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METR 2008–12 March 2008
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Note on the Continuity of M-convex and L-convex Functions in Continuous Variables

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March 2008

Abstract

M-convex and L-convex functions in continuous variables constitute subclasses of convex functions with nice combinatorial properties. In this note we give proofs of the fundamental facts that closed proper M-convex and L-convex functions are continuous on their effective domains.

1 Introduction

Two kinds of convexity concepts, called M-convexity and L-convexity, play primary roles in the theory of discrete convex analysis [6]. They are originally introduced for functions in integer variables by Murota [4, 5], and then for functions in continuous variables by Murota–Shioura [8, 10].

M-convex and L-convex functions in continuous variables constitute subclasses of convex functions with additional combinatorial properties such as submodularity and diagonal dominance (see, e.g., [6, 7, 8, 9, 10, 11]). Fundamental properties of M-convex and L-convex functions are investigated in [9], such as equivalent axioms, subgradients, directional derivatives, etc. Conjugacy relationship between M-convex and L-convex functions under the Legendre-Fenchel transformation is shown in [10]. Subclasses of M-convex and L-convex functions are investigated in [8] (polyhedral M-convex and L-convex functions) and in [11] (quadratic M-convex and L-convex functions).
As variants of M-convex and L-convex functions, the concepts of M^3-convex and L^3-convex functions are also introduced by Murota–Shioura [8, 10].

M-convex and L-convex functions in continuous variables appear naturally in various research areas. In inventory theory, a recent paper of Zipkin [13] sheds a new light on some classical results of Karlin–Scarf [2] and Morton [3] by pointing out that the optimal-cost function possesses L^3-convexity. Quadratic L^3-convex functions are exactly the same as the (finite dimensional case of) Dirichlet forms used in probability theory [1]. It is shown in [7, Section 14.8] that for (the finite dimensional distribution of) stochastic processes such as Gaussian processes and additive processes, cumulant generating functions and rate functions are M^3-convex and L^3-convex, respectively. The energy consumed in a nonlinear electrical network is an L^3-convex function when expressed as a function in terminal voltages, and is an M^3-convex function as a function in terminal currents [6, Section 2.2].

In this note, we discuss continuity issues of M-convex and L-convex functions in continuous variables. Although continuity is one of the most fundamental properties of functions, discussion on continuity is missing in the literature of M-convex and L-convex functions. The aim of this note is to give proofs of the facts that closed proper M-convex and L-convex functions are continuous on their effective domains. The main results of this note are summarized as follows, where the precise definitions of closed proper M-convex and L-convex functions are given in Section 2.1.

**Theorem 1.1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \).

(i) If \( f \) is closed proper M-convex, then it is continuous on \( \text{dom} \, f \).

(ii) If \( f \) is closed proper M^3-convex, then it is continuous on \( \text{dom} \, f \).

**Theorem 1.2.** Let \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \).

(i) If \( g \) is closed proper L-convex, then it is continuous on \( \text{dom} \, g \).

(ii) If \( g \) is closed proper L^3-convex, then it is continuous on \( \text{dom} \, g \).

It may be mentioned that our proof of Theorem 1.2 shows that an L-convex (L^3-convex) function is upper semi-continuous even if it is not closed.

## 2 Preliminaries

### 2.1 M-convex and L-convex Functions

A function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is said to be **M-convex** if it is convex and satisfies (M-EXC):

\[
\text{(M-EXC)} \forall x, y \in \text{dom} \, f, \forall i \in \text{supp}^+(x-y), \exists j \in \text{supp}^-(x-y), \exists \alpha_0 > 0 \text{ satisfying} \\
\quad f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in [0, \alpha_0]),
\]

\[\quad \text{where } \chi_i, \chi_j \text{ are elements of a lattice in } \mathbb{R}^n.\]
where $\chi_i \in \{0,1\}^n$ denotes the characteristic vector of $i \in N = \{1,2,\ldots,n\}$, and

\[
\begin{align*}
\text{dom } f &= \{x \in \mathbb{R}^n \mid f(x) < +\infty\}, \\
\text{supp}^+(x-y) &= \{i \in N \mid x(i) > y(i)\}, \\
\text{supp}^-(x-y) &= \{i \in N \mid x(i) < y(i)\}.
\end{align*}
\]

We call a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $M^\cdot$-convex if the function $\hat{f} : \mathbb{R}^{\hat{N}} \to \mathbb{R} \cup \{+\infty\}$ defined by

\[
\hat{f}(x_0, x) = \begin{cases} 
  f(x) & \text{if } (x_0, x) \in \mathbb{R}^{\hat{N}}, x_0 = -x(N), \\
  +\infty & \text{otherwise}
\end{cases}
\]

is $M$-convex, where $\hat{N} = \{0\} \cup N$ and $x(N) = \sum_{i \in N} x(i)$. An $M$-convex (resp., $M^\cdot$-convex) function is said to be closed proper $M$-convex (resp., closed proper $M^\cdot$-convex) if it is closed and proper, in addition.

A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be $L$-convex if it is convex and satisfies (LF1) and (LF2):

\[
\begin{align*}
(\text{LF1}) \quad &g(p) + g(q) \geq g(p \land q) + g(p \lor q) \quad (\forall p, q \in \text{dom } g), \\
(\text{LF2}) \quad &\exists r \in \mathbb{R} : g(p + r1) = g(p) + r \quad (\forall p \in \text{dom } g, \forall r \in \mathbb{R}),
\end{align*}
\]

where $p \land q, p \lor q \in \mathbb{R}^n$ are given by

\[
(p \land q)(i) = \min\{p(i), q(i)\}, \quad (p \lor q)(i) = \max\{p(i), q(i)\} \quad (i \in N).
\]

We call a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $L^\cdot$-convex if the function $\hat{g} : \mathbb{R}^{\hat{N}} \to \mathbb{R} \cup \{+\infty\}$ defined by

\[
\hat{g}(p_0, p) = g(p - p_01) \quad ((p_0, p) \in \mathbb{R}^{\hat{N}})
\]

is $L$-convex, where $\hat{N} = \{0\} \cup N$. An $L$-convex (resp., $L^\cdot$-convex) function is said to be closed proper $L$-convex (resp., closed proper $L^\cdot$-convex) if it is closed and proper, in addition.

### 2.2 Basic Facts from Convex Analysis

As technical preliminaries we describe some facts known in convex analysis. This also serves to illustrate the present issue.

**Theorem 2.1** ([12, Theorem 10.1]). *Any convex function is continuous on the relative interior of the effective domain.*

Theorem 2.1 implies, in particular, that a convex function is continuous on the effective domain if the effective domain is an open set.

On the other hand, a convex function is not necessarily continuous at relative boundary points of the effective domain, even if it is closed proper convex, as shown in the following example.
Example 2.2 ([12, Section 10]). Let $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be a function defined by

$$f(x, y) = \begin{cases} \frac{y^2}{2x} & (x > 0), \\ 0 & (x = y = 0), \\ +\infty & (\text{otherwise}), \end{cases}$$

which is closed proper convex since its epigraph $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq f(x, y)\}$ is a closed convex set. It is easy to see that $f$ is continuous at every point of $\text{dom } f$, except at the origin $(x, y) = (0, 0)$. For any positive number $\alpha$, we have

$$\lim_{y \to 0} f\left(\frac{y^2}{2\alpha}, y\right) = \lim_{y \to 0} \alpha = \alpha \neq 0 = f(0, 0),$$

which shows that $f$ is not continuous at the origin.

A sufficient condition for a closed proper convex function to be continuous on the effective domain is given in terms of “locally simplicial” sets. A subset $S$ of $\mathbb{R}^n$ is said to be locally simplicial if for each $x \in S$ there exists a finite collection of simplices $T_1, T_2, \ldots, T_m$ contained in $S$ such that

$$U \cap (T_1 \cup T_2 \cup \cdots \cup T_m) = U \cap S$$

for some neighborhood $U$ of $x$. The class of locally simplicial sets includes line segments, polyhedra, and relatively open convex sets.

Theorem 2.3 ([12, Theorem 10.2]). Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function. For a locally simplicial set $S \subseteq \text{dom } f$, the function $f$ is continuous on $S$. In particular, $f$ is continuous on $\text{dom } f$ if $\text{dom } f$ is locally simplicial.

3 Continuity of Closed Proper M-/L-convex Functions

We now consider the continuity of closed proper M-/L-convex functions.

The effective domains of closed proper M-/L-convex functions are “essentially polyhedral” in the sense that the closure of the effective domains are polyhedra (see Theorems 3.2 and 3.3 below). Hence, the continuity of closed proper M-/L-convex functions follows from Theorem 2.3 when the effective domains are closed sets. The effective domains of closed proper M-/L-convex functions, however, are not necessarily closed, as shown in the following example.

Example 3.1. Let $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function defined by

$$\phi(x) = \begin{cases} \frac{1}{x} & (0 < x \leq 1) \\ +\infty & (\text{otherwise}). \end{cases}$$
Then, $\varphi$ is a closed proper convex function such that the effective domain $\text{dom} \varphi$ is an interval $\{x \in \mathbb{R} \mid 0 < x \leq 1\}$, which is neither a closed set nor a relatively open set.

Using $\varphi$ we define functions $f, g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$f(x, y) = \begin{cases} \varphi(x) & (x + y = 0), \\ +\infty & (x + y \neq 0), \end{cases} \quad ((x, y) \in \mathbb{R}^2),$$

$$g(x, y) = \varphi(x - y) \quad ((x, y) \in \mathbb{R}^2).$$

Then, $f$ and $g$ are closed proper M-convex and L-convex functions, respectively. Neither $\text{dom} f$ nor $\text{dom} g$ is a closed set.

Although the effective domains are not always closed, they are well-behaved and almost polyhedral, as follows.

A polyhedron $S \subseteq \mathbb{R}^n$ is said to be M-convex (resp., $M^\mathbb{R}$-convex, L-convex, $L^\mathbb{R}$-convex) if the indicator function $\delta_S : \mathbb{R}^n \to \{0, +\infty\}$ defined by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S) \end{cases}$$

is M-convex (resp., $M^\mathbb{R}$-convex, L-convex, $L^\mathbb{R}$-convex). For any set $S \subseteq \mathbb{R}^n$ we denote by $\text{cl}(S)$ the closure of $S$, i.e., the smallest closed set containing $S$.

**Theorem 3.2.** For any closed proper M-convex (resp., $M^\mathbb{R}$-convex) function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the set $\text{cl}(\text{dom} f)$ is an M-convex (resp., $M^\mathbb{R}$-convex) polyhedron.

*Proof.* The proof is given in Section 4.1. \hfill \Box

**Theorem 3.3.** For any closed proper L-convex (resp., $L^\mathbb{R}$-convex) function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the set $\text{cl}(\text{dom} g)$ is an L-convex (resp., $L^\mathbb{R}$-convex) polyhedron.

*Proof.* The proof is given in Section 4.2. \hfill \Box

**Theorem 3.4.** The effective domain of a closed proper M-convex (resp., $M^\mathbb{R}$-convex) function is a locally simplicial set.

*Proof.* The proof is given in Section 4.3. \hfill \Box

**Theorem 3.5.** The effective domain of a closed proper L-convex (resp., $L^\mathbb{R}$-convex) function is a locally simplicial set.

*Proof.* The proof is given in Section 4.4. \hfill \Box

The continuity of closed proper M-/L-convex functions, as claimed in Theorems 1.1 and 1.2, follows from Theorems 2.3, 3.4, and 3.5.
4 Proofs

4.1 Proof of Theorem 3.2

For any closed proper convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), we define a function \( f^0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) by

\[
(f^0)(y) = \lim_{\lambda \to \infty} \frac{f(x + \lambda y) - f(x)}{\lambda} \quad (y \in \mathbb{R}^n),
\]

where \( x \in \mathbb{R}^n \) is any fixed vector in \( \text{dom} \ f \). The function \( f^0 \) is called the recession function of \( f \) (see [12] for the original definition of the recession function). The recession function \( f^0 \) is a positively homogeneous closed proper convex function. Our proof of Theorem 3.2 is based on the following fact.

**Theorem 4.1 ([12, Theorem 13.3]).** For any closed proper convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), the recession function \( f^0 \) is the support function of \( \text{dom} \ f^\bullet \).

It suffices to consider a closed proper \( M \)-convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \). Then, its conjugate function \( g = f^\bullet \) is a closed proper \( L \)-convex function [10, Theorem 1.1]. As shown below, the recession function \( g^0 \) of \( g \) is \( L \)-convex. This implies that the support function of (the closure of) \( \text{dom} \ f^\bullet \) is a positively homogeneous \( L \)-convex function, which in turn implies that \( \text{cl}(\text{dom} \ f^\bullet) \) is an \( M \)-convex polyhedron [8, Theorem 4.38].

We now show the \( L \)-convexity of the recession function \( g^0 \). Namely, we prove that \( g^0 \) satisfies (LF1) and (LF2).

Let \( p_0 \in \text{dom} \ g \) be any fixed vector. Then, the recession function \( g^0 \) is given as

\[
(g^0)(p) = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda} \quad (p \in \mathbb{R}^n).
\]

Since \( g \) satisfies (LF2), there exists \( r \in \mathbb{R} \) such that

\[
g(p + \alpha \mathbf{1}) = g(p) + \alpha r \quad (\forall p \in \text{dom} \ g, \forall \alpha \in \mathbb{R}). \tag{4.1}
\]

For any \( p \in \text{dom} \ g^0 \) and \( \alpha \in \mathbb{R} \), we have

\[
(g^0)(p + \alpha \mathbf{1}) = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda (p + \alpha \mathbf{1})) - g(p_0)}{\lambda} = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda p) + \lambda \alpha r - g(p_0)}{\lambda} = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda} + \alpha r = (g^0)(p) + \alpha r,
\]

where the second equality is by (4.1). Hence, (LF2) holds for \( g^0 \).
Let \( p, q \in \text{dom} \, g 0^+ \). For any \( \lambda \in \mathbb{R}_+ \), we have
\[
g(p_0 + \lambda p) + g(p_0 + \lambda q) \geq g(p_0 + \lambda (p \land q)) + g(p_0 + \lambda (p \lor q))
\]
by (LF1) for \( g \). Hence, we have
\[
g 0^+(p) + g 0^+(q) \leq \frac{\lim_{\lambda \to \infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda}}{\lambda} + \frac{\lim_{\lambda \to \infty} \frac{g(p_0 + \lambda q) - g(p_0)}{\lambda}}{\lambda} \geq g 0^+(p \land q) + g 0^+(p \lor q),
\]
i.e., (LF1) holds for \( g 0^+ \).

### 4.2 Proof of Theorem 3.3

It suffices to consider a closed proper \( L \)-convex function \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+, \infty\} \). The properties (LF1) and (LF2) for \( g \) imply that \( D = \text{dom} \, g \) satisfies the following properties:

- (LS1) \( p, q \in D \implies p \land q, p \lor q \in D \),
- (LS2) \( p \in D \implies p + 1 \in D \, (\forall \lambda \in \mathbb{R}) \).

Therefore, Theorem 3.3 follows immediately from the next theorem.

**Theorem 4.2.** For any nonempty set \( D \subseteq \mathbb{R}^n \), let
\[
\gamma_D(i, j) = \sup\{p(j) - p(i) \mid p \in D\} \quad (i, j \in N),
\]
\[
\tilde{D} = \{p \in \mathbb{R}^n \mid p(j) - p(i) \leq \gamma_D(i, j) \, (i, j \in N)\}.
\]
If \( D \) satisfies (LS1) and (LS2), then we have \( \text{cl}(D) = \tilde{D} \).

**Proof.** The inclusion \( \text{cl}(D) \subseteq \tilde{D} \) is easy to see. To prove the reverse inclusion, we show that \( q \in D \) holds for any vector \( q \) in the relative interior of \( \tilde{D} \).

We first show that for any \( i, j \in N \) there exists \( p_{ij} \in D \) such that
\[
p_{ij}(j) - p_{ij}(i) \geq q(j) - q(i).
\]
If \( -\gamma_D(j, i) = \gamma_D(i, j) \), then any vector in \( D \) can be chosen as \( p_{ij} \) since for any \( p \in D \) we have \( p(j) - p(i) = \gamma_D(i, j) = q(j) - q(i) \). Hence, we suppose that \( -\gamma_D(j, i) < \gamma_D(i, j) \) holds. Then, we have \( q(j) - q(i) < \gamma_D(i, j) \) since \( q \) is in the relative interior of \( \tilde{D} \). By the definition of \( \gamma_D(i, j) \), there exists some \( p_{ij} \in D \) such that \( q(j) - q(i) \leq p_{ij}(j) - p_{ij}(i) \leq \gamma_D(i, j) \).

By (LS2), we may assume that \( p_{ij}(i) = q(i) \) and \( p_{ij}(j) \geq q(j) \). For each \( i \in N \), the vector \( p_i = \bigvee_{j \in N} p_{ij} \) (\( \in D \)) satisfies \( p_i(i) = q(i) \), \( p_i(j) \geq q(j) \) for all \( j \in N \). Therefore, it holds that \( q = \bigwedge_{i \in N} p_i \in D. \)
4.3 Proof of Theorem 3.4

For any set \( S \subseteq \mathbb{R}^n \) and a vector \( x \in S \), we denote by \( \text{cone}(S, x) \) the conic hull of the vectors \( \{ y - x \mid y \in S \} \), i.e., \( \text{cone}(S, x) \) is the set of vectors \( d \in \mathbb{R}^n \) such that \( d = \sum_{k=1}^{m} \alpha_k (y_k - x) \) for some positive integer \( m \) and \( y_k \in S, \alpha_k > 0 \) \((k = 1, 2, \ldots, m)\). The following is immediate from the definition of locally simplicial sets.

**Lemma 4.3.** A convex set \( S \subseteq \mathbb{R}^n \) is locally simplicial if for each \( x \in S \), \( \text{cone}(S, x) \) is a polyhedral cone.

For the proof of Theorem 3.4 it suffices to consider an M-convex function. Then, Theorem 3.4 follows from Lemma 4.3 and the following lemma.

**Lemma 4.4.** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\} \) be a closed proper M-convex function. For any \( x \in \text{dom} f \), it holds that

\[
\text{cone}(\text{dom} f, x) = \text{cone}(R_x, x),
\]

where \( R_x \subseteq \mathbb{R}^n \) is a polyhedral cone given by

\[
R_x = \{ x_j - x_i \mid i, j \in \mathbb{N}, i \neq j, x + \alpha(x_j - x_i) \in \text{dom} f \text{ for some } \alpha > 0 \}.
\]

To prove Lemma 4.4 we use the following properties of M-convex functions.

**Lemma 4.5 ([10, Proposition 2.2]).** If \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\} \) is closed proper M-convex, then \( x(N) = y(N) \) for all \( x, y \in \text{dom} f \).

**Lemma 4.6 ([10, Theorem 3.11]).** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\} \) be a closed proper convex function. Then, \( f \) satisfies (M-EXC) if and only if it satisfies (M-EXC):

\[
(M\text{-EXC}_N) \forall x, y \in \text{dom} f, \forall i \in \text{supp}^+(x - y), \exists j \in \text{supp}^-(x - y):
\]

\[
f(x) + f(y) \geq f(x - \alpha(x_i - x_j)) + f(y + \alpha(x_i - x_j)) \quad (\forall \alpha \in [0, \alpha_0(x, y, i)]),
\]

where

\[
\alpha_0(x, y, i) = \frac{x(i) - y(i)}{2|\text{supp}^-(x - y)|}.
\]

**Proof of Lemma 4.4.** It is easy to see that \( \text{cone}(R_x, x) \subseteq \text{cone}(\text{dom} f, x) \). To prove the reverse inclusion, it suffices to show that \( y - x \in \text{cone}(R_x, x) \) for any \( y \in \text{dom} f \).

We will show that there exists a sequence of vectors \( y_k \) \((k = 0, 1, 2, \ldots)\) such that \( y_0 = y \) and

\[
y_k \in \text{dom} f, \ y_k \neq x, \ y - y_k \in \text{cone}(R_x, x) \quad (k = 0, 1, 2, \ldots), \quad (4.2)
\]

\[
||y_{k+1} - x||_1 \leq (1 - \frac{1}{2n^2})||y_k - x||_1. \quad (4.3)
\]
This implies that \( y - x = \lim_{k \to \infty} (y - y_k) \in \text{cone}(R_x, x) \), since \( \text{cone}(R_x, x) \) is a closed set.

We define the vectors \( y_k \) \((k = 0, 1, 2, \ldots)\) iteratively as follows. Suppose that \( y_k \) is already defined and satisfies the condition (4.2). Since \( y_k \neq x \), we have \( \text{supp}^+(y_k - x) \neq \emptyset \). Let \( i \in \text{supp}^+(y_k - x) \) be such that
\[
y_k(i) - x(i) = \max\{y_k(i') - x(i') \mid i' \in \text{supp}^+(y_k - x)\}.
\] (4.4)

By Lemma 4.6, there exists \( j \in \text{supp}^-(y_k - x) \) such that
\[
y_k - \alpha(\chi_i - \chi_j) \in \text{dom} f, \quad x + \alpha(\chi_i - \chi_j) \in \text{dom} f,
\]
where \( \alpha = (y_k(i) - x(i))/2n \). Then, \( y_{k+1} \) is defined as \( y_{k+1} = y_k - \alpha(\chi_i - \chi_j) \).

We now show that the vector \( y_{k+1} \) satisfies the conditions (4.2) and (4.3). Since \( y_{k+1}(i) > x(i) \), we have \( y_{k+1} \neq x \). Since \( x + \alpha(\chi_i - \chi_j) \in \text{dom} f \), we have \( \chi_i - \chi_j \in R_x \), which, together with \( y - y_k \in \text{cone}(R_x, x) \), implies
\[
y - y_{k+1} = (y - y_k) + \alpha(\chi_i - \chi_j) \in \text{cone}(R_x, x).
\]

Since \( y_k(N) = x(N) \) by Lemma 4.5, it holds that
\[
\|y_k - x\|_1 = 2 \sum \{y_k(i') - x(i') \mid i' \in \text{supp}^+(y_k - x)\}
\leq 2n(y_k(i) - x(i))
= 4n^2 \alpha,
\]
where the inequality is by (4.4). Hence, it holds that
\[
\|y_{k+1} - x\|_1 = \|y_k - x\|_1 - 2 \alpha \leq (1 - \frac{1}{2n^2})\|y_k - x\|_1.
\]

\(\square\)

### 4.4 Proof of Theorem 3.5

Theorem 3.5 follows from Lemma 4.3 and Lemma 4.7 below. Note that it suffices to consider an L-convex function.

**Lemma 4.7.** Let \( g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) be an L-convex function. For any \( p \in \text{dom} g \), it holds that
\[
\text{cone}(\text{dom} g, p) = \text{cone}(R_p, p),
\]
where \( R_p \subseteq \mathbb{R}^n \) is a polyhedral cone given by
\[
R_p = \{\chi_X \mid X \subset N, \quad p + \alpha \chi_X \in \text{dom} g \text{ for some } \alpha > 0\} \cup \{1, -1\}.
\]

To prove Lemma 4.7, we use the following property of L-convex functions.
Lemma 4.8 ([9, Proposition 3.10]). If $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is $L$-convex, then we have
\[
g(p) + g(q) \geq g(p + \lambda \chi_X) + g(q - \lambda \chi_X)
\]
for all $p, q \in \text{dom } g$ and $\lambda \in [0, \lambda_1 - \lambda_2]$, where $\chi_X \in \{0, 1\}^n$ denotes the characteristic vector of $X \subseteq N$, and
\[
\lambda_1 = \max \{q(i) - p(i) \mid i \in N\}, \\
X = \{i \in N \mid q(i) - p(i) = \lambda_1\}, \\
\lambda_2 = \max \{q(i) - p(i) \mid i \in N \setminus X\}.
\]

Proof of Lemma 4.7. It is easy to see that $\text{cone}(R_p, p) \subseteq \text{cone}(\text{dom } g, p)$, where it is noted that $p + \alpha \mathbf{1} \in \text{dom } g$ for all $\alpha \in \mathbb{R}$. To show the reverse inclusion, it suffices to show that $q - p \in \text{cone}(R_p, p)$ for any $q \in \text{dom } g$.

By (LF2) for $g$, we may assume that $p \leq q$ and $p(i_0) = q(i_0)$ for some $i_0 \in N$. We prove $q - p \in \text{cone}(R_p, p)$ by induction on the number $m$ of distinct values in $\{q(i) - p(i) \mid i \in N\}$.

If $m = 0$, then we have $q - p = 0 \in \text{cone}(R_p, p)$. Hence, we assume $m > 0$, which implies $q(i_1) > p(i_1)$ for some $i_1 \in N$. By Lemma 4.8, we have
\[
p + (\lambda_1 - \lambda_2) \chi_X \in \text{dom } g, \quad q - (\lambda_1 - \lambda_2) \chi_X \in \text{dom } g,
\]
where
\[
\lambda_1 = \max \{q(i) - p(i) \mid i \in N\}, \\
X = \{i \in N \mid q(i) - p(i) = \lambda_1\}, \\
\lambda_2 = \max \{q(i) - p(i) \mid i \in N \setminus X\}.
\]
We note that $\lambda_1$ and $\lambda_2$ are finite values and $X$ is a nonempty proper subset of $N$. Put $\tilde{q} = q - (\lambda_1 - \lambda_2) \chi_X$. Then, the number of distinct values in $\{\tilde{q}(i) - p(i) \mid i \in N\}$ is equal to $m - 1$. Therefore, the induction hypothesis implies $\tilde{q} - p \in \text{cone}(R_p, p)$. We also have $\chi_X \in R_p$ since $p + (\lambda_1 - \lambda_2) \chi_X \in \text{dom } g$. Hence, it holds that
\[
q - p = (\tilde{q} - p) + (\lambda_1 - \lambda_2) \chi_X \in \text{cone}(R_p, p).
\]

\[\square\]

References


