MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

Stable Computation of the Optimal Path for a Boat on a Water Stream and Its Applications

Kokichi Sugihara and Tetsushi Nishida

METR 2007–37 June 2007

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

Stable Computation of the Optimal Path for a Boat on a Water Stream and Its Applications

Kokichi Sugihara and Tetsushi Nishida

***Abstract*— A stable method for computing the optimal path for a boat in a water stream is constructed, and is applied to the estimation of water contamination phenomena. The proposed method is a modification of a marker-particle method. The conventional marker-partical method is unstable for our problem because of singular points in the field of the shortest reachable time. To circumvent the instability, we modify the way of estimating the optimal direction of a particle in such a way that we do not use the locations and the motions of neighbor particles, and thus construct a robust version of the marker-particle method, which we call an *independent marker-particle method*. This method can also be used to trace the time backward from the future to the past, and hence can estimate the location of the contamination source.**

I. Introduction

The change of contamination of undesired material such as oil leaked from a tanker in a ocean and polluted air in a windy sky can be modeled in a unified manner as a diffusion phenomenon in a stream. In this paper, we present a method for simulating this phenomenon, that is, a method for tracing the time-varying area of the diffusion in a stream.

If there is no stream, the diffusion phenomenon can be modeled in a manner similar to the light emission on the basis of the Huygens' principle. That is, the ratio of refraction corresponds to the speed of diffusion. The light emission in a given field of the ratio of refraction is characterized by an eikonal equation. This equation can be solved stably by established methods such as the fast marching method [7], [8] and the marker-particle method [1], [6]. These methods solve the equation along the time axis step by step, and consequently can be used in a simulation of contamination of material.

However, the situation changes drastically if the space behind has a stream. The material moves in different speeds at different points, and hence cause-and-effect relation becomes more complicate than the light emission. Neither the fast marching method nor the marker-particle method works; the fast marching method crashes because sometimes the future phenomena are computed earlier than the past phenomena, and the marker-particle method crashes because the estimation of the direction of the diffusion becomes incorrect due to singular points. Therefore, the conventional methods for eikonal equations cannot be used for our problem except for very simple case such as a homogeneous stream [9].

Recently, we formalized the boat-soil distance in a stream of water, found an equation called the *boat-sail distance equation*, whose solution gives the boat-sail distances, and proposed three robust methods for solving it. The first method is a variant of the fast marching method, in which the search directions are augumented so that the cause-and-effect constraints are always satisfied if the maximum speed of the water stream is not too large [2], [3]. The second method employs the Huygens' principle directly by approximating the speed of the boat in a stream by a cone [4]. The third method is a variant of the marker-particle method, in which the direction of the optimal path of a boat can be computed for each particle independently so that it is not disturbed by singular points of the solution of the boat-sail distance equation [5].

K. Sugihara and T. Nishida are with the University of Tokyo.

In this paper we review the third method and apply it to the simulation of contamination growth in a stream. We review the boat-sail distance equation in section II, and the independent marker-particle method to solve the equation in section III. In section IV, we present a mathematical model of diffusion in a stream, and show that this phenomena can be numerically simulated by the independent marker-particle method. We show some examples of computational experiments in section V, and give concluding remarks in section VI.

II. Boat-Sail Distance Equation

Let $\Omega \subset \mathbf{R}^2$ be a two-dimensional region. Suppose that we are given two-dimensional vector $f(x, y) \in \mathbf{R}^2$ at each point $(x, y) \in \Omega$. We consider Ω as water surface, and $f(x, y)$ as the velocity of the stream. We assume that $f(x, y)$ is continuously differentiable. We call $f(x, y)$ the *field of a water stream*.

We consider a boat that can run at speed F on still water. That is, the boat can move at speed F in any direction if there is no flow of water. Suppose that the boat is at point p_0 at time 0. Let $E(t)$ represent the boundary of the region which the boat can reach within time t in the flow field $f(x, y)$. We call $E(t)$ the *frontier curve*. $E(t)$ forms a closed curve if it does not cross the boundary of Ω. We introduce new parameter s to represent this curve, and represent each point in $E(t)$ as $p(s, t)$. The point $p(s, t)$ is the point which is reachable in time t by the boat that starts p_0 at time 0 and follows the shortest path through $p(s, t)$. We define that the point $p(s, t)$ moves along $E(t)$ counterclockwise as s increases.

Suppose that, as shown in Fig. 1, the boat is at point $p(s, t)$ on $E(t)$ at time t, and it is moving along the shortest path. Let $n(s, t)$ denote the unit normal at point $p(s, t)$ to the curve $E(t)$, where we choose the orientation of $n(s, t)$ so that it faces toward the unreached region. In order for the boat to reach a point on $E(t + \Delta t)$ in time $t + \Delta t$, the boat should try to move in the direction $n(s, t)$. If there were no flow, the boat would move by $Fn(s, t)\Delta t$, and in this time interval the flow will carry the boat by $f(p(s, t))\Delta t$. Therefore, in total the boat will reach

$$p(s, t + \Delta t) = p(s, t) + Fn(s, t)\Delta t + f(p(s, t))\Delta t. \quad (1)$$

Consequently, we get

$$\begin{aligned} \frac{\partial p(s,t)}{\partial t} &= \lim_{\Delta t \to 0} \frac{p(s, t + \Delta t) - p(s, t)}{\Delta t} \\ &= Fn(s, t) + f(p(s, t)). \end{aligned} \quad (2)$$

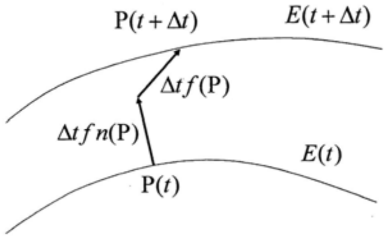


Fig. 1. Frontier curves at time t and $t + \Delta t$, and the optimal motion of a boat.

The basic idea to solve the equation (2) by the marker-particle method is as follows. We consider the start point p_0 as the circle with radius 0, and represent this circle by $p(s, 0)$. Next, we place many points called marker particles, along this circle in an equal distance, and represent them by $p(s_0, 0), p(s_1, 0), p(s_2, 0), \ldots, p(s_{n-1}, 0)$. Since the circle is with radius 0, the points $p(s_0, 0), p(s_1, 0), \ldots, p(s_{n-1}, 0)$ are at the same location, but their normals are different from each other. As shown in Fig. 2, we assume that these points have normals that divide the whole directions by n equal angles.

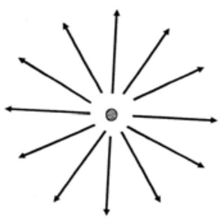
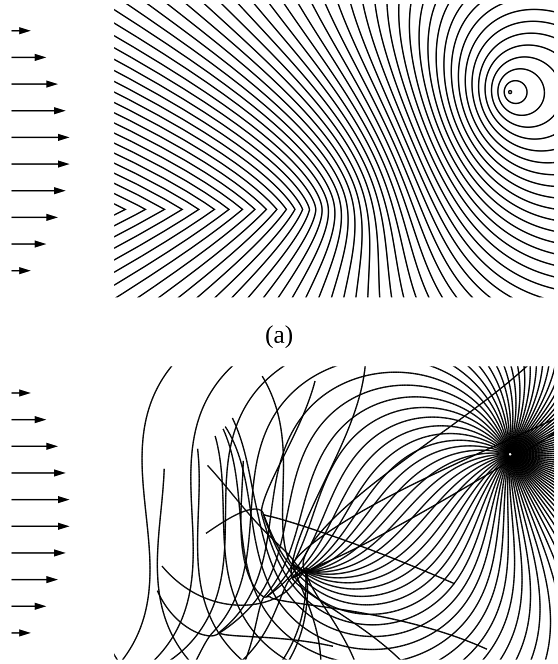


Fig. 2. Initial infinitesimally small circle and normal vectors at the particles.

Starting with these particles, we augment the time by Δt step by step and construct the frontier curves $E(\Delta t), E(2\Delta t), \ldots$. Suppose that we have already obtained the curve $E(t)$. To get the next frontier curve $E(t + \Delta t)$, we compute the location of a particle $p(s, t + \Delta t)$ from $p(s, t)$'s. For this purpose, we need the normal $n(s, t)$. It may seem that the normal $n(s_i, t)$ at point $p(s_i, t)$ can be obtained from the neighboring points $p(s_{i-1}, t)$ and $p(s_{i+1}, t)$. However, this does not work well, because the frontier curve has singular points.

For example, let us assume that, as shown by the arrows in Fig. 3(a), the water flows from the left to the right in a river, and that the flow is fast at the middle, and it becomes slower as we goes near to the river side. Then, the frontier curves for a boat starting from the point represented by the right upper dot are as shown by the curves in this figure. We can see that the frontier curves have non-smooth points; this is because there are two optimal paths, one going near the upper side and the other going near the lower side. Around these singular points, we cannot compute the normal direction stably if we employ the neighboring points.

Actually if we estimate the normal directions from the neighboring points, the traced paths of the marker particles are as shown in Fig. 3(b), which are incorrect around the singular points. Therefore, in order to trace the particles stably, we need to



(a)

(b)

Fig. 3. Singular points of the frontier curves and the resulting instability.

estimate the normal directions without consulting the neighboring points.

III. Robust Method for Tracing the Particles

Here we present a method for estimating the normal $n(s, t)$, which does not employ the neighboring points.

We denote by $p_s(s, t)$ and $p_t(s, t)$ the partial derivative of $p(s, t)$ with respect to s and t, respectively. Using these notations, we can rewrite the equation (2) by

$$p_t(s, t) = Fn(s, t) + f(p(s, t)). \tag{3}$$

On the other hand, let us choose the parameter s in such a way that s represents the arc length at some very early time t, say at $t = \Delta t$. We define $a(s, t)$ and $\tau(s, t)$ by

$$a(s, t) = \frac{1}{|p_s(s, t)|}, \tag{4}$$

and

$$\tau(s, t) = a(s, t)p_s(s, t). \tag{5}$$

The vector $p_s(s, t)$ represents the tangent vector of the frontier curve $E(t)$ at point $p(s, t)$, and $\tau(s, t)$ represents the unit tangent vector. Let us define 2×2 matrix J by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

J represents the counterclockwise rotation by $\pi/2$, and hence we get

$$n(p(s, t)) = J^{-1}\tau(s, t). \tag{7}$$

Therefore, once we get $\tau(s, t)$, we get $n(s, t)$, too. From now on, let consider how to compute $\tau(s, t)$ at each time t.

Partially differentiating the equation (5) by t, we get

$$\tau_t = ap_{st} + a_t p_s. \tag{8}$$

Hence, our goal is to compute a, p_s, p_{st} and a_t.

First, we concentrate on p_{st}. We denote $p(s,t)$ and $f(p(s,t))$ componentwise by

$$p(s,t) = (x(s,t), y(s,t))^{\mathrm{T}}, \tag{9}$$
$$f(x,y) = (g(x,y), h(x,y))^{\mathrm{T}}. \tag{10}$$

Furthermore let $\kappa(s,t)$ be the curvature of $E(t)$, that is,

$$\kappa(s,t) = \frac{y_{ss}x_s - x_{ss}y_s}{({x_s}^2 + {y_s}^2)^{3/2}}. \tag{11}$$

Then, we get

$$\begin{aligned} n_s(s,t) &= \frac{\partial}{\partial s}\left(\frac{y_s}{({x_s}^2+{y_s}^2)^{1/2}}, -\frac{x_s}{({x_s}^2+{y_s}^2)^{1/2}}\right)^{\mathrm{T}} \\ &= \left(\frac{y_{ss}x_s - x_{ss}y_s}{({x_s}^2+{y_s}^2)^{3/2}}x_s, \frac{y_{ss}x_s - x_{ss}y_s}{({x_s}^2+{y_s}^2)^{3/2}}y_s\right)^{\mathrm{T}} \\ &= \kappa(s,t)p_s(s,t). \end{aligned} \tag{12}$$

Let us define $\nabla^{\mathrm{T}} f$ by

$$\nabla^{\mathrm{T}} f = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}. \tag{13}$$

Then, we get

$$f_s(p(s,t)) = (\nabla^{\mathrm{T}} f)p_s(s,t). \tag{14}$$

Partially differentiating the equation (3) by s, and employing (12) and (14), we get

$$\begin{aligned} p_{ts}(s,t) &= F n_s(p(s,t)) + f_s(p(s,t)) \\ &= F\kappa(s,t)p_s(s,t) + (\nabla^{\mathrm{T}} f)p_s(s,t). \end{aligned} \tag{15}$$

Next, we consider how to compute a_t. From (4), we get

$$\begin{aligned} a_t(s,t) &= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{{x_s}^2+{y_s}^2}}\right) \\ &= -\frac{x_s x_{ts} + y_s y_{ts}}{({x_s}^2+{y_s}^2)^{3/2}} \\ &= -\frac{p_s(s,t)\cdot p_{ts}(s,t)}{({x_s}^2+{y_s}^2)^{3/2}} \\ &= -(F\kappa(s,t) + ((\nabla^{\mathrm{T}})\tau(s,t))\cdot\tau(s,t))a(s,t), \end{aligned} \tag{16}$$

where the last equality comes from eq. (15).

Substituting p_{st} and a_t in eq. (8), we get

$$\begin{aligned} \tau_t &= a p_{st} + a_t p_s \\ &= a[F\kappa p_s + (\nabla^{\mathrm{T}} f)p_s] - (F\kappa + ((\nabla^{\mathrm{T}} f)\tau)\cdot\tau)a p_s \\ &= (\nabla^{\mathrm{T}} f)\tau = (((\nabla^{\mathrm{T}} f)\tau)\cdot\tau)\tau \\ &= (I - \tau\tau^{\mathrm{T}})(\nabla^{\mathrm{T}} f)\tau. \end{aligned} \tag{17}$$

Noting that $\tau = Jn$ and $n = J^{\mathrm{T}}\tau$, we get from eq. (17)

$$n_t(s,t) = (I - n(s,t)n(s,t)^{\mathrm{T}})(\nabla^{\mathrm{T}} f)^{\mathrm{T}} n(s,t). \tag{18}$$

Eq. (18) implies that the change n_t of the normal is determined by the current normal $n(s,t)$ at $p(s,t)$ and the flow field f. In other words, we can compute n_t even if we do not know the locations of the neighbor particles.

Thus we have constructed the independent marker particle method. In this method, given the position $p(s,0)$ and the normal $n(s,0)$ at time 0, we get the location $p(s,\Delta t)$ and the normal $n(s,\Delta t)$ of the particle at time Δt by eqs. (3) and (18), respectively. For this purpose we can employ the Runge-Kutta method. We repeat the same procedure for time $2\Delta t, 3\Delta t, \ldots$, and thus can trace the particle for each fixed s independently.

For the flow field and the start point given in Fig. 3(a), we applied our method and obtained the tragectory of the particles as shown in Fig. 4.

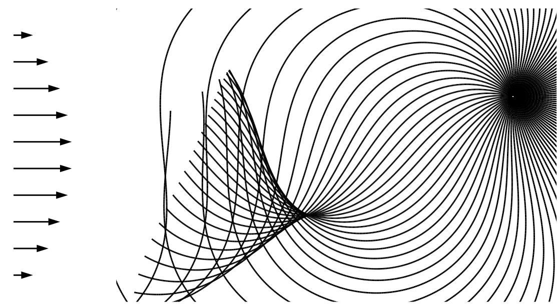


Fig. 4. Shortest paths for a boat in a flow field.

IV. Contamination Diffusion Model

In this section, we consider the phenomenon in which poluted material that start leaking from point p_0 at time 0 diffuses in a flow field.

Assume that if there is no flow of water, the material diffuses at the same speed in every direction. Therefore, if we know the speed F, we can apply our method for finding the frontier curves to estimate the contaminated region.

We can consider several different patterns of the diffusion speed F. The following are typical examples.

(a) Diffusion in constant speed

The simplest assumption is that the speed F of diffusion is constant. This might happen when the amount of the poluted material is very large, and plenty of material comes out during the time interval in consideration, and consequently the speed of diffusion does not decrease.

(b) Density-dependent diffusion

We can assume that the speed is proportional to the density of the contaminated material. If all the material is at p_0 at time 0 and it diffuses around, the speed $F(t)$ at each time t is proportional to the inverse of the area of the polluted region. Hence, if we denote the area surrounded by $E(t)$ by $S(t)$, we can express the speed $F(t)$ as

$$F(t) = c/S(t), \tag{19}$$

where c is a constant.

(c) Diffusion of pouring material

In some cases the polluted material comes out from the source point continuously. This can happen for example when the oil pours out from an oil tunker. Let $Q(t)$ be the amount of material that comes out from the source during a unit time interval, and let $S(t)$ be the area of the contaminated region. If the diffusion speed is proportional to be density, we get

$$F(t) = \frac{\int_0^t Q(u)\mathrm{d}u}{S(t)}. \tag{20}$$

In all the above three cases, the speed $F(t)$ can be computed at each time t from the area of the contaminated region. If the speed F is constant as in the diffusion model (a), we can apply our method in Section III directly to simulate the diffusion phenomena. On the other hand, if the speed F depends on time t as in the diffusion models (b) and (c), we need a slight modification of the boat-sail distance equation. We already succeeded in this modification, and are preparing another paper to present it, which can be used to the simulation of the diffusion phenomena for the models (b) and (c).

V. Computational Examples

5.1 Simulation of Contamination Diffusion

We computed the change of the contaminated area on the basis of the diffusion speed model (a) for various flow fields. We assume that the water flows from the left to the right, and the speed of the flow depends on y but does not depend on x.

Fig. 5 shows the result of the computed contaminated areas in four different patterns of the flow. The flow patterns are represented by the arrows at the left; the lengths of the arrows represent the relative speed of the flow. The source of the diffusion is shown by the dot. Fig. 5(a) shows the case where the flow is faster in the middle, and (b) shows a similar pattern with faster flow; on the other hand (c) shows the case where the flow is slower in the middle, and (d) shows a similar pattern with faster flow.

5.2 Estimation of the Source of Contamination

The boat-sail distance equation can be used for going back to the past from a given current state of the diffusion. This can be understood in the following manner. We define $\bar{n}(s,t)$ and $\bar{f}(p(s,t))$ by

$$\bar{n}(s,t) = -n(s,t), \quad \bar{f}(p(s,t)) = -f(p(s,t)), \tag{21}$$

and substitute them in eq. (3). Then, we get

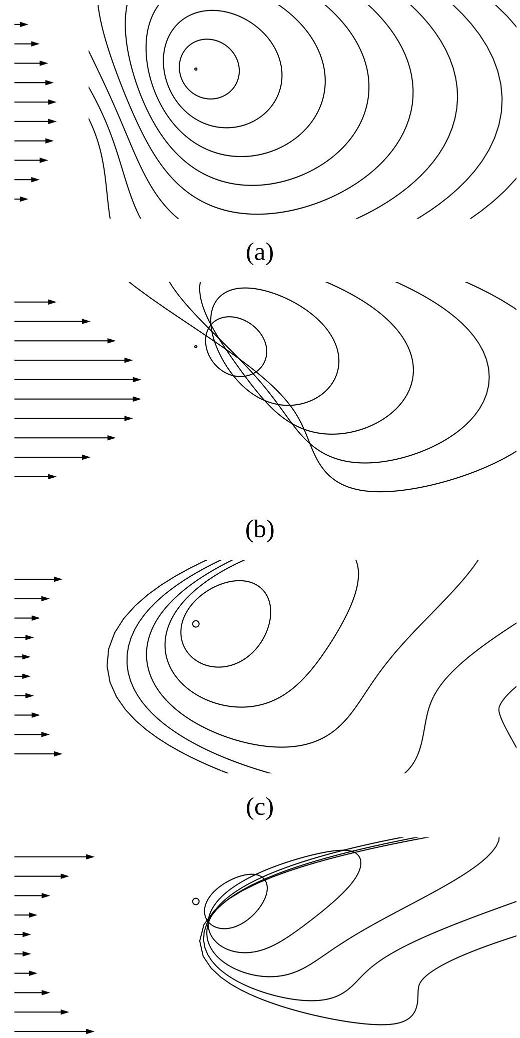
$$-p_t(s,t) = F\bar{n}(s,t) + \bar{f}(p(s,t)). \tag{22}$$

This equation implies that if we reverse both the orientation of the flow and the orientation of the normal, we can trace the particle toward the past by computing $-p_t(s,t)$. The equation (22) has the same form as eq. (3), and hence can be solved by the same method.

From this property, we can go back to the past from the current state of the contamination and thus identify the location of the source of the contamination as well as the time at which the contamination began.

Fig. 6 shows an example of tracing back to the past. We started with a state obtained at some time snap of Fig. 5(c), reverse the flow and the normals, and solve eq. (22). Fig. 6(a) shows the frontier curves and (b) shows the traced paths to the past. From this figure we can see that the state of the past, especially the location of the source of the contamination, can be recovered almost correctly.

Since we can trace each particle independently, the tracing back does not necessarily require all the particles in some time snap. We can trace back from a subset of particles at the current state. Fig. 7 shows the paths traced back from a subset of the particles at the current state in Fig. 6(a). We can see that the location of the source could be identified correctly from these particle information.



(a) (b) (c) (d)

Fig. 5. Simulation of the diffusion of polluted material in various patterns of flow from the left to the right.

Next we added artificial errors in the direction of the normals, and applied our method. Fig. 8 shows an example of the results. In this example, the direction of the normals were disturbed within ± 20 degrees by uniform random numbers. Fig. 8(a) shows the case where we used all the particles in the current state of the frontier, whereas (b) shows the case where only a subset of the particles were used for tracing. In both cases, the location of source could be estimated in a certain precision. Thus, we can see that our method is robust against numerical errors.

We have another idea to trace back to the past, in which the directions of the normals are not necessary. In this method, we consider each particle at current position as the source point, and simulate the diffusion process. We can expect that the frontier curves generated by this simulation have a common point of intersection at the actual source of the contamination.

VI. Concluding Remarks

We applied our independent marker-particle method to the simulation of contamination diffusion phenomena, and showed its performance by examples. This method can also be used to

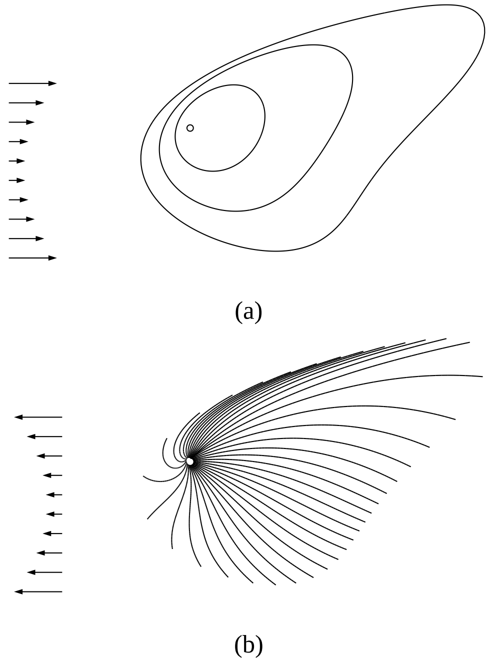


Fig. 6. Result of tracing back to the past from a frontier curve in Fig. 5(c).

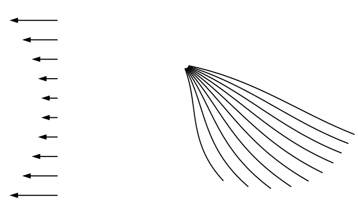


Fig. 7. Estimation of the source of the contamination from partial data of the current state.

trace back to the past, and hence can estimate the past situation, in particular, the location of the source of the contamination.

We are now planning to evaluate our method for actual data of ocean streams.

References

[1] D. Enright, R. Fedkiw, J. Ferziger and I. Mitchell: A hybrid particle level set method for improved interface capturing. *Journal of Computational Physics*, Vol. 183 (2002), pp. 83–116.
[2] T. Nishida and K. Sugihara: Voronoi diagram in the flow field. In T. Ibaraki, N. Katoh and H. Ono (eds.), *Algorithms and Computation, 14th International Symposium, ISAAC 2003*, Kyoto, Dec. 2003 (*Lecture Notes in Computer Science* 2906), Springer, 2003, pp. 26–35.
[3] T. Nishida and K. Sugihara: Boat-sail Voronoi diagram on a curved surface. *Japan Joural of Industrial and Applied Mathematics*, Vol. 22, No. 2 (2005), pp. 267–278.
[4] T. Nishida and K. Sugihara: Boat-sail Voronoi diagram and its computation based on a cone-approximation scheme. *Japan Joural of Industrial and Applied Mathematics*, Vol. 22, No. 3 (2005), pp. 367–383.
[5] T. Nishida, K. Sugihara and M. Kimura: Stable marker-particle method for a Voronoi diagram in a flow field. *Journal of Computational and Applied Mathematics*, Vol. 202 (2007), pp. 377–391.
[6] W. Rider and D. Kothe: A marker particle method for interface tracking. *Proceedings of 6th International Symposium on Comput. Fluid Dynamics*, 1995, pp. 976–981.

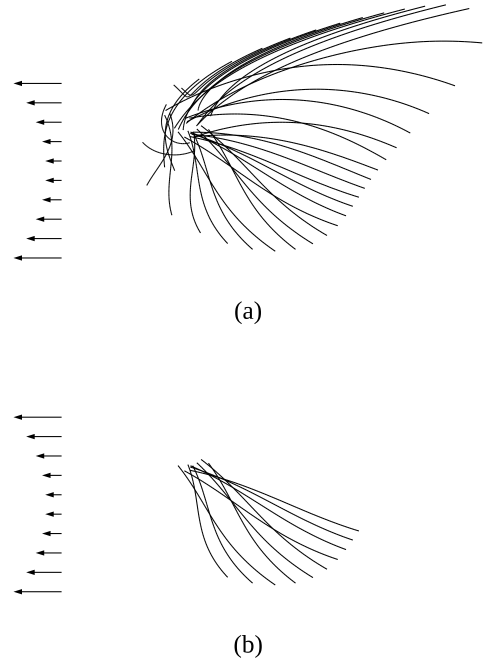


Fig. 8. Estimation of the source of the contamination using imprecise data.

[7] J. A. Sethian: Fast marching method. *SIAM Review*, Vol. 41 (1999), pp. 199–235.
[8] J. A. Sethian: *Level Set Methods and Fast Marching Methods, Second Edition*. Cambridge University Press, Cambridge, 1999.
[9] K. Sugihara: Voronoi diagrams in a river. *International Journal of Computational Geometry and Applications*, Vol. 2 (1992), pp. 29–48.