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Evaluation of the Precision for Exact Computation of a Circle Voronoi Diagram

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Abstract

An exact computation scheme for incremental construction of a circle Voronoi diagram is represented. The most important test for the incremental addition of a circle is to decide whether or not the new circle intersects the inscribing circle determined by three circles associated with a Voronoi vertex. For this purpose, the decision procedure is introduced using the sign of three polynomial functions. The proposed procedure requires about eight times higher precision than the input precision for the exact decision of the topological structure of a circle Voronoi diagram.

Keywords: circle Voronoi diagram, incremental algorithm, inscribing circle, circumscribing circle, exact computation

1. Introduction

The concept of the Voronoi diagrams is one of the most fundamental geometric tools and has a wide variety of application [1][3][11][14]. Hence, the stable construction of these diagrams is important from a practical point of view. This paper concentrates on the Voronoi diagram for circles in the two-dimensional space, and proposes a robust method for this diagram.

When we implement a geometric algorithm, we have to be careful of degeneracies and numerical errors. While many algorithms assume that neither degeneracy nor numerical error exists, both of them can happen in real computation. Actually if we ignore the robustness in the implementation of algorithms, the resulting computer programs easily break down due to inconsistency generated by numerical errors.

To overcome this difficulty, many methods have been proposed since late 1980's. They can be classified into three groups according to how much they rely on numerical values.

The first group is the exact computation approach, in which computation is carried out in high precision in such a way that the topological judgments are always done correctly [9][15][22]. This approach is usually accompanied with the symbolic perturbation [2][21] to avoid complicated branches of processing for degenerate cases and the floating-point acceleration [6] to decrease the cost of computation.

The second group is the moderate use of numerical values, in which every numerical computation is accompanied with error estimation and reliable numerical results only are used. Examples of this approach include epsilon geometry [5] and a hidden-variable method [10].

The third group is the topology-oriented approach [12][13][16], where higher priority is placed on topological consistency than on numerical values. Program codes generated by this approach run fast because floating-point arithmetic is used, and are simple because degeneracy need not be considered. However, it is not a beginners' method because we need some insight to individual problems in order to extract topological invariances.

Among these three groups, the exact computation approach seems most prevailing [9], because it can be applied to many problems in a unified manner. However, we still need careful consideration when we apply this approach to individual problems because naive application often requires very high precision in computation.

There are many variants of the Voronoi diagrams, and the difficulty of their computation mainly depends on what kind of boundaries appear in the diagram. The easiest class of the Voronoi diagram includes the Voronoi diagram for points and the Laguerre Voronoi diagram, where the boundaries of Voronoi regions consist of linear elements such as lines and planes. To these Voronoi diagrams, the application of the exact computation is rather straightforward. This is because the necessary precision can be estimated in a simple manner.

A circle Voronoi diagram in the two-dimensional space is a generalization of the Voronoi diagram where the generating points are replaced by circles. This is also one of the most basic Voronoi diagrams with practical applications [18]. From a robust computation point of view, this diagram is a little difficult to construct, because the boundaries contain curved lines. D.-S. Kim et al. [7][8] proposed a stable algorithm, in which they start with the point set Voronoi diagram and modify it by edge flip operations. This method is robust because it uses a robust method for constructing the point Voronoi diagram, and the flip is a topological operation and hence is not affected by numerical errors.

However, the construction via the point Voronoi diagram is a detour in a sense, and it might be desired to directly construct the circle Voronoi diagram. Gavrilova and Rokne [4] proposed the direct method in solving the problem of dynamic maintenance of a Voronoi diagram for a set of spheres moving independently. However, the proposed method has to obtain all the roots of fourth order polynomial in two dimension, and needs high cost to solve it. Moreover it is vulnerable to the numerical error. Our aim in this paper is to give an efficient and robust procedure to the direct construction of the circle Voronoi diagram, using exact computation scheme.

This paper is organized as follows. In Section 2, we review a circle Voronoi diagram and describe the outline of the incremental method for constructing it. In Section 3, we introduce some notations, definitions and the useful theorem for making the test function which checks topological information on the incremental addition of a circle. In incrementally adding the circle, the configuration of circles is very important, and hence we classify the configuration of circles into three cases. Section 4 introduces the effective and ingenious test functions to check topological information. In Section 5, we propose an elaborate algorithm to solve our problem. In Section 6 we estimate the amount of precision enough to compute one of these procedures exactly. Lastly, we give conclusion and future work in Section 7.

2. Circle Voronoi diagram

A circle Voronoi diagram is a Voronoi diagram whose generators are circles [11]. It is the division of the space from the viewpoint of the nearest circle according to some distance metric. Two most important metrics are the Euclidean metric and the Laguerre metric. Boundaries of the Voronoi diagram associated with the Laguerre metric consist of some line segments, while boundaries of the Voronoi diagram associated with the Euclidean metric comprise some curve segments, as shown in Figure 1. In this paper, we focus on the Euclidean metric.

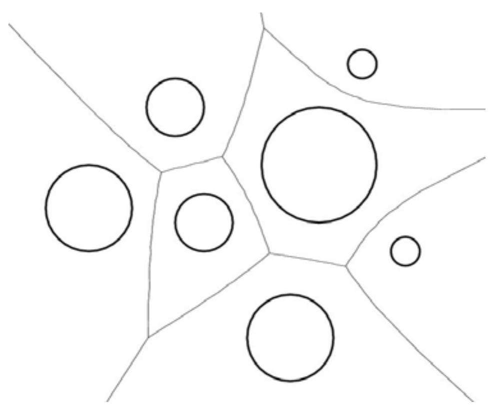


Figure 1. An example of the circle Voronoi diagram.

The incremental method is well-known for constructing the point Voronoi diagram, along with the divide-and-conquer algorithm [20] and the swapping algorithm [7][8]. We make use of it for constructing a circle Voronoi diagram, and design a robust algorithm based on exact computation.

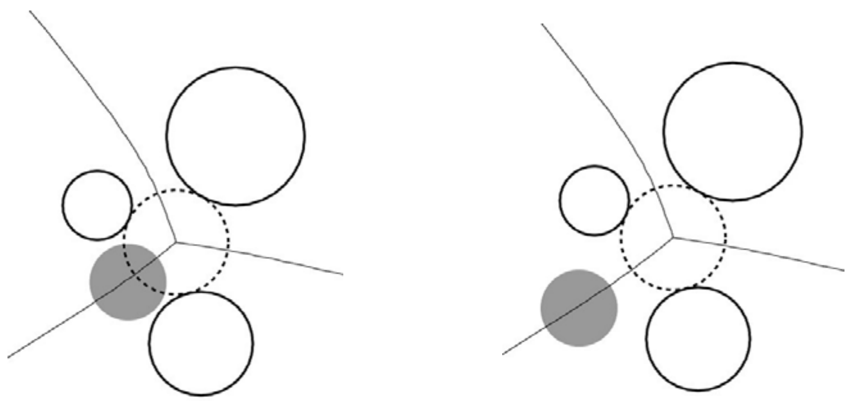
The incremental method of a point Voronoi diagram is the method that modifies topological information such as Voronoi vertices and Voronoi edges, every time a point is added. When we add the new generator point p_l, we have to check whether or not this new point is included in the common circle passing through three points $p_i.p_j$ and p_k associated with a Voronoi vertex. If the fourth point is included in the circle touching the three points, the existing Voronoi vertex vanishes and new Voronoi vertices are generated. For this purpose, we have to check the sign of the test function

$$\begin{vmatrix} 1 & x_i & y_i & x_i^2+y_i^2 \\ 1 & x_j & y_j & x_j^2+y_j^2 \\ 1 & x_k & y_k & x_k^2+y_k^2 \\ 1 & x_l & y_l & x_l^2+y_l^2 \end{vmatrix} \qquad (1)$$

for any other generator p_l. Here we will try the same approach to construct a circle Voronoi diagram.

We consider the inscribing circle, which corresponds to the common circle in the point Voronoi diagram, for the three circles associated with the Voronoi vertex. The detailed definition of the inscribing circles is described in Section 3. When we add a new circle, we have to check whether or not this new circle intersects the inscribing circle determined by the three circles associated with an existing Voronoi vertex. Figure 2 illustrates two typical situations of the forth circle, gray circle, with respect to the inscribing circle, represented by the circle with dashed line. If the fourth circle intersects the inscribing circle as shown in Figure 2(a), this Voronoi vertex vanishes; if the fourth circle does not intersect the inscribing circle as shown in Figure 2(b), this Voronoi vertex survives.

So our question is: which test functions can we use to check the topology of a circle Voronoi diagram?



(a) (b)

Figure 2. The illustration of whether the fourth circle, gray circle, intersects the inscribing circle, circle with dashed line: (a) the situation where the fourth circle intersects the inscribing circle; (b) the situation where it circle does not. Three circles with solid lines represent generators and the curves represent the Voronoi edges. The locations of all the objects except for the gray circle are the same.

3. Preliminaries

We give some notations and definitions: the circle Voronoi diagram, and the inscribing and circumscribing circles. In the incremental method, it is important to know the configuration of circles which construct the Voronoi vertices. In this section, we classify the characteristics of the configuration into three cases, using inscribing and circumscribing circles. We also introduce the theorem which gives our main test functions for the topological information of a circle Voronoi diagram.

3.1. Notations and definitions

Consider the two-dimensional Euclidean space. Let $\{p, r\}$ indicate the circle whose center is the point p, and whose radius is R, and $d(x, y)$ denote the Euclidean distance between a point x and a point y.

Definition 1. *A circle Voronoi diagram of n circles $P_i = \{p_i, r_i\}$ $(i = 1, \dots, n)$ is the set of Voronoi regions $V_i = \{p \in \boldsymbol{R}^2 \mid d(p, p_i) - r_i < d(p, p_j) - r_j,\ i \neq j\}$. P_i's are called the* generators *of this Voronoi diagram.*

A Voronoi vertex of a circle Voronoi diagram is the intersection of the closures of three or more Voronoi regions. If three circles $P_i = \{p_i, r_i\},\ i = 1, 2, 3$, determine a Voronoi vertex p_0, the equations $d(p_i, p_0) - r_i = d(p_j, p_0) - r_j,\ j = 1, 2, 3$, hold.

Definition 2. *A circle* $P_0 = \{p_0, r_0\}$ *is said to* inscribe *n circles* $P_i = \{p_i, r_i\}$ $(i = 1, \ldots, n)$ *if and only if the conditions* $d(p_0, p_i) = r_0 + r_i$ $(i = 1, \ldots, n)$ *hold. The circle* P_0 *is called* the inscribing circle.

A circle $P_s = \{p_s, r_s\}$ *is said to* circumscribe *n circles* $P_i = \{p_i, r_i\}$ $(i = 1, \ldots, n)$ *if and only if the conditions,* $d(p_s, p_i) = r_s - r_i$ $(i = 1, \ldots, n)$ *hold. The circle* P_s *is called* the circumscribing circle.

From this definition, a Voronoi vertex is the center of the inscribing circle.

In the incremental method, the inscribing circle determined by three circles which construct a Voronoi vertex is very important. For a set of three generators, three situations exist as shown in Figure 3:

Case (I): there are one inscribing circle and one circumscribing circle,

Case (II): there are two inscribing circles,

Case (III): there are two circumscribing circles.

In the case (III), a Voronoi vertex corresponding these three circles does not exist, and consequently we do not have to consider it for designing the algorithm. On the other hand, Voronoi vertices exist in the case (I) and (II), and we have to find the test functions for these cases.

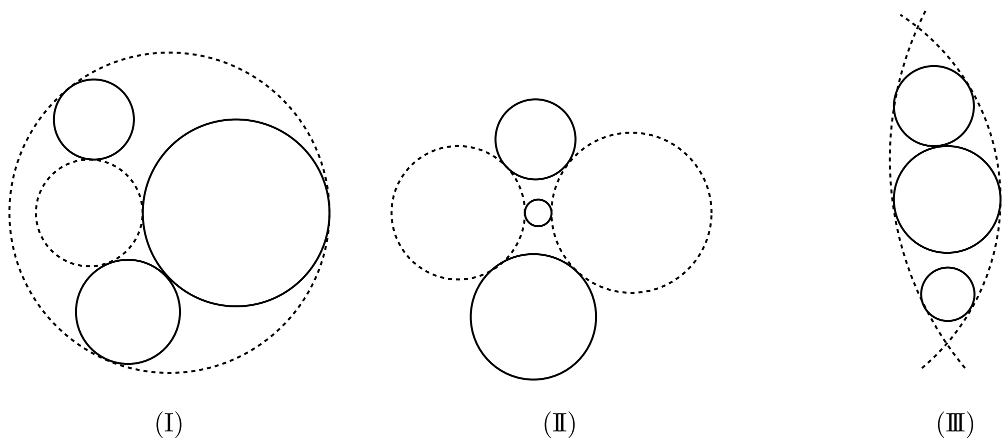


Figure 3. Three situations for a set of three generators: (I) there are one inscribing circle and one circumscribing circle; (II) there are two inscribing circles; (III) there are two circumscribing circles.

3.2. Generalization of Gavrilova and Rokne's theorem

We review the following theorem:

Theorem 1 (Gavrilova and Rokne [4]). *The time of the topological event in the Delaunay structure generated by* $d+2$ *moving spheres* $P_i = \{(x_{i1} = x_{i1}(t), \ldots, x_{id} = x_{id}(t)), r_i\}$*, in the Euclidean metric can be found as the minimum positive real root* t_0 *of the equation*

$$A_1^2 + A_2^2 + \cdots + A_d^2 = A_{d+1}^2, \tag{2}$$

where

$$A = \begin{vmatrix} x_{1,1} - x_{d+2,1} & \cdots & x_{1,d} - x_{d+2,d} & r_1 - r_{d+2} \\ x_{2,1} - x_{d+2,1} & \cdots & x_{2,d} - x_{d+2,d} & r_2 - r_{d+2} \\ \vdots & \cdots & \vdots & \cdots \\ x_{d+1,1} - x_{d+2,1} & \cdots & x_{d+1,d} - x_{d+2,d} & r_{d+1} - r_{d+2} \end{vmatrix},$$

and A_i *is obtained from* A *by replacing the* i*-th column with* $[w_j]_{j=1}^{d+1}$*, where*

$$w_j = \sum_{j=1}^{d} (x_{i,j} - x_{d+2,j})^2 - (r_i - r_{d+2})^2, \; i = 1, \ldots, d+1.$$

This theorem characterizes the topological change of the dynamic Voronoi diagram depending on spheres moving in the d-dimensional space, and aims at detecting the topological event, which occurs when $d+2$ moving spheres are inscribed by a common sphere in the d-dimensional space.

This theorem assumes that the sphere P_{d+2} is the smallest among all the spheres. In the theorem's proof [4], the sphere P_{d+2} is shrunk and translated to the origin, and at the same time all the other spheres are also shrunk and translated in the same way as the sphere P_{d+2}. The assumption is necessary in order for each sphere's radius to avoid becoming negative.

Fortunately, Theorem 1 is satisfied for any choice of the sphere to be shrunk, that is, the assumption is unnecessary. This can be verified in the following manner.

Choose any sphere P_k out of $d+2$ spheres and transform each sphere P_i in such a way that

$$x'_{i,j} = x_{i,j} - x_{k,j}, \quad r'_i = r_i - r_k, \quad 1 \le i \le d+2,\ 1 \le j \le d. \tag{3}$$

Let these spheres be denoted by $P'_i = \{p'_i = (x'_{i,1}, \dots, x'_{i,d}), r'_i\},\ i = 1, \dots, d+2$, where the sphere P'_k is the point at the origin. This operation approves that there are spheres which have negative radii. At the same time, let us transform the sphere P_0 inscribing $d+2$ spheres to

$$\xi'_{0,j} = \xi_{0,j} - x_{k,j} \quad j = 1, \dots, d, \quad r'_0 = r_0 + r_k, \tag{4}$$

and let this sphere be denoted by $P'_0 = \{p'_0 = (x'_{0,1}, \dots, x'_{0,d}), r'_0\}$. This transformation keeps the sphere P'_0 inscribing each sphere which has the positive radius; on the other hand, when the radius r_i is negative, we can geometrically interpret P'_0 as the sphere circumscribing the spheres with the positive radius $|r'_i|$.

Since the sphere P'_0 inscribes or circumscribes P'_i, the following equations hold:

$$(x'_{i,1} - x'_{0,1})^2 + \dots + (x'_{i,d} - x'_{0,d})^2 = (r'_i + r'_0)^2, \quad 1 \le i \ne k \le d+2. \tag{5}$$

$$(x'_{0,1})^2 + \dots + (x'_{0,d})^2 = (r'_0)^2, \quad i = k. \tag{6}$$

From Eqns. (5) and (6), we obtain

$$2x'_{i,1}x'_{0,1} + \dots + 2x'_{i,d}x'_{0,d} + 2r'_i r'_0 = w'_i \quad 1 \le i \ne k \le d+2, \tag{7}$$

where

$$w'_i = (x'_{i,1})^2 + \dots + (x'_{i,d})^2 - (r'_i)^2.$$

Now, Eqns. (7) are represented by a linear system

$$2\begin{pmatrix} x'_{1,1} & \cdots & x'_{1,d} & r'_1 \\ \vdots & & \vdots & \vdots \\ x'_{k-1,1} & \cdots & x'_{k-1,d} & r'_{k-1} \\ x'_{k+1,1} & \cdots & x'_{k+1,d} & r'_{k+1} \\ \vdots & & \vdots & \vdots \\ x'_{d+2,1} & \cdots & x'_{d+2,d} & r'_{d+2} \end{pmatrix} \begin{pmatrix} x'_{0,1} \\ x'_{0,2} \\ \vdots \\ \vdots \\ x'_{0,d} \\ r'_0 \end{pmatrix} = \begin{pmatrix} w'_1 \\ \vdots \\ w'_{k-1} \\ w'_{k+1} \\ \vdots \\ w'_{d+1} \end{pmatrix},$$

and Cramer's rule enables us to explicitly get the solution of the linear system:

$$x'_{0,j} = \frac{A_j}{2A} \ \ (j = 1 \dots d), \quad r'_0 = \frac{A_{d+1}}{2A}, \tag{8}$$

where

$$A = \begin{vmatrix} x'_{1,1} & \cdots & x'_{1,d} & r'_1 \\ \vdots & & \vdots & \vdots \\ x'_{k-1,1} & \cdots & x'_{k-1,d} & r'_{k-1} \\ x'_{k+1,1} & \cdots & x'_{k+1,d} & r'_{k+1} \\ \vdots & & \vdots & \vdots \\ x'_{d+2,1} & \cdots & x'_{d+2,d} & r'_{d+2} \end{vmatrix} = \begin{vmatrix} x_{1,1} - x_{k,1} & \cdots & x_{1,d} - x_{k,d} & r_1 - r_k \\ \vdots & & \vdots & \vdots \\ x_{k-1,1} - x_{k,1} & \cdots & x_{k-1,d} - x_{k,d} & r_{k-1} - r_k \\ x_{k+1,1} - x_{k,1} & \cdots & x_{k+1,d} - x_{k,d} & r_{k+1} - r_k \\ \vdots & & \vdots & \vdots \\ x_{d+2,1} - x_{k,1} & \cdots & x_{d+2,d} - x_{k,d} & r_{d+2} - r_k \end{vmatrix}$$

and A_j is the determinant obtained from A by replacing the i-th column with $(w'_1, \dots, w'_{k-1}, w'_{k+1}, \dots, w'_{d+2})^{\mathrm{T}}$.

Using well-known formulae for the determinant, we can rewrite A, A_j and A_{d+1} into the followings respectively:

$$A = (-1)^{k+1} \begin{vmatrix} 1 & x_{1,1} & \cdots & x_{1,d} & r_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{k,1} & \cdots & x_{k,d} & r_k \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{d+2,1} & \cdots & x_{d+2,d} & r_{d+2} \end{vmatrix}, \tag{9}$$

$$A_j = B_j - 2x_{j,k}A, \tag{10}$$

$$A_{d+1} = B_{d+1} + 2r_k A, \tag{11}$$

where B_j is obtained by replacing the $(j+1)$-th column of A with $(v_1, v_2, \ldots, v_{d+2})$, where $v_i = x_{i,1}^2 + \cdots + x_{i,d}^2 - r_i^2$.

Hence, from Eqns. (8), (9), (10) and (11), Eqn. (2) is satisfied for any P_k. Therefore, we obtain the generalized Theorem 1.

Hereafter, since we discuss the two-dimensional case, we rewrite the generalized Theorem 1 for the two dimensional case as follows. For simplicity, we always assume that k is equal to 4.

Corollary 1. *In the two-dimensional Euclidean space, if four circles* $P_i = \{(x_i, y_i), r_i\}$ $(i = 1, \ldots, 4)$ *is inscribed by a common circle, then the equation*

$$A_1^2 + A_2^2 = A_3^2 \tag{12}$$

holds, where

$$A_1 = B_1 - 2x_4 A, \tag{13}$$

$$A_2 = B_2 - 2y_4 A, \tag{14}$$

$$A_3 = B_3 + 2r_4 A, \tag{15}$$

where

$$A = - \begin{vmatrix} 1 & x_1 & y_1 & r_1 \\ 1 & x_2 & y_2 & r_2 \\ 1 & x_3 & y_3 & r_3 \\ 1 & x_4 & y_4 & r_4 \end{vmatrix}, \tag{16}$$

and B_i *is obtained by replacing the* $(i+1)$*-th column of* A *with* $(v_1, v_2, v_3, v_4)^{\mathrm{T}}$, *where* $v_i = x_i^2 + y_i^2 - r_i^2$ $(i = 1, \ldots, 4)$.

4. Test functions

In order to check whether or not the fourth circle intersects the inscribing circle determined by the first three circles, we need the test functions, as with the test function (1) used in the point Voronoi diagram. In this section, we propose three test functions that altogether can check the topology of the circle Voronoi diagram.

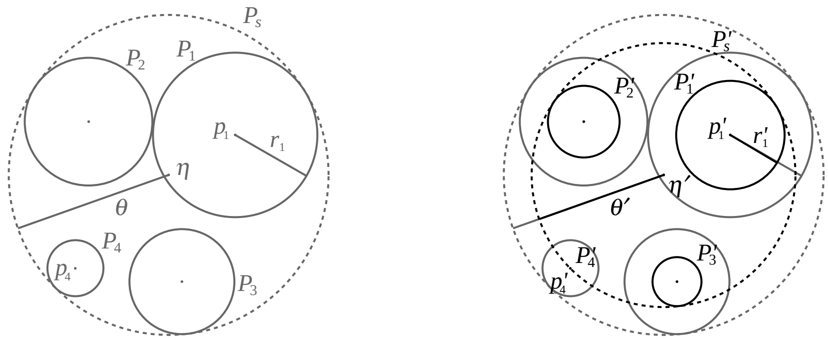
4.1. Main test function

Corollary 1 explains that the equation (12) holds at the special situation, at which four circles are inscribed by a common circle in the two-dimensional space. This situation implies that the fourth circle is inscribed by the inscribing circle determined by the other three generator circles. It is the critical situation in determining the topological structure when we carry out the incremental algorithm, so that a function generated from this equation is expected to distinguish the topological structure, which we want to know, in the procedure for constructing a circle Voronoi diagram.

Let the fourth circle be represented by $P_4 = \{(x, y), r\}$. Then, from the equation (12), we obtain

$$F(x, y, r) = A_1^2(x, y, r) + A_2^2(x, y, r) - A_3^2(x, y, r) \tag{17}$$

where $A_i(x, y, r)$ $(i = 1, 2, 3)$ represent the equations in which x_4, y_4 and r_4 in (13), (14) and (15) are replaced with x, y and r respectively. When a circle $\{(x_4, y_4), r_4\}$ is given specifically, if $F(x_4, y_4, r_4) = 0$, it indicates the possibility that the fourth circle touches the inscribing circle generated by the first given three circles.



(a) (b)

Figure 4. The shrink transformation for P_1, P_2, P_3, P_4 and P_s:(a) the condition before transformation and (b) the condition after transformation. The thick black circles illustrate the situation after transformation.

4.2. Second test function

We know from Corollary 1 that the value of the function F is 0 when the fourth circle is inscribed by the inscribing circle for the other three circles. However, the function F takes 0 also in another situation.

As shown in Figure 4(a), assume that four circles $P_i = \{p_i, r_i\}$ $(i = 1, \ldots, 4)$ are circumscribed by the common circle $P_s = \{\eta = (\eta_x, \eta_y), \theta\}$. The discussion on the rest of this section is similar to that of the section 3.2 . Now, we apply the following *shrink* transformation to these four circles:

$$x'_i = x_i - x_4,\ y'_i = y_i - y_4,\ r'_i = r_i - r_4 \quad (i = 1, \ldots, 4), \tag{18}$$

The fourth circle is transformed to the point at the origin, and the circumscribing circle is transformed to

$$\eta'_x = \eta_x - x_4, \eta'_y = \eta_y - y_4, \theta' = \theta - r_4. \tag{19}$$

Let the four circles and the circumscribing circle after the transformation be denoted by $P'_i = \{p'_i = (x'_i, y'_i), r'_i\}$ $(i = 1, \ldots, 4)$ and $P'_s = \{\eta' = (\eta'_x, \eta'_y), \theta'\}$ as shown in Figure 4(b). Since P'_s circumscribes P'_i $(i = 1, \ldots, 4)$, the next four equations hold:

$$d(p'_i, \eta') = \theta' - r'_i, \qquad i = 1, \ldots, 4. \tag{20}$$

These equations can be written as

$$(x'_i - \eta'_x)^2 + (y'_i - \eta'_y)^2 = (\theta' - r'_i)^2, \qquad i = 1, 2, 3, \tag{21}$$

$$(\eta'_x)^2 + (\eta'_y)^2 = (\theta')^2. \tag{22}$$

Note that the second-degree terms in (21) are canceled. Hence we get

$$2x'_i\eta'_x + 2y'_i\eta'_y - 2r'_i\theta' = w'_i, \qquad i = 1, 2, 3, \tag{23}$$

where $w'_i = (x'_i)^2 + (y'_i)^2 - (r'_i)^2$ $(i = 1, 2, 3)$. Now, the equations (23) are represented by a linear system as follows.

$$2\begin{pmatrix} x'_1 & y'_1 & r'_1 \\ x'_2 & y'_2 & r'_2 \\ x'_3 & y'_3 & r'_3 \end{pmatrix}\begin{pmatrix} \eta'_x \\ \eta'_y \\ -\theta' \end{pmatrix} = \begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \end{pmatrix}. \tag{24}$$

Assume that this linear system has a unique solution. By the Cramer's rule, this solution is written as

$$\eta'_x = \frac{A_1}{2A},\ \eta'_y = \frac{A_2}{2A},\ \theta' = -\frac{A_3}{2A}, \tag{25}$$

where A is the following determinant

$$A = \begin{vmatrix} x'_1 & y'_1 & r'_1 \\ x'_2 & y'_2 & r'_2 \\ x'_3 & y'_3 & r'_3 \end{vmatrix},$$

and A_i is the determinant of the matrix obtained from A by replacing the i-th column with $(w_1', w_2', w_3')^{\mathrm{T}}$. Hence, we substitute (25) into the equation (22), and we get

$$A_1^2 + A_2^2 = A_3^2.$$

This equation is the same as the equation (12), and the value of the function F is also 0 in this case. This trait prevents us from constructing the procedure to check the topological information of the circle Voronoi diagram with only computing the function F.

However, let us recall again the case where four circles are inscribed by the inscribing circle. Assume that four circles $P_1,\ P_2,\ P_3$ and P_4 are inscribed by a common circle $P_0 = \{\xi = (\xi_x, \xi_y), \rho\}$, which is the assumption of Corollary 1. We carry out the same shrink transformation as (18), but the radius of P_0 is increased:

$$\xi_x' = \xi_x - x_4,\ \xi_y' = \xi_y - y_4,\ \rho' = \rho + r_4. \tag{26}$$

After the discussion similar to the one mentioned above, we get the coordinates and the radius of P_0 represented by

$$\xi_x' = \frac{A_1}{2A},\ \xi_y' = \frac{A_2}{2A},\ \rho' = \frac{A_3}{2A}, \tag{27}$$

where A and A_i are the same as (25). Compared with (25), we can see ρ' and θ' have opposite signs. The sign of ρ' is positive when the fourth circle is inscribed by the inscribing circle, while the sign of θ' is negative when the fourth circle is circumscribed by the circumscribing circle. Hence, we can distinguish these situations.

Consequently, we can define a new test function

$$R(x, y, r) = \frac{A_3(x, y, r)}{2A(x, y, r)}, \tag{28}$$

where A and A_3 are the same as (15) and (16), where $x_4,\ y_4$ and r_4 are replaced by $x,\ y$ and r.

4.3. Third test function

From Section 4.1 and 4.2, when the main test function F equals 0 and the second test function R is positive, we can recognize the fourth circle touches the inscribing circle. However, when F is not 0, we can not recognize whether the forth circle crosses the inscribing circle or not. In order to overcome this difficulty, we propose another new test function.

Let four circles $P_i = \{(x_i, y_i), r_i\}$ $(i = 1, \dots, 4)$ be placed, where we assume that the position of $P_1,\ P_2$ and P_3 are mutually distinct, and let us introduce the circle $Q_4 = \{(x, y), r_4\}$, which has the same radius r_4 as the fourth circle P_4 and which travels along a straight line. Then, for the four circles P_1, P_2, P_3 and Q_4, consider the following determinant A:

$$\begin{aligned} A(x, y, r = r_4) &= -\begin{vmatrix} 1 & x_1 & y_1 & r_1 \\ 1 & x_2 & y_2 & r_2 \\ 1 & x_3 & y_3 & r_3 \\ 1 & x & y & r_4 \end{vmatrix} \\ &= \left| \begin{matrix} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{r} \end{matrix} \right| - x \left| \begin{matrix} \mathbf{1} & \boldsymbol{y} & \boldsymbol{r} \end{matrix} \right| + y \left| \begin{matrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{r} \end{matrix} \right| - r_4 \left| \begin{matrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{y} \end{matrix} \right|, \end{aligned} \tag{29}$$

where $\mathbf{1} = (1, 1, 1)^{\mathrm{T}}, \boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}}, \boldsymbol{y} = (y_1, y_2, y_3)^{\mathrm{T}}, \boldsymbol{r} = (r_1, r_2, r_3)^{\mathrm{T}}$. Here, let us redefine

$$L(x, y) = \left| \begin{matrix} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{r} \end{matrix} \right| - x \left| \begin{matrix} \mathbf{1} & \boldsymbol{y} & \boldsymbol{r} \end{matrix} \right| + y \left| \begin{matrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{r} \end{matrix} \right| - r_4 \left| \begin{matrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{y} \end{matrix} \right|, \tag{30}$$

and we call it the third test function.

When the function (30) equals to 0, we obtain the linear equation l,

$$\left| \begin{matrix} \mathbf{1} & \boldsymbol{y} & \boldsymbol{r} \end{matrix} \right| x - \left| \begin{matrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{r} \end{matrix} \right| y + \left| \begin{matrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{y} \end{matrix} \right| r_4 = \left| \begin{matrix} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{r} \end{matrix} \right|, \tag{31}$$

with respect to the variables x, y. That is, when the circle Q_4 moves on this line l, the determinant A is always equal to 0. In what follows, we scrutinize this line l.

If the fourth circle P_4 is inscribed by the inscribing circle and, at the same time, circumscribed by the circumscribing circle. Then, the next equations hold:

$$\left\{ \begin{array}{l} (x_i' - \xi_x')^2 + (y_i' - \xi_y')^2 = (r_i' + \rho')^2, \\ (x_i' - \eta_x')^2 + (y_i' - \eta_y')^2 = (\theta' - r_i')^2, \end{array} \right. \quad i = 1, 2, 3, \tag{32}$$

$$\left\{ \begin{array}{l} (\xi_x')^2 + (\xi_y')^2 = (\rho')^2, \\ (\eta_x')^2 + (\eta_y')^2 = (\theta')^2. \end{array} \right. \tag{33}$$

We can remove all quadratic terms from these equations, then we get

$$-2x_i'(\xi_x' - \eta_x') - 2y_i'(\xi_y' - \eta_y') = 2r_i'(\rho' + \theta') \quad (i = 1, 2, 3), \tag{34}$$

and rewrite these equations in the matrix form:

$$2 \begin{pmatrix} x_1' & y_1' & r_1' \\ x_2' & y_2' & r_2' \\ x_3' & y_3' & r_3' \end{pmatrix} \begin{pmatrix} \xi_x' - \eta_x' \\ \xi_y' - \eta_y' \\ \rho' + \theta' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

This equation indicates

$$\begin{vmatrix} x_1' & y_1' & r_1' \\ x_2' & y_2' & r_2' \\ x_3' & y_3' & r_3' \end{vmatrix} = 0 \quad \text{or} \quad \begin{pmatrix} \xi_x' - \eta_x' \\ \xi_y' - \eta_y' \\ \rho' + \theta' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The latter one means that the inscribing circle and circumscribing circle form concentric circles each other, that is, the four circles have the same radius. In this case, L equals 0 for any input (x, y), and is undecidable. We discuss this situation later in section 5.1.

In the test of this section, we assume that $(\xi_x' - \eta_x', \xi_y' - \eta_y', \rho' + \theta')^{\mathrm{T}} \neq (0, 0, 0)^{\mathrm{T}}$, that is, at least one of the circles, P_1, P_2, P_3 and Q_4 has the radius different from that of another circle. Then, we need

$$A(x, y, r = r_4) = L(x, y) = 0 \tag{36}$$

so as to satisfy the equation (35). Therefore, if Q_4 moves along the line l, there exist such positions that Q_4 is inscribed by the inscribing circle and circumscribed by the circumscribing circle at the same time.

The line l depends only on r_4 and just translates monotonously as r_4 changes monotonously. Hence, if circles with bigger radii than r_4 are simultaneously inscribed by the inscribing circle and circumscribed by the circumscribing circle, they are all located in one of two areas divided by the line l. We call this area area S. On the other hand, if their radii are smaller than r_4, they are in the opposite area. We call it area T.

Now, let the circle Q_4 move on the line l' which is parallel to the line l and passes through the center of the fourth circle P_4. Then we can discuss the following two cases: One is the case where P_4 is located in the area S; the other in the area T.

First, we discuss the former case. If on the line l' are a circle which is inscribed by the inscribing circle and circumscribed by the circumscribing circle at the same time, it has a bigger radius than P_4 as mentioned above. Hence, Q_4 does not crosses the inscribing circle and circumscribing circle simultaneously. That is, when Q_4 moves from the outside of the all circles through the inside, Q_4 is firstly circumscribed by the circumscribing circle, and secondly inscribed by the inscribing circle, as shown in Figure 5(a).

Next, we discuss the other case. If on the line l' are a circle which is inscribed by the inscribing circle and circumscribed by the circumscribing circle at the same time, it has smaller radius than that of P_4. When Q_4 touches the circumscribing circle from inside, it has already crossed the inscribing circle. Therefore Q_4 is first inscribed by the inscribing circle, and after that, circumscribed by the circumscribed circle as shown in Figure 5(b).

Finally, we should decide the side the forth is located in from the sigh of the function L. The easiest way to decide it is the following. We first select one of the first three circles: for example we select the smallest one. Next, we check the signs of L both for the smallest one and for the forth circle. Then, if these signs are same, the forth circle is at the same side with the smallest one, that is, it is at the area T; if these signs are different, it is at the opposite side, that is, it is at the area S.

4.4. Property of the function F

This section considers the property of the main test function (17), F, in more detail.

Firstly, the function F is invariant under the translation. The function F is made from the determinant of the matrix whose components are relative coordinates. Since the translation does not change relative coordinates, the function F does not change.

Secondly, the function F is invariant under the rotation of the coordinate system with respect to x-y plane. We can prove this proposition by directly applying the transformation.

Thirdly, the function F takes negative infinity at any points at infinity. Using the form in Eqns. (13), (14) and (15), we can rewrite Eqn. (17) into

$$\begin{aligned} F &= (B_1 - 2xA)^2 + (B_2 - 2yA)^2 - (B_3 + 2rA)^2 \\ &= (B_1^2 + B_2^2 - B_3^2) - 4\{xB_1 + yB_2 + rB_3 - (x^2 + y^2 - r^2)A\}A. \end{aligned} \tag{37}$$

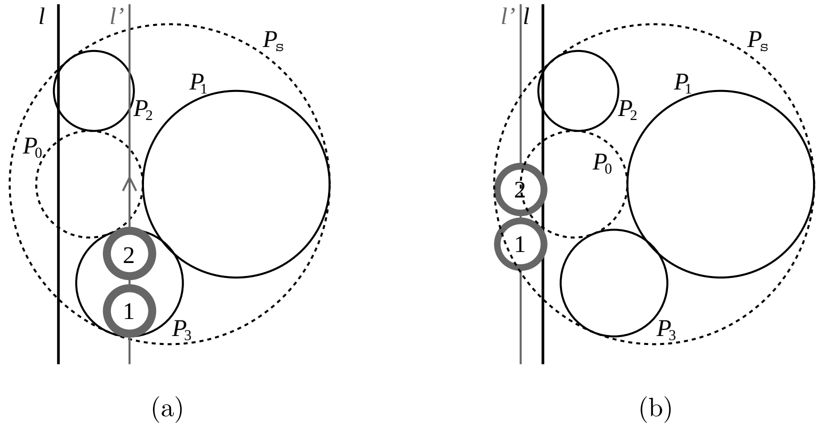


Figure 5. (a) The situation that l' passes through the same side for the line l as three generators. The number "1" and "2" in Q_4 are the order of the $F = 0$ points during the motion of Q_4 on the line l'; (b)the situation that l' passes through the opposite side for the line l as three generators.

Expanded in the same way as (29), A, B_1, B_2 and B_3 are represented respectively by

$$A = |\,\boldsymbol{x}\ \ \boldsymbol{y}\ \ \boldsymbol{r}\,| - |\,\boldsymbol{1}\ \ \boldsymbol{y}\ \ \boldsymbol{r}\,|x + |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{r}\,|y - |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{y}\,|r, \tag{38}$$

$$B_1 = |\,\boldsymbol{w}\ \ \boldsymbol{y}\ \ \boldsymbol{r}\,| - |\,\boldsymbol{1}\ \ \boldsymbol{y}\ \ \boldsymbol{r}\,|w + |\,\boldsymbol{1}\ \ \boldsymbol{w}\ \ \boldsymbol{r}\,|y - |\,\boldsymbol{1}\ \ \boldsymbol{w}\ \ \boldsymbol{y}\,|r, \tag{39}$$

$$B_2 = |\,\boldsymbol{x}\ \ \boldsymbol{w}\ \ \boldsymbol{r}\,| - |\,\boldsymbol{1}\ \ \boldsymbol{w}\ \ \boldsymbol{r}\,|x + |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{r}\,|w - |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{w}\,|r, \tag{40}$$

$$B_3 = |\,\boldsymbol{x}\ \ \boldsymbol{y}\ \ \boldsymbol{w}\,| - |\,\boldsymbol{1}\ \ \boldsymbol{y}\ \ \boldsymbol{w}\,|x + |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{w}\,|y - |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{y}\,|w, \tag{41}$$

where $w = x^2 + y^2 - r^2$ and $\boldsymbol{w} = (x_1^2 + y_1^2 - r_1^2, x_2^2 + y_2^2 - r_2^2, x_3^2 + y_3^2 - r_3^2)$. Assigning these equations into the function (37), we obtain the principal term of the function F:

$$\left(|\,\boldsymbol{1}\ \ \boldsymbol{y}\ \ \boldsymbol{r}\,|^2 + |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{r}\,|^2 - |\,\boldsymbol{1}\ \ \boldsymbol{x}\ \ \boldsymbol{y}\,|^2\right) w^2. \tag{42}$$

The coefficient of this term dominates the value at infinity.

From the invariance of the function F under rotations, assume without loss of generality that three circles are inscribed by the circle $P_0 = \{(0,0), a\}$, and are circumscribed by the circle $P_s = \{(c,0), b\}$, where $0 < a < b$ and $0 < c < a + b$. This is equivalent to the situation that the coordinate system is rotated in such a way that the line l obtained by Eqn. (31) becomes perpendicular to the x axis. Then, the three circles which are simultaneously inscribed by P_0 and circumscribed by P_s, are represented by

$$P_i = \{(x_i, y_i), r_i\} = \left\{\left(\frac{b+a}{2}\cos\theta_i + \frac{c}{2}, \frac{\sqrt{(b+a)^2 - c^2}}{2}\sin\theta_i\right), \frac{c}{2}\cos\theta_i + \frac{b-a}{2}\right\} \quad (i = 1, 2, 3), \tag{43}$$

where θ_1, θ_2 and θ_3 are mutually distinct (refer to the appendix A).

Let (43) be assigned to (42). Then the second term of (42) vanishes and we get

$$-\frac{\{(a+b)^2 - c^2\}^2}{16}\begin{vmatrix} 1 & \sin\theta_1 & \cos\theta_1 \\ 1 & \sin\theta_2 & \cos\theta_2 \\ 1 & \sin\theta_3 & \cos\theta_3 \end{vmatrix}^2 w^2 \tag{44}$$

Therefore, the function F takes negative infinity at any points at infinity.

Lastly, we note that the function F can be transformed into

$$F = (B_1^2 + B_2^2 - B_3^2) - 4AC, \tag{45}$$

where

$$C = \begin{vmatrix} w_1 & x_1 & y_1 & r_1 \\ w_2 & x_2 & y_2 & r_2 \\ w_3 & x_3 & y_3 & r_3 \\ w & x & y & r \end{vmatrix}.$$

Substituting the equation from (38) to (41) into the second term of righthand side of Eqn. (37), we can easily obtain the form above. This is very useful for the estimation of the range of values in exact computation.

5. Proposed procedure

The Gavrilova and Rokne's method [4] comprises the main test function and the second test function in the previous section. Their method detects that the forth circle touches the inscribing circle, while it can not sometimes discern whether the forth circle intersects the inscribing circle, as mentioned in Section 4.3. On the other hand, in this section, taking the third test function into account, we propose a new procedure.

5.1. Case (I)

In order to construct the procedure which verifies whether the forth circle intersects the inscribing circle associated with three generator circles, We firstly decide which side for the line l the center of the fourth circle is located in, using the function (30), L, and we check the intersection, using the function (17), F, and the function (28), R. The notations are the same as ones used in the previous section 4.3.

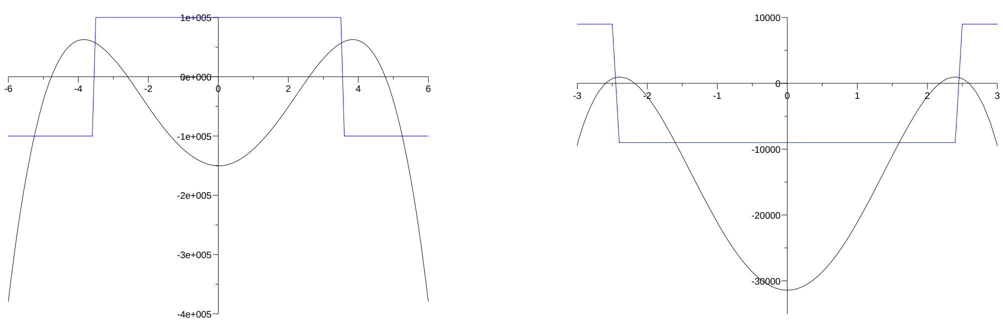
First, we discuss the condition in the area S. Since the function F is the fourth order polynomial and has negative infinity at infinity, the value of the function F depicts the curved graph as shown in Figure 6(a). In the section 4.3, we discuss that when the center of the circle Q_4 moves along the line l' in the area S, the Q_4 is first circumscribed by the circumscribing circle, and next inscribed by the inscribing circle. Of the four zero points of the fourth degree function, two inners indicate that Q_4 touches the inscribing circle, and that F is less than or equal to 0 must hold. The sign of the function R also changes like the step function, as shown in Figure 6(a). Hence, the condition that the fourth circle intersects the inscribing circle is represented by

$$F \leq 0 \text{ and } R > 0. \tag{46}$$

Note that this condition includes the case where Q_4 is included by the inscribing circle.

Likewise, we discuss the condition of the opposite side. When the center of the circle Q_4 moves along the line l' in the area T, the Q_4 is first inscribed by the inscribing circle, and next circumscribed by the circumscribing circle. Therefore, the value of function F and the sign of function R change as in Figure 6(b). This time, two outers indicate that Q_4 contacts the inscribing circle. Accordingly, that F is more than or equal to 0 must hold. In this case, while crossing the circumscribing circle, Q_4 continues to intersects the inscribing circle. In addition, considering the case where Q_4 is included by the inscribing circle, we can obtain the condition that the fourth circle intersects the inscribing circle:

$$F \geq 0 \text{ or } (F < 0 \text{ and } R < 0). \tag{47}$$



(a) (b)

Figure 6. The change of the value of the function F represented by the curve line and the sign of the function R represented by step function: (a) P_4 is located in the area S; (b) P_4 is located in the area T.

However, the previous conditions are not complete, and we discuss two special cases: one is that the center of the forth circle exists on the line l, namely $A = 0$, and the other is that all four circles have the same radius.

When $A = 0$, R is undecidable and F is always less than or equals to 0. Hence, we should decide whether the forth circles intersects or not using only A_3:

$$A_3(x, y, r = r_4) = -\begin{vmatrix} 1 & x_1 & y_1 & x_1^2 + y_1^2 - r_1^2 \\ 1 & x_2 & y_2 & x_2^2 + y_2^2 - r_2^2 \\ 1 & x_3 & y_3 & x_3^2 + y_3^2 - r_3^2 \\ 1 & x & y & x^2 + y^2 - r_4^2 \end{vmatrix} - 2r_4 \begin{vmatrix} 1 & x_1 & y_1 & r_1 \\ 1 & x_2 & y_2 & r_2 \\ 1 & x_3 & y_3 & r_3 \\ 1 & x & y & r_4 \end{vmatrix}.$$

Because the coefficients of the highest order of the variables x and y are the same, we recognize that the equation $A_3 = 0$ represents a circle, and the circle passes through the centers of two circles on the line l which are contacted by inscribing and circumscribing circles. This can be understood in the following way. First, let us shrink the three circles, as in (3) or (18) (see Figure 7(a)). Then, the inscribing circle is made enlarged by r_4 in the direction of radius, calling the circle P_0', and the circumscribing circle is made shrunk by r_4, calling the circle P_s'. As a result, P_0' and P_s' intersect each other at two points on the line l. The circle $A = 0$ must pass through the points, because, if it did not pass through the points, it would contradict the order in which Q_4 intersects the inscribing and circumscribing circles, when moving along the line other than the line l.

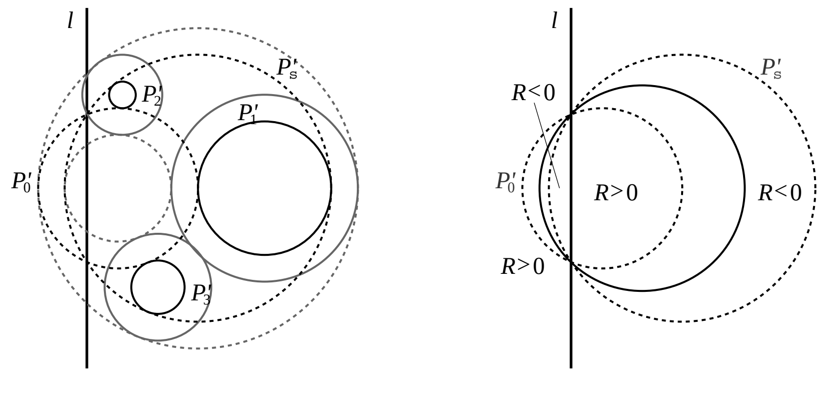


Figure 7. The relationship among three circles, inscribing circle, circumscribing circle and the straight line l after transformation: pale black curves represent original objects, while thick black curves represent the objects after the transformation; the solid curves stand for three circles and dotted ones the inscribing and circumscribing circles; the straight line represents the line which satisfies $A = 0$

From the whole argument in this section, we get information on the change of the sign of R as shown in Figure 7(b). The determinant A has the different sign between at one side of the area divided by the line l and the other side, and hence, A_3 has the same sign inside circle $A_3 = 0$ and the other sign outside.

As a result, the scheme to detect intersection becomes the following. When detecting $A = 0$, we compute A_3 of the forth circle and one of the other circles. If the forth circle obtains $A_3 = 0$, the forth circle touches the inscribing circle; if the two circles have the same sign, it intersects the inscribing circle; otherwise, it does not intersect.

Lastly, we discuss the case where the given four circles have the same radius, that is, $(\xi_x' - \eta_x', \xi_y' - \eta_y', \rho' + \theta')^{\mathrm{T}} = (0, 0, 0)^{\mathrm{T}}$ in section 4.3. In this case, the inscribing circle has as same center as the circumscribing circle does. Then, the problem of whether the forth circle intersects or not, reduces to the problem of whether or not the center of the forth circle is inside the circle which pass through the center of the first given three circles. We can judge it, using the following function:

$$\begin{vmatrix} 1 & x_1 & y_1 & x_1^2 + y_1^2 \\ 1 & x_2 & y_2 & x_2^2 + y_2^2 \\ 1 & x_3 & y_3 & x_3^2 + y_3^2 \\ 1 & x & y & x^2 + y^2 \end{vmatrix}. \qquad (48)$$

When the circle P_1, P_2 and P_3 are placed counterclockwise in this order, if this determinant (48) is positive, the forth circle crosses the inscribing circle; otherwise, it does not.

From the discussion in this section, we construct the following procedure.

Input: the three generators $P_1,\ P_2,\ P_3$ and the fourth circle P_4.

Output: "INTERSECT", if the fourth circle intersects the maximum void circle inscribing with $P_1,\ P_2,\ P_3$, and "NOT INTERSECT" otherwise.

Procedure:

1. If the given four circles have the same radius, go to the next; otherwise go to 3.
2. Renumber the three generators so that $P_1,\ P_2$ and P_3 touch the inscribing circle counterclockwise in this order. If the function (48) is positive, return "INTERSECT"; otherwise, "NOT INTERSECT"; stop.

3 Compute $L(x_4, y_4)$. If $L(x_4, y_4) = 0$, go to the next; otherwise, go to 5.

4 Compute $A_3(x_1, y_1)$ and $A_3(x_4, y_4)$. If the sign of $A_3(x_1, y_1)$ is the same as that of $A_3(x_4, y_4)$, return "INTERSECT"; otherwise, return "NOT INTERSECT"; stop.

5 Compute $L(x_1, y_1)$. If the sign of both $L(x_1, y_1)$ and $L(x_4, y_4)$ is same, go to 6; otherwise, go to 7.

6 Compute the function (17), F, and the function (28), R. If (46) is satisfied, return "INTERSECT"; otherwise, return "NOT INTERSECT"; stop.

7 Compute the function (17), F, and the function (28), R. If (47) is satisfied, return "INTERSECT"; otherwise, return "NOT INTERSECT"; stop.

5.2. Case (Ⅱ)

In this section, we give the way to solve the problem of the Case (Ⅱ); our basic idea is to reduce the problem of the Case (Ⅱ) to that of the Case (I) by making use of the reflection with respect to a circle.

Let P be a circle in the plane such that its center is (a, b) and its radius is ρ: $P = \{(a, b), \rho\}$. Suppose that a point p with coordinates (x_1, y_1) is mapped to the point p' with coordinates (x_2, y_2) by mapping such that

$$\sqrt{(x_1 - a)^2 + (y_1 - b)^2} \cdot \sqrt{(x_2 - a)^2 + (y_2 - b)^2} = \rho^2 \tag{49}$$

$$\frac{x_2 - a}{x_1 - a} = \frac{y_2 - b}{y_1 - b} = k > 0. \tag{50}$$

Then the point p' is called the mirror image of the point p with respect to the circle P. Points inside the circle P are mapped to the points outside it and vice versa. This is called the reflection of the circle.

This mapping has the following beautiful characteristics. Let us be given two points p_1 and p_2, and suppose that the distance from the center of the circle P to the point p_1 is larger than that from the center to the point p_2. Then after the mapping, the distance from the center to the mirror image of the point p_1 is smaller than that from the center to the mirror image of the point p_2. Another trait is the following: firstly a straight line passing through the center (a, b) is mapped to itself, secondly a straight line which does not pass through the center (a, b) to the circle which pass through the center and vice versa, and lastly a circle which does not pass through the center to the circle which does not.

Let P_1 be a circle which does not pass through the center of P: $P_1 = \{(p, q), r\}$. Then the mirror image of the circle P_1 is represented by

$$P' = \left\{\left(a + \frac{(p-a)\rho^2}{(p-a)^2 + (q-b)^2 - r^2}, b + \frac{(q-b)\rho^2}{(p-a)^2 + (q-b)^2 - r^2}\right), \frac{r\rho^2}{(p-a)^2 + (q-b)^2 - r^2}\right\} \tag{51}$$

These characteristics can lead to the reduction of the Case (Ⅱ) to the Case (I). Let P_s be a circle which is fully included in one of either inscribing circles. Then all the circles are mapped to the circles inside P_s. Clearly, the inscribing circle which includes P_s contains any other circles. From the continuity, the touching points between the four circles and the inscribing circles remain the touching points and the number does not increase. Hence after the transformation, the image of the inscribing circle with P_s becomes the circumscribing circle and the image of the other remains the inscribing circle.

Since there are two inscribing circles, we must conduct the above procedure two times. We select one of either inscribing circles, map all the circles as mentioned above and check whether of not the forth circle intersects the other inscribing circle. Likewise, we do the same thing for the other inscribing circles.

6. Precision necessary for exact computation

6.1. Case(I)

In the procedure constructed in Section 5.1, we have to compute the signs of the three functions. The first function is defined in (30). This function is computed twice to decide whether or not the fourth circle is located in the same side for the line l as three generating circles. The second function is F defined in (45). The third function is R defined in (28). If $A \neq 0$, then we can compute the sign of the function R as $\operatorname{sign} A \cdot \operatorname{sign} A_3$. All these functions use only addition and multiplication, and so all the signs of these functions are computed exactly if all the inputs x, y, r are represented by integers.

Let us assume that all the inputs are integers whose absolute values are not greater than a certain integer L, that is

$$-L \leq x_i, y_i, r_i \leq L \qquad (i = 1, \ldots, 4). \tag{52}$$

Now, we estimate the precision (i.e., the amount of memory) that is necessary to compute these functions without overflowing. Among these functions, the function F needs the highest precision. So we estimate the amount of memory enough to compute the function F exactly. From inequalities (52), we get

$$0 \le x_i^2, y_i^2, r_i^2 \le L^2 \qquad (i = 1, \dots, 4). \tag{53}$$

Then $w_i = x_i^2 + y_i^2 - r_i^2$ is bounded by

$$-L^2 \le w_i \le 2L^2. \tag{54}$$

From the Hadamard inequality (which claims that the absolute value of the determinant of a matrix is not bigger than the multiplication of the lengths of all the column vectors), the range of A, B_i and C are restricted respectively by

$$|A| \le \sqrt{\{1^2+1^2+1^2+1^2\}\cdot\{L^2+L^2+L^2+L^2\}\cdot\{L^2+L^2+L^2+L^2\}\cdot\{L^2+L^2+L^2+L^2\}} = 2^4L^3, \tag{55}$$

$$|B_i| \le \sqrt{\{4L^0\}\cdot\{4L^2\}\cdot\{4L^2\}\cdot\{4(2L^2)^2\}} = 2^5L^4, \tag{56}$$

$$|C| \le \sqrt{\{4(2L^2)^2\}\cdot\{4L^2\}\cdot\{4L^2\}\cdot\{4L^2\}} = 2^5L^5. \tag{57}$$

Hence, the range of function F is restricted by

$$-(2^{10}L^8) - 4(2^4L^3)(2^5L^5) = -3\cdot 2^{10}L^8 \le F \le 2(2^{10}L^8) + 4(2^4L^3)(2^5L^5) = 2^{12}L^8. \tag{58}$$

Now, suppose that the inputs are represented by $b+1$ bits, b bits for the absolute value and 1 bit for the sign. Then, $L = 2^b - 1$. From the inequality (58), we obtain

$$|F| \le 2^{12}(2^b-1)^8 < 2^{12}(2^{8b}-1) < 2^{8b+12} - 1. \tag{59}$$

So, using $8b+12$ bits for the absolute value, we can always compute the function F without overflowing.

6.2. Case(II)

As with the Case (I), the function F in the Case (II) needs the highest precision, and we focus on the estimation of the amount of memory enough to compute the function F without overflow.

For simplicity, let us first translate by (a, b) all the circles in Section 5.2. Then, from (51), the mirror images P_i' of the four circles $P_i = \{(x_i, y_i), r_i\}$ $(i = 1, \dots, 4)$ with respect to the circle $P = \{(0,0), \rho\}$ can be written by

$$\{(x_i', y_i'), r_i'\} = \left\{\left(\frac{x_i\rho^2}{x_i^2+y_i^2-r_i^2}, \frac{y_i\rho^2}{x_i^2+y_i^2-r_i^2}\right), \frac{r\rho^2}{x_i^2+y_i^2-r_i^2}\right\}. \tag{60}$$

From (45), we obtain

$$F = (B_1^2 + B_2^2 - B_3^2) - 4AC \tag{61}$$

where

$$A = \begin{vmatrix} 1 & x_1' & y_1' & r_1' \\ 1 & x_2' & y_2' & r_2' \\ 1 & x_3' & y_3' & r_3' \\ 1 & x_4' & y_4' & r_4' \end{vmatrix}, \quad C = \begin{vmatrix} w_1' & x_1' & y_1' & r_1' \\ w_2' & x_2' & y_2' & r_2' \\ w_3' & x_3' & y_3' & r_3' \\ w_4' & x_4' & y_4' & r_4' \end{vmatrix}$$

and B_j is obtained by replacing the $(j+1)$-th column of A with $(w_1', w_2', w_3', w_4')^{\mathrm{T}}$, where

$$w_i' = {x_i'}^2 + {y_i'}^2 - {r_i'}^2.$$

In what follows, we in fact estimate the amount of memory to compute F, but before that, note that ρ^2 does not affect the judgment of the sign of F, because all components x_i', y_i', r_i' commonly include ρ^2. Hence, we can neglect ρ^2 from the beginning, and can rewrite Eqn. (60) as

$$\{(x_i', y_i'), r_i'\} = \left\{\left(\frac{x_i}{w_i}, \frac{y_i}{w_i}\right), \frac{r_i}{w_i}\right\}, \tag{62}$$

where $w_i = x_i^2 + y_i^2 - r_i^2$. In fact, let us assign the values of (62) to the equation (61). Owing to

$$w_i' = \frac{1}{w_i},$$

we get

$$A = \begin{vmatrix} \frac{w_1}{w_1} & \frac{x_1}{w_1} & \frac{y_1}{w_1} & \frac{r_1}{w_1} \\ \frac{w_2}{w_2} & \frac{x_2}{w_2} & \frac{y_2}{w_2} & \frac{r_2}{w_2} \\ \frac{w_3}{w_3} & \frac{x_3}{w_3} & \frac{y_3}{w_3} & \frac{r_3}{w_3} \\ \frac{w_4}{w_4} & \frac{x_4}{w_4} & \frac{y_4}{w_4} & \frac{r_4}{w_4} \end{vmatrix} = \frac{1}{w_1 w_2 w_3 w_4} \begin{vmatrix} w_1 & x_1 & y_1 & r_1 \\ w_2 & x_2 & y_2 & r_2 \\ w_3 & x_3 & y_3 & r_3 \\ w_4 & x_4 & y_4 & r_4 \end{vmatrix}, \tag{63}$$

$$C = \begin{vmatrix} \frac{1}{w_1} & \frac{x_1}{w_1} & \frac{y_1}{w_1} & \frac{r_1}{w_1} \\ \frac{1}{w_2} & \frac{x_2}{w_2} & \frac{y_2}{w_2} & \frac{r_2}{w_2} \\ \frac{1}{w_3} & \frac{x_3}{w_3} & \frac{y_3}{w_3} & \frac{r_3}{w_3} \\ \frac{1}{w_4} & \frac{x_4}{w_4} & \frac{y_4}{w_4} & \frac{r_4}{w_4} \end{vmatrix} = \frac{1}{w_1 w_2 w_3 w_4} \begin{vmatrix} 1 & x_1 & y_1 & r_1 \\ 1 & x_2 & y_2 & r_2 \\ 1 & x_3 & y_3 & r_3 \\ 1 & x_4 & y_4 & r_4 \end{vmatrix}, \tag{64}$$

where B_j is obtained by replacing $(j+1)$-th column of A with $(1,1,1,1)^{\mathrm{T}}$.

Since we translate all the circles, $x_i,\ y_i,\ r_i$ are estimated as follows:

$$-2L \le x_i,\ y_i,\ r_i \le 2L \quad (i = 1, \dots, 4). \tag{65}$$

From the inequalities (65) and $x_i^2 + y_i^2 - r_i^2 > 0$, we get

$$-4L^2 \le x_i^2 + y_i^2 - r_i^2 \le 8L^2.$$

From the Hadamard inequality, the range of A, B_i and $C$are restricted respectively by

$$|A| \le \frac{\sqrt{\{4(8L^2)^2\} \cdot \{4(2L)^2\} \cdot \{4(2L)^2\} \cdot \{4(2L)^2\}}}{2^{12}L^8} = \frac{2^{10}L^5}{2^{12}L^8},$$

$$|B_i| \le \frac{\sqrt{\{4(8L^2)^2\} \cdot \{4(2L)^2\} \cdot \{4(2L)^2\} \cdot \{4(2L)^0\}}}{2^{12}L^8} = \frac{2^9L^4}{2^{12}L^8},$$

and

$$|C| \le \frac{\sqrt{\{4(2L)^2\} \cdot \{4(2L)^2\} \cdot \{4(2L)^2\} \cdot \{4(2L)^0\}}}{2^{12}L^8} = \frac{2^7L^3}{2^{12}L^8},$$

where the rightmost expression $\frac{aL^b}{cL^d}$ means that the numerator is bounded by aL^b and the denominator is bounded by cL^d. The range of the first term of the right hand side of Eqn. (61) is restricted by

$$|B_1^2 + B_2^2 - B_3^2| \le 2 \cdot \left(\frac{2^9L^4}{2^{12}L^8}\right)^2 = \cdot\frac{2^{19}L^8}{2^{24}L^{16}},$$

The range of the second term is restricted by

$$|4CA| \le 2^2 \cdot \frac{2^7L^3}{2^{12}L^8} \cdot \frac{2^{10}L^5}{2^{12}L^8} = \frac{2^{19}L^8}{2^{24}L^{16}}.$$

Consequently, we get

$$|F| \le \frac{2^{20}L^8}{2^{24}L^{16}}.$$

Since the denominator is positive, just computing the numerator enable us to obtain the sign of the function F. Set $L = 2^b - 1$ as in previous section. Then using $8b + 20$ bits for the absolute value, we can always compute the function F without overflowing.

6.3. Algorithm

First of all, we should discern whether the first three circles are stationed in the Case (I) or in the Case (II). In section 4.4, we proved that the coefficient of the principal term of the function F is negative in the Case (I). On the other hand, it becomes positive in the Case (II). We can prove it in the following way.

The principal term was represented by

$$\left(\begin{vmatrix} \mathbf{1} & \boldsymbol{y} & \boldsymbol{r} \end{vmatrix}^2 + \begin{vmatrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{r} \end{vmatrix}^2 - \begin{vmatrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{y} \end{vmatrix}^2\right) w^2,$$

and, assume also without loss of generality that the three circles are inscribed by both of the circle $P_0 = \{(0,0), a\}$ and the circle $P_s = \{(c,0), b\}$, where $0 < a, c$ and $0 < b < a + c$. Then the three circles which are simultaneously inscribed by the two inscribing circles, are represented by

$$P_i = \{(x_i, y_i), r_i\} = \left\{ \left(\frac{b-a}{2} \frac{1}{\cos\theta} + \frac{c}{2}, \frac{\sqrt{c^2-(b-a)^2}}{2} \tan\theta \right), -\frac{c}{2}\frac{1}{\cos\theta} + \frac{a+b}{2} \right\} \quad (i = 1,2,3) \tag{66}$$

Assigning (66) to the principal term, we obtain

$$\frac{\{(a+b)^2 - c^2\}^2}{16} \begin{vmatrix} 1 & \frac{1}{\cos\theta_1} & \tan\theta_1 \\ 1 & \frac{1}{\cos\theta_2} & \tan\theta_2 \\ 1 & \frac{1}{\cos\theta_3} & \tan\theta_3 \end{vmatrix}^2 w^2, \tag{67}$$

whose value is positive.

Consequently, we obtain the following algorithm:

Input: the three generators $P_1,\ P_2,\ P_3$ and the fourth circle P_4.

Output: "INTERSECT", if the fourth circle intersects the maximum void circle inscribing with $P_1,\ P_2,\ P_3$, and "NOT INTERSECT" otherwise.

Procedure: Compute the coefficient of the principal term (42). If (42) is negative, go to the Case (I); otherwise, go to the case (II).

7. Conclusion and Future Work

This paper constructed the procedure for checking the topological changes which occur at adding a new circle in the incremental method of constructing a circle Voronoi diagram. This procedure uses the function constructed from the equation which holds at the special configuration of the four circles, that is, they are inscribed by the common circle. For our purpose, additional functions are also used in order to make the procedure, because the above function alone is not sufficient for checking the topological information, Finally we evaluate the precision necessary for the exact computation.

Future work includes the following. First is to extend this procedure to the three dimensional case. Another work is to apply symbolic perturbation technique. They are future work to our final goal of designing a robust geometric algorithm for the circle Voronoi diagram. Last is to implement our procedure, and then to consider the acceleration of the computation. In this paper, we assume the exact computation is done in all the processes, but we do not have to use the exact computation, if we can retain the precision when we use the real value computation. Hence, we should use a hybrid method of the exact and real value computation. To this purpose, we have to devise a method to switch the exact and the real, using the precision-guaranteed computation.

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A. Inscribed circle

Assume that three circles are inscribed by both of the circle $P_0 = \{(0,0), a\}$ and the circle $P_s = \{(c,0), b\}$, where $0 < a, c$ and $0 < b < a + c$. Let a given circle $P = \{(x,y), r\}$ be simultaneously inscribed by P_0 and circumscribed P_s:

$$x^2 + y^2 = (a+r)^2 \tag{68}$$

$$(x-c)^2 + y^2 = (b-r)^2 \tag{69}$$

From Eqn. (68) and (69), we obtain

$$r = \frac{b^2 - a^2 - c^2 + 2cx}{2(b+a)} \tag{70}$$

Substituting Eqn. (70) into Eqn. (68), we get

$$\frac{(x-\frac{c}{2})^2}{\left(\frac{b+a}{2}\right)^2}+\frac{y^2}{\left(\frac{\sqrt{(b+a)^2-c^2}}{2}\right)^2}=1. \tag{71}$$

This means that the center of the given circle is on the locus of the ellipse (71). Moreover, when parametrically representing this ellipse with a polar coordinate system, we get

$$x=\frac{b+a}{2}\cos\theta+\frac{c}{2}, \tag{72}$$

$$y=\frac{\sqrt{(b+a)^2-c^2}}{2}\sin\theta, \tag{73}$$

and

$$r=\frac{c}{2}\cos\theta+\frac{b-a}{2} \tag{74}$$

by substituting Eqn. (72) into Eqn. (70).

Consequently, we obtain the circle which is inscribed by the inscribing circle and circumscribed by the circumscribing circle as

$$P=\{(x,y),r\}=\left\{\left(\frac{b+a}{2}\cos\theta_i+\frac{c}{2},\frac{\sqrt{(b+a)^2-c^2}}{2}\sin\theta_i\right),\frac{c}{2}\cos\theta_i+\frac{b-a}{2}\right\}. \tag{75}$$

B. Circumscribed circle

Assume that three circles are inscribed by both of the circle $P_0=\{(0,0),a\}$ and the circle $P_s=\{(c,0),b\}$, where $0<a,c$ and $0<b<a+c$. Let a given circle $P=\{(x,y),r\}$ be simultaneously inscribed by two circles

$$x^2+y^2=(a+r)^2 \tag{76}$$

$$(x-c)^2+y^2=(b+r)^2 \tag{77}$$

From Eqn. (76) and (77), we obtain

$$r=\frac{a^2-b^2+c^2-2cx}{2(b-a)} \tag{78}$$

Substituting Eqn. (78) into Eqn. (76), we get

$$\frac{(x-\frac{c}{2})^2}{\left(\frac{b-a}{2}\right)^2}-\frac{y^2}{\left(\frac{\sqrt{c^2-(b-a)^2}}{2}\right)^2}=1. \tag{79}$$

This means that the center of the given circle is on the locus of the ellipse (79). Moreover, when parametrically representing this ellipse with a polar coordinate system, we get

$$x=\frac{b-a}{2}\frac{1}{\cos\theta}+\frac{c}{2}, \tag{80}$$

$$y=\frac{\sqrt{c^2-(b-a)^2}}{2}\tan\theta, \tag{81}$$

and

$$r=-\frac{c}{2}\frac{1}{\cos\theta}+\frac{a+b}{2} \tag{82}$$

by substituting Eqn. (80) into Eqn. (78).

Consequently, we obtain the circle inscribed by two circle as

$$P=\{(x,y),r\}=\left\{\left(\frac{b-a}{2}\frac{1}{\cos\theta}+\frac{c}{2},\frac{\sqrt{c^2-(b-a)^2}}{2}\tan\theta\right),-\frac{c}{2}\frac{1}{\cos\theta}+\frac{a+b}{2}\right\}. \tag{83}$$

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