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Abstract

Modern modeling approaches for circuit analysis lead to differential-algebraic equations (DAEs). The index of a DAE is a measure of the degree of numerical difficulty. In general, the higher the index is, the more difficult it is to solve the DAE. The index of the DAE arising from the modified nodal analysis (MNA) is determined uniquely by the structure of the circuit. Instead, we consider a broader class of analysis method called the hybrid analysis. For linear time-invariant electric circuits, we devise a combinatorial algorithm for finding an optimal hybrid analysis in which the index of the DAE to be solved attains the minimum. The optimal hybrid analysis often results in a DAE with lower index than MNA.

1 Introduction

Circuit simulation is a useful tool to evaluate the behavior of an electric circuit prior to producing an actual prototype. Analysis methods in circuit simulation lead to differential-algebraic equations (DAEs), which consist of algebraic equations and differential operations. DAEs present numerical and analytical difficulties which do not occur with ordinary differential equations (ODEs).

Several numerical methods have been developed for solving DAEs. For example, Gear [9] proposed the backward difference formulae (BDF), which were implemented in the DASSL code by Petzold (cf. [4]). Hairer and Wanner [11] implemented an implicit Runge-Kutta method in their RADAU5 code.

The *index* concept plays an important role in the analysis of DAEs. The index is a measure of the degree of difficulty in the numerical solution. In general, the higher the index is, the more difficult it is to solve the DAE. While many different concepts exist to assign an index to a DAE such as the *differentiation index* [4, 6, 11], the *perturbation index* [5], and the *tractability*

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index [23], this paper focuses on the *nilpotency index*. In the case of linear DAEs with constant coefficients, all these indices are equal [5, 22].

The most commonly used analysis method for circuit simulation is the modified nodal analysis (MNA). The index of the DAE arising from MNA is determined uniquely by the structure of the circuit [23, 24]. Thus there is no room to reduce the index in MNA.

The hybrid analysis, which is essentially a generalization of MNA, was proposed by Kron [17] in 1939, and developed by Amari [1] and Branin [3] in 1960s. The hybrid analysis starts with selecting a *partition* of elements and a *reference tree* in the network. This selection determines a system of equations, called the *hybrid equations*, to be solved numerically. Thus it is natural to seek for an optimal selection that makes the hybrid equations easy to solve, among the exponential number of possibilities. In fact, the problem of obtaining the minimum size hybrid equations was solved [12, 16, 20] in 1968. This turns out to be an application of matroid intersection [13]. See also [19, 21] for matroid theoretic approach to circuit analysis.

Instead of the size of the hybrid equations, we focus on the index in this paper. We present a combinatorial algorithm for finding a partition and a reference tree which minimize the index of the hybrid equations. Our method first finds a *degree matrix*, which is defined by cofactors in the associated polynomial matrix. Then, it makes use of the satisfiability problem for 2-CNF (2SAT). The time complexity of this algorithm is $\mathrm{O}(n^6)$, where n is the number of elements in an electric circuit. We can improve the time complexity to $\mathrm{O}(n^3)$ under the assumption that the set of nonzero entries coming from the physical parameters is algebraically independent.

The organization of this paper is as follows. In Section 2, we explain matrix pencils and the definition of the nilpotency index. We describe the procedure of the hybrid analysis in Section 3. Section 4 is devoted to a characterization of the index of the DAE to be solved in the hybrid analysis. Section 5 presents an index minimization algorithm. Numerical examples are given in Section 6. Finally, Section 7 concludes this paper.

2 DAEs and Matrix Pencils

For a polynomial $a(s)$, we denote the degree of $a(s)$ by $\deg a$, where $\deg 0 = -\infty$ by convention. A polynomial matrix $A(s) = (a_{kl}(s))$ with $\deg a_{kl} \leq 1$ for all (k, l) is called a *matrix pencil*. Obviously, a matrix pencil $A(s)$ can be represented as $A(s) = A_0 + sA_1$ in terms of a pair of constant matrices A_0 and A_1. A matrix pencil $A(s)$ is said to be *regular* if $A(s)$ is square and $\det A(s)$ is a nonvanishing polynomial.

Consider a linear DAE with constant coefficients

$$A_0\boldsymbol{x}(t) + A_1\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}(t), \tag{1}$$

where A_0 and A_1 are constant matrices. With the use of the Laplace transformation, the linear DAE with constant coefficients in the form of (1) is expressed by the matrix pencil $A(s) = A_0+sA_1$ as $A(s)\tilde{\boldsymbol{x}}(s) = \tilde{\boldsymbol{f}}(s) + A_1\boldsymbol{x}(0)$, where s is the variable for the Laplace transform that corresponds to $\mathrm{d}/\mathrm{d}t$, the differentiation with respect to time.

Theorem 2.1 ([4, Theorem 2.3.1]). *The linear DAE with constant coefficients* (1) *is solvable if and only if* $A(s)$ *is a regular matrix pencil.*

The reader is referred to [4, Definition 2.2.1] for the precise definition of solvability. By Theorem 2.1, we assume that $A(s)$ is a regular matrix pencil throughout this paper. A regular matrix pencil is known to have the *Kronecker canonical form*, which determines the nilpotency index. Let N_μ denote a $\mu \times \mu$ matrix pencil defined by

$$N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & s \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

A matrix pencil $A(s)$ is said to be *strictly equivalent* to $\tilde{A}(s)$ if $A(s)$ can be brought into $\tilde{A}(s)$ by an equivalence transformation with nonsingular constant matrices.

Theorem 2.2 ([8, Chapter XII, Theorem 3]). *An* $n \times n$ *regular matrix pencil* $A(s)$ *is strictly equivalent to its Kronecker canonical form:*

$$\begin{pmatrix} sI_{\mu_0} + J & O & O & \cdots & O \\ O & N_{\mu_1} & O & \cdots & O \\ O & O & N_{\mu_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ O & O & \cdots & O & N_{\mu_b} \end{pmatrix},$$

where

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_b, \quad \mu_0 + \mu_1 + \mu_2 + \cdots + \mu_b = n,$$

and J *is a* $\mu_0 \times \mu_0$ *constant matrix.*

The matrices N_{μ_i} $(i = 1, \ldots, b)$ are called the *nilpotent blocks*. The maximum size μ_1 of them is the *nilpotency index*, denoted by $\nu(A)$. It is obvious that ODEs have index zero, and algebraic equations have index one.

We denote by $A[K, L]$ the submatrix of $A(s)$ with row set $K \subseteq R$ and column set $L \subseteq C$, where R and C are the row set and the column set of $A(s)$, respectively. For any square submatrix $A[K, L]$, we write $w(K, L) = \deg \det A[K, L]$, where $w(\emptyset, \emptyset) = 0$ by convention. Then, $w(K, L)$ enjoys the following property.

Lemma 2.3 ([18, pp. 287–289]). *Let* $A(s)$ *be a matrix pencil with row set* R *and column set* C*. Then, for any* $(K, L) \in \Lambda$*,* $(K', L') \in \Lambda$*, and* $l \in L \setminus L'$*, at least one of the following two assertions holds, where* $\Lambda = \{(K, L) \mid |K| = |L|, K \subseteq R, L \subseteq C\}$.

(a) $\exists k \in K \setminus K' : w(K,L) + w(K',L') \leq w(K \setminus \{k\}, L \setminus \{l\}) + w(K' \cup \{k\}, L' \cup \{l\})$,

(b) $\exists j \in L' \setminus L : w(K,L) + w(K',L') \leq w(K, L \setminus \{l\} \cup \{j\}) + w(K', L' \setminus \{j\} \cup \{l\})$.

Let $\delta_r(A)$ denote the highest degree of a minor of order r in $A(s)$:

$$\delta_r(A) = \max_{K,L}\{w(K,L) \mid |K| = |L| = r, K \subseteq R, L \subseteq C\}.$$

The index $\nu(A)$ can be determined from $\delta_r(A)$ as follows.

Theorem 2.4 ([18, Theorem 5.1.8])**.** *Let $A(s)$ be an $n \times n$ regular matrix pencil. The nilpotency index $\nu(A)$ is given by*

$$\nu(A) = \delta_{n-1}(A) - \delta_n(A) + 1.$$

3 Hybrid Analysis

In this section, we describe the procedure of the hybrid analysis. We focus on linear time-invariant electric circuits which are composed of resistances, capacitances, inductances, independent/dependent voltage sources, and independent/dependent current sources. For more complicated devices like transistors, there exist equivalent circuits which consist of the previously mentioned devices.

Let $\Gamma = (W, E)$ be a network graph with vertex set W and edge set E. An edge in Γ corresponds to a branch that contains one element in the circuit. We denote the set of edges corresponding to independent voltage sources and independent current sources by E_g and E_h, respectively. We split $E_* := E \setminus (E_g \cup E_h)$ into E_y and E_z, i.e., $E_y \cup E_z = E_*$ and $E_y \cap E_z = \emptyset$. Since the previous works [12, 16, 20] deal with circuits in the frequency domain, the hybrid analysis described therein can choose any partition (E_y, E_z). In order to deal with DAEs in the time domain, however, we need to consider a restricted class of partitions. A partition (E_y, E_z) is called an *admissible partition*, if E_y includes all the capacitances and dependent current sources, and E_z includes all the inductances and dependent voltage sources.

We now explain *circuit equations* for a linear time-invariant electric circuit. Let $\boldsymbol{\xi}$ denote the vector of currents through all branches of the circuit, and $\boldsymbol{\eta}$ the vector of voltages across all branches. We denote the *reduced cutset matrix* by Ψ and the *reduced loop matrix* by Φ. Using *Kirchhoff's current law* (KCL), which states that the sum of currents entering each node is equal to zero, we have $\Psi\boldsymbol{\xi} = \mathbf{0}$. Similarly, using *Kirchhoff's voltage law* (KVL), which states that the sum of voltages in each loop of the network is equal to zero, we have $\Phi\boldsymbol{\eta} = \mathbf{0}$. The physical characteristics of elements determine *constitutive equations*. Given an admissible partition (E_y, E_z), we split $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ into

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\xi}_g \\ \boldsymbol{\xi}_y \\ \boldsymbol{\xi}_z \\ \boldsymbol{\xi}_h \end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_g \\ \boldsymbol{\eta}_y \\ \boldsymbol{\eta}_z \\ \boldsymbol{\eta}_h \end{pmatrix},$$

where the subscripts correspond to the partition of E. Circuit equations, which consist of KCL, KVL, and constitutive equations, are described by

$$\left(\begin{array}{cccc|cccc}
\multicolumn{4}{c|}{\Psi} & \multicolumn{4}{c}{O} \\ \hline
\multicolumn{4}{c|}{O} & \multicolumn{4}{c}{\Phi} \\ \hline
O & I & \multicolumn{2}{|c|}{\multirow{2}{*}{$Z(s)$}} & \multicolumn{2}{c|}{\multirow{2}{*}{$Y(s)$}} & O & O \\
O & O & \multicolumn{2}{|c|}{} & \multicolumn{2}{c|}{} & I & O \\ \hline
O & O & O & O & I & O & O & O \\
O & O & O & I & O & O & O & O
\end{array}\right)
\left(\begin{array}{c}
\boldsymbol{\xi}_g \\ \boldsymbol{\xi}_y \\ \boldsymbol{\xi}_z \\ \boldsymbol{\xi}_h \\ \hline \boldsymbol{\eta}_g \\ \boldsymbol{\eta}_y \\ \boldsymbol{\eta}_z \\ \boldsymbol{\eta}_h
\end{array}\right)
=
\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{g}(s) \\ \boldsymbol{h}(s) \end{pmatrix} \tag{2}$$

after the Laplace transformation. The coefficient matrix $A(s)$ of the circuit equations is a matrix pencil. The row set and the column set of $A(s)$ are denoted by R and C, respectively.

We call a spanning tree T of Γ a *reference tree* if T contains all edges in E_g, no edges in E_h, and as many edges in E_y as possible. Note that T may contain some edges in E_z. The cotree of T is denoted by $\overline{T} = E \setminus T$.

Given an admissible partition (E_y, E_z), we denote the column sets of $A(s)$ corresponding to the current variables and the voltage variables for elements in E_g, E_y, E_z, E_h by I_g, I_y, I_z, I_h, and V_g, V_y, V_z, V_h, respectively. Moreover, given a reference tree T, we denote the column sets of $A(s)$ corresponding to the current variables and the voltage variables for elements in $E_y \cap T$ and $E_y \cap \overline{T}$ by I_y^τ, I_y^λ, and V_y^τ, V_y^λ, respectively. The superscripts τ and λ designate the tree T and the cotree $\overline{T}$. We define I_z^τ, I_z^λ, and V_z^τ, V_z^λ similarly. We also use $I^\tau = I_g \cup I_y^\tau \cup I_z^\tau$ and $V^\lambda = V_y^\lambda \cup V_z^\lambda \cup V_h$ for convenience. The row sets of $A(s)$ corresponding to KCL, KVL, and constitutive equations are denoted by R_I, R_V, and S, respectively.

Given an admissible partition (E_y, E_z) and a reference tree T, we transform $A(s)$ into $A_T(s)$ such that $A_T[R_I, I^\tau] = I$ and $A_T[R_V, V^\lambda] = I$ by row operations in $R_I \cup R_V$. This is possible because $A[R_I, I^\tau]$ and $A[R_V, V^\lambda]$ are nonsingular. Note that R_I and I^τ as well as R_V and V^λ have one-to-one correspondence. The row sets of $A_T(s)$ corresponding to I_g, I_y^τ, I_z^τ, and V_y^λ, V_z^λ, V_h are denoted by R_g, R_y^τ, R_z^τ, and R_y^λ, R_z^λ, R_h, where we have $A_T[K, L] = I$ if $K \subseteq R$ and $L \subseteq C$ have the same superscript and subscript. Similarly, the row sets corresponding to I_y, V_z, V_g, and I_h are denoted by S_y, S_z, S_g, and S_h. Let i_e and v_e denote the column corresponding to the current variable and the voltage variable for an element e. By the definition of a reference tree, $A_T(s)$ has the following property.

Lemma 3.1. *For a reference tree T, we have $A_T[R_z^\tau, I_y^\lambda] = O$ and $A_T[R_y^\lambda, V_z^\tau] = O$.*

Proof. Suppose to the contrary that there exists $e \in E_y \setminus T$ such that $A_T[R_z^\tau, \{i_e\}] \neq \mathbf{0}$. Then the unique cycle in $T \cup \{e\}$ is not contained in $E_y \cup E_g$. Hence, there exists an edge $f \in E_z \cap T$ such that $T \setminus \{f\} \cup \{e\}$ is a tree, which contradicts the assumption that T is a reference tree. Therefore, we have $A_T[R_z^\tau, I_y^\lambda] = O$. Similarly, we also have $A_T[R_y^\lambda, V_z^\tau] = O$. $\square$

Thus $A_T(s)$ is in the form of

$$A_T(s) = \begin{array}{c} \\ R_g \\ R_y^\tau \\ R_z^\tau \\ R_y^\lambda \\ R_z^\lambda \\ R_h \\ S_y \\ \\ S_z \\ \\ S_g \\ S_h \end{array} \begin{array}{c} \begin{array}{cccccccccccc} I_g & I_y^\tau & I_y^\lambda & I_z^\tau & I_z^\lambda & I_h & V_g & V_y^\tau & V_y^\lambda & V_z^\tau & V_z^\lambda & V_h \end{array} \\ \begin{pmatrix} I & O & * & O & * & * & O & O & O & O & O & O \\ O & I & * & O & * & * & O & O & O & O & O & O \\ O & O & O & I & * & * & O & O & O & O & O & O \\ O & O & O & O & O & O & * & * & I & O & O & O \\ O & O & O & O & O & O & * & * & O & * & I & O \\ O & O & O & O & O & O & * & * & O & * & O & I \\ O & I & O & ** & ** & ** & ** & ** & ** & O & O & O \\ O & O & I & ** & ** & ** & ** & ** & ** & O & O & O \\ O & O & O & ** & ** & ** & ** & ** & ** & I & O & O \\ O & O & O & ** & ** & ** & ** & ** & ** & O & I & O \\ O & O & O & O & O & O & I & O & O & O & O & O \\ O & O & O & O & O & I & O & O & O & O & O & O \end{pmatrix} \end{array}, \tag{3}$$

where $*$ means a constant matrix and $**$ means a matrix pencil. We can determine $A_T(s)$ only after being given both an admissible partition (E_y, E_z) and a reference tree T.

Let us denote $P = R \setminus (R_y^\tau \cup R_z^\lambda)$ and $Q = C \setminus (I_z^\lambda \cup V_y^\tau)$. We transform A_T into $\tilde{A}_T$ by row operations:

$$A_T = \begin{pmatrix} B & F \\ G & H \end{pmatrix} \rightarrow \tilde{A}_T = \begin{pmatrix} I & O \\ -GB^{-1} & I \end{pmatrix} \begin{pmatrix} B & F \\ G & H \end{pmatrix} = \begin{pmatrix} B & F \\ O & H - GB^{-1}F \end{pmatrix}, \tag{4}$$

where $B = A_T[P, Q], F = A_T[P, C \setminus Q], G = A_T[R \setminus P, Q]$, and $H = A_T[R \setminus P, C \setminus Q]$. We denote $H - GB^{-1}F$ by D.

Let $\hat{B}$, $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{D}$ denote the matrices obtained by replacing s with $\mathrm{d}/\mathrm{d}t$ in B, F, G, H, and D, respectively. Consider the DAE

$$\hat{B}\boldsymbol{x}_1(t) + \hat{F}\boldsymbol{x}_2(t) = \boldsymbol{f}_1(t), \tag{5}$$
$$\hat{G}\boldsymbol{x}_1(t) + \hat{H}\boldsymbol{x}_2(t) = \boldsymbol{f}_2(t). \tag{6}$$

By applying the transformation shown in (4), we obtain

$$\hat{B}\boldsymbol{x}_1(t) = \boldsymbol{f}_1(t) - \hat{F}\boldsymbol{x}_2(t), \tag{7}$$
$$\hat{D}\boldsymbol{x}_2(t) = \boldsymbol{f}_2(t) - \hat{G}\hat{B}^{-1}\boldsymbol{f}_1(t). \tag{8}$$

We call the resulting DAE (8) the *hybrid equations*. Let us denote the vectors of currents corresponding to I_g, I_y^τ, I_y^λ, I_z^τ, I_z^λ, I_h by $\boldsymbol{\xi}_g$, $\boldsymbol{\xi}_y^\tau$, $\boldsymbol{\xi}_y^\lambda$, $\boldsymbol{\xi}_z^\tau$, $\boldsymbol{\xi}_z^\lambda$, $\boldsymbol{\xi}_h$, and the vectors of voltages corresponding to V_g, V_y^τ, V_y^λ, V_z^τ, V_z^λ, V_h by $\boldsymbol{\eta}_g$, $\boldsymbol{\eta}_y^\tau$, $\boldsymbol{\eta}_y^\lambda$, $\boldsymbol{\eta}_z^\tau$, $\boldsymbol{\eta}_z^\lambda$, $\boldsymbol{\eta}_h$. The procedure of the hybrid analysis is as follows:

1. The values of $\boldsymbol{\xi}_h$ and $\boldsymbol{\eta}_g$ are obvious from the equations corresponding to S_h and S_g.

2. Find the values of $\boldsymbol{\xi}_z^\lambda$ and $\boldsymbol{\eta}_y^\tau$ by solving the *hybrid equations* (8).

3. Compute the values of $\boldsymbol{\xi}_z^\tau$ and $\boldsymbol{\eta}_y^\lambda$ by substituting the values obtained in Steps 1–2 into the equations corresponding to R_z^τ and R_y^λ.

4. Compute the values of $\boldsymbol{\xi}_y^\tau$, $\boldsymbol{\xi}_y^\lambda$, $\boldsymbol{\eta}_z^\tau$, and $\boldsymbol{\eta}_z^\lambda$ by substituting the values obtained in Steps 1–3 into S_y and S_z.

5. Compute the values of $\boldsymbol{\xi}_g$ and $\boldsymbol{\eta}_h$ by substituting the values obtained in Steps 1–4 into R_g and R_h.

In the case of $E_y = \emptyset$, the above procedure is called the *loop analysis* or the *tieset analysis*. In the case of $E_z = \emptyset$, the procedure is called the *cutset analysis*, which is essentially equivalent to MNA.

In order to ensure that the hybrid equations are a DAE, we require $D = H - GB^{-1}F$ to be a matrix pencil, which is not obviously satisfied because $B = A_T[P, Q]$ is a matrix pencil. Moreover, B needs to be an upper triangular matrix with diagonal ones so that we can compute the values in Steps 3–5 by only substituting the obtained values. The following lemma ensures this for admissible partitions.

Lemma 3.2. *If (E_y, E_z) is an admissible partition, then we can transform B into an upper triangular matrix with diagonal ones by permutations, and D is a matrix pencil.*

Proof. If (E_y, E_z) is an admissible partition, we can express $A_T(s)$ in the form of (3) for any reference tree T. By permutations of rows and columns, we can transform B into

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} I_g & V_h & I_y^\tau & I_y^\lambda & V_z^\tau & V_z^\lambda & I_z^\tau & V_y^\lambda & V_g & I_h \end{array} \\
\begin{array}{c} R_g \\ R_h \\ S_y \\ \\ S_z \\ \\ R_z^\tau \\ R_y^\lambda \\ S_g \\ S_h \end{array} &
\begin{pmatrix}
I & O & O & * & O & O & O & O & * & O \\
O & I & O & O & * & O & O & O & O & * \\
O & O & I & O & O & O & ** & ** & ** & ** \\
O & O & O & I & O & O & ** & ** & ** & ** \\
O & O & O & O & I & O & ** & ** & ** & ** \\
O & O & O & O & O & I & ** & ** & ** & ** \\
O & O & O & O & O & O & I & O & * & O \\
O & O & O & O & O & O & O & I & O & * \\
O & O & O & O & O & O & O & O & I & O \\
O & O & O & O & O & O & O & O & O & I
\end{pmatrix}
\end{array},$$

which is an upper triangular matrix with diagonal ones. It is easy and omitted to prove that D is a matrix pencil. $\square$

Since we only substitute the obtained values in Steps 3–5, the numerical difficulty is determined by the index of the hybrid equations (8).

4 Index of Hybrid Equations

In this section, we give a characterization of the index of the hybrid equations. Given an admissible partition (E_y, E_z) and a reference tree T, consider the transformation shown in (4). We now show that $\nu(D)$ can be expressed in terms of the degrees of minors in $A_T(s)$. For each $k \in R$ and $l \in C$, let d_{kl} denote the degree of $\det A_T[R \setminus \{k\}, C \setminus \{l\}]$. Then we have

$$d_{kl} = \deg\det \tilde{A}_T[R \setminus \{k\}, C \setminus \{l\}], \quad \forall k \in R \setminus P,\ \forall l \in C, \tag{9}$$

because we can transform $\tilde{A}_T[R \setminus \{k\}, C \setminus \{l\}]$ into $A_T[R \setminus \{k\}, C \setminus \{l\}]$ by row operations. The index $\nu(D)$ can be rewritten as follows.

Lemma 4.1. *Given an admissible partition (E_y, E_z) and a reference tree T, the index of D is given by*

$$\nu(D) = \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} - \delta_n(A_T) + 1. \tag{10}$$

Proof. We denote the size of D by m. By Theorem 2.4, we have $\nu(D) = \delta_{m-1}(D) - \delta_m(D) + 1$. Recall that $\tilde{A}_T = \begin{pmatrix} B & F \\ O & D \end{pmatrix}$. It follows from $\det A_T = \det \tilde{A}_T$ that

$$\delta_m(D) = \deg\det D = \deg\det \tilde{A}_T - \deg\det B = \deg\det A_T - \deg\det B.$$

Moreover, we have

$$\begin{aligned}
\delta_{m-1}(D) &= \max_{K,L}\{\deg\det D[K, L] \mid |K| = |L| = m - 1\} \\
&= \max_{K,L}\{\deg\det \tilde{A}_T[K, L] \mid |K| = |L| = n - 1, K \supseteq P, L \supseteq Q\} - \deg\det B \\
&= \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} - \deg\det B,
\end{aligned}$$

where the last step is due to (9). Thus we obtain (10). $\square$

By Lemma 4.1, the index $\nu(D)$ is determined by the maximum of d_{kl} such that $k \in R \setminus P$ and $l \in C \setminus Q$. The value of d_{kl} has the following property.

Lemma 4.2. *Given an admissible partition (E_y, E_z) and a reference tree T, we have*

$$\max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} = \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in I_z \cup V_y\}.$$

Proof. We may assume that $A_T[R_I, I_h] = O$, because adding a multiple of a row in S_h does not change the value of d_{kl} for any $k \in R \setminus P$ and $l \in C$. Similarly, we may assume that $A_T[R_V, V_g] = O$.

For any $k \in R \setminus P$ and $l \in I_z^\tau \cup V_y^\lambda$, we apply Lemma 2.3 to $(R_z^\tau \cup R_y^\lambda, I_z^\tau \cup V_y^\lambda)$ and $(R \setminus \{k\}, C \setminus \{l\})$. Since $R_z^\tau \cup R_y^\lambda \subseteq R \setminus \{k\}$, (a) does not hold. Then there exists $j \in C \setminus (I_z^\tau \cup V_y^\lambda)$ such that

$$d_{kl} \leq \deg\det A_T[R_z^\tau \cup R_y^\lambda, I_z^\tau \cup V_y^\lambda \setminus \{l\} \cup \{j\}] + d_{kj}.$$

If $d_{kl} > -\infty$, we have $\deg\det A_T[R_z^\tau \cup R_y^\lambda, I_z^\tau \cup V_y^\lambda \setminus \{l\} \cup \{j\}] > -\infty$. By the assumptions $A_T[R_I, I_h] = O$ and $A_T[R_V, V_g] = O$, this implies $j \in C \setminus Q$. Since $A_T[R_z^\tau \cup R_y^\lambda, I_z^\tau \cup V_y^\lambda \setminus \{l\} \cup \{j\}]$ is a nonsingular constant matrix, we have $d_{kl} \leq d_{kj}$. Hence, for any $k \in R \setminus P$, we have $\max_l\{d_{kl} \mid l \in C \setminus Q\} = \max_l\{d_{kl} \mid l \in I_z \cup V_y\}$. □

In order to prove that the index of the hybrid equations does not depend on a reference tree, we make use of the following lemma.

Lemma 4.3. *Let $A(s)$ be a matrix pencil with row set R and column set C. We transform $A(s)$ into $A'(s) = UA(s)$ with a nonsingular constant matrix U. For any $K \subseteq R$ and $l \in C$, if $U[K, R \setminus K] = O$, then*

$$\max_{k}\{\deg\det A[R \setminus \{k\}, C \setminus \{l\}] \mid k \in R \setminus K\} = \max_{k}\{\deg\det A'[R \setminus \{k\}, C \setminus \{l\}] \mid k \in R \setminus K\}$$

holds.

We are now ready to prove the following theorem.

Theorem 4.4. *Given an admissible partition (E_y, E_z), the index $\nu(D)$ is the same for any reference tree.*

Proof. Consider any pair of reference trees T and $\hat{T}$ such that $T \neq \hat{T}$. Then P, I_z, and V_y are the same for T and $\hat{T}$, while Q is not. We denote Q and d_{ij} with respect to $A_{\hat{T}}$ by $\hat{Q}$ and $\hat{d}_{ij}$. By applying Lemma 4.2 to T and $\hat{T}$, we have

$$\max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} = \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in I_z \cup V_y\},$$

and

$$\max_{k,l}\{\hat{d}_{kl} \mid k \in R \setminus P, l \in C \setminus \hat{Q}\} = \max_{k,l}\{\hat{d}_{kl} \mid k \in R \setminus P, l \in I_z \cup V_y\}.$$

Since there exists a nonsingular constant matrix U such that $U[P, R \setminus P] = O$ and $UA_T = A_{\hat{T}}$, we have $\max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in I_z \cup V_y\} = \max_{k,l}\{\hat{d}_{kl} \mid k \in R \setminus P, l \in I_z \cup V_y\}$ from Lemma 4.3. Thus we obtain $\max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} = \max_{k,l}\{\hat{d}_{kl} \mid k \in R \setminus P, l \in C \setminus \hat{Q}\}$, which implies that the indices for T and $\hat{T}$ are the same by Lemma 4.1 and $\delta_n(A_T) = \delta_n(A_{\hat{T}})$. □

By Theorem 4.4, the index of the hybrid equations is determined only by an admissible partition (E_y, E_z). However, all the values of d_{kl} are not invariant under row operations on the coefficient matrix $A(s)$ of the circuit equations, while we have to transform $A(s)$ into $A_T(s)$ with respect to an admissible partition (E_y, E_z) and a reference tree T. We now introduce a *degree matrix*, which consists of some invariants under row operations. Let us denote by I_* and V_* the sets corresponding to current and voltage variables for E_*, respectively.

Definition 4.5 (degree matrix)**.** *For each pair of $k \in I_* \cup V_*$ and $l \in I_* \cup V_*$, define*

$$\theta_{kl} = \deg\det \begin{pmatrix} A[R_I \cup R_V, C \setminus \{l\}] & A[R_I \cup R_V, \{k\}] \\ A[S, C \setminus \{l\}] & \mathbf{0} \end{pmatrix}.$$

Then the degree matrix is the matrix $\Theta = (\theta_{kl})$ whose row and column sets are both identical with $I_* \cup V_*$.

Note that the degree matrix is uniquely determined by the circuit, despite $A(s)$ is not unique. The relation between an entry of Θ and d_{kl} is as follows. Recall that row set $R_I \cup R_V$ and column set $I^\tau \cup V^\lambda$ have one-to-one correspondence by $A_T[R_I \cup R_V, I^\tau \cup V^\lambda] = I$.

Lemma 4.6. *Given an admissible partition (E_y, E_z) and a reference tree T, for any $k \in R_I \cup R_V$ and $l \in I_* \cup V_*$, we have $d_{kl} = \theta_{jl}$, where $j \in I^\tau \cup V^\lambda$ is the corresponding column to row k.*

Proof. Since we can transform any coefficient matrix of circuit equations into $A_T(s)$ by row operations, we may assume that Θ is defined in terms of $A_T(s)$. Then we have

$$\theta_{jl} = \deg\det \begin{pmatrix} A_T[R_I \cup R_V, C \setminus \{l\}] & A_T[R_I \cup R_V, \{j\}] \\ A_T[S, C \setminus \{l\}] & \mathbf{0} \end{pmatrix} = d_{kl},$$

because $A_T[R_I \cup R_V, \{j\}]$ has only one nonzero entry in row k. □

Lemma 4.6 implies that the index of the hybrid equations is expressed in terms of the degree matrix Θ.

Lemma 4.7. *Given an admissible partition (E_y, E_z) and a reference tree T, we have*

$$\nu(D) = \max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in I_z^\lambda \cup V_y^\tau\} - \delta_n(A) + 1, \tag{11}$$

where A is a coefficient matrix of the circuit equations.

Proof. By Lemma 4.6, we have

$$\max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} = \max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in C \setminus Q\}.$$

Since $\delta_n(A_T) = \delta_n(A)$ holds for any reference tree T, we obtain (11) by Lemma 4.1. □

We can rewrite the index $\nu(D)$ by the maximum of θ_{kl} such that $k \in I_y \cup V_z$ and $l \in I_z \cup V_y$.

Theorem 4.8. *Given an admissible partition (E_y, E_z), we have*

$$\nu(D) = \max_{k,l}\{\theta_{kl} \mid k \in I_y \cup V_z, l \in I_z \cup V_y\} - \delta_n(A) + 1, \tag{12}$$

where A is a coefficient matrix of the circuit equations.

Proof. Given an admissible partition (E_y, E_z) and a reference tree T, we may rewrite Lemma 4.2 in terms of the degree matrix Θ into

$$\max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in C \setminus Q\} = \max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in I_z \cup V_y\}. \tag{13}$$

Then, we intend to prove that

$$\max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in I_z \cup V_y\} = \max_{k,l}\{\theta_{kl} \mid k \in I_y \cup V_z, l \in I_z \cup V_y\}. \tag{14}$$

It follows from Lemma 4.7 and (13) that

$$\nu(D) = \max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in I_z \cup V_y\} - \delta_n(A) + 1. \tag{15}$$

For any $i_e \in I_y^\lambda$, there exists another reference tree $\hat{T}$ containing e by $e \in E_y$. Therefore, we have $\max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in I_z \cup V_y\} \geq \max_l\{\theta_{i_e l} \mid l \in I_z \cup V_y\}$ by Theorem 4.4.

Similarly, it follows that $\max_{k,l}\{\theta_{kl} \mid k \in I_y^\tau \cup V_z^\lambda, l \in I_z \cup V_y\} \geq \max_l\{\theta_{v_e l} \mid l \in I_z \cup V_y\}$ for any $v_e \in V_z^\tau$. Thus we obtain (14), which implies (12) by (15). $\square$

5 Index Minimization of Hybrid Equations

Let $\Theta = (\theta_{kl})$ be a degree matrix, where the row set and the column set are identical with $I_* \cup V_*$, and $A(s)$ be a coefficient matrix of the circuit equations. By Theorem 4.8, minimizing the index of the hybrid equations is equivalent to minimizing $\max\{\theta_{kl} \mid k \in I_y \cup V_z, l \in I_z \cup V_y\}$. In this section, we describe how to find an admissible partition (E_y, E_z) which minimizes this maximum value.

Theorem 5.1. *We have $\nu(D) < \alpha - \delta_n(A) + 1$ if and only if an admissible partition (E_y, E_z) satisfies* (i)–(iv) *for any pair of k and l with $\theta_{kl} \geq \alpha$.*

(i) *If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = i_f$, then $e \in E_z$ or $f \in E_y$.*

(ii) *If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = v_f$, then $e \in E_z$ or $f \in E_z$.*

(iii) *If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = i_f$, then $e \in E_y$ or $f \in E_y$.*

(iv) *If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = v_f$, then $e \in E_y$ or $f \in E_z$.*

Finding an admissible partition satisfying (i)–(iv) reduces to 2SAT as follows, using the boolean variable u_e to represent $e \in E_z$. First, in order to ensure that (E_y, E_z) is an admissible partition, we impose $u_e = 0$ if the element e is a capacitance or a dependent current source, and we impose $u_e = 1$ if e is an inductance or a dependent voltage source. Next, we rewrite (i) into $u_e \vee \overline{u}_f = 1$, (ii) into $u_e \vee u_f = 1$, (iii) into $\overline{u}_e \vee \overline{u}_f = 1$, and (iv) into $\overline{u}_e \vee u_f = 1$. Thus we obtain the following problem:

2SAT(α) Find u_e for any element e satisfying (1)–(6).

(1) If e is a capacitance or a dependent current source, then $u_e = 0$.

(2) If e is an inductance or a dependent voltage source, then $u_e = 1$.

(3) If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = i_f$, then $u_e \vee \overline{u}_f = 1$.

(4) If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = v_f$, then $u_e \vee u_f = 1$.

(5) If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = i_f$, then $\overline{u}_e \vee \overline{u}_f = 1$.

(6) If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = v_f$, then $\overline{u}_e \vee u_f = 1$.

We can solve 2SAT in linear time in the size of literals and clauses [2].

We describe the algorithm for finding an admissible partition which minimizes the index of the hybrid equations.

Algorithm for minimum index hybrid analysis

Step 1: Compute the degree matrix $\Theta = (\theta_{kl})$.

Step 2: Set $E_y \leftarrow \{e \mid e\colon \text{capacitance or dependent current source}\}$, $E_z \leftarrow E_* \setminus E_y$, and $\alpha \leftarrow \max\{\theta_{kl} \mid k \in I_* \cup V_*, l \in I_* \cup V_*\}$.

Step 3: Solve 2SAT(α) to obtain a feasible assignment u_e for $e \in E_*$. If 2SAT(α) is infeasible, then go to Step 5.

Step 4: Set $E_y \leftarrow \{e \mid u_e = 0\}$, $E_z \leftarrow \{e \mid u_e = 1\}$, and $\alpha \leftarrow \alpha - 1$. Go back to Step 3.

Step 5: Return (E_y, E_z) and α.

Algorithm for minimum index hybrid analysis finds an optimal admissible partition (E_y, E_z) together with the maximum value of α such that 2SAT(α) is infeasible. Therefore, Theorem 5.1 implies that the index of the resulting hybrid equations is $\alpha - \delta_n(A) + 1$ for any reference tree with respect to (E_y, E_z). Instead of the above decremental method, we may adopt the binary search on α.

Finally, we discuss the complexity of our algorithm. Let n be the size of the coefficient matrix of the circuit equations, i.e., the number of elements in the electric circuit is $n/2$. We can compute the degree of the determinant of a $\gamma \times \gamma$ matrix pencil in $O(\gamma^4)$ time [14]. By using this algorithm

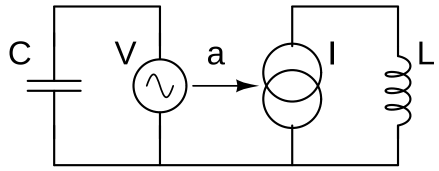

Figure 1: Linear circuit described by circuit equations with index three.

n^2 times, a degree matrix can be found in $O(n^6)$ time. Since 2SAT(α) in Step 3 has $O(n)$ literals and $O(n^2)$ clauses, we can solve it in $O(n^2)$ time. Thus the total time complexity of the algorithm is $O(n^6)$.

If one can compute a degree matrix faster, the total time complexity of the algorithm will be better. In [15], we discuss how to compute a degree matrix in $O(n^3)$ time under a genericity assumption that the set of nonzero entries coming from the physical parameters like resistances is algebraically independent, which implies that $A(s)$ is a mixed polynomial matrix [18]. Thus, we improve the time complexity of Algorithm for minimum index hybrid analysis to $O(n^3)$.

If the genericity assumption is not valid, the degree matrix obtained by the improved algorithm may have larger entries than the true values because of unlucky numerical cancellations. Relying on this degree matrix, we may fail to find the minimum index of hybrid equations.

6 Numerical Examples

In this section, we demonstrate the hybrid analysis in some numerical examples. We use RADAU5 [11] in Matlab as the DAE solver.

Example 6.1 (Electric circuit with index three [10])**.** Consider a circuit depicted in Figure 1, which is described by the circuit equations with index three:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & sC & 0 & 0 \\ 0 & 0 & 0 & sL & 0 & 0 & 0 & -1 \\ a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\xi}_V \\ \tilde{\xi}_C \\ \tilde{\xi}_I \\ \tilde{\xi}_L \\ \tilde{\eta}_V \\ \tilde{\eta}_C \\ \tilde{\eta}_I \\ \tilde{\eta}_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{V}(s) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{16}$$

While MNA results in a DAE with index three [10], one can obtain a DAE with lower index by

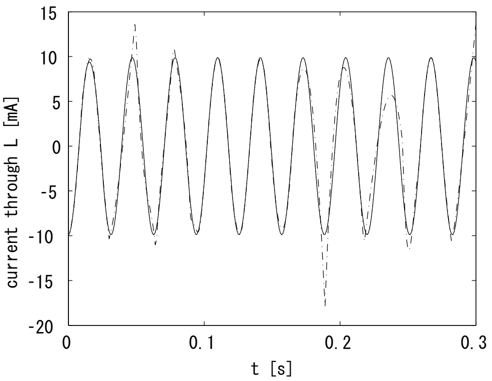

Figure 2: The current through the inductance in Example 6.1: numerical solutions of MNA (dash-dotted line), the hybrid analysis (solid line), and the exact solution (dotted line).

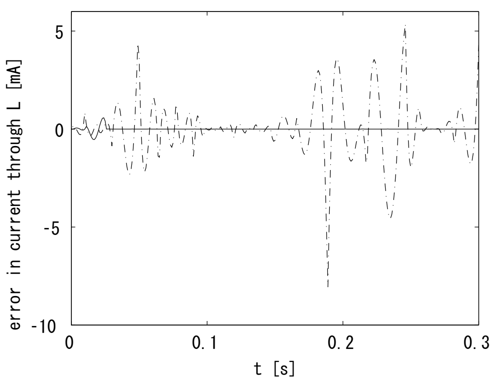

Figure 3: The error in the current through the inductance in Example 6.1: MNA (dash-dotted line) and the hybrid analysis (solid line).

the hybrid analysis. An admissible partition is uniquely determined and we have

$$E_g = \{V\}, \quad E_h = \emptyset, \quad E_y = \{C, I\}, \quad E_z = \{L\}. \tag{17}$$

By applying the hybrid analysis with respect to the partition (17) and the reference tree $T = \{V, I\}$, we obtain

$$D = \begin{pmatrix} 1 & 0 \\ -sL & 1 \end{pmatrix},$$

which has index two. The hybrid equations are $\tilde{\xi}_L = -saC\tilde{V}(s)$ and $-sL\tilde{\xi}_L + \tilde{\eta}_I = 0$.

Setting $C = 5$ [μF]$, L = 8$ [mH]$, a = 0.99$, and $V(t) = 10\sin(200t)$ [V], we numerically solve both DAEs arising from MNA and the hybrid analysis. Figure 2 presents these two numerical solutions and the exact solution, which can be obtained analytically. In Figure 2, the exact solution coincides with the solution of the hybrid analysis. Figure 3 shows the discrepancy of the two numerical solutions from the exact solution. It is observed that the index reduction effectively improves the accuracy of the numerical solution.

Example 6.2 (Electric circuit with index two). Consider another circuit given in Figure 4, which

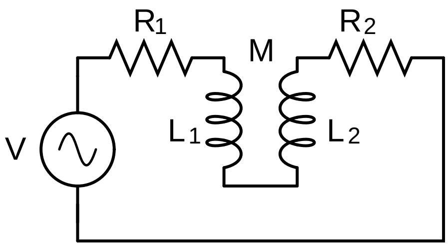

Figure 4: Linear circuit with mutual inductances described by circuit equations with index two.

is described by the circuit equations with index two:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & R_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & sL_1 & sM & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & sM & sL_2 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \tilde{\xi}_V \\ \tilde{\xi}_{R_1} \\ \tilde{\xi}_{R_2} \\ \tilde{\xi}_{L_1} \\ \tilde{\xi}_{L_2} \\ \tilde{\eta}_V \\ \tilde{\eta}_{R_1} \\ \tilde{\eta}_{R_2} \\ \tilde{\eta}_{L_1} \\ \tilde{\eta}_{L_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \tilde{V}(s) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

While MNA results in a DAE with index two, one can obtain a DAE with lower index by the hybrid analysis. The flow of Algorithm for minimum index hybrid analysis is traced below.

Step 1: The degree matrix Θ for this circuit is as follows:

$$\Theta = \begin{array}{c} \\ \tilde{\xi}_{R_1} \\ \tilde{\xi}_{R_2} \\ \tilde{\xi}_{L_1} \\ \tilde{\xi}_{L_2} \\ \tilde{\eta}_{R_1} \\ \tilde{\eta}_{R_2} \\ \tilde{\eta}_{L_1} \\ \tilde{\eta}_{L_2} \end{array} \begin{array}{c} \begin{array}{cccccccc} \tilde{\xi}_{R_1} & \tilde{\xi}_{R_2} & \tilde{\xi}_{L_1} & \tilde{\xi}_{L_2} & \tilde{\eta}_{R_1} & \tilde{\eta}_{R_2} & \tilde{\eta}_{L_1} & \tilde{\eta}_{L_2} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array},$$

which is computed by using the degree of minor of mixed polynomial matrices solver called VIAP [7] under the assumption that the set of nonzero entries coming from the physical parameters is algebraically independent.

Step 2: Set $E_y \leftarrow \emptyset$, $E_z \leftarrow \{R_1, R_2, L_1, L_2\}$, and $\alpha \leftarrow 2$.

Step 3: Solve 2SAT(2):

$$u_{L_1} = 1,\ u_{L_2} = 1,\ (u_{L_1} \vee u_{L_1}) \wedge (u_{L_1} \vee u_{L_2}) \wedge (u_{L_2} \vee u_{L_1}) \wedge (u_{L_2} \vee u_{L_2}) = 1.$$

There exists a feasible assignment. For example, we have $u_{R_1} = 0$, $u_{R_2} = 0$, $u_{L_1} = 1$, $u_{L_2} = 1$ as a solution.

Step 4: $E_y \leftarrow \{R_1, R_2\}$, $E_z \leftarrow \{L_1, L_2\}$, and $\alpha \leftarrow 1$.

Step 3: Solve 2SAT(1). Substituting $u_{L_1} = 1$ and $u_{L_2} = 1$, we can rewrite 2SAT(1) into

$$u_{L_1} = 1,\ u_{L_2} = 1,\ (u_{R_1} \vee \overline{u}_{R_1}) \wedge (u_{R_1} \vee u_{R_1}) \wedge (u_{R_2} \vee \overline{u}_{R_2}) \wedge (u_{R_2} \vee u_{R_2}) = 1.$$

We have a solution $u_{R_1} = 1$, $u_{R_2} = 1$, $u_{L_1} = 1$, $u_{L_2} = 1$.

Step 4: $E_y \leftarrow \emptyset$, $E_z \leftarrow \{R_1, R_2, L_1, L_2\}$, and $\alpha \leftarrow 0$.

Step 3: Solve 2SAT(0). There exists no feasible solution.

Step 5: Return $(E_y, E_z) = (\emptyset, \{R_1, R_2, L_1, L_2\})$ and $\alpha = 0$.

The algorithm finds an admissible partition $(E_y, E_z) = (\emptyset, \{R_1, R_2, L_1, L_2\})$ and $\alpha = 0$. Since $\deg \det A = 1$, the output suggests that the resulting hybrid equation has index zero. In fact, with respect to the partition $(E_y, E_z) = (\emptyset, \{R_1, R_2, L_1, L_2\})$ and the reference tree $T = \{V, R_1, R_2, L_1\}$, we obtain the hybrid equation

$$(R_1 + R_2 + sL_1 + sL_2 + 2sM)\tilde{\xi}_{L_2} = -\tilde{V}(s),$$

which has index zero.

7 Conclusion

We have proposed a combinatorial algorithm that minimizes the index of the hybrid equations in circuit simulation for linear time-invariant electric circuits. The optimal hybrid analysis often results in a DAE with lower index than MNA. We have shown some numerical examples to exhibit the effect of the index reduction.

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