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Global Stabilization at Arbitrary Eigenstates of N-dimensional Quantum Spin Systems via Continuous Feedback *

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Abstract

In this paper, a stabilization problem of quantum spin systems in general dimension under continuous measurement is considered and it is shown that the global stabilization at arbitrary eigenstates by continuous inputs is possible. Quantum states of spin systems under continuous measurement by mutual interference with laser beams can be estimated by quantum filtering and with its information, the intension of magnetic field, which is applied to atoms, can be controlled. Our proposing control input is the sum of two terms: a term which attracts the quantum states to a target and the other term which draws apart from the other equilibrium points.

1 Introduction

Quantum feedback control is indispensable for the realizations of many quantum technologies and the theory for it has been constructed. Belavkin [1] and others [21] showed that the time evolution of estimated quantum states under continuous measurement can be described by a classical stochastic differential equation in the early 1990's. After that, research on feedback control by using estimated quantum states has been actively investigated [20, 3, 22] and its effectiveness has been also demonstrated by actual experiments [5].

A recent notable result [18] is on feedback control of single spin 1/2 systems by using a continuous control rule. This result is important for a possibility of

*This paper is the technical report version of the conference paper [17] feedback control, however it is also limited with respect to its applications from the point of the generalization of the dimension.

This limitation has been solved recently by Mirrahimi & van Handel [10]. They proposed a switching control for a group of atoms to globally stabilize the angular moments at arbitrary eigenstates. The proof is done by the strict analysis on the sample paths of the quantum state. This is the first result to show the global stability for quantum spin systems in general dimensions.

With this result, our interest naturally moves to a question on the global stabilizability of the quantum system by continuous feedback. This problem is important from the viewpoint of realizability of apparatus or pure physics and mathematics, and it is the main subject of this paper.

Recently, Tsumura [15, 16] proved that global stabilizability is possible by a continuous control law. This control scheme was firstly considered in [8] and its effectiveness was demonstrated by numerical examples. However, the target state is limited to the maximum energy eigenstate. This paper solves this difficulty and proves that the global stability at arbitrary eigenstates is possible for N-dimensional quantum spin systems.

This paper is organized as follows. In section 2, we introduce the problem setting and some preliminaries. In section 3 we give the main result of this paper and its proof. The method for the proof is similar to that of Mirrahimi & van Handel [10] and Tsumura [15, 16]. In section 4, we show some examples in order to demonstrate the efficiency of our proposing control rule and in section 5, we conclude this paper.

2 Formulation

In this paper, we deal with the system in Fig. 1 [18, 19, 10] with *continuous measurement*.

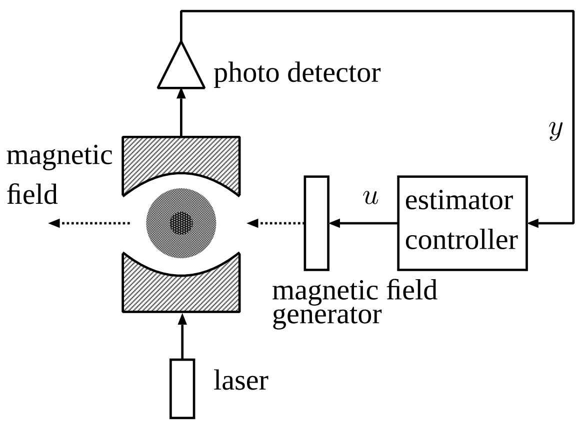

Fig.1: Quantum spin system under continuous measurement

A group of atoms is held in a cavity. When the number of atoms is n, the dimension of the quantum state on the angular moment is $N = 2J + 1$ where $J = \frac{1}{2}n$ is the absolute value of the moment. The mutual interaction between the laser beam and the atoms is observed by a photo detector where the intensity of the interference laser beam brings the information on the angular moment of the atoms. The observation of this indirect information causes a back action on the quantum state of the atoms.

Magnetic field is also applied to the group of the atoms and its intension is controlled. By using the history of the indirectly observed information, the conditional expectation of the observable can be calculated [1]. This is called *quantum filtering* and the time evolution of the estimated quantum state becomes a quantum version of a classical Kushner-Stratonovich equation [1, 2, 18].

When we observe the angular moment on z-axis and apply the magnetic field along y-axis, the corresponding nonlinear Itô stochastic differential equation is:

$$\begin{aligned} d\rho_t =& -iu_t[F_y, \rho_t]dt - \frac{1}{2}[F_z, [F_z, \rho_t]]dt \\ &+ \sqrt{\eta}(F_z\rho_t + \rho_t F_z - 2\mathrm{tr}\,(F_z\rho_t)\rho_t)dW_t, \quad (1) \\ dy =& 2\sqrt{\eta}\mathrm{tr}\,(F_z\rho)dt + dW_t \quad (2) \end{aligned}$$

where

$$\begin{aligned} \mathcal{S} &: \{\rho \in \mathcal{C}^{N\times N} : \rho = \rho^*,\ \mathrm{tr}\,(\rho) = 1,\ \rho \geq 0\} \\ \rho_t &: \rho_t \in \mathcal{S}, \text{a quantum state at time } t, \\ dW_t &: \text{an infinitesimal Wiener increment satisfying} \\ &\quad \mathrm{E}[(dW_t)^2] = dt,\ \mathrm{E}[dW_t] = 0, \\ u_t &: \text{control input } (u_t \in \mathcal{R}), \\ y_t &: \text{output } (y_t \in \mathcal{R}), \\ \eta &: \text{the detector efficiency } (0 < \eta \leq 1), \end{aligned}$$

F_y: the angular momentum along the axis y of the form [9]

$$\begin{aligned} F_y &= \frac{1}{2i}\begin{bmatrix} 0 & -c_1 & & & \\ c_1 & 0 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{2J-1} & 0 & -c_{2J} \\ & & & c_{2J} & 0 \end{bmatrix}, \\ c_m &= \sqrt{(2J+1-m)m}, \quad (3) \end{aligned}$$

F_z: the angular momentum along the axis z of the form [9]

$$F_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix}. \tag{4}$$

This is called SME (*stochastic master equation*) and it has been mainly investigated in the research field of quantum control. It should be noted that the solution of (1) is continuous in time [12] if u_t is continuous. We also define some notations:

$$\begin{aligned} \psi_i &:= [0\cdots 0 \underbrace{1}_{i-\text{th}} 0\cdots 0]^*, & (5)\\ \rho_{\psi_i} &:= \psi_i\psi_i^*, & (6)\\ V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) &:= 1-\mathrm{tr}\,(\rho\rho_{\mathrm{f}}), & (7)\\ V^{\mathrm{II}}_{\rho_{\mathrm{f}}}(\rho) &:= 1-(\mathrm{tr}\,(\rho\rho_{\mathrm{f}}))^2, & (8)\\ V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho) &:= \lambda_i-\mathrm{tr}\,(F_z\rho), & (9)\\ \lambda_i &:= J-(i-1), & (10) \end{aligned}$$

where $\rho_{\mathrm{f}} \in \mathcal{S}$ is one of eigenstates. The control objective is to globally stabilize the quantum state ρ_t on some desired state ρ_{f} by controlling the intensity u_t of the magnetic field, which is decided by ρ_t or its record. Note that $0 \le V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) \le 1$ ($0 \le V^{\mathrm{II}}_{\rho_{\mathrm{f}}}(\rho) \le 1$), and $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) = 0$ ($V^{\mathrm{II}}_{\rho_{\mathrm{f}}}(\rho) = 0$) iff $\rho = \rho_{\mathrm{f}}$. Moreover, for $\epsilon > 0$, define

$$\begin{aligned} \mathcal{S}^{<\epsilon}_{\rho_{\mathrm{f}}} &:= \left\{\rho \,|\, 0 \le V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) < \epsilon\right\}, & (11)\\ \mathcal{S}^{\epsilon}_{\rho_{\mathrm{f}}} &:= \left\{\rho \,|\, V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) = \epsilon\right\}, & (12)\\ \mathcal{S}^{\epsilon<}_{\rho_{\mathrm{f}}} &:= \left\{\rho \,|\, \epsilon < V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho)\right\}. & (13) \end{aligned}$$

We define the stochastic stability of (1) as follows.

Definition 2.1 *[7] Let ρ_{e} be an equilibrium point of* (1)*, i.e. $d\rho_t|_{\rho_t=\rho_{\mathrm{e}}} = 0$. Then*

1. *the equilibrium ρ_{e} is said to be stable in probability if*

$$\forall\epsilon > 0 \quad \lim_{\rho_0\to\rho_{\mathrm{e}}} \Pr\left(\sup_{0<t<\infty} \|\rho_t - \rho_{\mathrm{e}}\| \ge \epsilon\right) = 0, \tag{14}$$

   where $\|\cdot\|$ is an arbitrary norm of a matrix in $\mathcal{C}^{N\times N}$.

2. *The equilibrium ρ_{e} is globally stable if it is stable in probability and additionally*

$$\forall\rho_0 \in \mathcal{S} \quad \Pr\left(\lim_{t\to\infty} \rho_t = \rho_{\mathrm{e}}\right) = 1. \tag{15}$$

For showing the stochastic stabilities, a stochastic version of the Lyapunov theorem is available. At first define a nonnegative real-valued continuous function $V(\cdot)$ on $\mathcal{S}$. Also define $\rho_t^z := \rho_t$ such that $\rho_0 = z$, a level set Q_ϵ such that $Q_\epsilon := \{\rho \in \mathcal{S} : V(\rho) < \epsilon\}$, $\tau_\epsilon := \inf\{t : \rho_t^z \notin Q_\epsilon\}$ and $\tilde{\rho}_t^z = \rho_{t\wedge\tau_\epsilon}^z$, $t \wedge \tau_\epsilon = \min(t, \tau_\epsilon)$, $\mathcal{L}$: infinitesimal operator, $\mathcal{L}_\epsilon$: restriction of $\mathcal{L}$ on $\tilde{\rho}_t$. Then, we get the following propositions.

Proposition 2.1 *[7] Let $\mathcal{L}_\epsilon V \le 0$ in Q_ϵ. Then, the followings hold:*

1. $\lim_{t\to\infty} V(\tilde{\rho}_t^z)$ *exists a.s., so $V(\rho_t^z)$ converges for a.e. path remaining in Q_ϵ.*
2. $\mathrm{Pr}\text{–}\lim_{t\to\infty} \mathcal{L}_\epsilon V(\tilde{\rho}_t^z) = 0$*, so $\mathcal{L}_\epsilon V(\rho_t^z) \to 0$ in probability as $t \to \infty$ for almost all paths which never leave Q_ϵ.*
3. *For $z \in Q_\epsilon$ and $\alpha \le \epsilon$ we have the uniform estimate*
$$\begin{aligned} \mathrm{Pr}(\sup_{0\le t<\infty} V(\rho_t^z) \ge \alpha) &= \mathrm{Pr}(\sup_{0\le t<\infty} V(\tilde{\rho}_t^z) \ge \alpha) \\ &\le \frac{V(z)}{\alpha}. \end{aligned} \tag{16}$$
4. *If $V(\tilde{z}) = 0$ and $V(\rho) \ne 0$ for $\rho \ne \tilde{z}$, then $\tilde{z}$ ($\tilde{z} \in Q_\epsilon$) is stable in probability.*

Definition 2.2 *An invariant set C is defined as a set with the property that if system's initial state is in C then its whole path (forward and backward) lies in C.*

Proposition 2.2 *[10] Assume the followings:*

1. *Q_ϵ is bounded and that $\mathcal{L}_\epsilon V(\rho) \le 0$ within Q_ϵ.*
2. *For any bounded scalar continuous function $g(\rho)$ and a fixed t, $\mathrm{E}[g(\rho_t^z)]$ is continuous on $z = \rho_0$.*
3. *For any positive real number κ and $z \in Q_\epsilon$, $\mathrm{Pr}(\|\rho - z\| > \kappa) \to 0$, $t \to 0$.*

Let R be the set of all points within Q_ϵ where $\mathcal{L}_\epsilon V(\rho) = 0$, and let M be the largest invariant set in R. Then, every solution ρ_t in Q_ϵ tends to M as $t \to \infty$

Here we consider the control problem:

Problem 2.1 *For the controlled spin system (1), find a globally stabilizing controller u_t on an eigenstate $\rho_\mathrm{f} = \rho_{\psi_i}$.*

This is not a trivial problem from the following reasons: 1) (1) is a nonlinear stochastic system, 2) there exist plural locally stable equilibrium points when $u = 0$ because of the nonlinearity, 3) because of a kind of symmetry of the dynamics, many of locally stabilizing control scheme on one of above equilibrium points also preserve the other equilibrium points.

Mirrahimi & van Handel found a globally stabilizing control scheme on target eigenstates by introducing a switching rule in order to solve above difficulties [10]. Tsumura showed global stabilizability by using continuous control signal [15, 16], however, the target state is limited to the maximum energy eigenstate:

Proposition 2.3 *[15, 16] Consider the system* (1) *evolving in the set* $\mathcal{S}$*. Let* $\rho_{\rm f} = \rho_{\psi_1}$ *and* $\eta > 0$*. Then,*

$$\begin{aligned} u_t &= \alpha u_1(\rho_t) + \beta V^{\rm I}_{\rho_{\rm f}}(\rho_t) \\ &\quad \alpha,\ \beta > 0 \end{aligned} \tag{17}$$

globally stabilizes (1) *around* $\rho_{\rm f}$ *and* $\mathrm{E}[\rho_t] \to \rho_{\rm f}$ *as* $t \to \infty$ *when*

$$\frac{\beta^2}{8\alpha\eta} < 1. \tag{18}$$

The main purpose of this paper is to remove the limitation of the target state.

3 Main Result

In this section, we show the stabilizability of (1) by a continuous feedback and provide the strict proof for the global stabilizability at arbitrary eigenstates $\rho_{\rm f} = \rho_{\psi_i}$. We get the following theorem:

Theorem 3.1 *Consider the system* (1) *evolving in the set* $\mathcal{S}$*. Let* $\rho_{\rm f} = \rho_{\psi_i}$ *and* $\eta > 0$*. Then,*

$$\begin{aligned} u_t &= \alpha u_1(\rho_t) + \beta V^{\rm III}_{\rho_{\rm f}}(\rho_t) \\ &\quad \alpha,\ \beta > 0 \end{aligned} \tag{19}$$

globally stabilizes (1) *around* $\rho_{\rm f}$ *and* $\mathrm{E}[\rho_t] \to \rho_{\rm f}$ *as* $t \to \infty$ *when*

$$\frac{\beta^2}{8\alpha\eta} < 1. \tag{20}$$

Remark 3.1 This is the first result to show the global stabilizability of general finite dimensional quantum systems at arbitrary eigenstates by continuous feedback for the type of the master equation (1). Note that α and β are design parameters and we can always find them satisfying the condition (20) if $\eta > 0$.

We prove Theorem 3.1 in the followings. The procedure of the proof is similar to that in [10, 15, 16] and it is composed of the following three parts:

Step 1) $\rho_{\rm f} = \rho_{\psi_i}$ is stable in probability.

Step 2) there exists $0 < \gamma < 1$ and almost all sample paths which never leave the domain $\mathcal{S}^{<1-\gamma}_{\rho_{\rm f}}$ converge to $\rho_{\rm f}$.

Step 3) for almost all sample paths there exists a finite time T and after it, they never leave $\mathcal{S}^{<1-\gamma}_{\rho_{\rm f}}$.

Step 1)

In order to show the statement of the Step 1), we should find some Lyapunov function which satisfies the conditions of Proposition 2.1 around $\rho_{\rm f}$. We get a key lemma for it.

Lemma 3.1 *With the control input (19),*

$$\mathcal{L}_\epsilon V^{\rm II}_{\rho_{\rm f}} \quad \leq \quad 0 \tag{21}$$

is satisfied in the subsets $\mathcal{S}^{<1-\gamma_0}_{\rho_{\rm f}}$, where

$$\gamma_0 = \frac{\beta^2}{8\alpha\eta} < 1. \tag{22}$$

Proof By the direct calculation of $\mathcal{L}V^{\rm II}_{\rho_{\rm f}}$, we get the following:

$$\begin{aligned}
\mathcal{L}V^{\rm II}_{\rho_{\rm f}} &= -2{\rm tr}\,(\rho_t\rho_{\rm f})\, u_t\, {\rm tr}\,(-i[F_y,\rho_t]\rho_{\rm f}) \\
&\quad - 4\eta(\lambda_i - {\rm tr}\,(F_z\rho_t))^2({\rm tr}\,(\rho_t\rho_{\rm f}))^2 \\
&= -2{\rm tr}\,(\rho_t\rho_{\rm f})(\alpha u_1 + \beta V^{\rm III}_{\rho_{\rm f}})u_1 \\
&\quad - 4\eta(\lambda_i - {\rm tr}\,(F_z\rho_t))^2({\rm tr}\,(\rho_t\rho_{\rm f}))^2 \\
&= -2{\rm tr}\,(\rho_t\rho_{\rm f})\left\{(\alpha u_1 + \beta V^{\rm III}_{\rho_{\rm f}})u_1\right. \\
&\quad \left. + 2\eta(\lambda_i - {\rm tr}\,(F_z\rho_t))^2{\rm tr}\,(\rho_t\rho_{\rm f})\right\}.
\end{aligned} \tag{23}$$

The factor ${\rm tr}\,(\rho_t\rho_{\rm f})$ outside the curly brackets is always nonnegative, therefore, the factor in the brackets should be nonnegative for $\mathcal{L}V^{\rm II}_{\rho_{\rm f}}$ to be nonpositive. The factor is

$$\begin{aligned}
&\left(\alpha u_1 + \beta V^{\rm III}_{\rho_{\rm f}}\right) u_1 + 2\eta(\lambda_i - {\rm tr}\,(F_z\rho_t))^2{\rm tr}\,(\rho\rho_{\rm f}) \\
=\;& \alpha\left(u_1 + \frac{\beta}{\alpha}\frac{V^{\rm III}_{\rho_{\rm f}}}{2}\right)^2 - \frac{\beta^2}{\alpha}\frac{(V^{\rm III}_{\rho_{\rm f}})^2}{4} \\
&+ 2\eta{\rm tr}\,(\rho\rho_{\rm f})(V^{\rm III}_{\rho_{\rm f}})^2 \\
=\;& \alpha\left(u_1 + \frac{\beta}{\alpha}\frac{V^{\rm III}_{\rho_{\rm f}}}{2}\right)^2 \\
&+ 2\eta(V^{\rm III}_{\rho_{\rm f}})^2\left({\rm tr}\,(\rho\rho_{\rm f}) - \frac{\beta^2}{8\alpha\eta}\right).
\end{aligned} \tag{24}$$

Therefore, when

$$\frac{\beta^2}{8\alpha\eta} < 1 \tag{25}$$

is satisfied, we can set

$$\gamma_0 := \frac{\beta^2}{8\alpha\eta} \tag{26}$$

and for the case:

$$\gamma_0 < \rho_{ii} \leq 1, \tag{27}$$

we conclude $\mathcal{L}V^{\mathrm{II}}_{\rho_{\mathrm{f}}} \leq 0$. This means with the setting

$$\gamma = \gamma_0,\ \mathcal{S}^{<1-\gamma}_{\rho_{\mathrm{f}}} = \mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}, \tag{28}$$

$\mathcal{L}_{1-\gamma_0}V^{\mathrm{II}}_{\rho_{\mathrm{f}}} \leq 0$ for any $\rho \in \mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$.

□

With Lemma 3.1 and Proposition 2.1, $\rho = \rho_{\mathrm{f}}$ is stable in probability and in the subset $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$ around ρ_{f}, the statements in Proposition 2.1 are concluded.

Step 2)

From Lemma 3.1 and Lemma 3.4, the master equation (1) with the control input (19) satisfies the conditions in Proposition 2.2, therefore, the sample paths which never leave the subset $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$ converge to ρ_{f} in probability. Moreover, $V^{\mathrm{II}}_{\rho_{\mathrm{f}}}$ converges almost surely from Proposition 2.1. With this and Lebesgue's dominated convergence, we can show that almost all paths converge to ρ_{f} by employing the similar discussion in [10]:

Lemma 3.2 *ρ_t converges to ρ_{f} as $t \to \infty$ for almost all paths that never exit the set $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$.*

We omit the proof here.

Step 3)

We next show the behavior of the paths when they leave $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$ or the initial state is outside it. We get the following key lemma:

Lemma 3.3 *The solution ρ_t of (1) where $\rho(0) \in \mathcal{S}^{>1-\gamma_0}_{\rho_{\mathrm{f}}}$ satisfies*

$$\sup_{\rho_0 \in \mathcal{S}^{>1-\gamma_0}_{\rho_{\mathrm{f}}}} \mathrm{E}[\min\ t : \rho_t \notin \mathcal{S}^{>1-\gamma_0}_{\rho_{\mathrm{f}}}] < \infty. \tag{29}$$

At first introduce propositions which are used for the proof of Lemma 3.3.

Proposition 3.1 *[14, 6] Consider a Stratonovich's stochastic differential equation:*

$$d\varphi_t = f_0(\varphi_t, t)dt + \sum_{l=1}^{n} f_l(\varphi_t, t) \circ dW^l(t). \tag{30}$$

Assume that the coefficients $f_l(x,t)$, $l = 0, 1, 2, \ldots, n$ are of the class $C^{k+1,\delta}_b$ for some $k \geq 2$ and $\delta > 0$ (see Appendix for the definition of $C^{k+1,\delta}_b$). Let φ_t be the Brownian flow determined by (30). Then the support of $\varphi(t) = \varphi_t$ as the C^{k-1}-flow is equal to the closure $\{\varphi_t : \xi \in \Xi\}$ of

$$\frac{d\varphi_t}{dt} = f_0(\varphi_t, t) + \sum_{l=1}^{n} f_l(\varphi_t, t)\xi^l(t) \tag{31}$$

in the space W_{k-1}, where Ξ is the set of all deterministic piecewise smooth function and $W_k = C([0,T] : C^k)$.

Proposition 3.2 *[4] Consider diffusion process $x_t \in E$ starting from x where E is the domain of x_t. Let Γ be a subset of E and $\tau_x(\Gamma)$ be the first exit time of x_t from Γ. Then for all $T \geq 0, x \in E$,*

$$\mathrm{E}[\tau_x(\Gamma)] \leq \frac{T}{1 - \sup_{x\in E} \Pr\{\tau_x(\Gamma) > T\}}. \tag{32}$$

Proof of Lemma 3.3 At first, we claim that the support of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_t)$ contains $[0,\gamma]$ when $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_0) = \gamma$ by using Proposition 3.1.

The Stratonovich form of (1) is given as [13]

$$\begin{aligned} d\rho_t &= \mathcal{D}_{F_z}(\rho_t)dt \\ &\quad - \frac{1}{2}\eta\left(-2\mathcal{E}_{F_z}(\rho_t)\mathcal{H}_{F_z}(\rho_t) + \mathcal{K}_{F_z}(\rho_t)\right)dt \\ &\quad + u_t\mathcal{G}_{F_y}(\rho_t)dt + \sqrt{\eta}\mathcal{H}_{F_z}(\rho_t)\circ dW \end{aligned} \tag{33}$$

where

$$\begin{aligned} \mathcal{D}_{F_z}(\rho) &= -\frac{1}{2}[F_z,[F_z,\rho]] \\ \mathcal{E}_{F_z}(\rho) &= 2\mathrm{tr}\,(F_z\rho) \\ \mathcal{H}_{F_z}(\rho) &= F_z\rho + \rho F_z - 2\mathrm{tr}\,(F_z\rho)\rho \\ \mathcal{K}_{F_z}(\rho) &= F_z^2\rho + 2F_z\rho F_z^* + \rho(F_z^*)^2 \\ &\quad - \mathrm{tr}\,(F_z^2\rho + 2F_z\rho F_z^* + \rho(F_z^*)^2)\rho \\ \mathcal{G}_{F_y}(\rho) &= -i[F_y,\rho] \end{aligned} \tag{34}$$

and the corresponding deterministic differential equation is

$$\begin{aligned} \frac{d}{dt}\rho_t &= \mathcal{D}_{F_z}(\rho_t) \\ &\quad - \frac{1}{2}\eta\left(-2\mathcal{E}_{F_z}(\rho_t)\mathcal{H}_{F_z}(\rho_t) + \mathcal{K}_{F_z}(\rho_t)\right) \\ &\quad + u\mathcal{G}_{F_y}(\rho_t) + \sqrt{\eta}\mathcal{H}_{F_z}(\rho_t)\xi \end{aligned} \tag{35}$$

where ξ is an associated input. With this solution, we get

$$\begin{aligned} &\frac{d}{dt}V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) \\ = &\ -\mathrm{tr}\left(\frac{d\rho}{dt}\rho_{\mathrm{f}}\right) \\ = &\ -\mathrm{tr}\left(\left\{-\frac{1}{2}\eta(-2\mathcal{E}_{F_z}(\rho)\mathcal{H}_{F_z}(\rho) + \mathcal{K}_{F_z}(\rho))\right.\right. \\ &\quad \left.\left. + u\mathcal{G}_{F_y}(\rho) + \sqrt{\eta}\mathcal{H}_{F_z}(\rho)\xi\right\}\rho_{\mathrm{f}}\right). \end{aligned} \tag{36}$$

The term which includes ξ in (36) is

$$\begin{aligned}&\mathrm{tr}\,(\mathcal{H}_{F_z}(\rho)\xi\rho_{\mathrm{f}})\\ &\quad= \mathrm{tr}\,((F_z\rho+\rho F_z-2\mathrm{tr}\,(F_z\rho)\rho)\rho_{\mathrm{f}})\xi\\ &\quad= 2\,(\lambda_i-\mathrm{tr}\,(F_z\rho))\,\mathrm{tr}\,(\rho\rho_{\mathrm{f}})\xi\\ &\quad= 2V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho)\rho_{ii}\xi.\end{aligned} \tag{37}$$

The case (37) = 0 for $\xi \neq 0$ is when $\rho_{ii}=0$, $\rho_{ii}=1$ (i.e., $\rho=\rho_{\mathrm{f}}$), or $V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho)=0$.

When $\rho_{ii}=0$ and $V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho)\neq 0$, $u=V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho)$ and from [10], it is known that $\{\rho\,|\,V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho)=1\}$ is not an invariant set with $u_t\neq 0$. Next, it is known that $\{\rho\,|\,V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho)=0,\rho_{ii}\neq 1\}$ is not an invariant set of (1) from Lemma 3.4. Finally, when $\rho_{ii}=1$ ($V^{\mathrm{II}}_{\rho_{\mathrm{f}}}(\rho)=0$), $\rho=\rho_{\mathrm{f}}$ and it is the target point. In the other cases, (37) except for ξ is nonzero.

From above and Proposition 3.1, the assertion that the support of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_t)$ contains $[0,\gamma]$ when $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_0)=\gamma$ is proved. Therefore, we can get $\min_{t\in[0,T]}\mathrm{E}[V^{\mathrm{I}}_{\rho_{\mathrm{f}}}]<1-\gamma_0$ and finally with Proposition 3.2, we can conclude the statement [10]. □

Lemma 3.4 $\{\rho\,|\,V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho)=0,\rho_{ii}\neq 1\}$ *is not an invariant set of (1).*

Proof

$$\begin{aligned}&\frac{d}{dt}V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho)\\ &= -\mathrm{tr}\left(\frac{d\rho}{dt}F_z\right)\\ &= -\mathrm{tr}\left(\left\{-\frac{1}{2}\eta(-2\mathcal{E}_{F_z}(\rho)\mathcal{H}_{F_z}(\rho)+\mathcal{K}_{F_z}(\rho))\right.\right.\\ &\qquad \left.\left.+u\mathcal{G}_{F_y}(\rho)+\sqrt{\eta}\mathcal{H}_{F_z}(\rho)\xi\right\}F_z\right).\end{aligned} \tag{38}$$

The term which includes ξ in (38) is

$$\begin{aligned}&\mathrm{tr}\,(\mathcal{H}_{F_z}(\rho)\xi F_z)\\ &\quad= \mathrm{tr}\,((F_z\rho+\rho F_z-2\mathrm{tr}\,(F_z\rho)\rho)F_z)\xi\\ &\quad= 2\left(\mathrm{tr}\,F_z^2\rho-(\mathrm{tr}\,F_z\rho)^2\right)\xi.\end{aligned} \tag{39}$$

The case (39) = 0 for $\xi\neq 0$ is only when $\rho=\rho_{\psi_i}=\rho_{\mathrm{f}}$, however, when $\rho=\rho_{\psi_j}\neq\rho_{\mathrm{f}}$, it is known not to be an invariant set of (1) [10]. With the similar discussion in the proof of Lemma 3.3, we can conclude the statement of this lemma. □

By using Lemma 3.3 and employing the similar discussion of [10], we can derive the following lemma.

Lemma 3.5 *For almost every sample path of ρ_t there exists a time $T<\infty$ after which the path never exits the set $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$.*

We omit the proof.

Proof of Theorem 3.1 By unifying the results of Step 1) $\sim$ 3), we can conclude the convergence of the solution to the target point. The convergence of the expectation can be also derived by dominated convergence. $\square$

4 Numerical Example

We demonstrate the efficiency of the proposing continuous feedback by using numerical simulations. Here we consider spin systems where $N = 4$. The initial and the target states are

$$\rho_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \rho_{\mathrm{f}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

respectively. We simulate the solution ρ_t with Case 1) $\eta = 0.8$, $\alpha = 1$, $\beta = 1$ and Case 2) $\eta = 0.1$, $\alpha = 1$, $\beta = 1$, 10 times respectively. The former case satisfies the condition (20), on the other hand, the latter does not satisfy it. Fig. 2 and Fig. 3 show the average of the transitions of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}$, which indicates the gap between the target ρ_{f} and ρ_t, with the above two cases respectively.

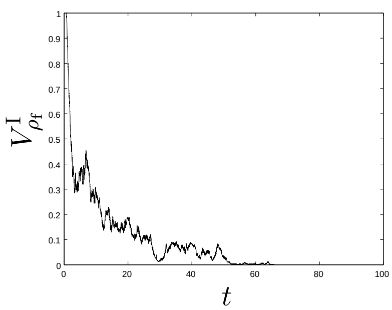

Fig.2: Average of transitions of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}$ with $\eta = 0.8$, $\alpha = 1$, $\beta = 1$

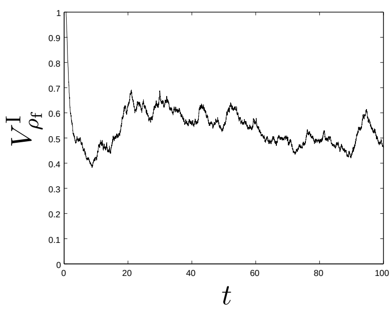

Fig.3: Average of transitions of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}$ with $\eta = 0.1$, $\alpha = 1$, $\beta = 1$

From the simulations, we can confirm the efficiency of our proposing continuous feedback. Note that (20) is a sufficient condition for the global stability, therefore, even if it is not satisfied, the system may be stable. However, we can see the significance of the condition (20) from these simulations.

5 Conclusion

In this paper, we considered control problem of N-dimensional quantum spin systems and showed that continuous feedback is possible to stochastically globally stabilize the systems on arbitrary eigenstates. The control scheme is composed of two distinctive terms and the stability is proved by following the sample paths of the stochastic master equation strictly.

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A Appendix

The notation $C_b^{m,\delta}$ is the set $\{f \in C^{k+1}, D^\alpha f$ $(|\alpha| = m)$: δ-Hölder continuous, $\|f\|_{m+\delta} < \infty$ $\}$ and

$$\|f\|_{m+\delta} := \|f\|_m + \sum_{|\alpha|=m} \sup \frac{D^\alpha f(x) - D^\alpha f(y)}{|x-y|^\delta}. \tag{40}$$

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