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# On the Fractional Chromatic Number of Monotone Self-Dual Boolean Functions*

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**Abstract**

We compute the exact fractional chromatic number for several classes of monotone self-dual Boolean functions. We characterize monotone self-dual Boolean functions in terms of the optimal value of a LP relaxation of a suitable strengthening of the standard IP formulation for the chromatic number. We also show that determining the self-duality of monotone Boolean function is equivalent to determining feasibility of a certain point in a polytope defined implicitly.

## 1 Introduction

A *Boolean function*, or a *function* in short, is a mapping $f : \{0,1\}^n \to \{0,1\}$, where $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ is called a *Boolean vector* (a *vector* in short). A Boolean function is said to be *monotone* if $f(x) \leq f(y)$ for all vectors $x$ and $y$ with $x \leq y$, where $x \leq y$ denotes $x_i \leq y_i$ for all $i \in \{1, \ldots n\}$. It is known that a Boolean function is monotone if and only if it can be represented by a formula that contains no negative literal. Especially, any monotone function $f$ has a unique prime disjunctive normal form (DNF) expression

$$f \quad = \quad \bigvee_{H \in \mathcal{H}} \Big( \bigwedge_{j \in H} x_j \Big), \tag{1}$$

where $\mathcal{H}$ is a *Sperner* (or *simple*) hypergraph on $V\,(= \{1, \ldots, n\})$, i.e., $\mathcal{H}$ is a subfamily of $2^V$ that satisfies $H \not\subseteq H'$ and $H \not\supseteq H'$ for all $H, H' \in \mathcal{H}$ with $H \neq H'$. It is well-known that $\mathcal{H}$ corresponds to the set of all prime implicants of $f$. Given a function $f$, we define its *dual* $f^d : \{0,1\}^n \to \{0,1\}$

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by $f^d(x) = \overline{f(\overline{x})}$ for all vectors $x \in \{0,1\}^n$, where $\overline{x}$ is the componentwise complement of $x$, i.e., $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$. As is well-known, the formula defining $f^d$ is obtained from that of $f$ by exchanging $\vee$ and $\wedge$ as well as the constants 0 and 1. A function $f$ is called *self-dual* if $f = f^d$ holds.

Monotone self-dual functions have been studied not only in Boolean algebra, but also in hypergraph theory [2, 21], distributed systems [13, 17], and game theory [30] under the names of strange hypergraphs, non-dominated coteries, and decisive games, respectively. For example, a Sperner hypergraph $\mathcal{H} \subseteq 2^V$ is called *strange* [21] (or *critical non-2-colorable* [1]) if it is intersecting (i.e., every pair in $\mathcal{H}$ has at least one element from $V$ in common) and not 2-colorable (i.e., the chromatic number $\chi(\mathcal{H})$ of $\mathcal{H}$ satisfies $\chi(\mathcal{H}) > 2$). It is known (e.g., [2, 8]) that a monotone function $f$ is self-dual if and only if it can be represented by (1) for a strange hypergraph $\mathcal{H}$. Here we note that there exists a one-to-one correspondence between monotone self-dual functions and strange hypergraphs. Strange hypergraphs can also be characterized in terms of transversals. For a hypergraph $\mathcal{H} \subseteq 2^V$, $T \subseteq V$ is a *transversal* of $\mathcal{H}$ if $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$, and the family of minimal transversals of $\mathcal{H}$ is called the *transversal hypergraph* of $\mathcal{H}$, denoted by $Tr(\mathcal{H})$. Then $\mathcal{H}$ is strange if and only if $Tr(\mathcal{H}) = \mathcal{H}$ holds (e.g., [1, 2, 8]).

Another characterization of self-duality (i..e, strangeness) appears in the literature of distributed systems [13, 17]. A *coterie* is an intersecting Sperner hypergraph. A coterie $\mathcal{H}$ *is dominated by* another coterie $\mathcal{H}'$ if for each $H \in \mathcal{H}$ there exists an $H' \in \mathcal{H}'$ such that $H' \subseteq H$, and is *non-dominated* if no such coterie $\mathcal{H}'$ exists. It is known (e.g., [13]) that $\mathcal{H}$ is strange if and only if it is a non-dominated coterie. In summery, the following equivalence is known.

**Theorem 1 (E.g., [1, 2, 8, 13, 17])** *Let $\mathcal{H}$ be a Sperner hypergraph, and $f$ be a monotone function defined by* (1). *Then the following statements are equivalent:*

1. *$f$ is self-dual.*
2. *$Tr(\mathcal{H}) = \mathcal{H}$.*
3. *$\mathcal{H}$ is strange (i.e., $\mathcal{H}$ is intersecting and $\chi(\mathcal{H}) > 2$).*
4. *$\mathcal{H}$ is a non-dominated coterie.*

Given a monotone function $f$ represented by (1), the *self-duality problem* is to determine whether $f^d = f$. By Theorem 1, the self-duality problem is to decide if a given hypergraph $\mathcal{H}$ satisfies $Tr(\mathcal{H}) = \mathcal{H}$, i.e., is strange (or a non-dominated coterie). Since it is known [3, 8] that the self-duality problem is polynomially equivalent to the monotone duality problem, i.e., given two monotone DNFs $\varphi$ and $\psi$, deciding if they are mutually dual (i.e., $\varphi^d \equiv \psi$), the self-duality problem has a host of applications in various areas such as database theory, machine learning, data mining, game theory, artificial intelligence, mathematical programming, and distributed systems (See surveys [11, 9] for example).

While the self-duality problem is in co-NP, since for a non-self-dual function $f$, there exists a succinct certificate $x \in \{0,1\}^n$ such that $f(x) \neq f^d(x)$ (i.e., $f(x) = f(\overline{x})$), the exact complexity of the self-duality is still open for more than 25 years now (e.g., [11, 20, 18, 27]). The best currently known upper time-bound is quasi-polynomial time [12, 14, 32]. It is also known that the self-duality problem can be solved in polynomial time by using poly-logarithmically many nondeterministic steps [10, 19]. These suggest that the self-duality problem is unlikely to be co-NP-complete, since it is widely believed that no co-NP-hard problem can be solved in quasi-polynomial time (without nondeterministic step) and in polynomial time with poly-logarithmically many nondeterministic steps. However the problem does not seem to lend itself to a polynomial time algorithm.

Much progress has been made in identifying special classes of monotone functions for which the self-duality problem can be solved in polynomial time (e.g., [4, 6, 7, 8, 10, 15, 20, 22, 23, 24, 28] and references therein). For example, Peled and Simeone [28] and Crama [6] presented polynomial time algorithms to dualize (and hence to determine the self-duality of) regular functions in polynomial time. Boros et al. [5] and Eiter and Gottlob [8] showed the self-duality for monotone $k$-DNFs (i.e, DNFs in which each term contains at most $k$ variables) can be determined in polynomial time, and Gaur and Krishnamurti [15] improved upon it to have a polynomial-time algorithm for the self-duality for monotone $O(\sqrt{\log n})$-DNFs.

**Our contributions:** Motivated by Theorem 1 (that $f$ is self-dual if and only if $\mathcal{H}$ is strange), we study the fractional chromatic number [31] of self-dual functions. We exactly characterize the fractional chromatic number of three classes of self-dual functions that arise in the context of distributed systems, namely, the functions associated with majority coteries [13], wheel coteries [25], and uniform Lovász coteries [26]. We also show that any threshold self-dual function has the fractional chromatic number greater than 2.

Since the fractional chromatic number of self-dual functions associated with uniform Lovász coteries is less than 2, it cannot be used to characterize self-dual functions, where we note that dual-minor (that corresponds to intersecting hypergraphs) and non-self-dual functions has chromatic number 2, and hence fractional chromatic number at most 2. Thus, by strengthening the standard integer programming formulation for chromatic number, we give another characterization of self-dual functions in terms of the optimal solution to an LP relaxation of the strengthening. This characterization also shows that the self-duality is equivalent to determining the feasibility of 'some' point in a suitably defined polytope.

## 2 Preliminaries

Let $\mathcal{H}$ be a hypergraph on vertex set $V$. A $k$-coloring of $\mathcal{H}$ is a partition $\{V_1, \ldots, V_k\}$ of $V$ (i.e., $V = \bigcup_{i=1}^{k} V_i$ and $V_i \cap V_j = \emptyset$ for all $i$ and $j$ with $i \neq j$) such that every edge $H \in \mathcal{H}$ intersects at least two subsets $V_i$ and $V_j$. Here the vertices that belong to $V_i$ are assigned the color $i$. For a hypergraph, we denote by $\chi(\mathcal{H})$ the smallest integer $k$ for which $\mathcal{H}$ admits a $k$-coloring. We define $\chi(\mathcal{H}) = +\infty$ if $\mathcal{H}$ contains a hyperedge $H$ of size 1 (i.e., $|H| = 1$). A vertex subset $W \subseteq V$ is called *independent* if it does not contain any edge $H \in \mathcal{H}$; otherwise, *dependent*. Let $\mathcal{I}$ denote the family of all the (inclusionwise) maximal independent sets of $\mathcal{H}$. Then the following integer programming problem determines the chromatic number $\chi(\mathcal{H})$ of $\mathcal{H}$.

$$
\begin{aligned}
\text{IP:}\ \ \text{minimize}\ \ & \sum_{I \in \mathcal{I}} x_I \\
\text{subject to}\ \ & \sum_{I : I \ni v} x_I \geq 1 \quad \text{for all} \quad v \in V && (2) \\
& x_I \in \{0, 1\} \quad \text{for all} \quad I \in \mathcal{I}, && (3)
\end{aligned}
$$

where $x_I$ takes 0/1 value (from constraint (3)) associated with maximal independent set $I \in \mathcal{I}$, constraint (2) ensures that each vertex is covered by some maximal independent set and the goal is to minimize the number of maximal independent sets needed. We note that $\chi(\mathcal{H})$ is the optimal value, since a $\chi(\mathcal{H})$-coloring can be constructed from a subfamily $\mathcal{I}^* = \{I \in \mathcal{I} \mid x_I = 1\}$.

Linear programming (LP) relaxation of the problem above is obtained by replacing (3) with non-negativity constraints:

$$
x_I \geq 0 \quad \text{for all} \quad I \in \mathcal{I}. \tag{4}
$$

The optimal value of the LP relaxation, denoted $\chi_f(\mathcal{H})$, is the fractional chromatic number (see Schneiderman and Ullman [31]). By definition, we have $\chi_f(\mathcal{H}) \leq \chi(\mathcal{H})$. Let us describe the dual (D) of the LP relaxation, where $y_v$ denotes a variable associated with $v \in V$.

$$
\begin{aligned}
\text{D:}\ \ \text{maximize}\ \ & \sum_{v \in V} y_v \\
\text{subject to}\ \ & \sum_{v \in I} y_v \leq 1 \quad \text{for all} \quad I \in \mathcal{I} && (5) \\
& y_v \geq 0 \quad \text{for all} \quad v \in V
\end{aligned}
$$

The subsequent sections make use of the strong and the weak duality in linear programming extensively. Weak duality states that the value of a feasible dual solution is a lower bound on the optimal value of the primal, where the primal is a minimization problem. Strong duality states that the feasibility of the primal and the dual problems implies that two problems have the same optimal values. For details see Vanderbei [33, Chapter 5], for example.

In general the number of maximal independent sets of a hypergraph $\mathcal{H}$ can be exponential in $|V|$ and $|\mathcal{H}|$, but by Theorem 1, we have the following nice characterization of maximal independent sets for strange hypergraphs, since $I$ is a maximal independent set if and only if $\overline{I}\,(= V \setminus I)$ is a minimal transversal.

**Lemma 1** *A hypergraph $\mathcal{H}$ is strange if and only if $\mathcal{I} = \{\overline{H} \mid H \in \mathcal{H}\}$ holds.*

This implies that the number of variables in the primal problem (the number of constraints (5) in the dual problem) is $|\mathcal{H}|$.

# 3 Fractional chromatic number of strange hypergraphs

In this section, we study the fractional chromatic number for well known classes of self-dual functions that have received considerable attention in the area of distributed systems.

## 3.1 Strange hypergraphs $\mathcal{H}$ with $\chi_f(\mathcal{H}) > 2$

We show that the majority, wheel, and threshold hypergraph have fractional chromatic number greater than 2.

Let us first consider the majority hypergraphs. For a positive integer $k$, let $\mathcal{M}_{2k+1}$ be a majority hypergraph on $V$ with $|V| = 2k+1$ defined by

$$\mathcal{M}_{2k+1} = \{M \subseteq V \mid |M| = k+1\}.$$

It is easy to see that $Tr(\mathcal{M}_{2k+1}) = \mathcal{M}_{2k+1}$ holds (i.e., $\mathcal{M}_{2k+1}$ is strange).

**Theorem 2** *For a positive integer $k$, we have $\chi_f(\mathcal{M}_{2k+1}) = 2 + \frac{1}{k}$.*

*Proof:* It follows from Lemma 1 that we have $\binom{2k+1}{k+1}$ maximal independent sets $I$, each of which satisfies $|I| = k$. We can see that each vertex $v$ in $V$ belongs to $\binom{2k}{k-1}$ maximal independent sets. We construct feasible primal and dual solutions with value $2 + \frac{1}{k}$ to complete the proof.

For the primal problem, we assign $1/\binom{2k}{k-1}$ to each maximal independent set. Then we note that this is a feasible solution, and the value is $\binom{2k+1}{k+1}/\binom{2k}{k-1} = 2 + \frac{1}{k}$. On the other hand, for the dual problem, we assign

$1/k$ to each vertex in $V$. This is again a feasible dual solution, and the value is $2 + 1/k$. □

Let us next consider wheel hypergraphs. For a positive integer $n\,(>3)$, $\mathcal{W}_n$ be a hypergraph on $V = \{1, \ldots, n\}$ defined by

$$\mathcal{W}_n = \{\{i, n\} \mid i = 1, \ldots, n-1\} \cup \{\{1, \ldots, n-1\}\}.$$

Clearly, $\mathcal{W}_n$ is strange, since $Tr(\mathcal{W}_n) = \mathcal{W}_n$ holds.

**Theorem 3** *For a positive integer $n\,(>3)$, we have $\chi_f(\mathcal{W}_n) = 2 + \dfrac{1}{n-2}$.*

*Proof:* We construct feasible primal and dual solutions with value $2 + 1/(n-2)$ to complete the proof.

For the primal problem, we assign 1 to maximal independent set $\{n\}$ and $1/(n-2)$ to all the other maximal independent sets. Then we can see that this is feasible whose value is $2 + 1/(n-2)$. On the other hand, for the dual problem, we assign $1/(n-2)$ to $y_i$, $i = 1, \ldots, n-1$, and 1 to $y_n$. Then this is a feasible dual solution with value $2 + 1/(n-2)$. □

A function $f$ is called *threshold* if it can be represented by

$$f(x) = \begin{cases} 1 & \text{if } \sum_i w_i x_i > 1 \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

for some nonnegative weights $w_1, \ldots, w_n$. We can see that functions $f_{\mathcal{M}_{2k+1}}$ and $f_{\mathcal{W}_n}$ associated with $\mathcal{M}_{2k+1}$ and $\mathcal{W}_n$ are threshold, since they can be represented by the following inequalities.

$$f_{\mathcal{M}_{2k+1}}(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{2k+1} \frac{1}{k} x_i > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\mathcal{W}_n}(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \frac{1}{n-2} x_i + x_n > 1 \\ 0 & \text{otherwise.} \end{cases}$$

As seen in Theorems 2 and 3, we have $\chi_f(\mathcal{M}_{2k+1}), \chi_f(\mathcal{W}_n) > 2$. The next theorem says that thresholdness ensures that the fractional chromatic number is greater than 2.

**Theorem 4** *The fractional chromatic number of any threshold self-dual function is greater than 2.*

*Proof:* Let $f$ be a threshold self-dual function defined by (6), and let $\mathcal{H}$ be a strange hypergraph corresponding to $f$. Let us consider the dual problem to have a lower bound on the fractional chromatic number. We assign the weights $w_i$ in (6) to dual variables $y_i$. Then by (1), any independent set $I$

satisfies $\sum_{i\in I} w_i \leq 1$, and hence it is feasible. We assume without loss of generality that there exists a vector $x \in \{0,1\}^n$ such that $\sum_i w_i x_i = 1$, i.e., a maximal independent set $I^*$ of $\mathcal{H}$ such that $\sum_{i\in I^*} w_i = 1$. By Lemma 1, we have $\overline{I^*} \in \mathcal{H}$ and hence $\sum_{i\in\overline{I^*}} w_i > 1$. Thus the objective value of $y_i$ is $\sum_{i=1}^n w_i = \sum_{i\in I^*} w_i + \sum_{i\in\overline{I^*}} w_i > 2$. $\square$

### 3.2 Strange hypergraphs $\mathcal{H}$ with $\chi_f(\mathcal{H}) \leq 2$

This section shows that not every strange hypergraph has the fractional chromatic number greater than 2. Especially, we show that there exists an infinite family of strange hypergraphs $\mathcal{H}$ with $\chi_f(\mathcal{H}) < 2$.

Let us first see that the following strange hypergraph $\mathcal{H}$ has $\chi_f(\mathcal{H}) = 2$.

**Example 1** Let $V = \{a,b\} \cup \{1,\ldots,7\}$, and $\mathcal{H}$ be a hypergraph on $V$ given by

$$\begin{aligned}\mathcal{H} &= \big\{\{a,b\}\big\} \cup \big\{\{1,2,3,c\},\{3,4,5,c\},\{1,5,6,c\},\\ &\qquad \{1,4,7,c\},\{2,5,7,c\},\{3,6,7,c\},\{2,4,6,c\} \mid c \in \{a,b\}\big\}.\end{aligned}$$

Note that this $\mathcal{H}$ satisfies $Tr(\mathcal{H}) = \mathcal{H}$ (i.e., $\mathcal{H}$ is strange), and hence we have $\chi(\mathcal{H}) = 3$. For its fractional chromatic number, we have a feasible primal solution with value 2 obtained by assigning $\frac{1}{2}$ to four maximal independent sets $\{1,2,6,7,b\}, \{2,3,5,6,b\}, \{4,5,6,7,a\}$, and $\{1,3,4,6,a\}$, and 0 to all the others. A dual solution of value 2 is obtained by assigning 1 to $x_a$ and $x_b$, and 0 to all the others. Thus we have $\chi_f(\mathcal{H}) = 2$.

We next show that there exist a strange hypergraph $\mathcal{H}$ with $\chi_f(\mathcal{H}) < 2$.

A finite projective plane of order $n$ is defined as a set of $n^2+n+1$ points with the properties that:

1. Any two points determine a line,
2. Any two lines determine a point,
3. Every point has $n+1$ lines on it, and
4. Every line contains $n+1$ points.

When $n$ is a power of a prime, finite projective planes can be constructed as follows, where the existence of finite projective planes when $n$ is not a power of a prime is an important open problem in combinatorics.

There are three types of points:

1. a single point $p$,
2. $n$ points $p(0), p(1)$, and $p(n-1)$,

3. $n^2$ points $p(i,j)$ for all $i,j \in \{0,\ldots,n-1\}$.

The lines are of the following types:

1. one line $\{p, p(0), p(1), \ldots, p(n-1)\}$,
2. $n$ lines of the type $\{p, p(0,c), p(1,c), \ldots, p(n-1,c)\}$ for all $c$'s,
3. $n^2$ lines of the type $\{p(c), p(0\times c+r \pmod{n}, 0), p(1\times c+r \pmod{n}, 1), \ldots, p((n-1)\times c+r \pmod{n}, n-1)\}$ for all $c$'s and $r$'s.

For example, the finite projective plane of order 2, called *Fano plane*, has 7 points

$$p, p(0), p(1), p(0,0), p(0,1), p(1,0), p(1,1),$$

and 7 lines

$$\{p,p(0),p(1)\}, \{p,p(0,0),p(1,0))\}, \{(p,p(0,1),p(1,1))\}, \{p(0),p(0,0),p(0,1)\},$$
$$\{(p(0),p(1,0),p(1,1)\}, \{p(1),p(0,0),p(1,1)\}, \{p(1),p(1,0),p(0,1)\}.$$

It is known [2] that Fano plane is a strange hypergraph, if we regard points and lines as vertices and hyperedges, respectively, but no finite projective plane of order $n\,(>2)$ is strange.

**Theorem 5** *Let $\mathcal{F}_n$ be a finite projective plane of order $n$. Then we have $\chi_f(\mathcal{F}_n) \le 1 + \dfrac{n+1}{n^2}$ if $n \ge 3$, and $\chi_f(\mathcal{F}_2) = \dfrac{7}{4}$.*

*Proof:* Let us first show that $\chi_f(\mathcal{F}_n) \le 1 + (n+1)/n^2$ for $n\,(\ge 2)$ by constructing a prime feasible solution with value $1 + (n+1)/n^2$. By the definition of finite projective plane, $F \in \mathcal{F}_n$ is a minimal transversal of $\mathcal{F}_n$, and hence $\overline{F}$ is a maximal independent set of $\mathcal{F}_n$. By assigning $1/n^2$ to each maximal independent set $\overline{F}$ with $F \in \mathcal{F}_n$, and 0 to each maximal independent set $I$ with $\overline{I} \notin \mathcal{F}_n$, we have a feasible primal solution with value $1 + (n+1)/n^2$.

We next prove $\chi_f(\mathcal{F}_2) = \frac{7}{4}$ by constructing a dual feasible solutions with value $7/4$. Since $\mathcal{F}_2$ is strange, Lemma 1 implies that $\mathcal{F}_2$ has 7 maximal independent sets, each of which has size 4. Thus by assigning $1/4$ to each vertex, we have a feasible solution with value $7/4$, which completes the proof. □

We now describe an infinite family of strange hypergraphs (obtained from Crumbling Walls coteries) whose fractional chromatic number goes to 1 as $n\,(=|V|)$ does.

Crumbling walls due to Peleg and Wool [29] are coteries that generalize the triangular coteries, grids, hollow grids, and wheel coteries. Let $V =$

$\{1, 2, \ldots, n\}$, and let $U_0, U_1, \ldots, U_d$ be a partition of $V$, where we denote $|U_i|$ by $n_i$. Then crumbling wall $\mathcal{H}$ is defined by $\mathcal{H} = \bigcup_{i=0}^{d} \mathcal{H}_i$ such that

$$\mathcal{H}_i \;=\; \big\{U_i \cup \{u_{i+1}, \ldots, u_d\} \mid u_j \in U_j \text{ for all } j = i+1, \ldots, n\big\}. \tag{7}$$

Note that $H \cap H' \neq \emptyset$ holds for all $H, H' \in \mathcal{H}$ (i.e., $\mathcal{H}$ is a coterie). It is known that a crumbling wall is strange if and only if $n_0 = 1$ and $n_i \geq 2$ for all $i \geq 1$ [29]. Crumbling walls with $n_0 = 1$ are also known as *Lovász* hypergraphs (or coteries), as the construction was first proposed by Lovász [21].

We consider the class of *uniform Lovász* hypergraphs, denoted by $\mathcal{L}_{k,d}$, which are crumbling walls with $n_0 = 1$ and $n_i = k \,(\geq 2)$ for all $i = 1, \ldots, d$. For example, if $k, d = 2$, then we have

$$\mathcal{L}_{2,2} = \big\{\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,5\},\{4,5\}\big\},$$

where $U_0 = \{1\}$, $U_1 = \{2,3\}$, and $U_2 = \{4,5\}$.

**Theorem 6** *Let $k$ and $d$ be positive integers with $k \geq 2$. Then we have*

$$\chi_f(\mathcal{L}_{k,d}) = 1 + \frac{k^d}{\displaystyle\sum_{i=1}^{d} (k-1)^i k^{d-i}}, \tag{8}$$

*which satisfies $\chi_f(\mathcal{L}_{k,d}) \to +1$ as $k, d \to +\infty$.*

*Proof:* Let us consider properties of coefficient matrix $A$ of the LP formulation of fractional chromatic number. Recall that the row set corresponds to vertex set $V = \bigcup_{i=0}^{d} U_i$ and the column set corresponds to the family $\mathcal{I}$ of maximal independent sets of $\mathcal{L}_{k,d}$. Let $\mathcal{I}_i = \{\overline{H} \mid H \in \mathcal{H}_i\}$, where $\mathcal{H}_i$ is given by (7). By Lemma 1 and the definition of Lovász hypergraphs, we have $\mathcal{I} = \bigcup_{i=0}^{d} \mathcal{I}_i$. Let us then partition matrix $A$ into submatrices $B_{i,j}$, $i, j = 0, \ldots, d$, which has row set $U_i$ and column set $\mathcal{I}_i$:

$$A = \begin{array}{c} \\ U_0 \\ U_1 \\ \vdots \\ U_d \end{array} \!\!\begin{array}{c} \begin{array}{cccc} \mathcal{I}_0 & \mathcal{I}_1 & \ldots & \mathcal{I}_d \end{array} \\ \begin{pmatrix} B_{0,0} & B_{0,1} & \ldots & B_{0,d} \\ B_{1,0} & B_{1,1} & \ldots & B_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ B_{d,0} & B_{d,1} & \ldots & B_{d,d} \end{pmatrix} \end{array}.$$

Then we have the following properties of $B_{ij}$.

- $B_{i,j}$ is a $k \times k^{d-j}$ matrix if $i \geq 1$, and a $1 \times k^{d-j}$ matrix, otherwise (i.e., $i = 0$).
- $B_{i,i} = 0$ holds for all $i$, where $0$ denotes the matrix (with appropriate size) whose elements are all 0's.

- $B_{i,j} = 1$ holds for all $i$ and $j$ with $i < j$, where 1 denotes the matrix (with appropriate size) whose elements are all 1's.
- For $i$ and $j$ with $i > j$, each row of $B_{i,j}$ has $(k^{d-j} - k^{d-j-1})$ 1's, and each column of $B_{i,j}$ has $(k-1)$ 1's.

We now construct a primal feasible solution in which the variables associated with independent sets in $\mathcal{I}_j$ are all assigned the same value of $x_j$, and each constraint is satisfied with equality.

Let us consider two primal constraints one from a row in $U_i$ and the other from row in $U_{i+1}$. Both these constraints agree in all assignment of the variables to all the blocks, except the blocks $B_{i,i+1}$ and $B_{i+1,i}$. The sum of the values assigned to the variables in the two blocks should be the same by our assumption. For all $i = 0, \ldots, d-1$, we have $k^{d-(i+1)}x_{i+1} = (k^{d-i} - k^{d-(i+1)})x_i$, i.e., $x_{i+1} = (k-1)x_i$. Thus we have

$$x_i \quad = \quad (k-1)^i x_0 \quad \text{for all} \quad i = 1, \ldots, d. \tag{9}$$

Furthermore, the first row constraint states that

$$\sum_{i=1}^{d} k^{d-i} x_i \quad = \quad 1. \tag{10}$$

By combining (9) with (10), we have

$$x_0 = \frac{1}{\sum_{i=1}^{d} (k-1)^i k^{d-i}},$$

Note that the solution computed is feasible by construction and has value

$$\begin{aligned} \sum_{i=0}^{d} k^{d-i} x_i \quad &= \quad 1 + k^d x_0 \\ &= \quad 1 + \frac{k^d}{\sum_{i=1}^{d} (k-1)^i k^{d-i}}, \end{aligned}$$

where the first equality follows from (10).

Next, we compute a feasible dual solution by using the same technique and show that the value of the dual solution is equal to the value of the primal solution. Let $y_j$ be the value assigned to dual variables in $U_j$ such that each dual constraint is satisfied with equality.

Let us consider two columns that belong to $\mathcal{I}_j$ and $\mathcal{I}_{j+1}$. The corresponding constraints agree in assignment of all the variables except in blocks

$B_{j,j+1}$ and $B_{j+1,j}$. By our assumption on $y_j$, we have $(k-1)y_1 = y_0$ and $(k-1)y_{j+1} = ky_j$ for all $j = 1, \ldots, d$. Therefore, we have

$$y_j \quad = \quad \frac{k^{j-1}}{(k-1)^j} y_0 \quad \text{for all} \quad j = 1, \ldots, d. \tag{11}$$

Furthermore, the last column states that

$$y_0 + k \sum_{j=1}^{d-1} y_j \quad = \quad 1. \tag{12}$$

By combining (11) with (12), we have

$$y_d \quad = \quad \frac{k^{d-1}}{\displaystyle\sum_{j=0}^{d-1} (k-1)^{d-j} k^j}. \tag{13}$$

We can see that this is a feasible solution with value

$$\begin{aligned} y_0 + k \sum_{j=1}^{d} y_j \quad &= \quad 1 + ky_d \\ &= \quad 1 + \frac{k^d}{\displaystyle\sum_{j=0}^{d-1} (k-1)^{d-j} k^j} \; = \; 1 + \frac{k^d}{\displaystyle\sum_{i=1}^{d} (k-1)^i k^{d-i}}, \end{aligned}$$

where the first and second equalities follow from (12) and (13), respectively. This completes the proof of (8).

It is not difficult to see that $\chi_f(\mathcal{L}_{k,d}) \to 1 + \frac{1}{d}$ as $k \to +\infty$ and $\chi_f(\mathcal{L}_{k,d}) \to +1$ as $k, d \to +\infty$. □

On the positive note, we show that at least half of strange hypergraphs have fractional chromatic number at least 2.

**Lemma 2** *Let $\mathcal{H}$ be an intersecting hypergraph having an hyperedge with exactly $k$ elements. Then we have $\chi_f(\mathcal{H}) \geq \frac{k}{k-1}$.*

*Proof:* Since the value of a feasible dual solution is a lower bound on $\chi_f(\mathcal{H})$, we construct a dual feasible solution with value $\frac{k}{k-1}$. Let $H$ be a hyperedge of size $k$. Assign $\frac{1}{k-1}$ to variables $y_j$ with $j \in H$, and 0 to $y_j$ with $j \notin H$. Suppose that the solution is not feasible. Then for some maximal independent set $I$, we have $\sum_{j \in I} y_j > 1$, which implies that $H$ is contained in the independent set $I$. This contradicts that $I$ is independent. □

It follows from the lemma that all the intersecting hypergraph with a hyperedge of size 2 has fractional chromatic number at least 2.

Let $\mathcal{H}$ be a hypergraph on $V$, and let $a$ and $b$ are new vertices, i.e., $a, b \notin V$. Define $\mathcal{H}_{a,b}$ by

$$\mathcal{H}_{a,b} = \big\{\{a,b\}\big\} \cup \big\{H \cup \{c\} \mid H \in \mathcal{H}, c \in \{a,b\}\big\}.$$

It is easy to see that the strangeness of $\mathcal{H}$ implies that of $\mathcal{H}_{a,b}$. We say that two hypergraphs are different, if they are not identical up to isomorphism (i.e., renaming of the vertices).

**Theorem 7** *At least half of strange hypergraphs have fractional chromatic number at least* 2.

*Proof:* Let $\mathbb{S}$ be the family of all strange hypergraphs (unique up to isomorphism). Let $\mathbb{S}_3 \subseteq \mathbb{S}$ be the family of strange hypergraphs such that all the hyperedges contain at least 3 vertices, and let $\mathbb{S}_2 = \mathbb{S} \setminus \mathbb{S}_3$. Then for each $\mathcal{H} \in \mathbb{S}_3$, we have $\mathcal{H}_{a,b} \in \mathbb{S}_2$, and $\mathcal{H}_{a,b}$ is different from $\mathcal{H}'_{a,b}$ if $\mathcal{H}$ and $\mathcal{H}'$ are different. Therefore, $|\mathbb{S}_2| \geq |\mathbb{S}_3|$ holds. By Lemma 2, every $\mathcal{H} \in \mathbb{S}_2$ has $\chi_f(\mathcal{H}) \geq 2$. This completes the proof. □

**Remark:** Let $\mathcal{H}$ be an intersecting Sperner hypergraph with a hyperedge of size 2. If $\mathcal{H}$ is not strange, then $\chi(\mathcal{H}) = 2$ holds by Theorem 1, and hence $\chi_f(\mathcal{H}) \leq 2$. By this, together with Lemma 2, we have $\chi_f(\mathcal{H}) = 2$, which implies that there exists an infinite family of non-strange hypergraphs $\mathcal{H}$ with $\chi_f(\mathcal{H}) = 2$. Moreover, by Theorem 6, we know the existence of an infinite family of strange hypergraphs $\mathcal{H}$ with $\chi_f(\mathcal{H}) < 2$. These imply that the fractional chromatic number cannot be used to separate strange hypergraphs from non-strange hypergraph. Hence, we need to strengthen the integer program by adding additional inequalities, which is discussed in the next section.

## 4 A LP characterization of strange hypergraphs

In this section, we show how to strengthen the LP-relaxation using derived constraints. The strengthened relaxation (which contains the derived constraints) has optimal value $\chi^*_f(\mathcal{H}) > 2$, provided that $\mathcal{H}$ is strange. Let us consider the following integer program SIP for the chromatic number. The linear programming relaxation to SIP is denoted by SLP.

$$
\begin{aligned}
\text{SIP: minimize} \quad & \sum_{I \in \mathcal{I}} x_I \\
\text{subject to} \quad & \sum_{I: I \ni v} x_I \geq 1 \quad \text{for all} \quad v \in V && (14) \\
& \sum_{I: I \cap H \neq \emptyset} x_I \geq 2 \quad \text{for all} \quad H \in \mathcal{H} && (15) \\
& x_I \in \{0, 1\} \quad \text{for all} \quad I \in \mathcal{I}, && (16)
\end{aligned}
$$

**Lemma 3** *$x$ is a feasible solution of SIP if and only it is a feasible solution of IP.*

*Proof:* We show the lemma by proving that constraint (15) is implied by constraints (14) and (16).

Let $H$ be a hyperedge of $\mathcal{H}$. Then from (14), the following inequality holds:

$$
\sum_{v \in H} \sum_{I: I \ni v} x_I \quad \geq \quad \sum_{v \in H} 1 \ (= |H|). \tag{17}
$$

We note that the coefficient $\alpha_I$ of variable $x_I$ in inequality (17) satisfies the following properties

1. $\alpha_I = 0$, if $I \cap H = \emptyset$,
2. $\alpha_I < |H| - 1$, if $I \cap H \neq \emptyset$.

Since each variable $x_i$ takes only 0/1, we can replace the previous constraint by (15). □

Here we prove analogue of Theorem 1.

**Theorem 8** *Let $\mathcal{H}$ be an intersecting Sperner hypergraph. Then $\mathcal{H}$ is strange if and only if $\chi_f^*(\mathcal{H}) > 2$.*

*Proof:* ($\Rightarrow$) We construct a dual solution to have a lower bound on the optimal value of the primal problem. Let us assign $1/(|\mathcal{H}| - 1)$ to dual variables associated with (15), and 0 to dual variables associated with (14). By lemma 1, $I \in \mathcal{I}$ if and only if $\overline{I} \in \mathcal{H}$. This implies that for each $I \in \mathcal{I}$, there exists a hyperedge $H \in \mathcal{H}$ such that $I \cap H = \emptyset$, where such a hyperedge is $\overline{I}$. Thus the assignment above constructs a feasible solution with value $2(1 + 1/(|E| - 1))$, which implies $\chi_f^*(\mathcal{H}) > 2$.

($\Leftarrow$) If $\chi_f^*(\mathcal{H}) > 2$, then we have $2 < \chi_f^*(\mathcal{H}) \leq \chi^*(\mathcal{H}) = \chi(\mathcal{H})$ by Lemma 3, where $\chi^*(\mathcal{H})$ denotes the optimal value of SIP. It follows from Theorem 1 that $\mathcal{H}$ is strange. □

**Remark:** The implicitly defined LP relaxation of SIP can be solved in polynomial time, provided that there exists a separation oracle for the LP relaxation. This would imply a polynomial time algorithm for the self-duality problem. However, the arguments used in the proof of Theorem 8 can be used to show that determining the feasibility of a point in the polytope associated with SLP is equivalent to the self-duality problem.

Let $\mathcal{P}$ be a polytope defined by the dual of the LP relaxation of SIP. Let $y$ be dual variables associated with constraint (14), and let $z$ be dual variables associated with constraint (15). Let $(y^*, z^*)$ be a vector obtained by assigning 0 to each $y_i$, and $\frac{1}{|\mathcal{H}|-1}$ to each $z_i$.

**Theorem 9** *Let $\mathcal{H}$ be an intersecting Sperner hypergraph. Then $(y^*, z^*) \in \mathcal{P}$ if and only if $\mathcal{H}$ is strange.*

*Proof:* ($\Leftarrow$) Suppose that $\mathcal{H}$ is strange. Then by the proof (of forward direction) of Theorem 8, $(y^*, z^*)$ is a feasible solution to the dual of SLP and hence it is contained in $\mathcal{P}$.

($\Rightarrow$) Suppose that $\mathcal{H}$ is not strange. Then by Lemma 1, there exists a maximal independent set $I$ such that $I \neq \overline{H}$ for all $H \in \mathcal{H}$. Since $\mathcal{H}$ is intersecting, each $\overline{H}$ with $H \in \mathcal{H}$ is independent. Thus such an $I$ satisfies $I \not\subseteq \overline{H}$ (i.e., $I \cap H \neq \emptyset$) for all $H \in \mathcal{H}$. This implies that, if we consider assignment $(y^*, z^*)$ to the dual variables, the dual constraint associated with $I$ is not satisfied, since $\frac{|\mathcal{H}|}{|\mathcal{H}|-1} > 1$. which implies $(y^*, z^*) \notin \mathcal{P}$. $\square$

# 5 Conclusion

In this paper, we characterize self-dual functions in terms of optimal value of a certain linear programming problem. The linear programming problem is a relaxation of a strengthened version of the standard IP formulation for the chromatic number and its dual is defined implicitly with exponentially many constraints. The linear programming problem could in principle be solved in polynomial time, if there exists a separation oracle. However, we also show that the problem for determining feasibility of a given point in the associated polytope is equivalent to the self-duality problem. We also compute the exact fractional chromatic number for well-known classes of self-dual functions arising from majority coteries, wheel coteries, and uniform Lovász coteries. The existence of a polynomial time algorithm for determining self-duality of monotone Boolean functions remains open.

## References


[1] C. Benzaken. Critical hypergraphs for the weak chromatic number. *Journal of Combinatorial Theory B*, 29: 328–338, 1980.

[2] C. Berge. *Hypergraphs: Combinatorics of Finite Sets*. North-Holland, 1989.

[3] C. Bioch and T. Ibaraki. Complexity of identification and dualization of positive Boolean functions. *Information and Computation*, 123:50–63, 1995.

[4] E. Boros, K. M. Elbassioni, V. Gurvich, and L. Khachiyan. Generating maximal independent sets for hypergraphs with bounded edge-intersections. In M. Farach-Colton, editor, *Proceedings of the 6th Latin American Symposium on Theoretical Informatics (LATIN 2004), Buenos Aires, Argentina, April 5-8, 2004*, volume 2976 of *Lecture Notes in Computer Science*, pages 488–498. Springer, 2004.

[5] E. Boros, V. Gurvich, and P.L. Hammer. Dual subimplicants of positive Boolean functions. *Optimization Methods and Software*, 10: 147–156, 1998.

[6] Y. Crama. Dualization of regular boolean functions. *Discrete Appl. Math.*, 16: 79–85, 1987.

[7] C. Domingo, N. Mishra, and L. Pitt. Efficient read-restricted monotone CNF/DNF dualization by learning with membership queries. *Machine Learning*, 37:89–110, 1999.

[8] T. Eiter and G. Gottlob. Identifying the minimal transversals of a hypergraph and related problems. *SIAM J. Comput.*, 24: 1278–1304, 1995.

[9] T. Eiter and G. Gottlob. Hypergraph transversal computation and related problems in logic and AI. In *JELIA '02: Proceedings of the European Conference on Logics in Artificial Intelligence*, pages 549–564, London, UK, 2002. Springer-Verlag.

[10] T. Eiter, G. Gottlob, and K. Makino. New results on monotone dualization and generating hypergraph transversals. *SIAM Journal on Computing*, 32: 514–537, 2003. Extended abstract appeared in *STOC-02*.

[11] T. Eiter, K. Makino, and G. Gottlob. Computational aspects of monotone dualization: A brief survey. KBS Research Report INFSYS RR-1843-06-01, Institute of Information Systems, Vienna University of Technology Favoritenstraße 9-11, A-1040 Vienna, Austria, 2006.

[12] M. L. Fredman and L. Khachiyan. On the complexity of dualization of monotone disjunctive normal forms. *J. Algorithms*, 21: 618–628, 1996.

[13] H. Garcia-Molina and D. Barbara. How to assign votes in a distributed system. *J. Assoc. Comput. Mach.*, 32: 841–860, 1985.

[14] D. Gaur. *Satisfiability and self-duality of monotone Boolean functions.* Dissertation, School of Computing Science, Simon Fraser University, January 1999.

[15] D. Gaur and R. Krishnamurti. Self-duality of bounded monotone boolean functions and related problems. In *ALT '00: Proceedings of the 11th International Conference on Algorithmic Learning Theory*, pages 209–223, London, UK, 2000. Springer-Verlag.

[16] D. Gaur and K. Makino. On the Fractional Chromatic Number of Monotone Self-Dual Boolean Functions. Self-duality of bounded monotone boolean functions and related problems. To appear in *FAW 2007: Proceedings of the Frontiers of Algorithmics WorkShop (FAW 2007)*, Lanzhou , China, 2007.

[17] T. Ibaraki and T. Kameda. A theory of coteries: Mutual exclusion in distributed systems. *IEEE Trans. Parallel Distrib. Syst.*, 4: 779–794, 1993.

[18] D. S. Johnson. Open and closed problems in NP-completeness. Lecture given at the International School of Mathematics "G. Stampacchia": Summer School "NP-Completeness: The First 20 Years", Erice (Sicily), Italy, 20 - 27 June 1991.

[19] D. J. Kavvadias and E. C. Stavropoulos. Monotone Boolean dualization is in co-NP[$\log^2 n$]. *Information Processing Letters*, 85:1–6, 2003.

[20] E. Lawler, J. Lenstra, and A. Rinnooy Kan. Generating all maximal independent sets: NP-hardness and polynomial-time algorithms. *SIAM Journal on Computing*, 9:558–565, 1980.

[21] L. Lovász. Coverings and coloring of hypergraphs. In *Proceedings of the 4th Southeastern Conference on Combinatorics, Graph Theory and Computing*, pages 3–12, 1973.

[22] K. Makino. Efficient dualization of $O(\log n)$-term monotone disjunctive normal forms. *Discrete Appl. Math.*, 126: 305–312, 2003.

[23] K. Makino and T. Ibaraki. The maximum latency and identification of positive boolean functions. *SIAM J. Comput.*, 26:1363–1383, 1997.

[24] K. Makino and T. Uno. New algorithms for enumerating all maximal cliques. In T. Hagerup and J. Katajainen, editors, *Proceedings 9th Scandinavian Workshop on Algorithm Theory on Algorithm Theory (SWAT 2004), Humlebaek, Denmark, July 8-10, 2004*, volume 3111 of *Lecture Notes in Computer Science*, pages 260–272. Springer, 2004.

[25] Y. Marcus and D. Peleg. Construction methods for quorum systems. Technical Report CS 92-33, The Weizmann Institute of Science, Rehovot, Israel, 1992.

[26] M. L. Neilsen. *Quorum structures in distributed systems.* PhD thesis, Manhattan, KS, USA, 1992.

[27] C. Papadimitriou. NP-completeness: A retrospective. In *Proceedings 24th International Colloquium on Automata, Languages, and Programming (ICALP '97)*, LNCS 1256, pages 2–6, 1997.

[28] U. N. Peled and B. Simeone. An O($nm$)-time algorithm for computing the dual of a regular boolean function. *Discrete Applied Mathematics*, 49: 309–323, 1994.

[29] D. Peleg and A. Wool. The availability of crumbling wall quorum systems. *Discrete Appl. Math.*, 74: 69–83, 1997.

[30] K. G. Ramamurthy. Coherent Structures and Simple Games. Kluwer Academic Publishers, 1990.

[31] E. R. Schneiderman and D. H. Ullman. *Fractional Graph Theory.* Wiley Interscience, 1997.

[32] H. Tamaki. *Space-efficient enumeration of minimal transversals of a hypergraph.* IPSJ-AL, 75 (2000), pp. 29–36.

[33] R. J. Vanderbei. *Linear programming: Foundations and extensions.* International Series in Operations Research & Management Science, 37. Kluwer Academic Publishers, Boston, MA, second edition, 2001.