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Abstract

This paper presents a sequential semidefinite programming (SDP) approach to maximize the minimal eigenvalue of the generalized eigenvalue problem, in which the two symmetric matrices defining the eigenvalue problem are supposed to be the polynomials in terms of the variables. An important application of this problem is found in the structural optimization which attempts to maximize the minimal eigenvalue of the free vibration. It is shown that the maximization of minimal eigenvalue of a structure can be formulated as the linear optimization over a polynomial matrix inequality (polynomial SDP). We propose a bisection method for the polynomial SDP, at each iteration of which we solve a maximization problem of a convex function over a linear matrix inequality. A sequential SDP method is proposed for the subproblem based on the DC (difference of convex functions) algorithm. Optimal topologies are computed for various framed structures to demonstrate that the algorithm presented can converge to optimal solutions with multiple lowest eigenvalues without any difficulty.

Keywords

Topology optimization, Semidefinite program, Multiple eigenvalue, Interior-point method, DC algorithm

1 Introduction

This paper discusses a technique for solving a class of nonlinear programming problems, in which we attempt to minimize a linear function over the constraint such that a symmetric matrix $Z(\boldsymbol{y})$ defined as a (matrix-valued) polynomial of the variables $\boldsymbol{y}$ should be positive semidefinite. This problem class is referred to as the *polynomial semidefinite program* (polynomial SDP), because it includes the *semidefinite program* (SDP) [13] as a particular case in which $Z(\boldsymbol{y})$ is an affine function of $\boldsymbol{y}$. It is known that the SDP problem is convex, while the polynomial SDP problem is nonconvex in general.

We have a motivation of studying the polynomial SDP in the design optimization of structures under the eigenvalue constraints of free vibration, which has been studied widely in structural engineering [33, 35]. Indeed, it was shown that the design optimization of truss structures under the frequency constraints can be formulated as an SDP problem [31]. In designing civil, mechanical and aerospace structures, the eigenvalues of free vibration, as well as the linear buckling load factor, have been used widely for decades as a performance measure of structures.

Let $\mathcal{S}^n \subset \mathbb{R}^{n\times n}$ denote the set of all $n \times n$ real symmetric matrices. The set of all positive semidefinite matrices is denoted by $\mathcal{S}^n_+ \subset \mathcal{S}^n$. We write $P \succeq O$ if $P \in \mathcal{S}^n_+$. Let $\boldsymbol{c} \in \mathbb{R}^m$ and $d \in \mathbb{R}$ be constant. In this paper, we solve the following polynomial SDP problem in the variables $(\boldsymbol{y}, \lambda) \in \mathbb{R}^m \times \mathbb{R}$:

$$\max_{\boldsymbol{y},\lambda} \left\{ \lambda : A(\boldsymbol{y}) - \lambda B(\boldsymbol{y}) \succeq O,\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} + d \leq 0,\ \boldsymbol{y} \geq \boldsymbol{0} \right\}. \tag{1}$$

Here, we suppose that $A : \mathbb{R}^m \to \mathcal{S}^n$ and $B : \mathbb{R}^m \to \mathcal{S}^n$ are the polynomial matrix-valued functions written as

$$A(\boldsymbol{y}) = \sum_{i=1}^{m}\sum_{j=1}^{p} A_i^{(j)} y_i^j + A^{(0)}, \tag{2}$$

$$B(\boldsymbol{y}) = \sum_{i=1}^{m}\sum_{j=1}^{p} B_i^{(j)} y_i^j + B^{(0)}, \tag{3}$$

where $A^{(0)}$, $B^{(0)}$, $A_i^{(j)}$, and $B_i^{(j)} \in \mathcal{S}^n$ $(i = 1,\dots,m;\ j = 1,\dots,p)$ are positive semidefinite constant matrices. The following problem is relevant to Problem (1) and is formulated in the variables $\boldsymbol{y} \in \mathbb{R}^m$

$$\min_{\boldsymbol{y}} \left\{ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} : A(\boldsymbol{y}) - \Lambda B(\boldsymbol{y}) \succeq O,\ \boldsymbol{y} \geq \boldsymbol{0} \right\}, \tag{4}$$

where $\Lambda \in \mathbb{R}$ is constant. Indeed, if Problem (4) is solvable, then Problem (1) can be solved by applying the bisection method, at each iteration of which Problem (4) is solved. Certainly, Problem (4) is nonconvex generally. Both Problems (1) and (4) are referred to as the polynomial SDP problems in this paper.

Suppose that A and B are affine mappings, i.e. $p = 1$. Then Problem (4) falls into the SDP problem, which can be solved by using the primal-dual interior-point method within the polynomial time [6, 13, 25]. Hence, in this case, Problem (1) can be solved efficiently by using the bisection method; see Remark 3.1 for more details.

Another interesting case seems to be

$$A(\boldsymbol{y}) = \sum_{i=1}^{m} A_i^{(1)} y_i + A^{(0)},$$

$$B(\boldsymbol{y}) = \sum_{i=1}^{m} B_i^{(2)} y_i^2 + \sum_{i=1}^{m} B_i^{(1)} y_i + B^{(0)},$$

for positive semidefinite $A_i^{(p)}$ and $B_i^{(p)}$, because Problem (4) becomes convex. However, for typical application of our interest, it is the case that $A(\boldsymbol{y})$ is a nonlinear function and hence Problem (4) is nonconvex. See the review paper [28] for various eigenvalue optimization problems and their application.

We give a brief review of eigenvalue optimization of structures. It is well known that optimum designs for maximization problem of the fundamental eigenvalue often have multiple (repeated) eigenvalues. Such an optimal structure was first presented by Olhoff and Rasmussen [32], in which necessary conditions for optimality are discussed and an optimal column under buckling constraint is found by using an optimality criteria approach. Cox and Overton [9], however, pointed out that the existence of two hinges in the optimal shape leads to serious computational difficulties.

It has been shown that the multiple eigenvalues are not differentiable continuously, and only directional derivatives with respect to the design variables may be calculated [7, 12]. Therefore, it is very difficult to obtain the optimal design related to eigenvalue optimization by using a gradient-based nonlinear programming algorithm for a large structure, especially for the topology optimization in which we allow some elements of the structure to vanish. Several computational approaches have been developed for sensitivity analysis of multiple eigenvalues of finite dimensional structures [35]. Khot [20] presented an optimality criteria approach for optimum design of trusses with multiple frequency constraints. Rodorigues *et al.* [34] developed necessary conditions for optimality for problems under constraints on the linear buckling load factor based on Clarke's generalized gradient. Nakamura and Ohsaki [29] proposed a parametric programming approach for generating a family of optimal trusses for specified frequency range.

An optimal topology may be obtained based on the conventional *ground structure method*, in which the locations of structural elements are fixed and the optimal topology is obtained by removing the elements with vanishing design variables. In the authors' previous paper [31], it has been shown that topology optimization of trusses under frequency constraints can be formulated as SDP, and an algorithm has been proposed based on the primal-dual interior-point method [25], which is applicable to cases with any multiplicity of the lowest eigenvalues. As a natural extension, we consider not only truss structures but general finite-dimensional structures in this paper. We show that the maximization problem of the minimal eigenvalue of a structure is formulated as a polynomial SDP problem.

Recently, in continuation of the interest in SDP, several extension models of SDP have been proposed, which are called *nonlinear SDP* problems. For distinction, a conventional SDP is sometimes called a *linear SDP*, in spite of the fact that the linear SDP is a nonlinear optimization. It is known that the nonlinear SDP has the application in control theory [10, 26, 30] as well as structural engineering [5, 16, 18, 22]. The structural optimization over the lower bound constraint on the linear buckling load factor can be formulated as a nonlinear SDP problem [5, 16, 22]. The polynomial optimization problem on positive semidefinite cones was studied by Kim *et al.* [21]. The first author showed that the robust optimal design of a structure subjected to uncertain loads can be formulated as a nonlinear SDP problem [18].

The interior-point methods for the linear SDP have been extended to the nonlinear SDP [14, 26]. As in the case of the sequential linear programming method for the differentiable nonlinear optimization, the sequential SDP method were proposed [18, 19], in which a nonlinear SDP problem is approximated as a linear SDP problem successively. Several algorithms for the nonlinear SDP have been inspired by the conventional sequential quadratic programming approach [8, 10, 11]. The augmented Lagrangian approach has been also applied to the nonlinear SDP [30, 37]. A penalty method for the convex optimization over the bilinear matrix inequality was proposed [23, 24].

Note that the existing algorithms [8, 10, 11, 18, 19] for the nonlinear SDP are based on an approximation of the nonlinear matrix inequality constraint into a linear matrix inequality. For example, such an approximation is realized in Problem (4) by approximating the nonlinear mappings A and B by affine mappings at the current solution. These algorithms are local methods and are not guaranteed theoretically to converge to the global solutions. While the algorithm presented in this paper is also a local approach, our idea differs from a direct linearization of A and B.

Indeed, we have examined a sequential SDP method based on a linearization of A and B to solve Problem (4). This problem is motivated by the optimization of framed structures over the lower bound constraint on the fundamental eigenvalue. We have found that such an algorithm often converges to local solutions that are unreasonable, or unnatural, from the practical view-point of structural engineering; see section 2.2 for the details and examples of unreasonable local solutions. As a consequence, we attempt to develop an algorithm that can often find a global (or, at least, possibly local but reasonable from the practical point of view) solution, by utilizing a methodology other than a local linearization of A and B. Note again that our algorithm does not guarantee the convergence to a global optimal solution. However, in numerical experiments reported, it seems that all solutions obtained are global optimal solutions.

We propose a bisection method for the polynomial SDP (1). At each iteration, we solve a maximization problem of a convex function over a linear matrix inequality, which can be embedded into a DC (difference of convex functions) optimization problem. The resulting problem is solved by using the so-called DC algorithm. For the DC optimization and the DC algorithm, see the review paper [4]. The DC algorithm is one of a few algorithms based on a local approach which has been successfully applied to large-scale DC optimization problems, and very often converges to the global optimal solution [3, 4].

This paper is organized as follows. In section 2.1, in order to make this paper self-contained, the (linear) SDP problem is briefly introduced. In section 2.2, preliminary numerical experiments for the polynomial SDP problem (4) are reported in order to explain the necessity of developing an algorithm which is not based on the linear matrix inequality approximation. Section 3 formulates the maximization problem of the minimal eigenvalue of a structure as a polynomial SDP problem (1). We modify Problem (1) slightly so that the feasibility problem becomes well-defined. In section 4, we present the framework of bisection method for polynomial SDP based on a maximization problem of a convex function over a linear matrix inequality, for which a sequential SDP method based on the DC algorithm is presented in section 5. Explicit formulations for framed structures are given in section 6. Numerical experiments are presented in section 7 for various structures by using the sequential SDP method, while conclusions are drawn in section 8.

2 Preliminaries

All vectors are assumed to be column vectors in this paper. The $(m+n)$-dimensional column vector $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ consisting of $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$ is often written simply as $(\boldsymbol{u}, \boldsymbol{v})$. For any $\boldsymbol{v} \in \mathbb{R}^n$, let $\|\boldsymbol{v}\|$ denote the standard Euclidean norm of $\boldsymbol{v}$, i.e. $\|\boldsymbol{v}\| = (\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v})^{1/2}$. For two sets $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{V} \subseteq \mathbb{R}^n$, their Cartesian product is defined by $\mathcal{U} \times \mathcal{V} = \{(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{m+n} | \ \boldsymbol{u} \in \mathcal{U}, \ \boldsymbol{v} \in \mathcal{V}\}$. Particularly, we write $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. The cardinality of the set $\mathcal{U}$ is denoted by $|\mathcal{U}|$. The empty set is denoted by $\emptyset$. Define the vector $\mathbf{1}$ with an appropriate size by $\mathbf{1} = (1, \dots, 1)^{\mathrm{T}}$. We write $\mathrm{Diag}(\boldsymbol{p})$ for the diagonal matrix with a vector $\boldsymbol{p} \in \mathbb{R}^n$ on its diagonal. For $\boldsymbol{p}_l \in \mathbb{R}^{n_l}$ $(l = 1, \dots, k)$, we simply write $\mathrm{Diag}(\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ instead of $\mathrm{Diag}((\boldsymbol{p}_1^{\mathrm{T}}, \dots, \boldsymbol{p}_k^{\mathrm{T}})^{\mathrm{T}})$. Define $\mathbb{R}^n_+ \subset \mathbb{R}^n$ and Define $\mathbb{R}^n_{++} \subset \mathbb{R}^n$ by $\mathbb{R}^n_+ = \{\boldsymbol{p} \in \mathbb{R}^n | \ \boldsymbol{p} \geq \mathbf{0}\}$ and $\mathbb{R}^n_{++} = \{\boldsymbol{p} = (p_i) \in \mathbb{R}^n | \ p_i > 0 \ (i - 1, \dots, n)\}$, respectively.

2.1 Introduction of semidefinite program

For any matrix $P \in \mathbb{R}^{n \times n}$, $\mathrm{tr}(P)$ denotes the *trace* of P, i.e. the sum of the diagonal elements of P. Let $A_i \in \mathcal{S}^n$ $(i = 1, \dots, m)$, $C \in \mathcal{S}^n$, and $b = (b_i) \in \mathbb{R}^m$ be constant matrices and a constant vector. The *semidefinite programming* (SDP) problem refers to the optimization problem having the form of [13]

$$\min \{\mathrm{tr}(CX) : \mathrm{tr}(A_i X) = b_i, \ i = 1, \dots, m, \ \mathcal{S}^n \ni X \succeq O\}, \tag{5}$$

where X is a variable matrix. The dual of Problem (5) is formulated in the variables $\boldsymbol{y} \in \mathbb{R}^m$ as

$$\max \left\{ \boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} : C - \sum_{i=1}^{m} A_i y_i \succeq O \right\}, \tag{6}$$

which is also an SDP problem.

Recently, SDP has received increasing attention for its wide fields of application [6, 17, 31]. The primal-dual interior-point method, which has been first developed for LP, has been naturally extended to SDP [13, 25]. It is theoretically guaranteed that the primal-dual interior-point method converges to optimal solutions of the primal-dual pair of SDP problems (5) and (6) within the number of arithmetic operations bounded by a polynomial of m and n.

The constraint in Problem (6) is referred to as a *linear matrix inequality*. It is easy to see that usual linear inequalities can be represented as a linear matrix inequality as a particular case in which A_i and C are diagonal matrices. This implies that the *linear program* (LP) is included in SDP as a particular case.

Let $\boldsymbol{a}_0 \in \mathbb{R}^m$, $A_1 \in \mathbb{R}^{m \times (n-1)}$, $c_0 \in \mathbb{R}$, and $\boldsymbol{c}_1 \in \mathbb{R}^{n-1}$ be constant. The inequality in the form of

$$c_0 - \boldsymbol{a}_0^{\mathrm{T}} \boldsymbol{y} \geq \|\boldsymbol{c}_1 - A_1^{\mathrm{T}} \boldsymbol{y}\|$$

is referred to as a *second-order cone* (or *conic quadratic*) inequality in $\mathbb{R}^n$. A minimization problem of a linear function of $\boldsymbol{y}$ over a direct product of some second-order cone inequalities is called a *second-order cone programming* (SOCP) problem [1]. It is known that a second-order cone inequality can be written as a linear matrix inequality [6], which implies that SOCP is included in SDP as a particular case. Besides this fact, SOCP itself has also received much attention for its wide fields of application [1, 6, 15].

2.2 Preliminary numerical experiments

We first report results of preliminary numerical experiments for the polynomial SDP problem (4), in order to explain the necessity of developing an algorithm which is not based on the linearization of A and B but is based on the feasibility problem.

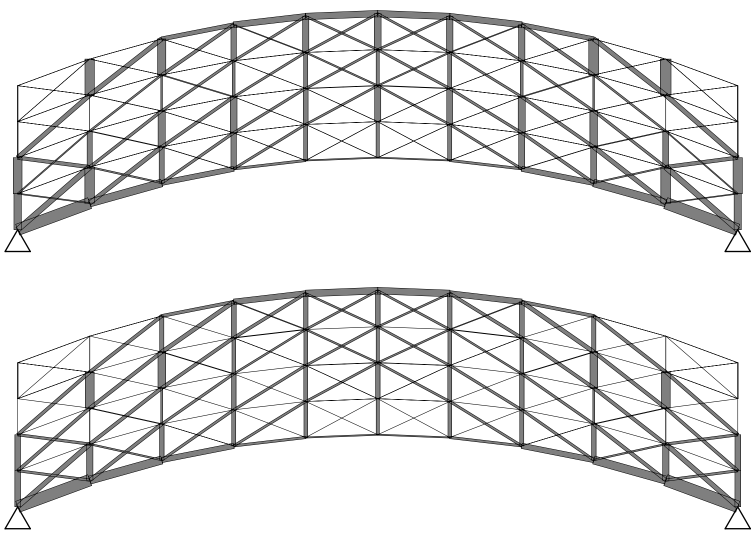


Figure 1: Two local solutions of the example in section 7.2 obtained by using the linear matrix inequality approximation (8).

A simple method to solve Problem (4) may be based on the sequential approximation of the nonlinear matrix inequality

$$A(\boldsymbol{y}) - \Lambda B(\boldsymbol{y}) \succeq O \tag{7}$$

as a linear matrix inequality. Roughly speaking, by using the linearization, we obtain a linear SDP problem as an approximation of Problem (4). More specifically, at $\boldsymbol{y} = \boldsymbol{y}^k$, the nonlinear matrix inequality (7) can be approximated as

$$\left[DA(\boldsymbol{y}^k)\Delta\boldsymbol{y} + A(\boldsymbol{y}^k)\right] - \Lambda\left[DB(\boldsymbol{y}^k)\Delta\boldsymbol{y} + B(\boldsymbol{y}^k)\right] \succeq O. \tag{8}$$

Here, $DA(\boldsymbol{y}^k)$ denotes the derivative of the mapping A at $\boldsymbol{y}^k \in \mathbb{R}^m$ defined such that $DA(\boldsymbol{y}^k)\boldsymbol{z} \in \mathcal{S}^n$ is a linear function of $\boldsymbol{z} = (z_i) \in \mathbb{R}^m$ given by

$$DA(\boldsymbol{y}^k)\boldsymbol{z} = \sum_{i=1}^{m} z_i \left.\frac{\partial A(\boldsymbol{y})}{\partial y_i}\right|_{\boldsymbol{y}=\boldsymbol{y}^k}.$$

Similarly, the derivative of B at $\boldsymbol{y}^k \in \mathbb{R}^m$ is denoted by $DB(\boldsymbol{y}^k)$. By using the localization (8) we obtain an SDP approximation of Problem (4) in the variables $\Delta\boldsymbol{y} \in \mathbb{R}^m$

$$\min_{\Delta\boldsymbol{y}} \left\{\boldsymbol{c}^{\mathrm{T}}\Delta\boldsymbol{y} : \left[DA(\boldsymbol{y}^k)\Delta\boldsymbol{y} + A(\boldsymbol{y}^k)\right] - \Lambda\left[DB(\boldsymbol{y}^k)\Delta\boldsymbol{y} + B(\boldsymbol{y}^k)\right] \succeq O,\ \Delta\boldsymbol{y} + \boldsymbol{y}^k \geq \boldsymbol{0}\right\}. \tag{9}$$

A sequential SDP method for Problem (4) may be designed, at each iteration of which the SDP problem (9) is to be solved as a subproblem. We may make further refinements on the subproblem (9), e.g. adding a trust-region constraint [18, 19], a convex quadratic model of the objective function [10], and a linearization based on the augmented Lagrangian [8, 30].

As a preliminary numerical experiment, we solved a structural optimization problem which minimizes the total structural volume over the lower bound constraint of the minimal eigenvalue. The framed structure illustrated in Fig.9 is considered; see section 7.2 for the details. The optimization problem is formulated in the form of Problem (4). The optimal solution was found by solving the approximation problem (9) successively. We added a trust-region constraint to Problem (9), but the details of the algorithm are omitted.

The typical solutions obtained are illustrated in Fig.1. These two solutions have different optimal values, and it seems that both solutions are not globally optimal but local optimal solutions. It should be emphasized that these solutions are not regarded as realistic designs from the practical point of view, because these solutions have apparently unnecessary members at upper-left and -right corners. On the contrary, the algorithm presented below converges to the solution illustrated in Fig.10, which is more natural and thus seems to be globally optimal, in spite of the fact that the algorithm has no guarantee of the convergence to the global optimal solution. In order to avoid the convergence to local solutions as Fig.1, we attempt to develop an algorithm that does not use a direct linearization of A and B.

3 Maximization of minimal eigenvalue of structures

Consider a finitely-discretized structure in the two- or three-dimensional space. Let n^{d} denote the number of degrees of freedom of displacements. The stiffness matrix is denoted by $K \in \mathcal{S}^{n^{\mathrm{d}}}$. In the free vibration problem of a structure, we consider the inertia forces caused by the masses both of the structural elements and the concentrated masses supported by the structure. The mass matrices due to the structural and concentrated nonstructural masses, respectively, are denoted by $M^{\mathrm{S}} \in \mathcal{S}^{n^{\mathrm{d}}}$ and $M_0 \in \mathcal{S}^{n^{\mathrm{d}}}$. Note that K and M^{S} are the functions of the design variables vector $\boldsymbol{y} \in \mathbb{R}^m_+$, while M_0 is a constant matrix.

The eigenvalue problem of free vibration is formulated as

$$K\boldsymbol{\phi}_r = \omega_r (M^{\mathrm{S}} + M_0)\boldsymbol{\phi}_r, \quad r = 1, \dots, n^{\mathrm{d}}. \tag{10}$$

Here, $\omega_1, \dots, \omega_{n^{\mathrm{d}}} \in \mathbb{R}$ are the eigenvalues arranged in the non-descending order, i.e.

$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_{n^{\mathrm{d}}}.$$

The corresponding eigenvector $\boldsymbol{\phi}_r \in \mathbb{R}^{n^{\mathrm{d}}}$ is normalized as

$$\boldsymbol{\phi}_r^{\mathrm{T}} (M^{\mathrm{S}} + M_0)\boldsymbol{\phi}_r = 1, \quad r = 1, \dots, n^{\mathrm{d}}. \tag{11}$$

Let $v : \mathbb{R}^m \to]0, +\infty[$ be the convex polynomial function which represents the structural volume. The upper bound of the structural volume is denoted by $V^0 \in \mathbb{R}_{++}$.

The following is the design problem to find the optimal design at which the minimal eigenvalue is maximized under the volume constraint:

$$\left. \begin{array}{ll} \max\limits_{\boldsymbol{y}} & \omega_1(\boldsymbol{y}) \\ \text{s.t.} & v(\boldsymbol{y}) \leq V^0, \\ & \boldsymbol{y} \geq \mathbf{0}. \end{array} \right\} \tag{12}$$

It is known that optimum solutions of Problem (12) often have multiple (repeated) eigenvalues [29, 31, 32, 35], and multiple eigenvalues are not differentiable continuously. The aim of the paper is to propose an algorithm solving (12) without using the sensitivity coefficients of ω_1 with respect to $\boldsymbol{y}$.

By using the Rayleigh principle, Problem (12) is reformulated into the problem

$$\left.\begin{array}{ll}\max\limits_{\boldsymbol{y},\omega} & \omega \\ \text{s.t.} & K(\boldsymbol{y}) - \omega\left(M^{\mathrm{S}}(\boldsymbol{y}) + M_0\right) \succeq O, \\ & v(\boldsymbol{y}) \le V^0, \\ & \boldsymbol{y} \ge \boldsymbol{0}, \end{array}\right\} \tag{13}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ and $\omega \in \mathbb{R}$ are the variables. See Ohsaki *et al.* [31].

Note that $K(\boldsymbol{y})$ and $M^{\mathrm{S}}(\boldsymbol{y})$ are matrix-valued polynomial functions of $\boldsymbol{y}$ for various classes of structures; see section 6 for examples. In sections 3, for simplicity of the presentation, we suppose that K and M^{S} are written as

$$K(\boldsymbol{y}) = \sum_{i=1}^{m} A_i^{(p)} y_i^p, \tag{14}$$

$$M^{\mathrm{S}}(\boldsymbol{y}) = \sum_{i=1}^{m} B_i^{(1)} y_i, \tag{15}$$

where $A_i^{(p)} \in \mathcal{S}^{n^{\mathrm{d}}}$ and $B_i^{(1)} \in \mathcal{S}^{n^{\mathrm{d}}}$ are positive semidefinite constant matrices, and $p \ge 2$ is the given natural number. It is immediate to generalize the results and the algorithm below to the general case in the forms of (2) and (3); see section 6. By using (14) and (15), Problem (13) is written explicitly as

$$\left.\begin{array}{ll}\max\limits_{\boldsymbol{y},\omega} & \omega \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} A_i^{(p)} y_i^p - \sum_{i=1}^{m} \omega B_i^{(1)} y_i - \omega M_0 \succeq O, \\ & v(\boldsymbol{y}) \le V^0, \\ & \boldsymbol{y} \ge \boldsymbol{0}, \end{array}\right\} \tag{16}$$

which is a polynomial SDP problem in the form of (1). Note that the optimal value of Problem (16) is nonnegative, because we have assumed that $A_i^{(p)}$ and $B_i^{(1)}$ $(i = 1, \dots, m)$ are positive semidefinite. Moreover, for $M_0 \ne O$, the optimal value is equal to zero only if the feasible set of Problem (16) is the singleton $\{(\boldsymbol{y}, \omega) \in \mathbb{R}^{m+1} \,|\, (\boldsymbol{y}, \omega) = \boldsymbol{0}\}$. Certainly, we are not interested in this trivial case. Hence, we assume that the optimal value of Problem (16) is positive in the remainder of the paper.

Remark 3.1. It is of interest to investigate Problem (16) in the particular case of $p = 1$, which corresponds to truss structures [31]. In this case, (16) can be solved easily by using the bisection method with respect to ω. Letting Ω be a current estimate of the optimal value of Problem (16), consider the feasibility problem

$$\left.\begin{array}{ll}\min\limits_{\boldsymbol{y},s} & s \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} A_i^{(1)} y_i - \sum_{i=1}^{m} \Omega B_i^{(1)} y_i - \Omega M_0 + sI \succeq O, \\ & v(\boldsymbol{y}) \le V^0, \\ & \boldsymbol{y} \ge \boldsymbol{0}, \end{array}\right\} \tag{17}$$

where $\boldsymbol{y}$ and s are the variables, and I denotes the identity matrix with an appropriate size. Note that (17) is a (linear) SDP problem, which can be solved by using the primal-dual interior-point method. Hence, we can easily check whether the global optimal value of Problem (17) is positive or negative. If the optimal value of (17) is negative, then there exists a feasible solution $(\boldsymbol{y}', \omega')$ of (16) satisfying $\omega' \geq \Omega$, which implies that the optimal value of (16) is not less than Ω. Conversely, if the optimal value of (17) is positive, then we can conclude that the optimal value of (16) is less than Ω. From this observation it follows that the global optimal solution of (16) with $p = 1$ is obtained by using the bisection method, at each iteration of which an SDP problem (17) is to be solved. In the case of $p \geq 2$, unfortunately, the feasibility problem of Problem (16) becomes nonconvex, which motivates us to reformulate (16) into a tractable form. □

We next investigate a reformulation of Problem (16), which prepares the algorithm presented in section 4. We introduce auxiliary variables $\boldsymbol{t} = (t_i) \in \mathbb{R}^m$ by replacing the nonlinear term of $\boldsymbol{y}$ in Problem (16) as

$$t_i = y_i^p, \quad i = 1, \dots, m. \tag{18}$$

We do not possess (18) as the equality constraint conditions directly in the algorithm. Note that Noll *et al.* [30] proposed an augmented Lagrangian algorithm for the linear optimization over the bilinear matrix inequality, in which the equations in (18) are dealt with as the nonlinear equality constraints. Our idea is not based on the linearization of (18) and quite different from the method proposed in [30]: roughly speaking, we rewrite the condition (18) into m convex inequalities and one reverse convex inequality, and we keep only convex inequalities explicitly.

Define the point-to-set mapping $\mathcal{F} : \mathbb{R} \to \mathcal{P}(\mathbb{R}^m \times \mathbb{R}^m)$ by

$$\mathcal{F}(\omega) = \left\{ (\boldsymbol{y}, \boldsymbol{t}) \in \mathbb{R}^m \times \mathbb{R}^m \;\middle|\; \begin{array}{l} \sum_{i=1}^m A_i^{(p)} t_i - \sum_{i=1}^m \omega B_i^{(1)} y_i - \omega M_0 \succeq O, \\ v(\boldsymbol{y}) \leq V^0, \\ \boldsymbol{y} \geq \boldsymbol{0}, \\ t_i \geq y_i^p, \quad i = 1, \dots, m \end{array} \right\}, \tag{19}$$

where $\mathcal{P}(\mathbb{R}^m \times \mathbb{R}^m)$ denotes the power set of $\mathbb{R}^m \times \mathbb{R}^m$. The definition (19) of $\mathcal{F}$ is motivated by the fact that the set

$$\{(\boldsymbol{y}, \boldsymbol{t}) \in \mathbb{R}^m \times \mathbb{R}^m \mid \boldsymbol{y} \geq \boldsymbol{0},\ t_i \geq y_i^p,\ i = 1, \dots, m\} \tag{20}$$

corresponds to the convex hull of the set

$$\{(\boldsymbol{y}, \boldsymbol{t}) \in \mathbb{R}^m \times \mathbb{R}^m \mid \boldsymbol{y} \geq \boldsymbol{0},\ (18)\},$$

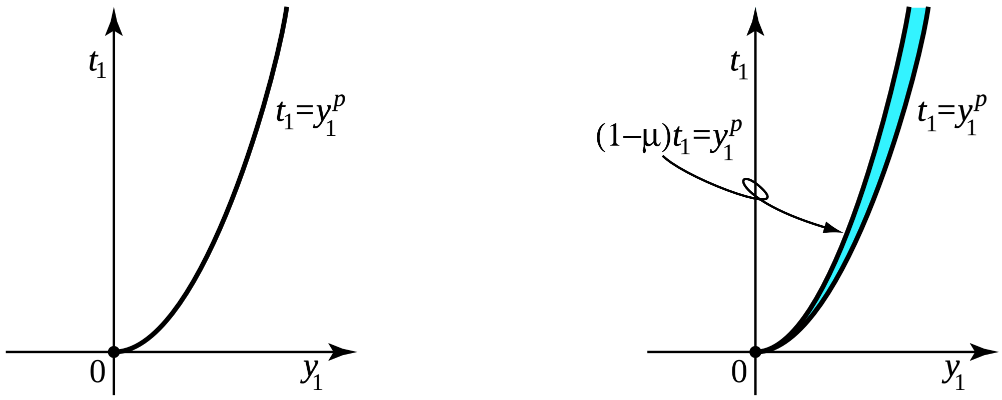
which is a part of the constraints of Problem (16). Moreover, the set (20) can be represented by a linear matrix inequality in terms of $\boldsymbol{y}$, $\boldsymbol{t}$, and (for $p \geq 3$) some auxiliary variables. Hence, we prefer to possess (20) as an explicit constraint.

Define the function $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by

$$g(\boldsymbol{y}, \boldsymbol{t}) = \sum_{i=1}^m (y_i^p)^+ - \sum_{i=1}^m t_i, \tag{21}$$

where $(y_i^p)^+$ is defined by

$$(y_i^p)^+ = \max\{y_i^p, 0\}, \quad i = 1, \dots, m.$$



(I) set (27) defined with g (II) set (28) defined with g_μ

Figure 2: Feasible sets defined with the functions g and g_μ in the case of $m = 1$.

Note that g is a convex smooth function, because we have supposed that $p \geq 2$. In Algorithm 5.2 presented below, $(y_i^p)^+$ can be replaced with y_i^p simply, because the subproblem solved in Algorithm 5.2 has the constraint $y_i \geq 0$.

Consider the following problem in the variables $(\boldsymbol{y}, \boldsymbol{t}, \omega) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$:

$$\max_{\boldsymbol{y},\boldsymbol{t},\omega} \{\omega \mid (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega),\ g(\boldsymbol{y}, \boldsymbol{t}) \geq 0\}. \tag{22}$$

Proposition 3.2 (reformulation of Problem (16)**).** $(\overline{\boldsymbol{y}}, \overline{\omega})$ *is an optimal solution of Problem* (16) *if and only if* $(\overline{\boldsymbol{y}}, \overline{\boldsymbol{t}}, \overline{\omega})$ *satisfying*

$$\bar{t}_i = \overline{y}_i^p, \quad i = 1, \ldots, m \tag{23}$$

is an optimal solution of Problem (22).

Proof. Observe that the condition (23) is equivalent to the $(m + 1)$ inequalities

$$t_i - y_i^p \geq 0, \quad i = 1, \ldots, m,$$
$$\sum_{i=1}^{m} (y_i^p - t_i) \geq 0.$$

Hence, the assertion immediately follows from the definitions (19) and (21) of $\mathcal{F}(\omega)$ and g. □

Proposition 3.2 justifies to solve Problem (22) instead of Problem (16). It should be emphasized that Problem (22) has only one nonconvex inequality $g(\boldsymbol{y}, \boldsymbol{t}) \geq 0$ in contrast to m nonconvex constraints in (18). Moreover, the nonconvex constraint of Problem (22) is the so-called *reverse convex* constraint [27]. The idea of the algorithm presented in section 4 is essentially based on the bisection method exploiting feasibility problems of Problem (22); see also Remark 3.1. However, another difficulty arises when we employ this approach directly. We first investigate this difficulty briefly, and then propose a slightly modified version of Problem (22).

Let Ω be an estimate of the optimal value of Problem (22). Consider the following problem in the variables $(\boldsymbol{y}, \boldsymbol{t}) \in \mathbb{R}^m \times \mathbb{R}^m$:

$$\max_{\boldsymbol{y},\boldsymbol{t}} \{g(\boldsymbol{y}, \boldsymbol{t}) \mid (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\Omega)\}, \tag{24}$$

which can be regarded as a feasibility problem of Problem (22). Indeed, if the optimal value of Problem (24) is negative, then there exists no feasible solution of Problem (22) satisfying $\omega \geq \Omega$, which implies that the optimal value of Problem (22) is less than Ω. On the other hand, if the optimal value of Problem (24) is equal to zero, then there exists a feasible solution $(\boldsymbol{y}', \boldsymbol{t}', \omega')$ of Problem (22) satisfying $\omega' = \Omega$, which implies that the optimal value of Problem (22) is not less than Ω. Note that any feasible solution of Problem (24) satisfies $g(\boldsymbol{y}, \boldsymbol{t}) \leq 0$. Thus, we may perform the bisection method by checking whether the objective value of Problem (24) is negative or equal to zero. However, this procedure is not recommended from the view point of numerical computation, because we cannot expect that a numerically obtained optimal value becomes exactly equal to zero if the exact value is equal to zero. We prefer to perform the bisection method based on a (slightly modified, if necessary) feasibility problem, whose objective value becomes strictly positive if the optimal value of Problem (22) is greater than Ω; see also Remark 4.3. This motivates us to propose a slightly modified version of Problem (22).

Letting $\mu \in \mathbb{R}_{++}$ be a sufficiently small constant, define the function $g_\mu : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by

$$g_\mu(\boldsymbol{y}, \boldsymbol{t}) = g(\boldsymbol{y}, \boldsymbol{t}) + \mu \sum_{i=1}^{m} t_i, \tag{25}$$

where g has been defined in (21). Consider the optimization problem

$$\omega^* := \max_{\boldsymbol{y}, \boldsymbol{t}, \omega} \{\omega \mid (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega),\ g_\mu(\boldsymbol{y}, \boldsymbol{t}) \geq 0\}, \tag{26}$$

where $\boldsymbol{y}$, $\boldsymbol{t}$, and ω are the variables. We solve Problem (26) instead of Problem (22). For sufficiently small μ, we may regard that the solution of Problem (26) accurately approximates the solution of the original problem (22). Theoretically, an arbitrary small μ is acceptable. In the numerical experiments of section 7, for example, we choose $\mu = 10^{-3}$.

Remark 3.3. We attempt to give an intuitive interpretation of the modification introduced in (25) and (26). Suppose $m = 1$ for simplicity. In association with the feasible set of the original problem (22), consider the set

$$\{(y_1, t_1) \mid y_1 \geq 0,\ t_1 \geq y_1^p,\ g(y_1, t_1) \geq 0\}, \tag{27}$$

by neglecting the positive semidefinite constraint and the volume constraint of Problem (22). The modified version of (27) defined with g_μ is written as

$$\{(y_1, t_1) \mid y_1 \geq 0,\ t_1 \geq y_1^p,\ g_\mu(y_1, t_1) \geq 0\}, \tag{28}$$

which is related to the feasible set of Problem (26). Fig.2 depicts the sets (27) and (28). It is observed in Fig.2 (I) that the two inequalities $t_1 \geq y_1^p$ and $g(y_1, t_1) \geq 0$ reduce to the equality constraint $g(y_1, t_1) = 0$. This is the reason why the feasibility problem (24) of Problem (22) cannot have positive optimal value. On the contrary, the feasible set satisfying $g_\mu(y_1, t_1) > 0$ is not empty in Fig.2 (II). This allows us to formulate a feasibility problem of Problem (26) which attains a positive optimal value if the feasibility problem possesses a feasible solution of Problem (26). This assertion is shown rigorously in Theorem 4.4; see also Remark 4.3. □

4 Framework of bisection method

Define $\Omega^{\max} \in \mathbb{R}_{++}$ by

$$\Omega^{\max} := \max_{\boldsymbol{y},\boldsymbol{t},\omega} \{\omega \mid (\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega)\}. \tag{29}$$

For the given $\Omega \in]0, \Omega^{\max}[$, consider the following problem in the variables $(\boldsymbol{y},\boldsymbol{t},\omega) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$:

$$g_\mu^*(\Omega) := \max_{\boldsymbol{y},\boldsymbol{t},\omega} \{g_\mu(\boldsymbol{y},\boldsymbol{t}) \mid (\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega),\ \omega \geq \Omega\}, \tag{30}$$

where $\mathcal{F}$ and g_μ have been defined in (19) and (25), respectively. Note that Problem (30) is obtained by exchanging the objective function and the nonconvex constraint of Problem (26), and by introducing a parameter Ω. From the definition (19) of $\mathcal{F}$ it follows that Problem (30) is equivalently rewritten as

$$g_\mu^*(\Omega) = \max_{\boldsymbol{y},\boldsymbol{t}} \{g_\mu(\boldsymbol{y},\boldsymbol{t}) \mid (\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\Omega)\}, \tag{31}$$

where $\boldsymbol{y}$ and $\boldsymbol{t}$ are the variables. It is of interest to note that Problem (31) is the maximization problem of the convex function over the convex set. In section 5, we utilize this property to develop an algorithm for Problem (31).

We next investigate the relation between Problems (26) and (31). The following two lemmas should be prepared.

Lemma 4.1. *For any* $\Omega \in]0, \Omega^{\max}[$,

$$\lim_{\epsilon \to +0} \sup_{\boldsymbol{y},\boldsymbol{t},\omega} \{\omega \mid (\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega),\ g_\mu(\boldsymbol{y},\boldsymbol{t}) \geq \epsilon\} = \omega^* > -\infty, \tag{32}$$

$$\lim_{\Omega' \to \Omega+0} \sup_{\boldsymbol{y},\boldsymbol{t},\omega} \left\{g_\mu(\boldsymbol{y},\boldsymbol{t}) \,\middle|\, (\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega),\ \omega \geq \Omega'\right\} = g_\mu^*(\Omega) > -\infty. \tag{33}$$

Proof. In order to utilize the result of Tuy [38, Lemma 4.1], we first rewrite the assertions (32) and (33) of the lemma equivalently as

$$\lim_{\epsilon \to +0} \inf_{\boldsymbol{y},\boldsymbol{t},\omega} \{-\omega \mid (\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega),\ g_\mu(\boldsymbol{y},\boldsymbol{t}) \geq \epsilon\} = -\omega^* < +\infty, \tag{34}$$

$$\lim_{-\Omega' \to -\Omega-0} \sup_{\boldsymbol{y},\boldsymbol{t},\omega} \left\{g_\mu(\boldsymbol{y},\boldsymbol{t}) \,\middle|\, (\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega),\ -\omega \leq -\Omega'\right\} = g_\mu^*(-\Omega) > -\infty. \tag{35}$$

Observing that g_μ defined in (25) is a continuous function, we can show (34) with the aid of Lemma 4.1 in [38], which implies that it suffices to show that (i) ω^* defined by (26) satisfies $-\omega^* < +\infty$ and (ii) 0 is not a local maximum of g_μ over the constraint $(\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega)$. The definition (19) of the set $\mathcal{F}(\omega)$ implies $\mathbf{0} \in \mathcal{F}(0)$. Moreover, the definition (25) of g_μ implies $g_\mu(\mathbf{0}) = 0$. Consequently, $(\boldsymbol{y},\boldsymbol{t},\omega) = \mathbf{0}$ is feasible for Problem (26), which guarantees the condition (i). To show the condition (ii) we start by seeing that there exists $\hat{i} \in \{1,\dots,m\}$ such that $t'_{\hat{i}} > y'_{\hat{i}}$ if $(\boldsymbol{y}',\boldsymbol{t}',\omega')$ satisfies $(\boldsymbol{y}',\boldsymbol{t}') \in \mathcal{F}(\omega')$, $\omega' > 0$ and $g_\mu(\boldsymbol{y}',\boldsymbol{t}') = 0$. Recall that $A_i^{(p)}$, $B_i^{(1)}$ and M_0 in (19) are positive definite. Hence, we can choose $\omega'' \in [0, \omega'[$ such that there exists a vector $(\boldsymbol{y}'',\boldsymbol{t}'') \in \mathcal{F}(\omega'')$ satisfying $t''_{\hat{i}} < t'_{\hat{i}}$. For example, by letting $y''_i = y'_i$ and $t''_i = (y'_i)^p$, we see that $(\boldsymbol{y}'',\boldsymbol{t}'')$ satisfies $t''_{\hat{i}} < t'_{\hat{i}}$ and $(\boldsymbol{y}'',\boldsymbol{t}'') \in \mathcal{F}(0)$. Then we obtain $g_\mu(\boldsymbol{y}'',\boldsymbol{t}'') > 0$, and thus the condition (ii) is satisfied. Similarly, the assertion (35) can be proved by showing that (iii) $g_\mu^*(\Omega)$ defined by (30) satisfies $g_\mu^*(\omega) > -\infty$ and (iv) $-\Omega$ is not a local minimum of $-\omega$ over the constraint $(\boldsymbol{y},\boldsymbol{t}) \in \mathcal{F}(\omega)$. The conditions (iii) and (iv) are immediate from the definition (29) of $\Omega^{\max}$ and the assumption of Lemma 4.1. $\square$

Lemma 4.2. *For any* $\Omega \in]0, \Omega^{\max}[$, $\omega^* \leq \Omega$ *if and only if* $g_\mu^*(\Omega) \leq 0$.

Proof. We start by showing the 'only if' part. If ω^* defined by (26) satisfies $\omega^* \leq \Omega$, then

$$\big\{(\boldsymbol{y}, \boldsymbol{t}, \omega) \,\big|\, (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega),\ g_\mu(\boldsymbol{y}, \boldsymbol{t}) \geq 0,\ \omega > \Omega' \big\} = \emptyset$$

for any Ω' satisfying $\Omega' > \Omega$. Hence, for any $\Omega' > \Omega$, we see that the inequality

$$g_\mu^*(\Omega') = \max_{\boldsymbol{y}, \boldsymbol{t}, \omega} \big\{ g_\mu(\boldsymbol{y}, \boldsymbol{t}) \,\big|\, (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega),\ \omega \geq \Omega' \big\} \leq 0$$

holds, from which and Lemma 4.1 we obtain $g_\mu^*(\Omega) \leq 0$. Similarly, if g_μ^* defined by (30) satisfies $g_\mu^*(\Omega) \leq 0$, then

$$\{(\boldsymbol{y}, \boldsymbol{t}, \omega) \,|\, (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega),\ \omega \geq \Omega,\ g_\mu(\boldsymbol{y}, \boldsymbol{t}) \geq \epsilon\} = \emptyset$$

for any $\epsilon > 0$. Hence, for any $\epsilon > 0$, the inequality

$$\max_{\boldsymbol{y}, \boldsymbol{t}, \omega} \{\omega \,|\, (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega),\ g_\mu(\boldsymbol{y}, \boldsymbol{t}) \geq \epsilon\} \leq \omega$$

holds, from which and Lemma 4.1 we obtain $\omega^* \leq \Omega$. □

Remark 4.3. We give some explanations regarding the motivation of working on the perturbed problem (26) instead of the original problem (22). In association with the condition (ii) investigated in the proof of Lemma 4.1, observe that 0 is the global maximum of g under the constraint $(\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega)$. Indeed, we see

$$-\infty = \lim_{\epsilon \to +0} \sup_{\boldsymbol{y}, \boldsymbol{t}, \omega} \{\omega \,|\, (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\omega),\ g(\boldsymbol{y}, \boldsymbol{t}) \geq \epsilon\} \neq \omega^*,$$

which implies that Problem (22) does not have the property similar to (32). Hence, we cannot show Theorem 4.4 below, which plays a key role in the bisection method (Algorithm 4.5). In this manner, the perturbation (or regularization) (25) on g is necessary for the condition (32). Certainly, such a perturbation is not unique. An intuitive interpretation of g_μ defined by (25) has been given in Remark 3.3. It is important that g_μ in (25) inherits the convexity and smoothness properties of g. □

The following is a key result providing the optimality condition of Problem (26):

Theorem 4.4. *Problems* (26) *and* (31) *have the following relations:*

(i) $\omega^* < \Omega$ *if* $g_\mu^*(\Omega) < 0$*;*

(ii) $\omega^* > \Omega$ *if* $g_\mu^*(\Omega) > 0$.

Furthermore,

(iii) *A feasible solution* $(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{t}}, \widehat{\omega})$ *for Problem* (26) *is optimal if and only if* $g_\mu^*(\widehat{\omega}) = 0$.

Proof. Recall that Problem (31) is equivalent to Problem (30), and hence we show this theorem with respect to Problem (30). Firstly, observe that the assertion (i) follows (iii) and the 'if' part of Lemma 4.2; the assertion (ii) follows (iii) and the fact that (iv) $g_\mu^*(\Omega) \geq 0$ implies $\omega^* \geq \Omega$. Thus, it suffices to show the assertions (iii) and (iv). We start with showing (iv). Observe that $g_\mu^*(\Omega) \geq 0$ implies that there exists a feasible solution $(\boldsymbol{y}', \boldsymbol{t}', \omega')$ of Problem (30) satisfying

$$\omega' \geq \Omega \tag{36}$$

and $g_\mu(\boldsymbol{y}', \boldsymbol{t}') \geq 0$. Then, we easily see that $(\boldsymbol{y}', \boldsymbol{t}', \omega')$ is also feasible for Problem (26) satisfying (36), which proves the assertion (iv). We next show the assertion (iii), i.e. show that $\omega^* = \Omega$ if and only if $g_\mu^*(\Omega) = 0$. Note that the 'if' part of (iii) follows (iv) and the 'if' part of Lemma 4.2. Hence, suppose $\omega^* = \Omega$. Then there exists a feasible solution $(\boldsymbol{y}'', \boldsymbol{t}'', \Omega)$ of Problem (26). Since $(\boldsymbol{y}'', \boldsymbol{t}'', \Omega)$ is also feasible for Problem (30), the constraint $g_\mu(\boldsymbol{y}'', \boldsymbol{t}'') \geq 0$ of Problem (26) implies $g^*(\Omega) \geq 0$ in Problem (30). On the other hand, the 'only if' part of Lemma 4.2 and the condition $\omega^* = \Omega$ imply $g^*(\Omega) \leq 0$. Thus, we obtain $g^*(\Omega) = 0$ if $\omega^* = \Omega$. □

Let $(\boldsymbol{y}^*, \boldsymbol{t}^*)$ denote an optimal solution of Problem (31) for a given Ω. Recall that ω^* has been defined in (26). If $g^*(\Omega) \geq 0$, then $(\boldsymbol{y}^*, \boldsymbol{t}^*)$ is a feasible solution of Problem (26), which implies $\omega^* \geq \Omega$. On the contrary, if $g^*(\Omega) < 0$, then $\omega^* < \Omega$. As a consequence, we see that the following bisection method solves Problem (26):

Algorithm 4.5 (prototype of bisection method for Problem (26)).

Step 0: Choose $\underline{\Omega}^0$ and $\overline{\Omega}^0$ satisfying $0 < \underline{\Omega}^0 \leq \omega^* \leq \overline{\Omega}^0 < \Omega^{\max}$, and the small tolerance $\epsilon > 0$. Set $l := 0$.

Step 1: If $\overline{\Omega}^l - \underline{\Omega}^l \leq \epsilon$, then stop. Otherwise, set $\Omega := (\underline{\Omega}^l + \overline{\Omega}^l)/2$.

Step 2: Find an optimal solution $(\boldsymbol{y}^*, \boldsymbol{t}^*)$ of Problem (31).

Step 3: If $g_\mu^*(\Omega) \geq 0$, then set $\underline{\Omega}^{l+1} := \Omega$ and $\overline{\Omega}^{l+1} := \overline{\Omega}^l$. Otherwise, set $\overline{\Omega}^{l+1} := \Omega$ and $\underline{\Omega}^{l+1} := \underline{\Omega}^l$.

Step 4: Set $l := l + 1$, and go to Step 1.

We investigate a solution process of Problem (31) in the following section.

5 Sequential SDP algorithm for polynomial SDP

In the previous section, we have seen that a solution of the nonlinear SDP problem (26) is obtained by Algorithm 4.5, provided that Problem (31) can be solved efficiently at Step 2. The aim of this section is to propose an algorithm for Problem (31). We pay attention to the special property of Problem (31), which is a maximization problem of the convex function over the convex constraints that are represented via a linear matrix inequality. Based on this property, in order to solve Problem (31) we employ the *DC algorithm* [4], which has been developed for the *DC* (difference of convex functions) *programming* problems. It has been observed that the DC algorithm quite often converges to global optimal solutions of various nonconvex optimization problems in practice; see, e.g. [2–4].

For a given $\Omega \in \mathbb{R}_{++}$, let $I_{\mathcal{F}}(\cdot;\Omega) : \mathbb{R}^m \times \mathbb{R}^m \to (-\infty, +\infty]$ be the indicator function of the feasible set of Problem (31), i.e.

$$I_{\mathcal{F}}(\boldsymbol{y}, \boldsymbol{t}; \Omega) = \begin{cases} 0, & \text{if } (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Letting $\rho \in \mathbb{R}_{++}$ be constant, define $h_1 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ and $h_2 : \mathbb{R}^m \times \mathbb{R}^m \to (-\infty, +\infty]$ by

$$h_1(\boldsymbol{y}, \boldsymbol{t}) = \frac{\rho}{2}\left(\|\boldsymbol{y}\|^2 + \|\boldsymbol{t}\|^2\right) + g_\mu(\boldsymbol{y}, \boldsymbol{t}), \tag{37}$$

$$h_2(\boldsymbol{y}, \boldsymbol{t}; \Omega) = I_{\mathcal{F}}(\boldsymbol{y}, \boldsymbol{t}; \Omega) + \frac{\rho}{2}\left(\|\boldsymbol{y}\|^2 + \|\boldsymbol{t}\|^2\right). \tag{38}$$

Note that h_1 and h_2 are strictly convex. Problem (31) is equivalently rewritten as

$$\max_{\boldsymbol{y},\boldsymbol{t}} \left\{ h_1(\boldsymbol{y}, \boldsymbol{t}) - h_2(\boldsymbol{y}, \boldsymbol{t}; \Omega) \mid (\boldsymbol{y}, \boldsymbol{t}) \in \mathbb{R}^m \times \mathbb{R}^m \right\}, \tag{39}$$

which is a DC programming problem.

Remark 5.1. The choice of the pair of h_1 and h_2 in (37) and (38) is not unique. Indeed, it is known that there exist infinitely many pairs of strictly convex functions h_1 and h_2 such that Problem (39) becomes equivalent to Problem (31) [4]. Certainly, for example, we may choose any $\rho \in \mathbb{R}_{++}$ in (37) and (38), while the convergence property of the algorithm presented below may depend on the choice of ρ. In our numerical examples, we choose $\rho = 0.1$; see section 7. □

It is known that the DC programming problem is often solved globally by using the *DC algorithm* [4]. Indeed, the DC algorithm is one of a few algorithms based on a local approach that has been successfully applied to large-scale DC programming problems. The DC algorithm for Problem (39) sequentially solves the convex optimization problem in the form of

$$\max_{\boldsymbol{y},\boldsymbol{t}} \left\{ \left[\langle \boldsymbol{y}_*^k, \boldsymbol{y} - \boldsymbol{y}^k\rangle + \langle \boldsymbol{t}_*^k, \boldsymbol{t} - \boldsymbol{t}^k\rangle + h_1(\boldsymbol{y}^k, \boldsymbol{t}^k)\right] - h_2(\boldsymbol{y}, \boldsymbol{t}; \Omega) \,\middle|\, (\boldsymbol{y}, \boldsymbol{t}) \in \mathbb{R}^m \times \mathbb{R}^m \right\}, \tag{40}$$

where $(\boldsymbol{y}^k, \boldsymbol{t}^k)$ and $(\boldsymbol{y}_*^k, \boldsymbol{t}_*^k)$ correspond to current solutions for Problem (39) and its dual, respectively. Particularly, the so-called *simplified* DC algorithm is designed based on the update scheme [3, section 2.3]

$$\begin{pmatrix} \boldsymbol{y}_*^k \\ \boldsymbol{t}_*^k \end{pmatrix} := \partial h_1(\boldsymbol{y}^k, \boldsymbol{t}^k). \tag{41}$$

Then it is known that the generated sequence $\{h_1(\boldsymbol{y}^k, \boldsymbol{t}^k) - h_2(\boldsymbol{y}^k, \boldsymbol{t}^k; \Omega)\}$ of the objective function of Problem (39) increases monotonically [4]. From the definition (37) of h_1, we obtain

$$\partial h_1(\boldsymbol{y}^k, \boldsymbol{t}^k) = \nabla h_1(\boldsymbol{y}^k, \boldsymbol{t}^k) = \begin{pmatrix} \mathrm{Diag}(p(y_i^k)^{p-1})\mathbf{1} \\ (\mu - 1)\mathbf{1} \end{pmatrix} + \rho \begin{pmatrix} \boldsymbol{y}^k \\ \boldsymbol{t}^k \end{pmatrix} \tag{42}$$

at any point $(\boldsymbol{y}^k, \boldsymbol{t}^k)$ satisfying $h_2(\boldsymbol{y}^k, \boldsymbol{t}^k) < +\infty$. Substitution of (41) and (42) into Problem (40) yields

$$\left.\begin{array}{ll} \displaystyle\max_{\boldsymbol{y},\boldsymbol{t}} & \left\langle \mathrm{Diag}(p(y_i^k)^{p-1})\mathbf{1} + \rho\boldsymbol{y}^k, \boldsymbol{y} - \boldsymbol{y}^k \right\rangle + \left\langle \rho\boldsymbol{t}^k + (\mu-1)\mathbf{1}, \boldsymbol{t} - \boldsymbol{t}^k \right\rangle - \frac{\rho}{2}\left(\|\boldsymbol{y}\|^2 + \|\boldsymbol{t}\|^2\right) \\ \mathrm{s.t.} & (\boldsymbol{y}, \boldsymbol{t}) \in \mathcal{F}(\Omega). \end{array}\right\} \tag{43}$$

Note that Problem (43) can be embedded into the SDP problem. See section 6 for the explicit reformulation. We claim here only the fact that the objective function of Problem (43) is a strictly convex quadratic function and the constraint condition can be represented as a linear matrix inequality in terms of $\boldsymbol{y}$, $\boldsymbol{t}$, and some auxiliary variables.

The following is the full-description of the algorithm for Problem (26), which solves linear SDP problems sequentially:

Algorithm 5.2 (sequential SDP for Problem (26)).

Step 0: Choose $\underline{\Omega}^0$ and $\overline{\Omega}^0$ satisfying $0 < \underline{\Omega}^0 \leq \omega^* \leq \overline{\Omega}^0 < \Omega^{\max}$, $(\boldsymbol{y}^0, \boldsymbol{t}^0) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$, $\rho > 0$, and the small tolerances $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Set $l := 0$.

Step 1: If $\overline{\Omega}^l - \underline{\Omega}^l \leq \epsilon_1$, then stop. Otherwise, set $\Omega := (\underline{\Omega}^l + \overline{\Omega}^l)/2$. Set $k := 0$.

Step 2: Find the unique optimal solution $(\boldsymbol{y}^{k+1}, \boldsymbol{t}^{k+1})$ of the SDP problem (43).

Step 3: If $\|(\boldsymbol{y}^{k+1}, \boldsymbol{t}^{k+1}) - (\boldsymbol{y}^{k+1}, \boldsymbol{t}^{k+1})\| \leq \epsilon_2$, then set $(\boldsymbol{y}^*, \boldsymbol{t}^*) := (\boldsymbol{y}^{k+1}, \boldsymbol{t}^{k+1})$. Otherwise, set $k \leftarrow k+1$, and go to Step 2.

Step 4: If $g_\mu(\boldsymbol{y}^*, \boldsymbol{t}^*) \geq 0$, then set $\underline{\Omega}^{l+1} := \Omega$ and $\overline{\Omega}^{l+1} := \overline{\Omega}^l$. Otherwise, set $\underline{\Omega}^{l+1} := \underline{\Omega}^l$ and $\overline{\Omega}^{l+1} := \Omega$.

Step 5: Set $l := l+1$, $(\boldsymbol{y}^0, \boldsymbol{t}^0) := (\boldsymbol{y}^*, \boldsymbol{t}^*)$, and go to Step 1.

The following proposition guarantees that Algorithm 5.2 is well-defined in the sense that the subproblem (43) solved at each iteration has the unique solution.

Proposition 5.3. *Problem* (43) *has the unique optimal solution.*

Proof. We see that the objective function of Problem (43) is strongly convex since $\rho > 0$. Recall that $\Omega^{\max}$ has been defined by (29). The construction of Ω at Step 1 of Algorithm 5.2 implies that $0 < \Omega < \Omega^{\max}$ is satisfied at each iteration, from which it follows that the set $\mathcal{F}(\Omega)$ is nonempty and convex. Accordingly, Problem (43) is the minimization of the strongly convex function over the nonempty bounded convex set, which implies that the optimal solution exists uniquely. □

6 Explicit SDP formulations of subproblem for framed structures

In this section, we investigate the stiffness and mass matrices of framed structures with various cross-sections, and give the explicit formulation of the SDP subproblem (43) for each particular case.

Consider the framed structures in the two- or three-dimensional space. Let $n^{\rm m}$ denote the number of members of the frame. For each member, a_i and I_i denote the cross-sectional area and the moment of inertia, respectively. The stiffness matrix is written as

$$K = \sum_{i=1}^{n^{\rm m}} K_i^{\rm a} a_i + \sum_{i=1}^{n^{\rm m}} K_i^{\rm I} I_i, \tag{44}$$

where $K_i^{\rm a}$ and $K_i^{\rm I}$ are constant and positive semidefinite matrices. Note that a_i and I_i are generally supposed to be dependent variables. We define the member stiffness matrix $K_i^{\rm a} a_i + K_i^{\rm I} I_i$ by using the Euler–Bernoulli beam element. The mass matrix due to the structural mass can be written as

$$M^{\rm S} = \sum_{i=1}^{n^{\rm m}} M_i a_i, \tag{45}$$

where M_i $(i = 1, \dots, n^{\rm m})$ are constant and positive semidefinite matrices. Thus, $M^{\rm S}$ is the linear function of $\boldsymbol{a}$.

6.1 Circular solid cross-sections

Suppose that each member of the frame have a circular solid cross-section with the radius r_i. The cross-sectional area and the moment of inertia are written in terms of r_i as

$$a_i = \pi r_i^2, \quad I_i = \frac{\pi r_i^4}{4}. \tag{46}$$

Thus, the dimension of each member is determined by any one of the parameters a_i, I_i, and r_i.

Choosing a_i $(i = 1, \dots, n^{\mathrm{m}})$ as the design variables, we put

$$y_i := a_i, \quad m := n^{\mathrm{m}}$$

in Problem (12). The structural volume is a linear function of $\boldsymbol{y}$ written as

$$v(\boldsymbol{y}) = \boldsymbol{l}^{\mathrm{T}} \boldsymbol{y}, \tag{47}$$

where l_i denotes the length of the ith member, which is a positive constant.

Consequently, from (44)–(47) it follows that the optimization problem (13) which maximizes the fundamental eigenvalue is formulated explicitly as

$$\left.\begin{array}{ll}\displaystyle\max_{\boldsymbol{y},\omega} & \omega \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n^{\mathrm{m}}} \left(A_i^{(2)} y_i^2 + A_i^{(1)} y_i \right) - \omega \left(\sum_{i=1}^{n^{\mathrm{m}}} B_i^{(1)} y_i + M_0 \right) \succeq O, \\ & \boldsymbol{l}^{\mathrm{T}} \boldsymbol{y} \leq V^0, \\ & \boldsymbol{y} \geq \boldsymbol{0}, \end{array}\right\} \tag{48}$$

where

$$A_i^{(2)} := \frac{1}{4\pi} K_i^{\mathrm{I}}, \quad A_i^{(1)} := K_i^{\mathrm{a}}, \quad B_i^{(1)} := M_i,$$

and M_0 denotes the mass matrix for the nonstructural mass. Note that $A_i^{(1)}$, $A_i^{(2)}$, $B_i^{(1)}$, and M_0 are positive semidefinite matrices from their definitions. Thus, for the framed structures with circular cross-sections, the structural optimization problem (13) is embedded into the form of the polynomial SDP problem (1).

We next investigate the explicit formulation of the subproblem, which is to be solved in Algorithm 5.2. The new variables $\boldsymbol{t} \in \mathbb{R}^{n^{\mathrm{m}}}$ are introduced according to (18) so that

$$t_i = y_i^2, \quad i = 1, \dots, n^{\mathrm{m}} \tag{49}$$

should be satisfied at the optimal solution. The subproblem solved at Step 2 of Algorithm 5.2 is obtained as

$$\left.\begin{array}{ll}\displaystyle\max_{\boldsymbol{y},\boldsymbol{t}} & \left\langle (\rho+2)\boldsymbol{y}^k, \boldsymbol{y} - \boldsymbol{y}^k \right\rangle + \left\langle \rho \boldsymbol{t}^k + (\mu - 1)\boldsymbol{1}, \boldsymbol{t} - \boldsymbol{t}^k \right\rangle - \frac{\rho}{2} \left(\|\boldsymbol{y}\|^2 + \|\boldsymbol{t}\|^2 \right) \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n^{\mathrm{m}}} A_i^{(2)} t_i + \sum_{i=1}^{n^{\mathrm{m}}} \left(A_i^{(1)} - \Omega B_i^{(1)} \right) y_i - \Omega M_0 \succeq O, \\ & \boldsymbol{l}^{\mathrm{T}} \boldsymbol{y} \leq V^0, \\ & \boldsymbol{y} \geq \boldsymbol{0}, \\ & t_i \geq y_i^2, \quad i = 1, \dots, n^{\mathrm{m}}. \end{array}\right\} \tag{50}$$

By multiplying a positive constant $2/\rho$ and eliminating the constant terms, the objective function of Problem (50) is simplified as

$$-\left\|\boldsymbol{y}-\left(1+\frac{2}{\rho}\right)\boldsymbol{y}^k\right\|^2-\left\|\boldsymbol{t}-\left(\boldsymbol{t}^k+\frac{\mu-1}{\rho}\mathbf{1}\right)\right\|^2$$

without changing the optimal solution. Observe that the condition

$$\xi \geq \left\|\boldsymbol{y}-\left(1+\frac{2}{\rho}\right)\boldsymbol{y}^k\right\|^2+\left\|\boldsymbol{t}-\left(\boldsymbol{t}^k+\frac{\mu-1}{\rho}\mathbf{1}\right)\right\|^2$$

holds if and only if

$$\xi+\frac{1}{4} \geq \left\|\begin{pmatrix} \xi-(1/4) \\ \boldsymbol{y}-[1+(2/\rho)]\boldsymbol{y}^k \\ \boldsymbol{t}-\left[\boldsymbol{t}^k+(\mu-1)\mathbf{1}/\rho\right] \end{pmatrix}\right\|$$

is satisfied, which is the second-order cone constraint.

Consequently, by introducing an auxiliary variable $\xi \in \mathbb{R}$, Problem (50) can be rewritten without changing the optimal solution as

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{y},\boldsymbol{t},\xi} & \xi \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n^{\rm m}} A_i^{(2)} t_i+\sum_{i=1}^{n^{\rm m}}\left(A_i^{(1)}-\Omega B_i^{(1)}\right) y_i-\Omega M_0 \succeq O, \\ & V^0-\boldsymbol{l}^{\rm T}\boldsymbol{y} \geq 0, \\ & \boldsymbol{y} \geq \mathbf{0}, \\ & t_i+(1/4) \geq \left\|\begin{pmatrix} t_i-(1/4) \\ y_i \end{pmatrix}\right\|, \quad i=1,\dots,n^{\rm m}, \\ & \xi+(1/4) \geq \left\|\begin{pmatrix} \xi-(1/4) \\ \boldsymbol{y}-[1+(2/\rho)]\boldsymbol{y}^k \\ \boldsymbol{t}-\left[\boldsymbol{t}^k+(\mu-1)\mathbf{1}/\rho\right] \end{pmatrix}\right\|, \end{array}\right\} \quad (51)$$

which is a linear SDP problem. We can solve Problem (51) by using the primal-dual interior-point method for SDP. Some of such softwares, e.g. SeDuMi [36], are designed to solve the linear optimization over the symmetric cone $\mathcal{K} \subset \mathbb{R}^n$, where $\mathcal{K}$ is the direct product of some cones expressed by linear inequalities, second-order cone constraints, and positive semidefinite constraints. Notice here that Problem (51) has (i) $(2n^{\rm m}+1)$ variables; (ii) $(n^{\rm m}+1)$ linear inequalities; (iii) $n^{\rm m}$ second-order cone constraints in $\mathbb{R}^3$; (iv) one second-order cone constraint in $\mathbb{R}^{2n^{\rm m}+2}$; (v) one linear matrix inequality constraint in $\mathcal{S}^{n^{\rm d}}$.

6.2 Rectangular solid cross-sections with fixed widths

Suppose that each member of the frame have a rectangular solid cross-section with the width b_i and the hight h_i. The cross-sectional area and the moment of inertia are written in the terms of b_i and h_i as

$$a_i = b_i h_i, \quad I_i = \frac{b_i h_i^3}{12}. \quad (52)$$

Throughout this section, we assume that b_i is fixed. Thus, the dimension of each member is determined only by h_i, and we regard h_i $(i = 1, \dots, n^{\mathrm{m}})$ as the design variables by putting

$$y_i := h_i, \quad m := n^{\mathrm{m}}$$

in Problem (12). The structural volume is written as

$$v(\boldsymbol{y}) = \boldsymbol{l}^{\mathrm{T}} \mathrm{Diag}(\boldsymbol{b})\boldsymbol{y}, \tag{53}$$

where l_i denotes the length of the ith member.

Consequently, from (44), (45), (52), and (53) it follows that the optimization problem (13) which maximizes the fundamental eigenvalue is formulated explicitly as

$$\left.\begin{array}{ll} \displaystyle\max_{\boldsymbol{y},\omega} & \omega \\ \mathrm{s.t.} & \displaystyle\sum_{i=1}^{n^{\mathrm{m}}} \left(A_i^{(3)} y_i^3 + A_i^{(1)} y_i \right) - \omega \left(\sum_{i=1}^{n^{\mathrm{m}}} B_i^{(1)} y_i + M_0 \right) \succeq O, \\ & \boldsymbol{l}^{\mathrm{T}} \mathrm{Diag}(\boldsymbol{b})\boldsymbol{y} \leq V^0, \\ & \boldsymbol{y} \geq \boldsymbol{0}, \end{array}\right\} \tag{54}$$

where

$$A_i^{(3)} := \frac{b_i}{12} K_i^{\mathrm{I}}, \quad A_i^{(1)} := b_i K_i^{\mathrm{a}}, \quad B_i^{(1)} := b_i M_i,$$

and M_0 denotes the mass matrix for the nonstructural mass. Note that $A_i^{(1)}$, $A_i^{(3)}$, $B_i^{(1)}$, and M_0 are positive semidefinite matrices from their definitions. Thus, for the framed structures with rectangular cross-sections, the structural optimization problem (13) is embedded into the form of the nonlinear SDP problem (1).

The remainder of this section is devoted to an explicit formulation of the subproblem, which is to be solved in Algorithm 5.2. According to (18), we introduce the auxiliary variables $\boldsymbol{t} \in \mathbb{R}^{n^{\mathrm{m}}}$ so that

$$t_i = y_i^3, \quad i = 1, \dots, n^{\mathrm{m}} \tag{55}$$

is satisfied at the optimal solution. The subproblem solved at Step 2 of Algorithm 5.2 is obtained as

$$\left.\begin{array}{ll} \displaystyle\max_{\boldsymbol{y},\boldsymbol{t}} & \left\langle \rho \boldsymbol{y}^k + 3\,\mathrm{Diag}(\boldsymbol{y}^k)\boldsymbol{y}^k, \boldsymbol{y} - \boldsymbol{y}^k \right\rangle + \left\langle \rho \boldsymbol{t}^k + (\mu - 1)\boldsymbol{1}, \boldsymbol{t} - \boldsymbol{t}^k \right\rangle - \dfrac{\rho}{2} \left(\|\boldsymbol{y}\|^2 + \|\boldsymbol{t}\|^2 \right) \\ \mathrm{s.t.} & \displaystyle\sum_{i=1}^{n^{\mathrm{m}}} A_i^{(3)} t_i + \sum_{i=1}^{n^{\mathrm{m}}} \left(A_i^{(1)} - \Omega B_i^{(1)} \right) y_i - \Omega M_0 \succeq O, \\ & V^0 - \boldsymbol{l}^{\mathrm{T}} \mathrm{Diag}(\boldsymbol{b})\boldsymbol{y} \geq 0, \\ & \boldsymbol{y} \geq \boldsymbol{0}, \\ & t_i \geq y_i^3, \quad i = 1, \dots, n^{\mathrm{m}}. \end{array}\right\} \tag{56}$$

By multiplying a positive constant $2/\rho$ and eliminating the constant terms, the objective function of Problem (56) is simplified as

$$-\left\| \boldsymbol{y} - \left(\boldsymbol{y}^k + \frac{3}{\rho} \mathrm{Diag}(\boldsymbol{y}^k)\boldsymbol{y}^k \right) \right\|^2 - \left\| \boldsymbol{t} - \left(\boldsymbol{t}^k + \frac{\mu - 1}{\rho} \boldsymbol{1} \right) \right\|^2$$

without changing the optimal solution.

The following proposition prepares the reformulation of Problem (56) into the SDP formulation:

Proposition 6.1. (y_i, t_i) *satisfies*

$$t_i \geq y_i^3, \quad y_i \geq 0 \tag{57}$$

if and only if there exists $\zeta_i \in \mathbb{R}$ *satisfying*

$$\zeta_i + \frac{1}{4} \geq \left\| \begin{pmatrix} \zeta_i - 1/4 \\ y_i \end{pmatrix} \right\|, \tag{58}$$

$$y_i + t_i \geq \left\| \begin{pmatrix} y_i - t_i \\ 2\zeta_i \end{pmatrix} \right\|. \tag{59}$$

Proof. Observe that the condition (57) is equivalent to

$$y_i t_i \geq y_i^4, \quad y_i \geq 0,$$

which holds if and only if there exists ζ_i such that

$$\zeta_i \geq y_i^2, \tag{60}$$

$$y_i t_i \geq \zeta_i^2, \quad y_i \geq 0. \tag{61}$$

We easily see that the condition (60) is equivalent to (58). Note that (59) implies $y_i + t_i \geq 0$, from which it follows that (59) is equivalent to (61). □

Consequently, by introducing an auxiliary variables $\xi \in \mathbb{R}$ and $\boldsymbol{\zeta} \in \mathbb{R}^{n^{\mathrm{m}}}$, and by using Proposition 6.1, Problem (56) can be rewritten without changing the optimal solution as

$$\left. \begin{array}{ll} \displaystyle\min_{\boldsymbol{y},\boldsymbol{t},\boldsymbol{\zeta},\xi} & \xi \\ \mathrm{s.t.} & \displaystyle\sum_{i=1}^{n^{\mathrm{m}}} A_i^{(3)} t_i + \sum_{i=1}^{n^{\mathrm{m}}} \left(A_i^{(1)} - \Omega B_i^{(1)} \right) y_i - \Omega M_0 \succeq O, \\ & V^0 - \boldsymbol{l}^{\mathrm{T}} \mathrm{Diag}(\boldsymbol{b}) \boldsymbol{y} \geq 0, \\ & \zeta_i + 1/4 \geq \left\| \begin{pmatrix} \zeta_i - 1/4 \\ y_i \end{pmatrix} \right\|, \quad i = 1, \dots, n^{\mathrm{m}}, \\ & y_i + t_i \geq \left\| \begin{pmatrix} y_i - t_i \\ 2\zeta_i \end{pmatrix} \right\|, \quad i = 1, \dots, n^{\mathrm{m}}, \\ & \xi + 1/4 \geq \left\| \begin{pmatrix} \xi - 1/4 \\ \boldsymbol{y} - \boldsymbol{y}^k - (3/\rho)\,\mathrm{Diag}(\boldsymbol{y}^k)\boldsymbol{y}^k \\ \boldsymbol{t} - \left[\boldsymbol{t}^k + (\mu - 1)\boldsymbol{1}/\rho \right] \end{pmatrix} \right\|, \end{array} \right\} \tag{62}$$

which is a linear SDP problem. It should be emphasized that the constraints $\boldsymbol{y} \geq \boldsymbol{0}$ are redundant in Problem (62), and thus are omitted. Note that Problem (62) has (i) $(3n^{\mathrm{m}} + 1)$ variables; (ii) one linear inequality; (iii) $2n^{\mathrm{m}}$ second-order cone constraints in $\mathbb{R}^3$; (iv) one second-order cone constraint in $\mathbb{R}^{2n^{\mathrm{m}}+2}$; (v) one linear matrix inequality constraint in $\mathcal{S}^{n^{\mathrm{d}}}$. Thus, Problem (62) for rectangular cross-sections with fixed widths has larger numbers of variables and constraints than Problem (51) formulated for circular solid cross-sections. This is because Problem (54) has cubic terms of y_i, while Problem (48) has quadratic terms.

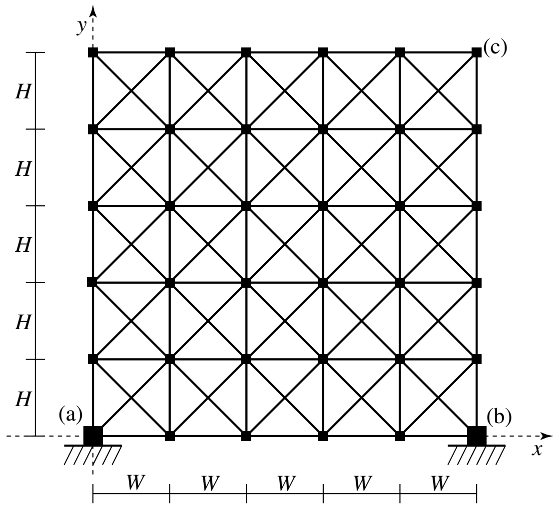


Figure 3: 5×5 plane grid frame.

Table 1: Problem sizes and upper bounds V^0 for structural volumes of $k \times k$ grid frames.

	n^{m}	n^{d}	V^0 (m^3)
2×2	20	21	2.7976×10^{-2}
3×3	42	42	5.9347×10^{-2}
4×4	72	69	1.0231×10^{-1}
5×5	110	102	1.5685×10^{-1}
6×6	156	141	2.2299×10^{-1}

7 Numerical experiments

Optimal designs are computed for various framed structures by using Algorithm 5.2. At Step 2 we solve the SDP problem (43) by using SeDuMi Ver. 1.05 [36], which implements the primal-dual interior-point method for the linear programming problems over symmetric cones. Computation has been carried out on Pentium M (1.2 GHz with 1.0 GB memory) with MATLAB Ver. 7.0.1 [39].

In the following examples, the elastic modulus and the mass density of members are 200.0 GPa and 7.86×10^{-3} kg/cm^2, respectively.

7.1 Plane square grids

Optimal topologies are found for plane frames with 2×2, 3×3, 4×4, 5×5, and 6×6 grids. The ground structure for 5×5 grid frame is shown in Fig.3, where $W = 2.0$ m and $H = 2.0$ m. The intersecting pair of diagonals are not connected at their center. The nodes (a) and (b) are the fixed-supports. The nonstructural mass of 2.0×10^4 kg is located at the node (c), i.e. at the the upper-right corner. The ground structures and nonstructural masses are defined similarly for other grids. The number n^{m} of members and the number n^{d} of degrees of freedom for each case are listed in Table 1.

Table 2: Optimization results of $k \times k$ grid frames.

		case (I)				case (II)			
2×2	ω_1^0 (rad^2/s^2)	827.3				230.3			
	ω_i^* (rad^2/s^2)	1743.8	5228.8			1744.7	8180.0		
	CPU (sec.)	7.7				22.7			
	# of SDPs	23				57			
3×3	ω_1^0 (rad^2/s^2)	633.8				212.5			
	ω_i^* (rad^2/s^2)	1638.2	2211.4			1639.7	3988.6		
	CPU (sec.)	22.7				80.2			
	# of SDPs	25				80			
4×4	ω_1^0 (rad^2/s^2)	531.0				201.7			
	ω_i^* (rad^2/s^2)	1581.2	1651.6			1583.0	1589.4		
	CPU (sec.)	67.5				346.4			
	# of SDPs	27				115			
5×5	ω_1^0 (rad^2/s^2)	466.4				197.2			
	ω_i^* (rad^2/s^2)	1541.8	1541.8	1558.7		1544.7	1604.0	2555.6	
	CPU (sec.)	188.8				982.1			
	# of SDPs	27				127			
6×6	ω_1^0 (rad^2/s^2)	421.5				193.4			
	ω_i^* (rad^2/s^2)	1511.6	1511.6	1511.6	1513.4	1514.8	1515.2	1544.5	1655.4
	CPU (sec.)	482.2				3085.1			
	# of SDPs	28				164			

For each grid, optimal topologies are computed for circular cross-sections (case (I)) and rectangular cross-sections (case (II)) by using the formulations investigated in sections 6.1 and 6.2, respectively. For the case (I), the initial solution $(\boldsymbol{a}^0, \boldsymbol{t}^0)$ for Algorithm 5.2 is given as

$$a_i^0 = 6.0 \times 10^2 \text{ mm}^2, \quad t_i^0 = (a_i^0)^2, \quad i = 1, \dots, n^{\mathrm{m}};$$

for the case (II), the width of each cross-section is fixed as

$$b_i = 50.0 \text{ mm},$$

while we choose the initial solution as

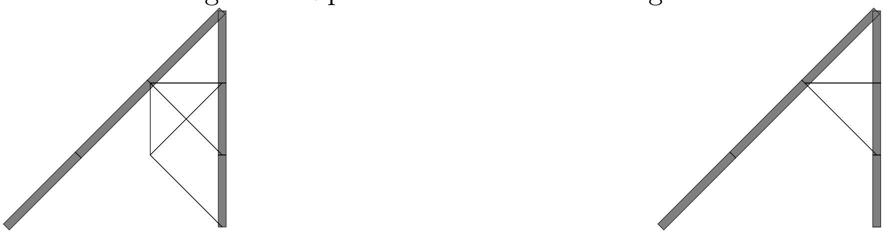
$$a_i^0 = 12.0 \text{ mm}, \quad t_i^0 = \frac{b_i (a_i^0)^3}{12}, \quad i = 1, \dots, n^{\mathrm{m}}.$$

Note that the initial solutions of both cases share the same structural volume. For each problem, the upper bound of structural volume listed in Table 1 is equal to the volume at the initial solution. We set $\mu = 10^{-3}$ in (25), and choose the parameters for Algorithm 5.2 as $\underline{\Omega} = \omega_1^0$, $\rho = 0.1$, $\epsilon_2 = 10^{-4}$, $\epsilon_1 = 10^{-5}\omega_1^0$. Note that $\omega_1^* - \omega_1^0$ of the case (II) is larger than that of the case (I). Hence, we choose $\overline{\Omega} = 4\omega_1^0$ and $\overline{\Omega} = 30\omega_1^0$ for the cases (I) and (II), respectively.

The optimal solutions obtained are shown in Figs. 4–7, where the width of each member is proportional to its cross-sectional area. Extremely slender member satisfying $a_i < 10^{-1}$ mm^2 for the case (I) and $a_i < 10^{-4}$ mm for the case (II) are removed in Figs. 4–7. However, the optimal

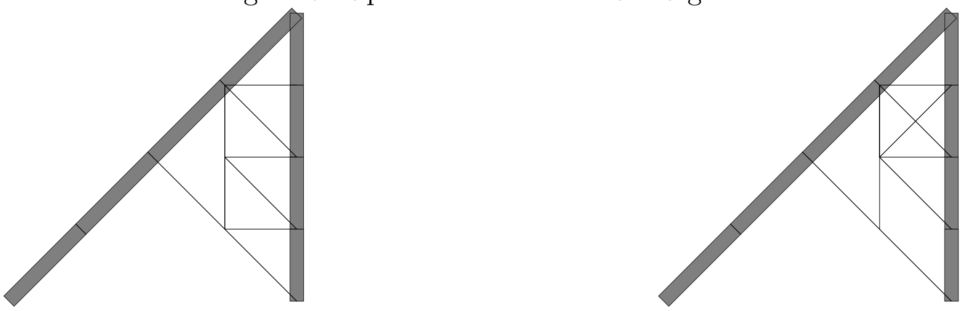
(I) circular cross-sections (II) rectangular cross-sections

Figure 4: Optimal solutions of 2×2 grid.



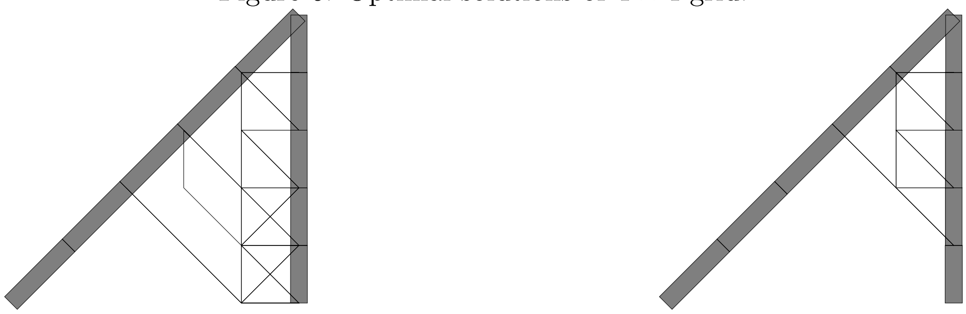
(I) circular cross-sections (II) rectangular cross-sections

Figure 5: Optimal solutions of 3×3 grid.



(I) circular cross-sections (II) rectangular cross-sections

Figure 6: Optimal solutions of 4×4 grid.



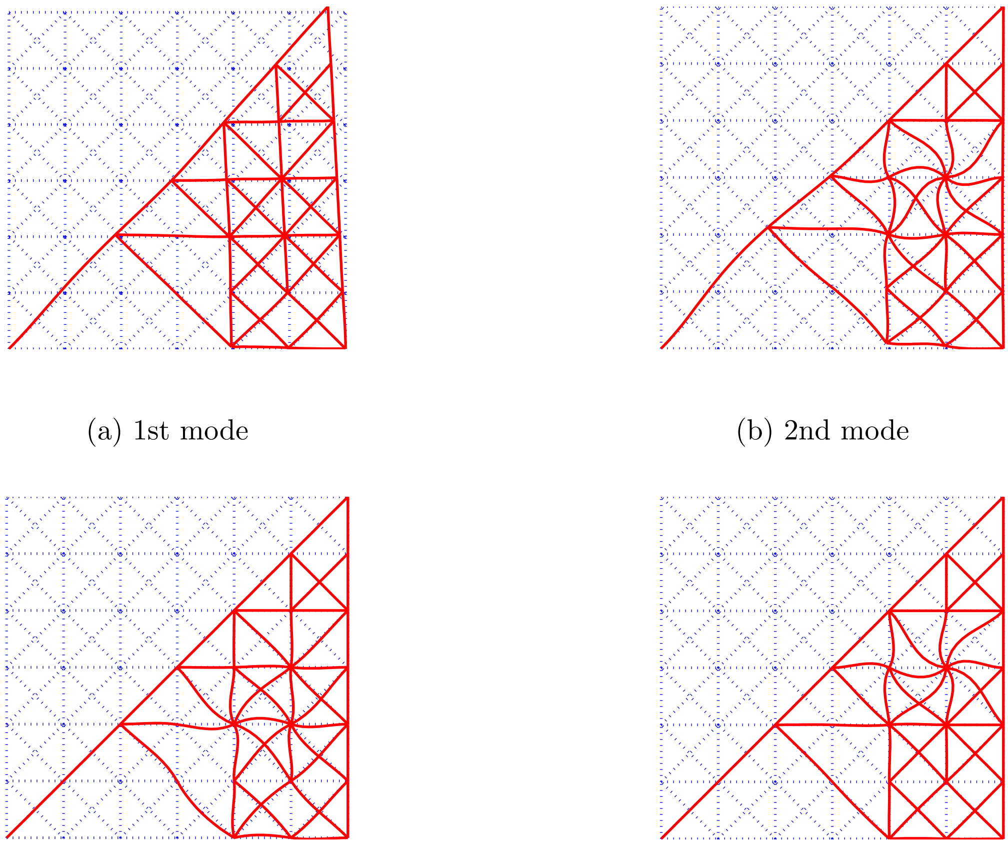
(I) circular cross-sections (II) rectangular cross-sections

Figure 7: Optimal solutions of 5×5 grid.

solutions still have some slender members supporting stiff members that form a long column and a diagonal beam.

The results are summarized in Table 2, where ω_1^0 denotes the lowest eigenvalue at the initial solution; ω_i^* are some of the lowest eigenvalues at the optimal solution. The computational time required by Algorithm 5.2 and the number of SDP problems solved are also listed.

In Table 2, it is of interest to note that the multiplicity of the lowest eigenvalues of the 5×5 frame (case (I)) is two, and the third eigenvalue is very close to the first two eigenvalues. For the 6×6



(a) 1st mode

(b) 2nd mode

(c) 3rd mode

(d) 4th mode

Figure 8: Eigenmodes of optimal 6×6 grid (case (I)).

frame (case (I)), we see that the multiplicity of the lowest eigenvalues is three, and forth eigenvalue is also very close to the first three ones. For the 6×6 frame (case (II)), the first two eigenvalues are very close. Thus, a large structure may possibly have large multiplicity of the lowest eigenvalues at its optimal design. The proposed algorithm converges optimal solutions with the multiple lowest eigenvalues without any difficulty. This is because the optimal solution of the SDP subproblem (43) in Algorithm 5.2 can be found by using the primal-dual interior-point method which do not resort to the derivatives of the minimal eigenvalue at all.

Fig.8 depicts the modes corresponding to the first four eigenvalues of the 6×6 grid (case (I)). In the mode (a), we see that the displacement of the node (c) at which the nonstructural mass is located is very large, while the local flexural deformations of the diagonal thick members dominate the mode (b). The local flexural deformations of the slender members dominate the modes (c) and (d). It is observed that many slender members are necessary at the optimal solution in order to prevent the mode (b) from having the smaller eigenvalue than that of the mode (a). Consequently, we may conclude that the algorithm presented successfully finds the global optimal solutions of these problems, which are regarded to be benchmark problems.

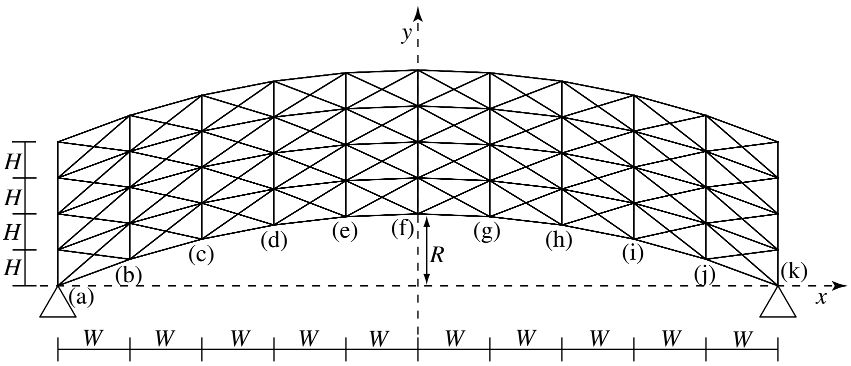
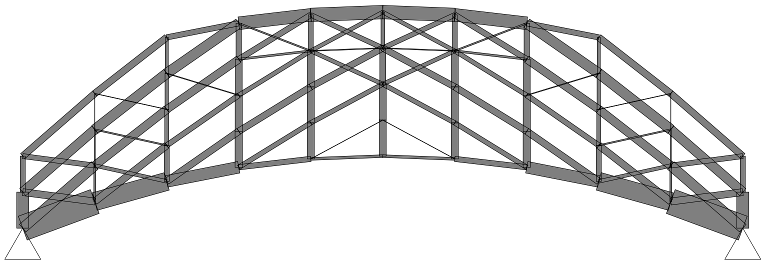
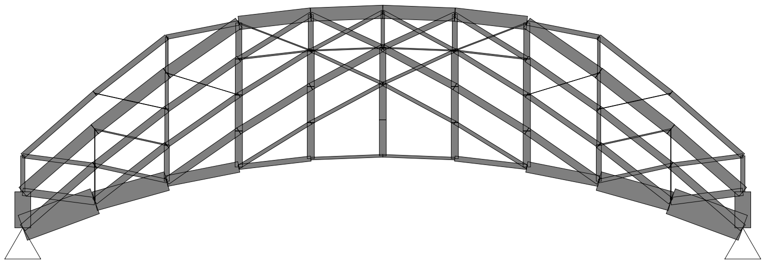


Figure 9: Plane arch frame.



(I) circular cross-sections



(II) rectangular cross-sections

Figure 10: Optimal solutions of the plane arch frame.

7.2 Plane arch grid

Consider a plane circular arch grid shown in Fig.9, where $W = 2.0$ m, $H = 1.0$ m, and $R = 2.0$ m. Nonstructural masses of 4.0×10^4 kg are located at the nodes (b)–(j). The nodes (a) and (k) are pin-supported, i.e. $n^{\mathrm{d}} = 161$ and $n^{\mathrm{m}} = 174$.

Optimal topologies are computed for circular cross-sections (case (I)) and rectangular cross-sections (case (II)) by using Algorithm 5.2. For the case (I), the initial solution $(\boldsymbol{a}^0, \boldsymbol{t}^0)$ for Algorithm 5.2 is given as

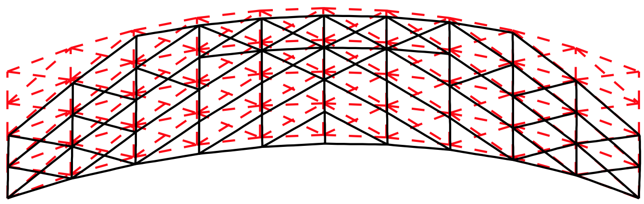
$$a_i^0 = 3.0 \times 10^3 \ \mathrm{mm}^2, \quad t_i^0 = (a_i^0)^2, \quad i = 1, \dots, n^{\mathrm{m}};$$

for the case (II), the width of each cross-section is fixed as

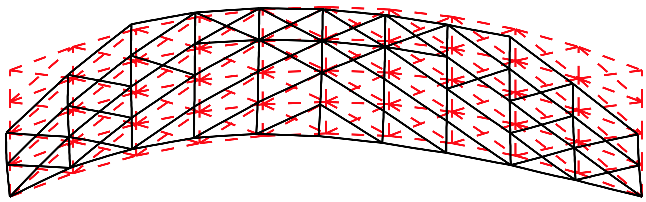
$$b_i = 50.0 \ \mathrm{mm},$$

Table 3: Results.

	case (I)			case (II)		
ω_1^0 (rad^2/s^2)	581.9			581.2		
ω_i^* (rad^2/s^2)	1127.5	1127.5	1874.1	1139.8	1139.8	1792.7
CPU (sec.)	732.1			8898.8		
# of SDPs	34			341		



(a) symmetric mode



(b) antisymmetric mode

Figure 11: Fundamental eigenmodes of the optimal arch frame (case (I)).

while we choose the initial solution as

$$a_i^0 = 60.0 \text{ mm}, \quad t_i^0 = \frac{b_i (a_i^0)^3}{12}, \quad i = 1, \dots, n^{\mathrm{m}}.$$

Note that the initial solutions of both cases share the same structural volume. The upper bound of structural volume $V^0 = 9.8572 \times 10^{-1}$ m^3 is equal to the volume at the initial solution. The minimal eigenvalues at the initial solutions are listed in Table 3. We set $\mu = 10^{-3}$ in (25), and choose the parameters for Algorithm 5.2 as $\underline{\Omega} = \omega_1^0$, $\rho = 0.1$, $\epsilon_2 = 10^{-4}$, $\epsilon_1 = 10^{-5}\omega_1^0$. The upper bounds for ω^* are chosen as $\overline{\Omega} = 4\omega_1^0$ and $\overline{\Omega} = 100\omega_1^0$ for the cases (I) and (II), respectively.

The optimal solutions obtained by using Algorithm 5.2 are shown in Fig.10. The eigenvalues and the computational costs are listed in Table 3. It is observed in Table 3 that the multiplicity of the fundamental eigenvalues of the optimal truss is two for each case, which is same as the results of the truss structure reported in [29, 31]. The corresponding fundamental modes (case (I)) are shown in Fig.11, which are symmetric and antisymmetric with respect to the y-axis in Fig.9. No local flexural modes are observed to be significant in these cases. Similarly, the two fundamental modes of case (II) are also symmetric and antisymmetric with respect to the y-axis.

8 Conclusions

In this paper, we have proposed a numerical technique for solving a class of nonlinear semidefinite programming problems, which correspond to the maximization problems of the lowest eigenvalues of structures. A sequential semidefinite programming (SDP) approach is proposed based on the bisection method. The method presented is applicable to the cases in which the optimal design has the multiple lowest eigenvalues.

A generalized eigenvalue problem has been formulated for the free vibration of a finitely discretized structure, in which the two symmetric matrices defining the eigenvalue problem are supposed to be polynomials in terms of the design variables. The maximization problem of the fundamental eigenvalue of the structure has been formulated as a polynomial SDP problem, which maximizes a linear function over a polynomial matrix inequality. We have proposed a bisection method for the polynomial SDP, at each iteration of which we should solve the maximization problem of a convex function over a linear matrix inequality.

In order to solve the convex maximization problem over a linear matrix inequality, we have embedded it into the DC (difference of convex functions) optimization problem, which is solved by using the so-called DC algorithm. The DC algorithm is one of a few algorithms based on a local approach which has been successfully applied to large-scale DC optimization problems, and quite often converges to the global optimal solution. For our problem, the (linear) SDP problem is to be solved at each iteration of the DC algorithm.

We showed in the numerical examples that the proposed algorithm can find the optimal topologies of framed structures. The optimal solutions with multiple lowest eigenvalues can be computed without any difficulty. For benchmarking examples, it seems that the obtained solutions are probably globally optimal, on the contrary to the fact that the sequential SDP method based on a linear matrix inequality approximation often converged to local optimal solutions in our preliminary numerical experiments.

References

[1] Alizadeh, F. and Goldfarb, D., Second-order cone programming, *Mathematical Programming*, **B95** (2003) 3–51.

[2] An, L.T.H., An efficient algorithm for globally minimizing a quadratic function under convex quadratic constraints, *Mathematical Programming*, **87** (2000) 401–426.

[3] An, L.T.H. and Tao, P.D., Solving a class of linearly constrained indefinite quadratic problems by D.C. Algorithms, *Journal of Global Optimization*, **11** (1997) 253–285.

[4] An, L.T.H. and Tao, P.D., The DC (difference of convex functions) programming and DCA revisited with DC models of real world nonconvex optimization problems, *Annals of Operations Research*, **133** (2005) 23–46.

[5] Ben-Tal, A., Jarre, F., Kočvara, M., Nemirovski, A. and Zowe, J., Optimal design of trusses under a nonconvex global buckling constraint, *Optimization and Engineering*, **1** (2000) 189–213.

[6] Ben-Tal, A. and Nemirovski, A., *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, Philadelphia, PA, 2001.

[7] Choi, K.K. and Kim, N.H., *Structural Sensitivity Analysis and Optimization: Linear Systems*, Springer, New York, NY, 2006.

[8] Correa, R. and Ramirez, H.C., A global algorithm for nonlinear semidefinite programming, *SIAM Journal on Optimization*, **15** (2004) 303–318.

[9] Cox, S. and Overton M.L., On the optimal design of columns against buckling, *SIAM Journal on Mathematical Analysis*, **23** (1992) 287–325.

[10] Freund, R.W., Jarre, F. and Vogelbusch, C., A sequential semidefinite programming method and an application in passive reduced-order modeling, *Technical Report*, 2005.

[11] Fares, B., Noll, D. and Apkarian, P., Robust control via sequential semidefinite programming, *SIAM Journal on Control and Optimization*, **40** (2002) 1791–1820.

[12] Haug, E.J., Choi, K.K. and Komkov, V., *Design Sensitivity Analysis of Structural Systems*, Academic Press, 1986.

[13] Helmberg, C., Semidefinite programming, *European Journal of Operational Research*, **137** (2002) 461–482.

[14] Jarre, F., An interior method for nonconvex semidefinite programs, *Optimization and Engineering*, **1** (2000) 347–372.

[15] Kanno, Y. and Ohsaki, M., Contact analysis of cable networks by using second-order cone programming, *SIAM Journal on Scientific Computing*, **27** (2006) 2032–2052.

[16] Kanno, Y., Ohsaki, M., and Katoh, N., Sequential semidefinite programming for optimization of framed structures under multimodal buckling constraints, *International Journal of Structural Stability and Dynamics*, **1** (2001) 585–602.

[17] Kanno, Y. and Takewaki, I., Confidence ellipsoids for static response of trusses with load and structural uncertainties, *Computer Methods in Applied Mechanics and Engineering*, **196** (2006) 393–403.

[18] Kanno, Y. and Takewaki, I., Sequential semidefinite program for maximum robustness design of structures under load uncertainties, *Journal of Optimization Theory and Applications*, **130** (2006) 265–287.

[19] Kanzow, C., Nagel, C., Kato, H. and Fukushima, M., Successive linearization methods for nonlinear semidefinite programs, *Computational Optimization and Applications*, **31** (2005) 251–273.

[20] Khot, N.S., Optimization of structures with multiple frequency constraints, *Computers & Structures*, **20** (1985) 869–876.

[21] Kim, S., Kojima, M. and Waki, H., Generalized Lagrangian duals and sums of squares relaxations of sparse polynomial optimization problems, *SIAM Journal on Optimization*, **15** (2005) 697–719.

[22] Kočvara, On the modeling and solving of the truss design problem with global stability constraints, *Structural and Multidisciplinary Optimization*, **23** (2002) 189–203. Also: Levy, R., Su, H.-H., and Kočvara, M., *Structural and Multidisciplinary Optimization*, **26** (2004) 367–368.

[23] Kočvara, M. and Stingl, M., PENNON: a code for convex nonlinear and semidefinite programming, *Optimization Methods and Software*, **18** (2003) 317–333.

[24] Kočvara, M. and Stingl, M., PENBMI User's Guide. Version 2.0., December 2004. `http://www.penopt.com/`.

[25] Kojima, M., Shindoh, S. and Hara, S., Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices, *SIAM Journal on Optimization*, **7** (1997) 86–125.

[26] Leibfritz, F. and Mostafa, M.E., An interior point constrained trust region method for a special class of nonlinear semidefinite programming problems, *SIAM Journal on Optimization*, **12** (2002) 1048–1074.

[27] Lemaire, B., Duality in reverse convex optimization, *SIAM Journal on Optimization*, **8** (1998) 1029–1037.

[28] Lewis A.S. and Overton, M.L., Eigenvalue optimization, *Acta Numerica*, **5** (1996) 149–190.

[29] Nakamura, T. and Ohsaki, M., Sequential optimal truss generator for frequency ranges, *Computer Methods in Applied Mechanics and Engineering*, **67** (1988) 189–209.

[30] Noll, D., Torki, M. and Apkarian, P., Partially augmented Lagrangian method for matrix inequality constraints, *SIAM Journal on Optimization*, **15** (2004) 161–184.

[31] Ohsaki, M., Fujisawa, K., Katoh, N. and Kanno, Y., Semi-definite programming for topology optimization of trusses under multiple eigenvalue constraints, *Computer Methods in Applied Mechanics and Engineering*, **180** (1999) 203–217.

[32] Olhoff, N. and Rasmussen, S.H., On single and bimodal optimum buckling loads of clamped columns, *International Journal of Solids and Structures*, **13** (1977) 605–614.

[33] Pedersen, N.L., Maximization of eigenvalues using topology optimization, *Structural and Multidisciplinary Optimization*, **20** (2000) 2–11.

[34] Rodorigues, H.C., Guedes, J.M. and Bendsøe, M.P., Necessary conditions for optimal design of structures with a nonsmooth eigenvalue based criterion, *Structural Optimization*, **9** (1995) 52–56.

[35] Seyranian, A.P., Lund, E. and Olhoff, N., Multiple eigenvalues in structural optimization problem, *Structural Optimization*, **8** (1994) 207–227.

[36] Sturm, J.F., Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, *Optimization Methods and Software*, **11/12** (1999) 625–653.

[37] Sun, D., Sun, J. and Zhang, L., The rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming, *Technical Report*, January 2006.

[38] Tuy, H., Convex programs with an additional reverse convex constraint, *Journal of Optimization Theory and Applications*, **52** (1987) 463–486.

[39] *Using MATLAB*, The MathWorks, Inc., Natick, MA, 2002.