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An Extension of the Disjoint Paths Problem

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Abstract

For a graph G and a collection of vertex pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$, the disjoint paths problem is to find k vertex-disjoint paths $P_1, \ldots, P_k$, where P_i is a path from s_i to t_i for each $i = 1, \ldots, k$. This problem is one of the classic problems in algorithmic graph theory and has many applications, for example in VLSI-design.

As an extension of the disjoint paths problem, we introduce a new problem which we call the stable paths problem. In this problem we are given a graph G and a collection of vertex pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$. The objective is to find k paths $P_1, \ldots, P_k$ such that P_i is a path from s_i to t_i and P_i and P_j have neither common vertices nor adjacent vertices for any distinct i, j.

The stable paths problem has several variants depending on whether k is a fixed constant or a part of the input, whether the graph is directed or undirected, and whether the graph is planar or not. We investigate the computational complexity of several variants of the stable paths problem. We show that the stable paths problem is (i) solvable in polynomial time when k is fixed and G is a directed (or undirected) planar graph, (ii) NP-hard when $k = 2$ and G is an acyclic directed graph, (iii) NP-hard when $k = 2$ and G is an undirected general graph.

1 Introduction

1.1 Disjoint paths problem and basic definitions

Suppose that we are given a graph G and a collection of vertex pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$. The disjoint paths problem is to find k vertex-disjoint paths $P_1, \ldots, P_k$, where P_i is a path from s_i to t_i for each $i = 1, \ldots, k$. This problem is one of the classic problems in algorithmic graph theory and has many applications, for example in VLSI-design.

The disjoint paths problem has several variants depending on whether k is a fixed constant or a part of the input, whether the graph is directed or undirected, etc. The disjoint paths problem was shown to be NP-hard by Knuth (described in [5]) when k is a part of the input. Fortune–Hopcroft–Wyllie [4] proved that the directed version of the problem (DDPP) is NP-hard even if $k = 2$, whereas the problem can be solved in polynomial time when the given digraph is acyclic and k is fixed. However, it was shown that the disjoint paths problem in undirected graphs (DPP) is solvable in polynomial time when $k = 2$ [10], [11], [12]. Then, in 1995, Robertson–Seymour [7] gave a polynomial time algorithm based on the graph minor theory for the DPP when k is fixed. On the other hand, Schrijver [8] gave a polynomial time algorithm for the DDPP when G is a directed planar graph and k is fixed. We summarize the known results on the problem in Table 1 (see [9] for more results).

Table 1: Complexity of DDPP and DPP.

	DDPP	DPP
k: constant	NP-hard (Planar digraph : P [8]) (Acyclic digraph : P [4])	P [7]
k: variable	NP-hard (see [5]) (Planar digraph : NP-hard [6]) (Acyclic digraph : NP-hard [3])	NP-hard (see [5]) (Planar graph: NP-hard [6])

For an undirected graph (or simply a graph) $G = (V, E)$, let uv denote an edge connecting u and v. For $V' \subseteq V$, the *subgraph induced by* V' is a subgraph $G' = (V', E')$, where E' consists of all edges of G spanned by V'. *Contracting* an edge $e = uv$ means deleting e and identifying u and v. A vertex set V' is called a *stable set* if no two vertices in V' share an edge. For a path P traveling vertices $v_0, v_1, \ldots, v_l$ in this order, $v_i v_j \in E$ is called a *shortcut* of P if $0 \leq i < j \leq l$ and $i + 2 \leq j$.

For a directed graph (or a digraph) $D = (V, A)$, let (u, v) denote an arc which starts in u and ends in v, and for an arc $a = (u, v)$ we define $a^{-1} = (v, u)$. For vertices $v_0, v_1, \ldots, v_l$ and arcs $a_1, \ldots, a_l$, a sequence $P = (v_0, a_1, v_1, a_2, \ldots, a_l, v_l)$ is called a *directed path* (or a *dipath*) if $a_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, k$. If no confusion may arise, we sometimes denote $P = (a_1, \ldots, a_l)$ or identify P with its arc set $\{a_1, \ldots, a_l\}$.

Suppose that $D = (V, A)$ is an embedded planar digraph and $\mathcal{F}$ is the set of faces of D. For $a \in A$, let left(a) and right(a) be the faces of D at the left-hand side and the right-hand side of a, respectively. The *dual digraph* D^* of D is a digraph $D^* = (\mathcal{F}, A^*)$ whose arc set A^* is defined by $A^* = \{a^* \mid a \in A\}$, where a^* is an arc from left(a) to right(a).

Let $(G_k, \cdot)$ be the free group generated by $g_1, g_2, \ldots, g_k$, and let 1 denote its unit element. More precisely, G_k consists of all words $b_1 \cdots b_t$, where $t \geq 0$ and $b_1, \ldots, b_t \in \{g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$ such that $b_i b_{i+1} \neq g_j g_j^{-1}$ and $b_i b_{i+1} \neq g_j^{-1} g_j$ for $i = 1, \ldots, t-1$ and $j = 1, \ldots, k$. The product $x \cdot y$ of two words is obtained from the concatenation xy by deleting iteratively all $g_j g_j^{-1}$ and $g_j^{-1} g_j$. A word y is called a *segment* of a word w if $w = xyz$ for certain words x, z. A subset $\Gamma \subseteq G_k$ is called *hereditary* if for each word $y \in \Gamma$ each segment of y belongs to Γ.

1.2 Stable paths problem

As a generalization of the disjoint paths problem, we introduce a new problem called *stable paths problem*. Suppose we are given a graph G and $P_1, \dots, P_k$ are paths in G. We say that $P_1, \dots, P_k$ are *stable* if P_i and P_j have neither common vertices nor adjacent vertices for any distinct i, j. In other words, $P_1, \dots, P_k$ are stable paths if the following two conditions hold:

- Any pair of paths have no common vertices.
- Let H be the graph obtained by contracting all edges in $P_1, \dots, P_k$. For each $i = 1, \dots, k$, let p_i be the vertex of H that corresponds to all vertices on P_i. Then $\{p_1, p_2, \dots, p_k\}$ is a stable set in H.

The stable paths problem is the following problem.

Stable paths problem (SPP)

Input: A graph $G = (V, E)$ and a collection of vertex pairs $\{(s_1, t_1), \dots, (s_k, t_k)\}$.

Output: Stable paths $P_1, \dots, P_k$ in G, where P_i is a path whose end vertices are s_i and t_i for each $i = 1, \dots, k$.

For a digraph $D = (V, A)$, we also introduce a new problem called *directed stable paths problem*. Let $P_1, \dots, P_k$ be dipaths in D. As with undirected graphs, we say that $P_1, \dots, P_k$ are *stable* if P_i and P_j have neither common vertices nor adjacent vertices for any distinct i, j. The directed stable paths problem is defined as follows.

Directed stable paths problem (DSPP)

Input: A directed graph $D = (V, A)$ and a collection of vertex pairs $\{(s_1, t_1), \dots, (s_k, t_k)\}$.

Output: Stable dipaths $P_1, \dots, P_k$ in D, where P_i is a dipath from s_i to t_i for each $i = 1, \dots, k$.

The stable paths problem is an extension of the disjoint paths problem, because any instance of the disjoint paths problem can be reduced to an instance of the stable paths problem as follows. Consider an instance of the disjoint paths problem in a graph $G = (V, E)$ with respect to a collection of vertex pairs $\{(s_1, t_1), \dots, (s_k, t_k)\}$. Let G' be the graph obtained from G by subdividing each edge into two edges, that is, we replace each edge $e = uv \in E$ by a new vertex v_e and two edges uv_e, v_ev. Then solving the stable paths problem in G' with respect to $\{(s_1, t_1), \dots, (s_k, t_k)\}$ corresponds to solving the original disjoint paths problem. Similarly, the directed stable paths problem is an extension of the directed version of the disjoint paths problem.

By the above reduction, we see that the variants of the (directed) stable paths problem which correspond to NP-hard variants of the (directed) disjoint paths problem are NP-hard, that is, we

Table 2: Complexity of SPP and DSPP.

	DSPP	SPP
k: constant	NP-hard (Planar digraph : P) (Acyclic digraph : NP-hard)	NP-hard (Planar graph : P)
k: variable	NP-hard (Planar digraph : NP-hard) (Acyclic digraph : NP-hard)	NP-hard (Planar graph: NP-hard)

obtain the following results:

- When k is a part of the input, the SPP is NP-hard even if the given graph is planar.
- When k is a part of the input, the DSPP is NP-hard even if the given digraph is acyclic or planar.
- The DSPP is NP-hard even if $k = 2$.

In this paper, we reveal the time complexity of several variants of the SPP and the DSPP as shown in Table 2. The rest of the paper is organized as follows. In Section 2, which is the main part of this paper, we show that the DSPP is solvable in polynomial time when the given digraph is planar and k is fixed. This result implies that the SPP is also solvable in polynomial time when the given graph is planar and k is fixed. In Section 3, we give some applications of our algorithm to finding certain structures, called a "hole" and a "theta", in a planar graph. In Section 4, we present NP-hardness results saying that the SPP is NP-hard even if $k = 2$, and the DSPP is NP-hard even if the given digraph is acyclic and $k = 2$.

2 Stable paths problem in planar graphs

When the given digraph D is planar and k is fixed, Schrijver gave a polynomial time algorithm for the directed disjoint paths problem [8]. As a generalization of this result, when the given digraph D is planar and k is fixed we give a polynomial time algorithm for the directed stable paths problem, which is based on Schrijver's algorithm.

Theorem 1. *The directed stable paths problem is solvable in polynomial time, if the given digraph $D = (V, A)$ is planar and k is a fixed constant.*

Before giving the proof of Theorem 1, we show that a polynomial time algorithm for the undirected version is obtained from Theorem 1.

Corollary 2. *The stable paths problem is solvable in polynomial time, if the given graph is planar and k is a fixed constant.*

Proof. Consider an instance of the stable paths problem in a planar undirected graph $G = (V, E)$. Let $G' = (V, A)$ be the directed graph obtained from G by replacing every edge $uv \in E$ with two arcs (u, v) and (v, u). Then solving the stable paths problem in G corresponds to solving the directed stable paths problem in G', and hence the original stable paths problem is solvable in polynomial time by Theorem 1. □

The rest of this section is devoted to the proof of Theorem 1, which is based on Schrijver's algorithm.

2.1 Preliminaries for the proof

Let $D = (V, A)$ be a directed planar graph, and $\{(s_1, t_1), \ldots, (s_k, t_k)\}$ be a collection of vertex pairs. The vertices $s_1, \ldots, s_k, t_1 \ldots, t_k$ are called *terminals*. Without loss of generality, we assume that D is weakly connected and each terminal is incident to exactly one arc. We fix a planar embedding of D. Let $\mathcal{F}$ be the set of all faces of D, and $R \in \mathcal{F}$ be the unbounded face of D.

We say that a function $\phi : A \to G_k$ is a *flow* if the following three conditions hold.

- For $i = 1, \ldots, k$, the arc a leaving s_i satisfies that $\phi(a) = g_i$.
- For $i = 1, \ldots, k$, the arc a entering t_i satisfies that $\phi(a) = g_i$.
- For each vertex $v \in V \setminus \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$,

$$\phi(a_1)^{\epsilon_1} \cdot \phi(a_2)^{\epsilon_2} \cdot \cdots \cdot \phi(a_l)^{\epsilon_l} = 1,$$

  where $a_1, \ldots, a_l$ are the arcs incident with v, in clockwise order, and $\epsilon_i = +1$ if a_i leaves v and $\epsilon_i = -1$ if a_i enters v.

Note that $\phi(a)$ represents the dipaths which go through the arc a. For example, if $\phi(a) = g_1 g_2$ then dipaths P_1 and P_2 go through the arc a and P_1 is to the left of P_2, and if $\phi(a) = g_3^{-1}$ then a dipath P_3 goes through the arc a in the reverse direction of a. The definition of flows means that no pair of dipaths cross at any vertices. We note here the relation between directed stable paths (or directed disjoint paths) and flows. Given a solution $\Pi = (P_1, \ldots, P_k)$ of the DSPP (or the DDPP), we define a function $\psi_\Pi : A \to G_k$ by

$$\psi_\Pi(a) = \begin{cases} g_i & \text{(if } a \text{ is an arc on } P_i\text{)}, \\ 1 & \text{(otherwise)}. \end{cases}$$

Then ψ_Π is obviously a flow.

We say that two functions $\phi, \psi : A \to G_k$ are *R-homologous* if there exists a function $f : \mathcal{F} \to G_k$ such that

- $f(R) = 1$,
- $f(\text{left}(a))^{-1} \cdot \phi(a) \cdot f(\text{right}(a)) = \psi(a)$ for each arc $a \in A$.

It can be easily seen that if ϕ is a flow and ψ is R-homologous to ϕ, then ψ is also a flow.

2.2 Proof of Theorem 1

Schrijver's algorithm is obtained from the following two propositions for the DDPP in an embedded planar digraph.

Proposition 3 (Schrijver [8])**.** *For each fixed k, we can find in polynomial time a collection of flows $\phi_1, \ldots, \phi_N$ with the property that for each solution Π of the DDPP, ψ_Π is R-homologous to at least one of $\phi_1, \ldots, \phi_N$.*

Proposition 4 (Schrijver [8])**.** *There exists a polynomial time algorithm that, for any flow ϕ, either finds a solution Π of the DDPP such that ψ_Π is R-homologous to ϕ or concludes that such a solution does not exist.*

Proposition 3 implies the following as a corollary, because stable paths are a special case of disjoint paths.

Proposition 5. *For each fixed k, we can find in polynomial time a collection of flows $\phi_1, \ldots, \phi_N$ with the property that for each solution Π of the DSPP, ψ_Π is R-homologous to at least one of $\phi_1, \ldots, \phi_N$.*

For the proof of Theorem 1, we need the following proposition, which is the DSPP version of Proposition 4. The proof is given in Section 2.3.

Proposition 6. *There exists a polynomial time algorithm that, for any flow ϕ, either finds a solution Π of the DSPP such that ψ_Π is R-homologous to ϕ or concludes that such a solution does not exist.*

Our algorithm for the DSPP is obtained from Proposition 5 and Proposition 6 as follows.

Proof of Theorem 1. By Proposition 5, we can find a collection of flows $\phi_1, \ldots, \phi_N$ such that for each solution Π of the DSPP, ψ_Π is R-homologous to at least one of $\phi_1, \ldots, \phi_N$. By Proposition 6, we can either find a solution Π of the DSPP such that ψ_Π is R-homologous to ϕ_i or conclude that such a solution does not exist, for each $i = 1, \ldots, N$. Thus we can solve the DSPP in polynomial time when the given digraph is planar and k is a fixed constant. $\square$

2.3 Proof of Proposition 6

In order to show Proposition 4, Schrijver introduced a new problem called cohomology feasibility problem (CFP), and gave a polynomial time algorithm for it. He showed that Proposition 4 can be derived from the polynomial time algorithm for the CFP. In this section, we describe the CFP and show that Proposition 6 can also be obtained from the polynomial time algorithm for the CFP.

Let $D = (V, A)$ be a weakly connected digraph, which may have parallel arcs, and let $r \in V$. Two functions $\phi, \psi : A \to G_k$ are called *r-cohomologous* if there exists a function $f : V \to G_k$ such that

- $f(r) = 1$,
- $\psi(a) = f(u)^{-1} \cdot \phi(a) \cdot f(v)$ for each arc $a = (u, v) \in A$.

Schrijver introduced the following problem called *cohomology feasibility problem (CFP)*, and showed that it can be solved in polynomial time.

Cohomology feasibility problem (CFP)

Input: A weakly connected digraph $D = (V, A)$, a vertex $r \in V$, a function $\phi : A \to G_k$, a hereditary subset $\Gamma(a) \subseteq G_k$ for each arc $a \in A$.

Output: A function $\psi : A \to G_k$ such that ψ is r-cohomologous to ϕ and $\psi(a) \in \Gamma(a)$ for each arc $a \in A$.

Theorem 7 (Schrijver [8])**.** *The CFP is solvable in polynomial time of $|A|, \sigma$, and k, where $\sigma = \max\{|\Gamma(a)| \mid a \in A\}$.*

We are now ready to show Proposition 6.

Proof of Proposition 6. Let $D^* = (\mathcal{F}, A^*)$ be the dual digraph of D with respect to the planar embedding of D. Let A_1 be the set of all chords in all faces of D^*. More precisely, we consider all vertex pairs $F, F' \in \mathcal{F}$ which are on the boundary of a face of D^*, and define A_1 as the set of all arcs $a_{F,F'}$ from F to F'. For each arc $a \in A$, let a^* denote the arc in A^* from left(a) to right(a), and $D^* - a^*$ be the digraph obtained by removing a^* from D^*. Then, two faces left(a) and right(a) of D^* make up a new face v_a of $D^* - a^*$. Let A_a be the set of all chords in v_a which are not chords in left(a) or right(a), and let $A_2 = \bigcup_{a \in A^*} A_a$. We construct a new graph $D^+ = (\mathcal{F}, A^+)$, where $A^+ = A^* \cup A_1 \cup A_2$.

We define $\phi^+ : A^+ \to G_k$ as follows:

- $\phi^+(a^*) = \phi(a)$ for each arc $a \in A$.

- Let $\pi = ((a_1^*)^{\epsilon_1}, (a_2^*)^{\epsilon_2}, \dots, (a_l^*)^{\epsilon_l})$ be the dipath traveling clockwise from F to F' on the boundary of the face of D^* or $D^* - a^*$, where $\epsilon_i \in \{+1, -1\}$. Then $\phi^+(a_{F,F'}) = \phi(a_1)^{\epsilon_1} \cdot \phi(a_2)^{\epsilon_2} \cdot \dots \cdot \phi(a_l)^{\epsilon_l}$ for each $a_{F,F'} \in A_1 \cup A_2$.

We say that ϕ^+ is the *extended function* of ϕ. Note that ϕ^+ can also be defined along a dipath traveling counterclockwise, since ϕ is a flow.

For each arc $a \in A^+$, we define $\Gamma^+(a) \subseteq G_k$ as follows:

- $\Gamma^+(a) = \{1, g_1, \dots, g_k\}$ for each arc $a \in A^*$,
- $\Gamma^+(a_{F,F'}) = \{1, g_1, g_1^{-1}, \dots, g_k, g_k^{-1}\}$ for each chord $a_{F,F'} \in A_1$, and
- $\Gamma^+(a_{F,F'}) = \{1, g_1, g_1^{-1}, \dots, g_k, g_k^{-1}, g_1^2, g_1^{-2}, \dots, g_k^2, g_k^{-2}\}$ for each chord $a_{F,F'} \in A_2$.

Then finding a solution Π of the DSPP in D such that ψ_Π is R-homologous to ϕ corresponds to solving the CFP in D^+ with respect to ϕ^+ and Γ^+. We now show this fact.

Suppose that $\psi_\Pi : A \to G_k$ corresponds to a solution Π of the DSPP which is R-homologous to ϕ. Then its extended function $\psi_\Pi^+ : A^+ \to G_k$ is R-cohomologous to ϕ^+ by Lemma 8 below. Since no pair of dipaths in Π have common arcs or common vertices, we have $\psi_\Pi^+(a) \in \Gamma^+(a)$ ($\forall a \in A^* \cup A_1$). Furthermore, since no pair of dipaths in Π have adjacent vertices, we have $\psi_\Pi^+(a) \in \Gamma^+(a)$ ($\forall a \in A_2$). Hence ψ_Π^+ is a solution of the CFP.

Conversely, suppose that $\psi^+ : A^+ \to G_k$ is a solution of the CFP. Define $\psi : A \to G_k$ by $\psi(a) = \psi^+(a^*)$ for each $a \in A$. Then ψ is R-homologous to ϕ and ψ^+ is the extended function of ψ by Lemma 9 below. For each $i = 1, \dots, k$, define $P_i = \{a \in A \mid \psi(a) = g_i\}$. Since ψ is a flow and $\psi^+(a) \in \Gamma^+(a)$ ($\forall a \in A^*$), P_i consists of a dipath from s_i to t_i and some dicycles. Hence we may assume that P_i is a dipath from s_i to t_i, and $P_1, \dots, P_k$ are arc-disjoint by the definition of P_i. We now show that $\Pi = (P_1, \dots, P_k)$ is stable.

Suppose that two dipaths P_i and P_j have a common vertex v for some distinct i, j. Then there exist arcs a_1 and a_2 of D such that both a_1 and a_2 are incident to v, $\psi(a_1) = g_i^{\pm 1}$, and $\psi(a_2) = g_j^{\pm 1}$. Let π be the dipath in D^* whose first and last arcs are a_1^* and a_2^*, respectively, along the boundary of the face of D^* corresponding to v. We may assume that we have chosen a_1 and a_2 such that π is as short as possible. Then $g_i^{\pm 1}$ and $g_j^{\pm 1}$ are segments of $\psi^+(a_{F,F'})$ for an arc $a_{F,F'} \in A_1$, where π is the dipath from F to F', which contradicts the assumption that ψ^+ is a solution of the CFP. Hence no pair of Π have common vertices.

Suppose that P_i has a vertex v_1, P_j has a vertex v_2, and $a = (v_1, v_2) \in A$ for some distinct i, j. Then there exist arcs a_1 and a_2 of D such that a_1 is incident to v_1, a_2 is incident to v_2, $\psi(a_1) = g_i^{\pm 1}$, and $\psi(a_2) = g_j^{\pm 1}$. Let π be the dipath in D^* whose first and last arcs are a_1^* and a_2^*, respectively, along the boundary of the face of $D^* - a^*$. We may assume that we have chosen a_1 and a_2 such that π is as short as possible. Then $g_i^{\pm 1}$ and $g_j^{\pm 1}$ are segments of $\psi^+(a_{F,F'})$ for an arc $a_{F,F'} \in A_2$, where π is the dipath from F to F', which contradicts the assumption that ψ^+ is a solution of the CFP.

By the above arguments and Theorem 7, we can find a solution Π of the DSPP such that ψ_Π is R-homologous to ϕ in polynomial time by solving the CFP. $\square$

To complete the proof of Proposition 6, we show the following two lemmas.

Lemma 8. *If $\phi, \psi : A \to G_k$ are R-homologous in D, then their extended functions $\phi^+, \psi^+ : A^+ \to G_k$ are R-cohomologous in D^+.*

Proof. Since $\phi, \psi : A \to G_k$ are R-homologous, there exists a function $f : \mathcal{F} \to G_k$ such that

- $f(R) = 1$,
- $f(\mathrm{left}(a))^{-1} \cdot \phi(a) \cdot f(\mathrm{right}(a)) = \psi(a)$ for each arc $a \in A$.

It is enough to show that f satisfies that

$$\psi^+(a) = f(u)^{-1} \cdot \phi^+(a) \cdot f(v) \tag{$\bigstar$}$$

for each arc $a = (u, v) \in A^+$. If $a \in A^*$ then ($\bigstar$) is obvious. Suppose that $a_{F,F'} \in A_1 \cup A_2$ is a chord and $\pi = ((a_1^*)^{\epsilon_1}, (a_2^*)^{\epsilon_2}, \ldots, (a_l^*)^{\epsilon_l})$, where $\epsilon_i \in \{+1, -1\}$, is the dipath from F to F' which appeared in the definition of the extended function. Then $\phi^+(a_{F,F'}) = \phi(a_1)^{\epsilon_1} \cdot \phi(a_2)^{\epsilon_2} \cdot \cdots \cdot \phi(a_l)^{\epsilon_l}$ and $\psi^+(a_{F,F'}) = \psi(a_1)^{\epsilon_1} \cdot \psi(a_2)^{\epsilon_2} \cdot \cdots \cdot \psi(a_l)^{\epsilon_l}$. Let $F_0(= F), F_1, \ldots, F_l(= F') \in \mathcal{F}$ be the vertices lying on π in this order. Then we have

$$\begin{aligned}
\psi^+(a_{F,F'}) &= \psi(a_1)^{\epsilon_1} \cdot \psi(a_2)^{\epsilon_2} \cdot \cdots \cdot \psi(a_l)^{\epsilon_l} \\
&= (f(F_0)^{-1} \cdot \phi(a_1)^{\epsilon_1} \cdot f(F_1)) \cdot (f(F_1)^{-1} \cdot \phi(a_2)^{\epsilon_2} \cdot f(F_2)) \cdot \cdots \cdot (f(F_{l-1})^{-1} \phi(a_l)^{\epsilon_l} f(F_l)) \\
&= f(F_0)^{-1} \cdot \phi(a_1)^{\epsilon_1} \cdot \phi(a_2)^{\epsilon_2} \cdot \cdots \cdot \phi(a_l)^{\epsilon_l} \cdot f(F_l) \\
&= f(F)^{-1} \cdot \phi^+(a_{F,F'}) \cdot f(F'),
\end{aligned}$$

which means that ($\bigstar$) holds for $a_{F,F'}$. $\square$

Lemma 9. *Let ϕ^+ be the extended function of $\phi : A \to G_k$. Suppose that $\psi^+ : A^+ \to G_k$ is R-cohomologous to ϕ^+ and $\psi : A \to G_k$ is the function defined by $\psi(a) = \psi^+(a^*)$ for each $a \in A$. Then ψ is R-homologous to ϕ and ψ^+ is the extended function of ψ.*

Proof. Suppose that ψ^+ is R-cohomologous to ϕ^+ with respect to $f : \mathcal{F} \to G_k$. Then it is obvious that ψ is R-homologous to ϕ with respect to f.

Suppose that $a_{F,F'} \in A_1 \cup A_2$ is a chord and $\pi = ((a_1^*)^{\epsilon_1}, (a_2^*)^{\epsilon_2}, \ldots, (a_l^*)^{\epsilon_l})$, where $\epsilon_i \in \{+1, -1\}$, is the dipath from F to F' which appeared in the definition of the extended function. Let $F_0(=$

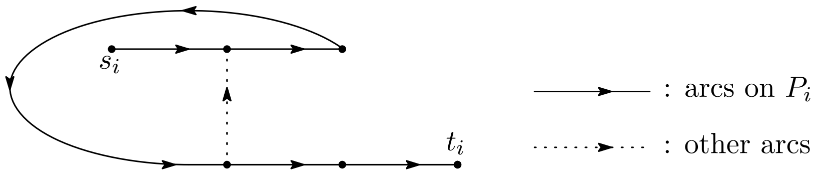

Figure 1: Remark on Γ^+.

$F), F_1, \ldots, F_l (= F') \in \mathcal{F}$ be the vertices lying on π in this order. Then we have

$$\begin{aligned}\psi^+(a_{F,F'}) &= f(F)^{-1} \cdot \phi^+(a_{F,F'}) \cdot f(F') \\ &= f(F_0)^{-1} \cdot \phi(a_1)^{\epsilon_1} \cdot \phi(a_2)^{\epsilon_2} \cdot \cdots \cdot \phi(a_l)^{\epsilon_l} \cdot f(F_l) \\ &= (f(F_0)^{-1} \cdot \phi(a_1)^{\epsilon_1} \cdot f(F_1)) \cdot (f(F_1)^{-1} \cdot \phi(a_2)^{\epsilon_2} \cdot f(F_2)) \cdot \cdots \cdot (f(F_{l-1})^{-1} \phi(a_l)^{\epsilon_l} f(F_l)) \\ &= \psi(a_1)^{\epsilon_1} \cdot \psi(a_2)^{\epsilon_2} \cdot \cdots \cdot \psi(a_l)^{\epsilon_l},\end{aligned}$$

which means that ψ^+ is the extended function of ψ. $\square$

Here, we explain the reason why we define $\Gamma^+(a_{F,F'}) = \{1, g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}, g_1^2, g_1^{-2}, \ldots, g_k^2, g_k^{-2}\}$ for each chord $a_{F,F'} \in A_2$ in the proof of Proposition 6. Our definition of directed stable paths forbids pairs of dipaths to have adjacent vertices, but allows each dipath to have adjacent vertices as in Fig. 1. In this case, $\psi^+(a_{F,F'})$ may possibly be equal to g_i^2 or g_i^{-2}.

3 Applications

In this section, we apply Corollary 2 for the SPP to finding certain structures, called a "hole" and a "theta", in planar graphs.

Given a graph G, we say that a cycle C is a *hole* (or an *induced cycle*) if C is a cycle of G induced by some set of vertices. In other words, C is called a hole if C has no chords. We remark here that holes with an odd number of edges play an important role in the strong perfect graph theorem.

Given a graph $G = (V, E)$ and two distinct vertices $u, v \in V$, we consider the problem of finding a hole in G that passes through u and v. Although this problem is known to be NP-hard [1] in general graphs, we can solve it in planar graphs in polynomial time by applying our algorithm for the stable paths problem.

Corollary 10. *Suppose that we are given a planar graph $G = (V, E)$ and two distinct vertices $u, v \in V$. Then we can find a hole that passes through u and v in polynomial time.*

Proof. Let uu^- and uu^+ be edges incident to u, and let vv^- and vv^+ be edges incident to v. It is enough to show that we can find a hole traveling u^-, u, u^+, v^-, v, and v^+ in this order, because the number of choices of u^-, u^+, v^-, and v^+ is at most $|V|^4$.

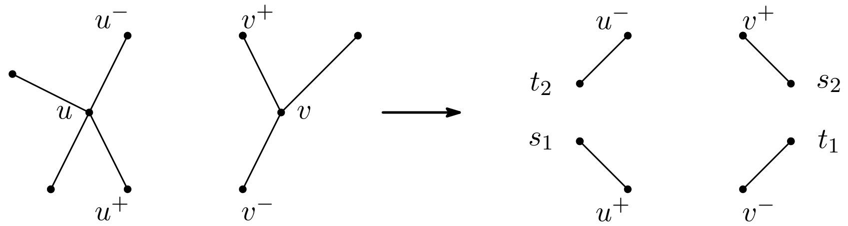

Figure 2: Construction of G'.

As in Fig. 2, construct G' from G by replacing u, v and all edges incident to them with new vertices s_1, s_2, t_1, t_2 and edges $s_1u^+, v^-t_1, s_2v^+, u^-t_2$. Note that G' is planar since G is planar.

Then there exists a hole traveling u^-, u, u^+, v^-, v, and v^+ in this order in G if and only if there exist stable paths in G' with respect to terminals s_1, s_2, t_1, and t_2, because a pair of stable paths with no shortcuts in G' corresponds to a desired hole in G. Hence, by solving the stable paths problem in G', we can find a desired hole. Since G' is planar, by Corollary 2, it can be done in polynomial time. □

In a similar way as Corollary 10, we can find in polynomial time a hole that passes through k given vertices if the given graph is planar and k is fixed.

Corollary 11. *Suppose that we are given a planar graph $G = (V, E)$ and k distinct vertices $v_1, v_2, \dots, v_k \in V$, where k is a fixed constant. Then we can find a hole that passes through $v_1, \dots, v_k$ in polynomial time.*

Proof. Let $v_hv_h^-$ and $v_hv_h^+$ be edges incident to v_h for $h = 1, \dots, k$. It is enough to show that we can find a hole traveling $v_1^-, v_1, v_1^+, v_2^-, v_2, v_2^+, v_3^-, v_3, \dots, v_k^-, v_k$, and v_k^+ in this order, because the number of choices of $v_1^-, v_1^+, v_2^-, v_2^+, \dots, v_k^-$, and v_k^+ is at most $|V|^{2k}$ and that of permutations of $v_1, \dots, v_k$ is $k!$.

Construct G' from G by replacing a vertex v_h and all edges incident to v_h with new vertices s_h, t_{h-1} and edges $s_hv_h^+, t_{h-1}v_h^-$ for every $h = 1, \dots, k$. Note that G' is planar since G is planar.

Then there exists a desired hole in G if and only if there exist stable paths in G' with respect to terminals $s_1, \dots, s_k, t_1, \dots, t_k$, where we define $t_k = t_0$, because a collection of k stable paths with no shortcuts in G' corresponds to a desired hole in G. Hence, by solving the stable paths problem in G', we can find a hole that passes through $v_1^-, v_1, v_1^+, v_2^-, v_2, v_2^+, v_3^-, v_3, \dots, v_k^-, v_k$, and v_k^+ in this order. Since G' is planar, by Corollary 2, it can be done in polynomial time. □

A *theta* is a graph consisting of two nonadjacent vertices u, v and three paths P_1, P_2, P_3 connecting u and v, such that P_1, P_2, P_3 are pairwise vertex-disjoint except for u, v, and the union of every pair of P_1, P_2, P_3 is a hole. We call u and v *end vertices* of the theta.

Chudnovsky–Seymour [2] gave a polynomial time algorithm for the problem of finding a theta in a given graph. However, the problem of finding a theta that has specified end vertices is NP-hard, because finding a hole which passes through two specified vertices is NP-hard [1]. We show that this problem is solvable in polynomial time in planar graphs.

Corollary 12. *Suppose that we are given a planar graph $G = (V, E)$ and nonadjacent vertices $u, v \in V$. Then we can find a theta with u, v as its end vertices in polynomial time.*

Proof. Let uu_1, uu_2, and uu_3 be edges incident to u, and let vv_1, vv_2, and vv_3 be edges incident to v. It is enough to show that we can find a theta such that uu_i and vv_i are the edges on P_i for $i = 1, 2, 3$, because the number of choices of u_1, u_2, u_3, v_1, v_2, and v_3 is at most $|V|^6$.

Construct G' from G by replacing u, v and all edges incident to them with new vertices $s_1, s_2, s_3, t_1, t_2, t_3$ and new six edges s_iu_i, v_it_i for $i = 1, 2, 3$. Note that since G is planar, G' is also planar.

Then there exists a desired theta in G if and only if there exist stable paths in G' with respect to terminals s_1, s_2, s_3, t_1, t_2, and t_3, because a collection of three stable paths with no shortcuts in G' corresponds to a desired theta in G. Hence, by solving the stable paths problem in G', we can find a desired theta. Since G' is planar, by Corollary 2, it can be done in polynomial time. □

4 Hardness Results

In this section, we give two NP-hardness results. These results indicate that the (directed) stable paths problem is essentially different from the (directed) disjoint paths problem.

First, we show that the SPP is NP-hard even if $k = 2$, whereas the DPP is solvable in polynomial time if k is fixed.

Theorem 13. *The stable paths problem (SPP) is NP-hard, even if $k = 2$.*

Proof. The problem of finding a hole that passes through two given vertices is NP-hard [1]. As in the proof of Corollary 10, this problem can be reduced to the SPP with $k = 2$, and hence the SPP is NP-hard, even if $k = 2$. □

We next show that the DSPP is NP-hard even if the given digraph is acyclic and $k = 2$, whereas the DDPP is solvable in polynomial time when the given digraph is acyclic and k is fixed.

Theorem 14. *The directed stable paths problem (DSPP) is NP-hard, even if the given digraph $D = (V, A)$ is acyclic and $k = 2$.*

Proof. It is enough to show that 3-SAT can be reduced to the DSPP in acyclic digraphs with $k = 2$. Let $C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be an instance of 3-SAT with n variables $x_1, \ldots, x_n$.

We construct an acyclic digraph $D = (V, A)$ as follows (see Fig. 3). Let $W = \{w_1, \bar{w}_1, \ldots, w_n, \bar{w}_n\}$ be a set of vertices, where w_i and $\bar{w}_i$ correspond to the variable x_i for each $i = 1, \ldots, n$. For

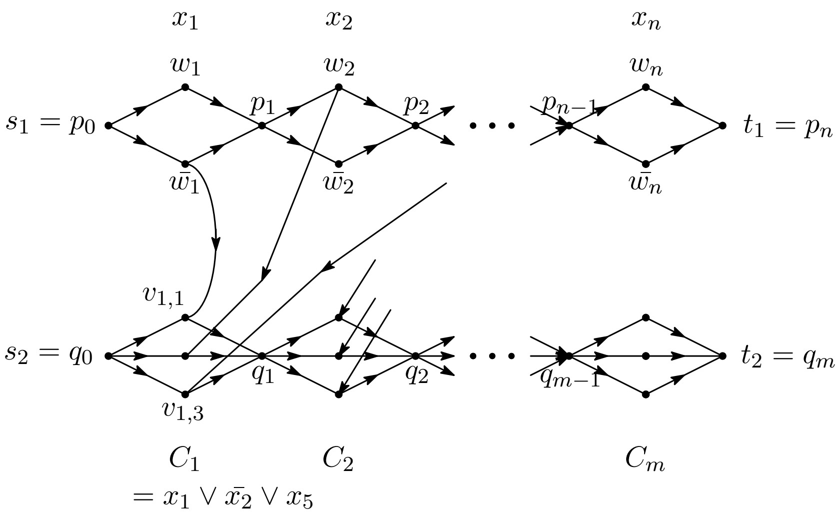

Figure 3: Construction of D.

each $j = 1, \dots, m$, let $v_{j,1}, v_{j,2}, v_{j,3}$ be vertices, which correspond to the literal C_j, and define $V_j = \{v_{j,1}, v_{j,2}, v_{j,3}\}$. Let $P = \{p_0, p_1, \dots, p_n\}$ and $Q = \{q_0, q_1, \dots, q_m\}$ be sets of vertices, and define the vertex set V by $V = W \cup (\bigcup_{j=1}^{m} V_j) \cup P \cup Q$.

Define arc sets A_p and A_q by

$$A_p = \{(p_{i-1}, w_i), (p_{i-1}, \bar{w}_i), (w_i, p_i), (\bar{w}_i, p_i) \mid i = 1, \dots, n\},$$
$$A_q = \{(q_{j-1}, v_{j,i}), (v_{j,i}, q_j) \mid j = 1, \dots, m,\ i = 1, 2, 3\},$$

and let A_x be the arc set defined as follows: $(\bar{w}_i, v_{j,l}) \in A_x$ if the l-th element of C_j is x_i and $(w_i, v_{j,l}) \in A_x$ if the l-th element of C_j is $\bar{x}_i$. The arc set A is defined by $A = A_p \cup A_q \cup A_x$.

We now show that solving the DSPP in $D = (V, A)$ with respect to $s_1 = p_0,\ t_1 = p_n,\ s_2 = q_0$, and $t_2 = q_m$ is equivalent to solving the original 3-SAT.

For each $i = 1, \dots, n$, every dipath from s_1 to t_1 goes through exactly one of w_i and $\bar{w}_i$. Then, we can see with the following observation that assigning "true" or "false" to x_j in the 3-SAT problem corresponds to deciding that P_1 goes through w_i or $\bar{w}_i$ in the DSPP, respectively.

Suppose that the l-th element of C_j is x_i (resp. $\bar{x}_i$). Then, if we assign "true" (resp. "false") to x_i then C_j is satisfied in 3-SAT, which corresponds to the fact that if P_1 goes through w_i (resp. $\bar{w}_i$) then P_2 can go through $v_{j,l}$ from q_{j-1} to q_j in the DSPP. Thus, C_j is satisfied for every $j = 1, \dots, m$ in the 3-SAT problem if and only if P_2 can go from q_0 to q_m in the DSPP.

By the above arguments, 3-SAT can be reduced to the DSPP in an acyclic digraph with $k = 2$. $\square$

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