1} & -I_{m_1} & 0 \\ \hline C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{array}\right] \begin{bmatrix} x \\ x_a \\ \hline w \\ u \end{bmatrix}, \tag{15}$$

$$\begin{bmatrix} \dot{x} \\ 0 \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ 0 & I_{m_2} & 0 & -I_{m_2} \\ \hline C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{array}\right] \begin{bmatrix} x \\ x_b \\ \hline w \\ u \end{bmatrix}. \tag{16}$$

Using (12), we know that n_{b0} for realization (15) is the same as the bound in [22]. However, we can see that n_{b0} for realization (16) is

$$n_{b0} = \min_{\mathrm{Re}\,[\lambda]\geq 0} \operatorname{rank} \begin{bmatrix} -\lambda I_n + A \\ C_1 \end{bmatrix}, \tag{17}$$

which is greater than or equal to the bound in [22].

Thus, for a given descriptor realization, the bound in Proposition 1 may fail to reveal the existence of a reduced-order controller. Indeed, from the two numerical examples in this paper, we can see that a reduced-order controller cannot be shown to exist by using Proposition 1. Let us consider the following generalized plant, $G(s)$, taken from [15], which considered the H_∞ control problem for $G^{\mathrm{T}}(s)$ with $\gamma = 1$.

$$G(s) = \left[\begin{array}{cc|c} \dfrac{2s}{s+1} & s & \dfrac{s^2+2s-1}{s+1} \\ \hline 2 & s+2 & s+1 \end{array}\right]. \tag{18}$$

A descriptor form representation of this plant is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix},\quad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix},\ D_{12} = 1,\ D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Takaba *et al.* [15] gave the following controller, for which $||T_{zw}||_\infty = 0.934$:

$$C(s) = \frac{-(s+1.807)(s+1.452)}{2(s+1.301)(s+2.043)}. \tag{19}$$

Since $G_{12}(s)$ has an unstable transmission zero at $s = -1+\sqrt{2}$, based on the fact of controller order reduction owing to an unstable invariant zero for state-space systems, we expect the existence of a reduced-order controller for (18). However, from (12) in Proposition 1, we obtain $n_{b0} = 2$, which fails to show that a reduced-order controller exists.

Therefore, *a direct application of the result for the state-space realization to the descriptor system is not good enough*; and further investigation is needed to obtain a refined result, which is discussed in the next section.

4 A Refined Bound on the Order of H_∞ Controllers and A Design Algorithm

4.1 A Refined Bound on the Order of H_∞ Controllers

By examining some special features of the LMI-based solvability conditions for the H_∞ control problem for descriptor systems, we can *separate* EY^{T} and $E^{\mathrm{T}}X$, which determine the order of the H_∞ controller, *from* other matrix variables in the LMIs of Lemma 1. This yields more freedom in the choice of matrix variables EY^{T} and $E^{\mathrm{T}}X$ for obtaining a refined bound on the order of H_∞ controllers. This bound is expressed in terms of the original parameter matrices of a descriptor realization of the generalized plant, and is invariant under the allowed transformations on the system matrix of the descriptor realization.

First, we present the following theorem.

Theorem 1 For a given $\gamma > 0$, suppose that the H_∞ control problem for the generalized plant (3) is solvable. Then, there exists a proper H_∞ controller whose order, r_c, satisfies

$$r_c \leq n_b := \min\left\{ \min_{\mathrm{Re}\,[\lambda]\geq 0} \rho(\lambda),\ \rho_\infty \right\}, \tag{20}$$

where

$$\rho(\lambda) : \quad = \quad \mathrm{rank}\, E + \mathrm{rank} \begin{bmatrix} -\lambda E + A & B_2 \\ C_1 & D_{12} \end{bmatrix} - \mathrm{rank} \begin{bmatrix} E & 0 \\ A & B_2 \\ C_1 & D_{12} \end{bmatrix}, \tag{21}$$

$$\rho_\infty := \mathrm{rank} \begin{bmatrix} 0 & E & 0 \\ E & A & B_2 \\ 0 & C_1 & D_{12} \end{bmatrix} - \mathrm{rank} \begin{bmatrix} E & 0 \\ A & B_2 \\ C_1 & D_{12} \end{bmatrix}. \tag{22}$$

Moreover, the following inequalities hold:

$$\rho(\lambda) \leq \rho_0(\lambda) \leq r,\ \rho_\infty \leq \rho_{0\infty} \leq r,\ n_b \leq n_{b0}, \tag{23}$$

where $\rho_0(\lambda)$, $\rho_{0\infty}$, and n_{b0} are defined in (13), (14), and (12), respectively. ♢

Proof: See Appendix D. ■

For the special case $E = I_n$, Theorem 1 reduces to the one in [22] for state-space systems. Moreover, the values of n_b for the two descriptor realizations (15) and (16) are the same as the one in [22] for state-space systems.

Next, we present the following theorem to reveal a prominent difference between n_{b0} in (12) and n_b in (20).

Theorem 2 For the generalized plant $G(s)$ with its descriptor realization in (3), the inequality in (20) holds with n_b being invariant under the allowed transformations on the system matrix of (3). $\diamondsuit$

Proof: See Appendix E. ■

We present a dual result of Theorems 1 and 2.

Theorem 3 For a given $\gamma > 0$, suppose that the H_∞ control problem for the generalized plant (3) is solvable. Then, there exists a proper H_∞ controller whose order, r_c, satisfies

$$r_c \leq \widetilde{n}_b := \min\left\{ \min_{\mathrm{Re}\,[\lambda]\geq 0} \widetilde{\rho}(\lambda),\ \widetilde{\rho}_\infty \right\}, \tag{24}$$

where

$$\widetilde{\rho}(\lambda) := \mathrm{rank}\, E + \mathrm{rank} \begin{bmatrix} -\lambda E + A & B_1 \\ C_2 & D_{21} \end{bmatrix} - \mathrm{rank} \begin{bmatrix} E & A & B_1 \\ 0 & C_2 & D_{21} \end{bmatrix}, \tag{25}$$

$$\widetilde{\rho}_\infty := \mathrm{rank} \begin{bmatrix} 0 & E & 0 \\ E & A & B_1 \\ 0 & C_2 & D_{21} \end{bmatrix} - \mathrm{rank} \begin{bmatrix} E & A & B_1 \\ 0 & C_2 & D_{21} \end{bmatrix}. \tag{26}$$

Moreover, the inequality in (24) holds with $\widetilde{n}_b$ being invariant under the allowed transformations on the system matrix of (3). $\diamondsuit$

The following corollary yields from Theorems 1 and 3.

Corollary 1 Suppose the H_∞ control problem for the generalized plant (3) is solvable. If the system matrix of $G_{12}(s)$ or $G_{21}(s)$ induced from (3), which is

$$\Gamma_{12}(s) := \begin{bmatrix} -sE + A & B_2 \\ C_1 & D_{12} \end{bmatrix}, \quad \text{or} \tag{27}$$

$$\Gamma_{21}(s) := \begin{bmatrix} -sE + A & B_1 \\ C_2 & D_{21} \end{bmatrix}, \tag{28}$$

has unstable finite zeros or infinite zeros, then there exists a proper H_∞ controller whose order, r_c, is strictly less than $\mathrm{rank}\, E$. $\diamondsuit$

The system matrices of $G_{12}(s)$ and $G_{21}(s)$ play an important role in GARE (generalized algebraic Riccati equation) based solutions of the H_∞ control problem for the descriptor system [15, 20]. However, their role is somewhat unclear in the LMI-based approach to this problem. We clarified their role in LMI-based solutions by establishing the relationship between the bound on the order of the H_∞ controller, and the (unstable finite or infinite) zeros of the system matrices.

Finally, we present a remark to reveal some special features of the LMI-based solvability conditions for the H_∞ control problem for descriptor systems in [11, 12].

Remark 1 Using the projection lemma for variable elimination in LMIs (C8) and (C9), as we illustrated in (D3) and (D4) for LMI (C8), we can show that the LMI-based solvability conditions in [11, 12] are equivalent to the negative definiteness of ten constant matrices, and the feasibility of three LMIs which contain only the matrix variables determining the controller order and have the same forms as those for the H_∞ control problem for state-space systems (see (B1)–(B3) in the Appendix B). $\diamondsuit$

4.2 A Constructive Design Algorithm

Using the results in Appendices C and D, we provide an algorithm for computing a reduced-order H_∞ controller. The algorithm assumes that $\Gamma_{12}(s)$ in (27) has unstable finite zeros or infinite zeros and $G(s)$ has the structure of (C2).

Algorithm 1

1). Solve LMIs (C4), (C9), (D3), and (D4) to obtain a solution, which is denoted $(Y_{p11}, Y_{p22}, Z_{p2}, X_p, W_p)$.

2). Let λ_0 be an unstable finite zero or infinite zero of $\Gamma_{12}(s)$. Using $J_2^\perp N_1$, $J_2^\perp J_1$ (which is related to LMI (D3)), λ_0, X_{p11}, and Y_{p11}, apply Algorithm 1 in [22] to obtain $\bar{Y}_{p11}$ such that $\text{rank}\,(X_{p11} - \bar{Y}_{p11}) \leq \text{rank}\,\rho(\lambda_0) < r$.

3). *Solve (C8) for* Y_{12} *with* $Y_{11} = \bar{Y}_{p11}$, $Y_{22} = Y_{p22}$, *and* $Z_2 = Z_{p2}$. *Let a solution be* $\bar{Y}_{p12}$.

4). Construct a reduced-order proper controller of order $\text{rank}\,(X_{p11} - \bar{Y}_{p11})$ using the algorithm in [11, 12], and the solution $(\bar{Y}_{p11}, \bar{Y}_{p12}, Y_{p22}, Z_{p2}, X_p, W_p)$ to LMIs (C4), (C8), (C9), (D3), and (D4).

We apply Theorem 1 and Algorithm 1 to the generalized plant (18), for which $\Gamma_{12}(s)$ has an unstable finite zero at $s = \lambda_0 = -1 + \sqrt{2}$. From (20) in Theorem 1, we have $n_b = \text{rank}\,\rho(\lambda_0) = 1$. This means that a first-order H_∞ controller exists. Using Algorithm 1 with the aid of the LMI control toolbox [4], we obtain

$$X_{p11} = \begin{bmatrix} 2.4313 & -1.8865 \\ -1.8865 & 4.4750 \end{bmatrix},\ Z_{p2} = \begin{bmatrix} 0.0000 \\ 1.0000 \end{bmatrix},$$

$$Y_{p11} = \begin{bmatrix} 4.2950 & 1.5860 \\ 1.5860 & 1.0833 \end{bmatrix},$$

$$X_{p21} = \begin{bmatrix} -9.7506 & -4.6344 \end{bmatrix},\ X_{p22} = -7.2085,$$

$$Y_{p22} = 9.1729,\ W_{p2} = \begin{bmatrix} -1.6530 & -12.8281 \end{bmatrix},$$

$$\bar{Y}_{p11} = \begin{bmatrix} 4.2599 & 1.6706 \\ 1.6706 & 0.8792 \end{bmatrix}, \quad \bar{Y}_{p12} = \begin{bmatrix} -1.0420 \\ 4.5950 \end{bmatrix}.$$

The eigenvalues of $X_{p11} - Y_{p11}$ are 3.3711 and 1.0189, while those of $X_{p11} - \bar{Y}_{p11}$ are 1.5226 and 6×10^{-16}. We thus obtain

$$C(s) = \frac{-0.4152(s + 1.643)}{s + 1.721}. \tag{29}$$

Under this controller, the closed-loop system is impulsive-free and has the poles -3.2946, -1.7569, and -1; and $||T_{zw}||_\infty = 0.8029$. In addition, $\gamma_{opt} = 0.7678$, which was obtained by solving (C4), (C8), and (C9), provides the optimal H_∞ performance for $G(s)$.

5 Efficient Algorithm for Designing Reduced-Order H_∞ Controllers

For the H_∞ control problem for state-space systems, Skelton *et al.* [14] (p. 167) suggested the use of $\min \operatorname{trace}(X + Y)$ in combination with the three LMIs as a heuristic method of constructing a reduced-order H_∞ controller; and Xin [22] proved that such linear objective minimization always yields a controller of order not greater than the bound determined by the unstable invariant zeros or infinite zeros of $G_{12}(s)$ or $G_{21}(s)$. We can extend this result to descriptor systems as shown in the following theorem.

Theorem 4 For a given $\gamma > 0$, suppose that the H_∞ control problem for the generalized plant (3) is solvable. Then, there exist scalars $\epsilon_B > 0$ and $\epsilon_C > 0$ such that

$$L_B(Y, Z) + \epsilon_B I \leq 0, \quad L_C(X, W) + \epsilon_C I \leq 0, \tag{30}$$

where $L_B(Y, Z)$ and $L_C(X, W)$ are defined in (9) and (10), respectively. Let

$$(X_m, Y_m, W_m, Z_m) = \arg \min_{X,Y,W,Z} \operatorname{trace}(E^{\mathrm{T}} X + E Y^{\mathrm{T}}) \quad \text{subject to (5), (6), and (30).} \tag{31}$$

Then, X_m and Y_m satisfy

$$\operatorname{rank} \begin{bmatrix} E^{\mathrm{T}} X_m & E^{\mathrm{T}} \\ E & E Y_m^{\mathrm{T}} \end{bmatrix} - r \leq \min(n_b,\ \widetilde{n}_b), \tag{32}$$

where n_b and $\widetilde{n}_b$ are defined in (20) and (24), respectively. ♢

Note that, if (7) and (8) rather than (30) are used in the optimization problem (31), then the optimal solution may lie on the boundary of $L_B(Y, Z) < 0$ or $L_C(X, W) < 0$.

Proof: Let (X_p, Y_p, W_p, Z_p) be a solution of (5)–(8). Then, (30) holds for any scalars ϵ_B and ϵ_C satisfying $0 < \epsilon_B < \lambda_{\min}(-L_B(Y_p, Z_p))$ and $0 < \epsilon_C < \lambda_{\min}(-L_C(X_p, W_p))$, where $\lambda_{\min}(\Psi)$ denotes the minimal eigenvalue of matrix $\Psi > 0$.

Without loss of generality, we assume that the generalized plant (3) has the structure of (C2). Owing to (C3), we have $\text{trace}\,(E^{\mathrm{T}}X + EY^{\mathrm{T}}) = \text{trace}\,(X_{11} + Y_{11})$. The rest of the proof is similar to that of Theorem 4 in [22]. ∎

We now provide an algorithm for computing a reduced-order H_∞ controller based on Theorem 4.

Algorithm 2

1). Solve LMIs (5)–(8) to obtain a solution, which is denoted (X_p, Y_p, W_p, Z_p).

2). Choose the scalars ϵ_B and ϵ_C such that $0 < \epsilon_B < \lambda_{\min}(-L_B(Y_p, Z_p))$ and $0 < \epsilon_C < \lambda_{\min}(-L_C(X_p, W_p))$.

3). Solve (31) to obtain (X_m, Y_m, W_m, Z_m).

4). Construct a reduced-order proper controller using the algorithm in [11, 12], and the solution (X_m, Y_m, W_m, Z_m).

We apply Algorithm 2 to the following $G(s)$, for which the H_∞ control problem was studied in [11, 12]:

$$G(s) = \left[\begin{array}{cc:c} \dfrac{s^2+3s+5}{s^2+2s+3} & 0 & \dfrac{s^3+8s^2+14s+4}{(s^2+2s+3)(s^2+5s+2)} \\ 0 & 0 & 1 \\ \hdashline \dfrac{s^2}{s^2+2s+3} & 1 & \dfrac{s(s^4+8s^3+21s^2+21s+6)}{(s^2+2s+3)(s^2+5s+2)} \end{array}\right] \tag{33}$$

with the descriptor form representation of this plant being $E = \text{diag}(I_5, 0)$ and

$$A = \left[\begin{array}{ccccc:cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -2 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right], B_2 = \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \hdashline 0 \\ 1 \end{array}\right],$$

$$B_1 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hdashline 1 & 0 \\ 0 & 0 \end{array}\right], \; C_1^{\mathrm{T}} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 0 \\ 1 & 0 \\ \hdashline -2 & 0 \\ 0 & 0 \end{array}\right], C_2^{\mathrm{T}} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 1 \end{array}\right],$$

$$D_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Masubuchi [11, 12] stated that $\gamma_{opt} = 1.7064$ provides the optimal H_∞ performance.

From (12), we obtain $n_{b0} = 5$. This fails to show that a reduced-order controller exists. Notice that $G_{12}(s)$ and $G_{21}(s)$ are both proper and have neither finite zeros nor infinite zeros; and note that $G_{11}(s)$ and $G_{22}(s)$ are stable. This means that *the system matrices* of $G_{12}(s)$ and $G_{21}(s)$ have no unstable finite zeros. However, since $G_{22}(s)$ is improper (that is, it has a pole at infinity), the system matrices of $G_{12}(s)$ and $G_{21}(s)$ inherit that pole as their infinite zero. Indeed,

$$\begin{bmatrix} A_{22} & B_{22} \\ C_{12} & D_{12} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} A_{22} & B_{12} \\ C_{22} & D_{21} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

are not full-rank matrices. From (20)–(22) in Theorem 1, we obtain $n_b = 4 < 5$. This shows that a reduced-order H_∞ controller exists.

Setting $\gamma = 1.8 > \gamma_{opt} = 1.7064$ and using Algorithm 1, we obtain the following fourth-order controller:

$$C(s) = \frac{-0.001(s-315.4)(s+4.562)(s-1.74)(s+0.4383)}{(s+4.226)(s+0.3838)(s^2+2.844s+2.192)}, \tag{34}$$

under which the closed-loop system is impulsive-free and has the poles -721.28, -4.5576, -4.4998, $-1.7756 \pm 1.6896i$, $-1.2197 \pm 0.2362i$, -0.4384, and -0.3693; and $||T_{zw}||_\infty = 1.7294$.

Now, by using Algorithm 2 with $\epsilon_B = \epsilon_C = 0.02$, we obtain (X_m, Y_m, W_m, Z_m), with the largest eigenvalue of $X_{m11} - Y_{m11}^{-1}$ being 0.8647 and the others being less than 8×10^{-7}. This yields the following first-order controller:

$$C(s) = \frac{-0.0344(s+3.899)}{s+0.2612}. \tag{35}$$

Under this controller, the closed-loop system is impulsive-free and has the poles -33.654, -4.5701, $-1.0369 \pm 1.3359i$, -0.4433, and -0.2337; and $||T_{zw}||_\infty = 1.7489 < 1.8$. For this particular plant and the performance $\gamma = 1.8$, minimizing the trace of the linear combination of the matrix variables yields a first-order H_∞ controller for which the order is less than that determined by the bound n_b.

6 Conclusions

This paper concerns the existence and design of reduced-order proper H_∞ controllers for descriptor systems with the same level of performance as that of full-order controllers. Using the projection lemma for variable elimination, we *separated* the matrix variables determining the order of the H_∞ controller *from* other matrix variables in Lemma 1. This yields more freedom in choosing the matrix variables determining the controller order to obtain the refined bound given in (32) on the controller order, which is expressed in terms of the original parameter matrices of the system matrices of $G_{12}(s)$ and $G_{21}(s)$. We showed that a prominent feature of this bound is the invariance under the allowed transformations on a descriptor realization of the generalized plant. When the H_∞ control problem for descriptor systems is solvable, a reduced-order controller can be shown to exist if either of these two system matrices has unstable finite zeros or infinite zeros. Two numerical examples demonstrated the validity of the theoretical results obtained in this paper. We not only extended in a satisfying way the results on reduced-order H_∞ controllers for state-space systems to descriptor systems, but also provided insight into the mechanism by which the order of H_∞ controllers for descriptor systems can be reduced.

References

[1] B. R. Copeland and M. G. Safonov, "A generalized eigenproblem solution for singular H_2 and H_∞ problems," In C. T. Leondes (Ed.), *Advances in Robust Control Systems Techniques* & *Applications*, Academic Press, New York, 1992.

[2] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831–847, 1989.

[3] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *International Journal of Robust and Nonlinear Control*, vol. 4, pp. 421–448, 1994.

[4] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI control toolbox - for use with MATLAB*, the Math Works Inc., 1995.

[5] K. Glover, D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally, and M. G. Safonov, "A Characterization of All Solutions to the Four Block General Distance Problem," *SIAM Journal on Control and Optimization*, vol. 29, no. 2, pp. 283–324, 1991.

[6] S. Hara, T. Sugie, and R. Kondo, "H_∞ control problem with $j\omega$-axis zeros," *Automatica*, vol. 28, pp. 55–70, 1992.

[7] T. Iwasaki and R. E. Skelton, "All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas," *Automatica*, vol. 30, no. 8, pp. 1307–1317, 1994.

[8] F. L. Lewis “A survey of linear singular systems,” *Circuits Systems and Signal Processing*, vol. 5, no 1, pp. 3–36, 1986.

[9] G. Meinsma, “Unstable and nonproper weights in H_∞ control,” *Automatica*, vol. 31, pp. 1655–1658, 1995.

[10] I. Masubuchi, Y. Kamitane, A. Ohara, and N. Suda, “H_∞ control for descriptor systems: a matrix inequalities approach,” *Automatica*. vol. 33, pp. 669–673, 1997.

[11] I. Masubuchi, “Synthesis of output feedback controllers for descriptor systems satisfying closed-loop dissipativity,” *Proc. of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005*, pp. 5012–5017, 2005.

[12] I. Masubuchi, “H_∞ control of descriptor systems –Solutions based on realization-independent LMIs,” *Transactions of the Institute of Systems, Control and Information Engineers*, vol. 19, no. 5, pp. 203–209, 2006. (In Japanese)

[13] H. H. Rosenbrock, Structural properties of linear dynamical systems, *International Journal of Control*, vol. 20, no. 2, pp. 191–202, 1974.

[14] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis, *A unified algebraic approach to linear control design*. Taylor & Francis, 1997.

[15] K. Takaba, N. Morihira, and T. Katayama, “H_∞ control for descriptor systems –A J-spectral factorization approach –,” *Proc. of the 31st Conference of Decision and Control*, pp. 2251–2256, 1994.

[16] E. Uezato, M. Ikeda, “Strict LMI conditions for stability, robust stabilization, and H_∞ control of descriptor systems,” *Proc. of the 38th Conference of Decision and Control*, pp. 4092–4097, 1999.

[17] G. Verghese, P. Van Dooren, and T. Kailath, “Properties of the system matrix of a generalized state-space system,” *International Journal of Control*, vol. 30, no. 2, pp. 235–243, 1979.

[18] G. Verghese, B. Lévy and T. Kailath, “A generalized state-space for singular systems,” *IEEE Transactions on Automatic Control*, vol. 26, no. 4, pp. 811–831, 1981.

[19] T. Watanabe and A. A. Stoorvogel, “Plant zero structure and further order reduction of a singular H_∞ controller,” *International Journal of Robust and Nonlinear Control*, vol. 12, pp. 591–619, 2002.

[20] H. S. Wang, C. F. Yung, and F. R. Chang, “Bounded real lemma and H_∞ control for descriptor systems,” *IEE Proceeding D: Control Theory and Its Applications*, vol. 145, no. 3, pp. 316–322, 1998.

[21] X. Xin, L. Guo, and C. B. Feng, "Reduced-order controllers for continuous and discrete time singular H_∞ control problems based on LMI," *Automatica*, vol. 32, pp. 581–585, 1996.

[22] X. Xin, "Reduced-order controllers for the H_∞ control problem with unstable invariant zeros," *Automatica*, vol. 40, pp. 319–326, 2004.

Appendix A RSE transformations and allowed transformations

Let $\Gamma(s)$ be the system matrix of system Σ in (1), that is,

$$\Gamma(s) = \begin{bmatrix} -sE + A & B \\ C & D \end{bmatrix}. \tag{A1}$$

The *restricted system equivalence (RSE) transformations [13]* on the system matrix of (1) is

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \Gamma(s) \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -s\bar{E} + \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}, \tag{A2}$$

where T and P are nonsingular.

Second, the following three operations are termed allowed transformations [18] (p. 187):
i). The operation of strong equivalence, which is defined as:

$$\begin{cases} \begin{bmatrix} T & 0 \\ S & I \end{bmatrix} \Gamma(s) \begin{bmatrix} P & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} -s\bar{E} + \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}, \\ SE = 0 \quad \text{and} \quad ER = 0, \end{cases} \tag{A3}$$

where T and P are nonsingular;
ii). The trivial augmentation of (A1) to

$$\begin{bmatrix} -sE + A & 0 & B \\ 0 & I & 0 \\ C & 0 & D \end{bmatrix}; \tag{A4}$$

iii). The trivial deflation of (A4) to (A1).

Appendix B Reduced-order H_∞ controllers for state-space systems

From [3] and [7], the H_∞ control problem for a generalized plant in the state-space model is solvable if and only if $\mathcal{L} \neq \emptyset$, where

$$\mathcal{L} := \left\{ (X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} : X \in \mathcal{L}_1,\ Y \in \mathcal{L}_2,\ \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \right\}, \tag{B1}$$

where

$$\mathcal{L}_1 := \left\{X \in \mathbb{R}^{n\times n} : X > 0,\ \mathbf{He}\, H_1 X M_1^{\mathrm{T}} + Q_1 < 0\right\}, \tag{B2}$$

$$\mathcal{L}_2 := \left\{Y \in \mathbb{R}^{n\times n} : Y > 0,\ \mathbf{He}\, H_2 Y M_2^{\mathrm{T}} + Q_2 < 0\right\}, \tag{B3}$$

where H_i, M_i, and Q_i $(i = 1, 2)$ are constant matrices. Based on the result in Section 2.2 of [22], we obtain the following:

Lemma B1 *[22] Suppose $\mathcal{L} \neq \emptyset$, and let $(X_p, Y_p) \in \mathcal{L}$. Then,*

1) *there exists X_r such that $(X_r, Y_p) \in \mathcal{L}$ and*

$$\operatorname{rank}(X_r - Y_p^{-1}) \leq \min\left\{\min_{\operatorname{Re}[\lambda]\geq 0} \operatorname{rank}(-\lambda H_1 + M_1), \operatorname{rank} M_1\right\}; \tag{B4}$$

2) *there exists Y_r such that $(X_p, Y_r) \in \mathcal{L}$ and*

$$\operatorname{rank}(X_p - Y_r^{-1}) \leq \min\left\{\min_{\operatorname{Re}[\lambda]\geq 0} \operatorname{rank}(-\lambda H_2 + M_2), \operatorname{rank} M_2\right\}; \tag{B5}$$

♢

It is worth mentioning that Lemma B1 holds for any $\mathcal{L}$ defined in (B1)–(B3) as long as H_i, M_i, and Q_i are constant matrices, because the special structure of H_i, M_i, and Q_i $(i = 1, 2)$, which is inherited from the H_∞ control problem, is not used in the proof of Lemma B1.

Next, we recall the projection lemma as follow:

Lemma B2 *[3, 7]* Given a symmetric matrix $\Psi \in \mathbb{R}^{n\times n}$ and two matrices $\Gamma \in \mathbb{R}^{n\times k}$ and $\Xi \in \mathbb{R}^{n\times k}$, there exists $\Theta \in \mathbb{R}^{k\times k}$ satisfying

$$\Gamma\Theta\Xi^{\mathrm{T}} + \Xi\Theta^{\mathrm{T}}\Gamma^{\mathrm{T}} + \Psi < 0 \tag{B6}$$

if and only if

$$\Gamma^{\perp}\Psi\Gamma^{\perp\mathrm{T}} < 0, \quad \Xi^{\perp}\Psi\Xi^{\perp^{\mathrm{T}}} < 0. \tag{B7}$$

♢

Appendix C Proof of Proposition 1

First, let a singular-value decomposition of E be $E = U\operatorname{diag}\{\Sigma, 0_{q\times q}\}V^{\mathrm{T}}$, where U, $V \in \mathbb{R}^{n\times n}$ are unitary matrices, and $\Sigma > 0$. Letting $M = U\operatorname{diag}\{\Sigma, I_q\}$, we obtain

$$\operatorname{diag}\{I_r, 0_{q\times q}\} = M^{-1}EV. \tag{C1}$$

Letting $\bar{x} := V^{-1}x$ and using M and V, we decompose compatibly the matrices in (3) into

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ 0 \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} M^{-1}AV & M^{-1}B_1 & M^{-1}B_2 \\ \hline C_1V & D_{11} & D_{12} \\ C_2V & D_{21} & 0 \end{array}\right] \begin{bmatrix} \bar{x} \\ \hline w \\ u \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{21} \\ A_{21} & A_{22} & B_{12} & B_{22} \\ \hline C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & 0 \end{array}\right] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \hline w \\ u \end{bmatrix}. \tag{C2}$$

With a little abuse of notations, we still use the same notations X, Y, W, and Z in this new coordinate system; for (C2), we can see from [11, 12] that X, Y, W, and Z satisfying (5) and (6) have the following structures:

$$\begin{cases} X = \begin{bmatrix} X_{11} & 0_{r\times q} \\ X_{21} & X_{22} \end{bmatrix},\ Y = \begin{bmatrix} Y_{11} & Y_{21} \\ 0_{q\times r} & Y_{22} \end{bmatrix}, \\ W = \begin{bmatrix} 0_{r\times m_1} \\ W_2 \end{bmatrix},\ Z = \begin{bmatrix} 0_{m_1\times r} & Z_2 \end{bmatrix}, \end{cases} \tag{C3}$$

where $X_{11} = X_{11}^{\mathrm{T}} \in \mathbb{R}^{r\times r}$, $Y_{11} = Y_{11}^{\mathrm{T}} \in \mathbb{R}^{r\times r}$, and $q = n - r$. Moreover, (5) is equivalent to

$$\begin{bmatrix} X_{11} & I_r \\ I_r & Y_{11} \end{bmatrix} \geq 0, \tag{C4}$$

and (11) is equivalent to

$$r_c \leq \operatorname{rank}(X_{11} - Y_{11}^{-1}) \leq r. \tag{C5}$$

Now, from $E^{+} = V\operatorname{diag}\{\Sigma^{-1}, 0\}U^{\mathrm{T}}$ and $E^{+}E = V\operatorname{diag}\{I_r, 0\}V^{\mathrm{T}}$, it follows that (13) and (14) are equal, respectively, to

$$\rho_0(\lambda) = \operatorname{rank}\begin{bmatrix} -\lambda I_r + A_{11} & B_{21} \\ A_{21} & B_{22} \\ C_{11} & D_{12} \end{bmatrix} - \operatorname{rank}\begin{bmatrix} B_{21} \\ B_{22} \\ D_{12} \end{bmatrix}, \tag{C6}$$

$$\rho_{0\infty} = r + \operatorname{rank}\begin{bmatrix} B_{22} \\ D_{12} \end{bmatrix} - \operatorname{rank}\begin{bmatrix} B_{21} \\ B_{22} \\ D_{12} \end{bmatrix}. \tag{C7}$$

Thus, below we only need to show that, *for the generalized plant* (C2), (12) holds, with (13) and (14) being given by (C6) and (C7), respectively.

Second, for (C2), by using the structure of Y and Z in (C3), we rewrite (7) as

$$\mathbf{He}\, N_1Y_{11}J_1^{\mathrm{T}} + \left\{\mathbf{He}\,(N_1Y_{12} + N_2Y_{22} + FZ_2)J_2^{\mathrm{T}}\right\} + Q < 0, \tag{C8}$$

where

$$\left[\begin{array}{c:c} N_1 & N_2 \end{array}\right] := \begin{bmatrix} B_{21} \\ B_{22} \\ D_{12} \\ 0_{m_1\times m_2} \end{bmatrix}^{\perp} \left[\begin{array}{c:c} I_r & 0 \\ 0 & I_q \\ 0 & 0 \\ 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{c:c:c} J_1 & J_2 & F \end{array}\right] := \begin{bmatrix} B_{21} \\ B_{22} \\ D_{12} \\ 0_{m_1\times m_2} \end{bmatrix}^{\perp} \left[\begin{array}{c:c:c} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ C_{11} & C_{12} & 0 \\ 0 & 0 & I_{m_1} \end{array}\right],$$

$$Q := \begin{bmatrix} B_{21} \\ B_{22} \\ D_{12} \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} 0 & 0 & 0 & B_{11} \\ 0 & 0 & 0 & B_{12} \\ 0 & 0 & -\gamma I & D_{11} \\ B_{11}^{\mathrm{T}} & B_{12}^{\mathrm{T}} & D_{11}^{\mathrm{T}} & -\gamma I \end{bmatrix} \begin{bmatrix} B_{21} \\ B_{22} \\ D_{12} \\ 0 \end{bmatrix}^{\perp\mathrm{T}}.$$

Dually, (8) can be rewritten in an equivalent form as

$$\mathbf{He}\,\widetilde{N}_1 X_{11}\widetilde{J}_1^{\mathrm{T}} + \left\{\mathbf{He}\,(\widetilde{N}_1 X_{21}^{\mathrm{T}} + \widetilde{N}_2 X_{22}^{\mathrm{T}} + \widetilde{F} W_2^{\mathrm{T}})\widetilde{J}_2^{\mathrm{T}}\right\} + \widetilde{Q} < 0, \qquad \text{(C9)}$$

where the expressions $\widetilde{N}_1$, $\widetilde{J}_1$, $\widetilde{N}_2$, $\widetilde{J}_2$, $\widetilde{F}$, and $\widetilde{Q}$ are omitted due to space limitations.

Third, since LMIs (C4), (C8), and (C9) are feasible, and since *the second and third terms of the left-hand side of (C8) do not contain* Y_{11}, based on Lemma B1, we know that there exists a solution satisfying

$$\operatorname{rank}(X_{11} - Y_{11}^{-1}) \le \min\left\{\min_{\operatorname{Re}[\lambda]\ge 0} \operatorname{rank}(-\lambda N_1 + J_1), \operatorname{rank} N_1\right\}. \qquad \text{(C10)}$$

Indeed, let (Y_{p11}, Y_{p12}, Y_{p22}, Z_{p2}), and (X_p, W_p) with the decomposition (C3) be a solution of LMIs (C4), (C8), and (C9). Consider the following LMI instead of LMI (C8):

$$N_1 Y_{11} J_1^{\mathrm{T}} + J_1 Y_{11} N_1^{\mathrm{T}} + Q_{p1} < 0, \qquad \text{(C11)}$$

where $Q_{p1} := \left\{\mathbf{He}\,(N_1 Y_{p12} + N_2 Y_{p22} + F Z_{p2}) J_2^{\mathrm{T}}\right\} + Q$ is *a constant matrix*. Since LMIs (C4), (C9), and (C11) are feasible, (C10) follows from Statement 2) in Lemma B1.

Using the following equality which can be shown via direct manipulation,

$$\operatorname{rank} B^{\perp} H = \operatorname{rank}\begin{bmatrix} H & B \end{bmatrix} - \operatorname{rank} B, \qquad \text{(C12)}$$

where the matrices H and B have the same number of rows, we can show that $\operatorname{rank}(-\lambda N_1 + J_1)$ and $\operatorname{rank} N_1$, which are less than r, are equal to the right-hand terms of (C6) and (C7), respectively. This proves (12). ■

Appendix D Proof of Theorem 1

First, from (C1) and (C2), we obtain

$$\rho(\lambda) = \operatorname{rank} \begin{bmatrix} -\lambda I_r + A_{11} & A_{12} & B_{21} \\ A_{21} & A_{22} & B_{22} \\ C_{11} & C_{12} & D_{12} \end{bmatrix} - \operatorname{rank} \begin{bmatrix} A_{12} & B_{21} \\ A_{22} & B_{22} \\ C_{12} & D_{12} \end{bmatrix}, \quad \text{(D1)}$$

$$\rho_\infty = r + \operatorname{rank} \begin{bmatrix} A_{22} & B_{22} \\ C_{12} & D_{12} \end{bmatrix} - \operatorname{rank} \begin{bmatrix} A_{12} & B_{21} \\ A_{22} & B_{22} \\ C_{12} & D_{12} \end{bmatrix}. \quad \text{(D2)}$$

Thus, below we only need to show that, *for the generalized plant (C2)*, (20) holds, with $\rho(\lambda)$ and ρ_∞ being given by (D1) and (D2), respectively.

Second, for the generalized plant (C2), we consider LMI (C8). Note that Y_{11} and Y_{12} have the same coefficient matrix, N_1, and that Y_{12}, Y_{22}, and Z_2 have the same coefficient matrix, J_2 in LMI (C8). Thus, using the projection lemma (Lemma B2) enables us to eliminate Y_{12} from (C8) and show that LMI (C8) is feasible if and only if the following LMI is solvable for Y_{11}:

$$L_{B1}(Y_{11}) := \mathbf{He}\, J_2^\perp N_1 Y_{11} (J_2^\perp J_1)^{\mathrm{T}} + J_2^\perp Q J_2^{\perp\mathrm{T}} < 0, \quad \text{(D3)}$$

and the following LMI is solvable for Y_{22} and Z_2:

$$L_{B2}(Y_{22}, Z_2) := \left\{ \mathbf{He}\, (N_1^\perp N_2 Y_{22} + N_1^\perp F Z_2)(N_1^\perp J_2)^{\mathrm{T}} \right\} + N_1^\perp Q N_1^{\perp\mathrm{T}} < 0. \quad \text{(D4)}$$

Since Y_{11} and (Y_{22}, Z_2) are in two different LMIs, we have more freedom in choosing Y_{11} so as to further reduce the rank of $X_{11} - Y_{11}^{-1}$. Since LMIs (C4), (C9), and (D3) are feasible, based on Lemma B1, from the left-hand coefficient matrices of Y_{11} in (D3) (i.e. $J_2^\perp N_1$ and $J_2^\perp J_1$), we know that there exists a solution satisfying

$$\operatorname{rank}(X_{11} - Y_{11}^{-1}) \le \left\{ \min_{\mathrm{Re}\,[\lambda] \ge 0} \operatorname{rank} J_2^\perp(-\lambda N_1 + J_1)\, , \operatorname{rank} J_2^\perp N_1 \right\}. \quad \text{(D5)}$$

Using (C12) repeatedly shows that $\operatorname{rank} J_2^\perp(-\lambda N_1 + J_1)$ and $\operatorname{rank} J_2^\perp N_1$ are equal to the right-hand terms of (D1) and (D2), respectively. This proves (20). Finally, from

$$\operatorname{rank} J_2^\perp(-\lambda N_1 + J_1) \le \operatorname{rank}(-\lambda N_1 + J_1),$$
$$\operatorname{rank} J_2^\perp N_1 \le \operatorname{rank} N_1,$$

it follows that (23) holds. This completes the proof of Theorem 1. ■

Appendix E Proof of Theorem 2

Using (A1), (A3), and (A4), we can show via a straightforward verification that n_b in (20) is invariant under the allowed transformations.

From (A3), it follows that a standard assumption of $D_{22} = 0$ in (3), which is used in Lemma 1, can not be preserved under some allowed transformations. However, this does not disappoint us. Indeed, through a loop shifting transformation, we can show that the existence of one proper H_∞ controller with order r_c for (3) with $D_{22} \neq 0$, *is equivalent to*, that of another proper H_∞ controller with the same order r_c for (3) with y being replaced by $y - D_{22}u = C_2x + D_{21}w$. A detailed proof given below for this statement is similar to the case of the H_∞ control problem for state-space systems (see for example [7] (p.1308)).

Suppose there exists a proper H_∞ controller for (3) with y being replaced by $\widehat{y}$, which is expressed as:

$$\begin{bmatrix} E_c\dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} \widehat{A}_c & \widehat{B}_c \\ \widehat{C}_c & \widehat{D}_c \end{bmatrix} \begin{bmatrix} x_c \\ \widehat{y} \end{bmatrix}, \tag{E1}$$

where $-sE_c + \widehat{A}_c$ is impulsive-free and $r_c = \operatorname{rank} E_c$. We can assume that $D_0 := I + \widehat{D}_c D_{22}$ is nonsingular (if not, we can make this assumption hold by using a small perturbation to D_c). From (E1), it follows that

$$\begin{bmatrix} E_c\dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix}, \tag{E2}$$

is an H_∞ controller for (3), where $C_c := D_0^{-1}\widehat{C}_c$, $D_c := D_0^{-1}\widehat{D}_c$, $A_c := \widehat{A}_c - \widehat{B}_c D_{22} C_c$, and $B_c := \widehat{B}_c(I - D_{22}D_c)$. Since $-sE_c + A_c$ is impulsive-free (if not, we can make $-sE_c + A_c$ impulsive-free by using a small perturbation to A_c), the controller in (E2) is proper.

The reverse process, namely going from (E2) to (E1), can be performed similarly.

Therefore, the inequality (20) holds with n_b being invariant under the allowed transformations. ■