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Sensitivity Analysis of Networked Control Systems via an Information Theoretic Approach

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Abstract

This paper deals with an MIMO feedback control system that has two channels with additive noise and studies the effects of the noise on the input and output signals of the plant. We derive integral-type limitations for sensitivity-like properties of the feedback system based on an information theoretic approach. It is shown that they are generalizations of Bode's integral formula for the case that the feedback system includes nonlinear elements.

1 Introduction

The rapid development in communication technologies and the growth of computer networks have enabled us to implement feedback control systems utilizing communication channels. Research on such networked control systems has recently attracted much attention; see, e.g., [1] and the references therein. A typical system setup can be found in Fig. 1. Here, for the communication between the plant and the controller, various constraints may arise including time delays, data losses, quantization/coding errors. Such constraints may be harmful and cause degradation in performance and even instability of the closed-loop system.

To deal with these issues, it is important to evaluate the amount of information that the communication signals contain regarding the plant and the controller. This view has motivated analyses of networked control systems based on notions and results from information theory. For example, channels can be characterized by their capacity and rate of communication, which represent the numbers of bits that can be transferred at each time

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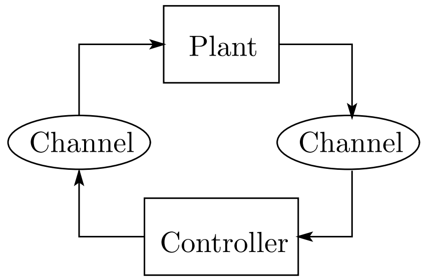

Figure 1: Networked control system.

step. The results in [6, 13] give conditions on the channels in terms of the communication rate for the existence of stabilizing controllers, encoders, and decoders.

Furthermore, in the approach based on information theory, the focus is on signals rather than on systems representing input-output relations. Hence, in certain cases, we may relax the assumptions on systems and extend prior results in control theory which have been limited to linear time-invariant systems to systems with nonlinear elements.

One such result can be found in [5]; in this work, a sensitivity property of feedback systems with linear plant and nonlinear controllers is analyzed by measuring the entropy of the signals. In particular, it provides a lower bound on the gain of a sensitivity-like function which is expressed by the unstable poles of the plant. On the other hand, a well-known relation between a sensitivity property and the unstable poles of the plant is given by Bode's integral formula [2, 4, 11]. While Bode's integral formula deals only with linear systems, the result in [5] extends it to systems with nonlinear controllers. In our prior work [7], this approach has been followed to characterize a complementary sensitivity property by the unstable zeros and the direct feedthrough term of the plant. Though only linear systems are considered there, employing an information theoretic approach, it has shown the possibility of extending Bode's integral formula for complementary sensitivity functions [12] to nonlinear systems. From the viewpoint of networked control systems, these results show certain limitations on the reduction of the effects of channel noises.

In this paper, we consider networked control systems based on the approach of [5, 7]. Specifically, we analyze the effects of channel noises on the input and output signals of the plant in a unified manner. These effects are measured by four sensitivity-like functions including those for the sensitivity and the complementary sensitivity properties. We derive constraints on these functions described by the plant properties such as the unstable poles/zeros and direct feedthrough terms. Moreover, the constraints are extensions of the results in [5, 7] to multi-input multi-output (MIMO) systems and a class of nonlinear controllers.

This paper is organized as follows: We first introduce some notions and results in information theory in Section 2. Then, in Section 3, we formulate the problem of the paper and introduce related previous works. The main results of the paper are given in Section 4. This is followed by Section 5, where important properties required to derive the results are presented. In Section 6, we illustrate the results through numerical examples. Finally, we state concluding remarks in Section 7.

In this paper, we adopt the following notation.

- We represent random variables using boldface letters such as $\mathbf{x}$.
- Consider a discrete-time stochastic process $\{\mathbf{x}(k)\}_{k=0}^{\infty}$. We represent a sequence of random variables from $k = l$ to $k = m$ ($m \geq l$) as $\mathbf{x}_l^m := \{\mathbf{x}(k)\}_{k=l}^{m}$. In particular, when $l = 0$, we write $\mathbf{x}_l^m$ simply as $\mathbf{x}^m$.
- We use $\mathbf{x}$ instead of $\{\mathbf{x}(k)\}_{k=0}^{\infty}$ when it is clear from the context.
- The operation $\mathrm{E}[\cdot]$ denotes the expectation of a random variable.

2 Entropy and mutual information

In this section, we introduce some notation and basic results from information theory that we use in the paper.

Entropy is a notion widely used as a measure of uncertainty contained in a random variable. It is defined as follows.

Definition 1. *The (differential) entropy $h(\mathbf{x})$ of a continuous random variable $\mathbf{x} \in \mathbb{R}^m$ with the (joint) probability density $p_{\mathbf{x}}$ is defined as*

$$h(\mathbf{x}) := -\int_{\mathbb{R}^m} p_{\mathbf{x}}(\xi) \log p_{\mathbf{x}}(\xi) \mathrm{d}\xi.$$

Next, we introduce mutual information, which is a measure of the amount of information that one random variable possesses about another random variable.

Definition 2. *The mutual information $I(\mathbf{x};\mathbf{y})$ between $\mathbf{x} \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}$ is defined as*

$$I(\mathbf{x};\mathbf{y}) = h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}),$$

where $h(\cdot|\cdot)$ represent the conditional entropy.

Note that in the above definitions, we assume the existence of the probability density functions and the joint probability density functions of the random variables.

We now list some of the basic properties of entropy and mutual information which are required in the paper. Their proofs can be found in, e.g., [3, 8, 9].

- *Symmetry and nonnegative property:*

$$\begin{aligned} I(\mathbf{x};\mathbf{y}) &= I(\mathbf{y};\mathbf{x}) \\ &= h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}|\mathbf{x}) \geq 0 \end{aligned} \tag{1}$$

- *Entropy and conditional entropy:* From the above property, the following holds:

$$h(\mathbf{x}|\mathbf{y}) \leq h(\mathbf{x}). \tag{2}$$

- *Chain rule:*

$$h(\mathbf{x},\mathbf{y}) = h(\mathbf{x}) + h(\mathbf{y}|\mathbf{x}) \tag{3}$$

- *Data processing inequality:* Suppose that f is a measurable function on the appropriate space. Then, the following holds:

$$h(\mathbf{x}|\mathbf{y}) \leq h(\mathbf{x}|f(\mathbf{y})). \tag{4}$$

  We have equality if f is invertible.

- *Transformations of random variables and their entropy:* Suppose that f is a piecewise C^1-class injective function and $\mathbf{x}$ and $\mathbf{y} = f(\mathbf{x})$ take continuous values. Then, the following holds:

$$h(\mathbf{y}) = h(\mathbf{x}) + \mathrm{E}\left[\log |J_f(\mathbf{x})|\right], \tag{5}$$

  where J_f is the Jacobian of the transformation f.

- Suppose that f is any given function on the appropriate space. Then, the following holds:

$$h(\mathbf{x} - f(\mathbf{y})|\mathbf{y}) = h(\mathbf{x}|\mathbf{y}). \tag{6}$$

Next, we turn our attention to stochastic processes and introduce some notions. The entropy rate is an asymptotic time average of the entropy of a process and plays an important role in our analysis.

Definition 3. *The entropy rate* $h_\infty(\mathbf{x})$ *of a stochastic process* $\mathbf{x}$ *is defined as*

$$h_\infty(\mathbf{x}) := \limsup_{k\to\infty} \frac{h(\mathbf{x}^{k-1})}{k}.$$

The stochastic processes appearing in this paper are of weak stationarity as defined below. For such processes, their power spectral densities can be obtained.

Definition 4. *A zero-mean stochastic process* $\mathbf{x}$ *(*$\mathbf{x}(k) \in \mathbb{R}^m$*) is weakly stationary if for every* $\gamma \in \mathbb{Z}$*, the following equations hold:*

$$\mathrm{E}[\mathbf{x}(k+\gamma)] = \mathrm{E}[\mathbf{x}(\gamma)],$$
$$\mathrm{E}[\mathbf{x}(k+\gamma)\mathbf{x}(k)^\top] = \mathrm{E}[\mathbf{x}(\gamma)\mathbf{x}(0)^\top].$$

For a weakly stationary process $\mathbf{x}$*, we can define the power spectral density* $\Phi_\mathbf{x}$ *using* $R_\mathbf{x}(\gamma) := \mathrm{E}[\mathbf{x}(\gamma)\mathbf{x}(0)^\top] = \mathrm{E}[\mathbf{x}(k+\gamma)\mathbf{x}(k)^\top]$ *as*

$$\Phi_\mathbf{x}(\omega) := \sum_{\gamma=-\infty}^{\infty} R_\mathbf{x}(\gamma)\mathrm{e}^{-\mathrm{j}\gamma\omega}.$$

The following lemma gives an explicit relation between the entropy rate and the power spectral density.

Lemma 1 ([8, 14])**.** *If* $\mathbf{x}$ *is a weakly stationary process, then the following equation holds:*

$$h_\infty(\mathbf{x}) = \frac{1}{2}\log(2\pi\mathrm{e})^m + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log\det\Phi_\mathbf{x}(\omega)\mathrm{d}\omega.$$

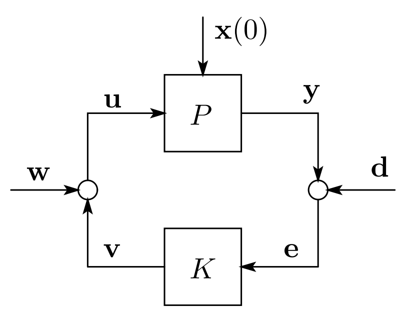

Figure 2: Model of a networked control system.

3 Problem Setting

In this section, we formulate the problem considered in this paper and present several related works in the literature.

As a model of networked control systems in Fig. 1, we consider the feedback system depicted in Fig. 2. The system has two analog channels with additive noises $\mathbf{w}$ and $\mathbf{d}$. We assume that the noises $\mathbf{w}$ and $\mathbf{d}$ are weakly stationary stochastic processes. The plant P is an m-input m-output discrete-time linear time-invariant system which consists of a discretized continuous-time system and some time delays. Let a state-space representation of P be given by

$$P: \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{y}(k) \end{bmatrix} = \begin{bmatrix} A_P & B_P \\ C_P & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{bmatrix},$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state, $\mathbf{u}(k) \in \mathbb{R}^m$ is the input, and $\mathbf{y}(k) \in \mathbb{R}^m$ is the output. It is assumed that there exists $\nu \in \mathbb{N}$ such that

$$\begin{aligned} C_P A_P^{i-1} B_P = 0, \quad \text{for} \quad i = 1, 2, \cdots, \nu - 1, \\ \det C_P A_P^{\nu-1} B_P \neq 0. \end{aligned} \tag{7}$$

Then, let $D_P := C_P A_P^{\nu-1} B_P$. It is known that when a continuous-time system is discretized with sampling period τ, this assumption is satisfied with $\nu = 1$ for almost all τ. We should take an appropriate ν when P has time delays due to communication and/or computation in the controller.

The controller K is an m-input m-output dynamical nonlinear system represented as

$$K: \quad \begin{aligned} \mathbf{z}(k+1) &= f(\mathbf{z}(k), \mathbf{e}(k)), \\ \mathbf{v}(k) &= g(\mathbf{z}(k)) + \phi(\mathbf{e}(k)), \end{aligned} \tag{8}$$

where $\mathbf{z}(k) \in \mathbb{R}^{n_K}$ is the state, $\mathbf{e}(k) \in \mathbb{R}^m$ is the input, and $\mathbf{v}(k) \in \mathbb{R}^m$ is the output. Here, f and g are arbitrary nonlinear functions, and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a piecewise C^1-class injective function.

Regarding the random variables in the system, we assume that $\mathbf{x}(0)$, $\mathbf{w}^k$, and $\mathbf{d}^k$ are mutually independent for every $k \in \mathbb{Z}_+$, and $|h(\mathbf{x}(0))| < \infty$. In particular, the assumption $|h(\mathbf{x}(0))| < \infty$ implies that the initial condition $x(0)$ is neither completely known nor completely unknown. As we will see later, this assumption is characteristic to the information theoretic approach.

In this paper, we analyze the effects of the noises $\mathbf{w}$ and $\mathbf{d}$ on the input $\mathbf{u}$ and the output $\mathbf{y}$ of the plant. These effects are measured by the sensitivity-like functions defined in the following.

Definition 5. *Consider the system depicted in Fig. 2. Assume that* $\mathbf{u}$ *and* $\mathbf{y}$ *are weakly stationary processes. Denote by* $\Phi_{\mathbf{s}}$ *and* $\Phi_{\mathbf{t}}$ *the power spectral densities of the signals* $\mathbf{s} \in \{\mathbf{w}, \mathbf{d}\}$ *and* $\mathbf{t} \in \{\mathbf{u}, \mathbf{y}\}$*. The sensitivity-like function* $\overline{T}_{\mathbf{ts}}(\omega)$ *from* $\mathbf{s}$ *to* $\mathbf{t}$ *is the ratio of these power spectral densities given by*

$$\overline{T}_{\mathbf{ts}}(\omega) := \sqrt{\frac{\det \Phi_{\mathbf{t}}(\omega)}{\det \Phi_{\mathbf{s}}(\omega)}}, \quad \mathbf{s} = \mathbf{w}, \mathbf{d}, \ \mathbf{t} = \mathbf{u}, \mathbf{y}. \tag{9}$$

We now show that the sensitivity-like function $\overline{T}_{\mathbf{ts}}$ is closely related to the transfer function from $\mathbf{s}$ to $\mathbf{t}$. If the controller K is linear, we can define the corresponding transfer functions $T_{\mathbf{ts}}(z)$ as

$$\begin{aligned} T_{\mathbf{uw}}(z) &:= (I - K(z)P(z))^{-1}, \\ T_{\mathbf{yw}}(z) &:= P(z)(I - K(z)P(z))^{-1}, \\ T_{\mathbf{ud}}(z) &:= (I - K(z)P(z))^{-1}K(z), \\ T_{\mathbf{yd}}(z) &:= P(z)(I - K(z)P(z))^{-1}K(z). \end{aligned}$$

When we consider a transfer function, we implicitly assume that its initial state is zero. We note that, under this assumption, if a transfer function $G(z)$ is stable, then the following equation holds:

$$\left(\det G(\mathrm{e}^{\mathrm{j}\omega})\right)^2 = \frac{\det \Phi_{\mathbf{y}}(\omega)}{\det \Phi_{\mathbf{x}}(\omega)},$$

where $\mathbf{x}$, $\mathbf{y}$ are the input and output signals of $G(z)$, and $\Phi_{\mathbf{x}}$, $\Phi_{\mathbf{y}}$ are their power spectral densities, respectively [8]. Thus, in the special case that, in the system in Fig. 2, the initial state $\mathbf{x}(0)$ is zero, the following relation holds:

$$\overline{T}_{\mathbf{ts}}(\omega) = |\det T_{\mathbf{ts}}(\mathrm{e}^{\mathrm{j}\omega})|, \quad \mathbf{s} = \mathbf{w}, \mathbf{d}, \ \mathbf{t} = \mathbf{u}, \mathbf{y}. \tag{10}$$

In general, this equality is important since it clarifies the relation between the ratio of power spectral densities and the transfer function.

However, we emphasize that, in the problem setting of this paper, this equality does not hold. This is because the assumption that the initial state $x(0)$ of the plant P is zero implies $h(\mathbf{x}(0)) = -\infty$. Hence, $\mathbf{x}(0)$ does not satisfy the assumption $|h(\mathbf{x}(0))| < \infty$. Therefore, within the scope of this paper, the relation (10) does not hold.

In the analysis of this paper, the plant properties of poles and zeros are important.

Definition 6. *Let* $\mathcal{UP}_P$ *and* $\mathcal{UZ}_P$ *represent the sets of unstable poles and unstable zeros of* P*, respectively:*

$$\begin{aligned} \mathcal{UP}_P &:= \{z \,|\, |z| \geq 1, \, \det(A_P - zI) = 0\}, \\ \mathcal{UZ}_P &:= \left\{z \mid |z| \geq 1, \, F_P(z) < \max_{\lambda \in \mathbb{C}} F_P(\lambda)\right\}, \end{aligned}$$

where

$$F_P(z) := \det \begin{bmatrix} A_P - zI & B_P \\ C_P A_P^{\nu} & D_P \end{bmatrix}.$$

We next present previous works related to the analysis of this paper. First, in the work of Martins *et al.* [5], for the case $m = 1$, the sensitivity property $\overline{T}_{\mathbf{uw}}$ is analyzed based on an information theoretic approach. They have focused on the entropy and mutual information of $\mathbf{w}$ and $\mathbf{u}$ and have clarified a relation between $\overline{T}_{\mathbf{uw}}$ and the unstable poles of P. The following theorem holds by applying the result in [5] to our problem.

Theorem 1 ([5])**.** *Consider the system depicted in Fig. 2. Suppose that P is a single-input single-output (SISO) discrete-time linear time-invariant system, and K is an arbitrary causal system. If $\mathbf{u}$ and $\mathbf{y}$ are weakly stationary and $\sup_k \mathrm{E}[\mathbf{x}(k)^\top \mathbf{x}(k)] < \infty$, then the following relation holds:*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \overline{T}_{\mathbf{uw}}(\omega) \right| \mathrm{d}\omega \geq \sum_{\lambda \in \mathcal{UP}_P} \log |\lambda| .$$

Next, in our prior paper [7], we have followed the approach of [5] and have analyzed the complementary sensitivity property $\overline{T}_{\mathbf{yd}}$ for the special case of $m = 1$ and linear K, i.e., the controller K is in the form as

$$K: \begin{bmatrix} \mathbf{z}(k+1) \\ \mathbf{v}(k) \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} \mathbf{z}(k) \\ \mathbf{e}(k) \end{bmatrix}. \tag{11}$$

The following theorem has been shown.

Theorem 2 ([7])**.** *Consider the system depicted in Fig. 2. Suppose that P and K are SISO discrete-time linear time-invariant systems. If $\sup_k \mathrm{E}[\mathbf{x}(k)^\top \mathbf{x}(k)] < \infty$, then the following relation holds:*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \overline{T}_{\mathbf{yd}}(\omega) \right| \mathrm{d}\omega \geq \sum_{\beta \in \mathcal{UZ}_P} \log |\beta| + \log |D_K|.$$

We remark that Theorems 1 and 2 are similar to Bode's integral formula for the sensitivity and complementary sensitivity functions, $T_{\mathbf{uw}}(z)$ and $T_{\mathbf{yd}}(z)$ (see [11, 12], and Theorem 4 in this paper). The formula has given a well-known trade-off property, i.e., the water-bed effects on the gains of $T_{\mathbf{uw}}(z)$ and $T_{\mathbf{yd}}(z)$ in the frequency domain. In our problem, the gain of $T_{\mathbf{uw}}(z)$ corresponds to the effect of the noise $\mathbf{w}$ on the input $\mathbf{u}$ of plant, and the gain of $T_{\mathbf{yd}}(z)$ corresponds to the effect of the noise $\mathbf{d}$ on the output $\mathbf{y}$ of P.

Theorems 1 and 2 have been derived based on an information theoretic approach. In contrast, Bode's integral formula has been shown using complex analysis. Moreover, Theorem 1 extends Bode's formula in the sense that the theorem deals with an arbitrary causal nonlinear controller, while Bode's formula treats only linear controllers.

In this paper, we follow the approach in [5, 7], and extend Theorems 1 and 2 to square MIMO systems which includes a nonlinear controller K represented in (8). In addition, we discuss all four properties of the feedback system for the noises defined in (9) in a unified manner.

4 Sensitivity analysis of networked control systems

4.1 Main result

In this subsection, we present the main result of this paper. Let J_ϕ be the Jacobian of the function ϕ in the controller K in (8), and let

$$\mathcal{D}_K := \liminf_{k\to\infty} \frac{\sum_{i=0}^{k} \mathrm{E}\left[\log |J_\phi(\mathbf{e}(i))|\right]}{k}.$$

Here, $\mathcal{D}_K$ represents (the logarithm of) the gain of K. If K is a linear system such as that in (11), this $\mathcal{D}_K$ is reduced to the simpler form $\mathcal{D}_K = \log|\det D_K|$, which is determined by the direct feedthrough term D_K.

The following theorem provides integral-type constraints on the sensitivity-like functions $\overline{T}_{\mathbf{ts}}$.

Theorem 3. *Consider the system depicted in Fig. 2. If* $\mathbf{u}$ *and* $\mathbf{y}$ *are weakly stationary and* $\sup_k \mathrm{E}[\mathbf{x}(k)^\top \mathbf{x}(k)] < \infty$*, then the following relations hold:*

$$\text{i)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\overline{T}_{\mathbf{uw}}(\omega)\right| \mathrm{d}\omega \geq \sum_{\lambda\in\mathcal{UP}_P} \log|\lambda|, \tag{12}$$

$$\text{ii)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\overline{T}_{\mathbf{yw}}(\omega)\right| \mathrm{d}\omega \geq \sum_{\beta\in\mathcal{UZ}_P} \log|\beta| + \log|\det D_P|, \tag{13}$$

$$\text{iii)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\overline{T}_{\mathbf{ud}}(\omega)\right| \mathrm{d}\omega \geq \sum_{\lambda\in\mathcal{UP}_P} \log|\lambda| + \mathcal{D}_K, \tag{14}$$

$$\text{iv)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\overline{T}_{\mathbf{yd}}(\omega)\right| \mathrm{d}\omega \geq \sum_{\beta\in\mathcal{UZ}_P} \log|\beta| + \log|\det D_P| + \mathcal{D}_K. \tag{15}$$

Proof. We immediately obtain this theorem from Proposition 1 and Lemma 3, which are presented in Section 5. □

Several remarks regarding this result are in order. First, Theorem 3 shows that there are lower bounds on the reduction of the effects of the noises **w** and **d**. Furthermore, these bounds can be expressed by the unstable poles/zeros of P and the direct feedthrough terms of P and K. Second, in Theorem 3, the relations in (12) and (15) extend those in Theorems 1 and 2 to a more general setup and in particular to a class of m-input m-output nonlinear controllers. It is also noted that similarly to Theorem 1, the bound in (12) holds for any nonlinear controller that is causal and hence is not restricted to K in the form of (8). Third, we obtain the constraints on $\overline{T}_{\mathbf{yw}}$ and $\overline{T}_{\mathbf{ud}}$ in (13) and (14), respectively, which are similar to the those on $\overline{T}_{\mathbf{uw}}$ and $\overline{T}_{\mathbf{yd}}$.

4.2 Comparison with Bode's integral formula

As we have described in Section 3, the sensitivity-like function $\overline{T}_{\mathbf{ts}}$ is the ratio of power spectral densities and corresponds to the transfer function $T_{\mathbf{ts}}$ from $\mathbf{s}$ to $\mathbf{t}$. In this subsection, we present a result on $T_{\mathbf{ts}}(z)$ based on complex analysis. This is a generalization of Bode's integral formula.

Suppose that in Fig. 2, the plant is SISO ($m = 1$) and also that K is a linear time-invariant system as in (11). The following theorem provides the constraint on $T_{\mathbf{ts}}(z)$.

Theorem 4. *Consider the system depicted in Fig. 2. If the feedback system is stable, then the following equalities hold:*

$$\text{i)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|T_{\mathbf{uw}}(\mathrm{e}^{\mathrm{j}\omega})\right| \mathrm{d}\omega = \sum_{\lambda\in\mathcal{UP}_P} \log|\lambda| + \sum_{\lambda\in\mathcal{UP}_K} \log|\lambda|, \tag{16}$$

$$\text{ii)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|T_{\mathbf{yw}}(\mathrm{e}^{\mathrm{j}\omega})\right| \mathrm{d}\omega = \sum_{\beta\in\mathcal{UZ}_P} \log|\beta| + \sum_{\lambda\in\mathcal{UP}_K} \log|\lambda| + \log|D_P|, \tag{17}$$

$$\text{iii)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|T_{\mathbf{ud}}(\mathrm{e}^{\mathrm{j}\omega})\right| \mathrm{d}\omega = \sum_{\lambda\in\mathcal{UP}_P} \log|\lambda| + \sum_{\beta\in\mathcal{UZ}_K} \log|\beta| + \log|D_K|, \tag{18}$$

$$\text{iv)}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|T_{\mathbf{yd}}(\mathrm{e}^{\mathrm{j}\omega})\right| \mathrm{d}\omega = \sum_{\beta\in\mathcal{UZ}_P} \log|\beta| + \sum_{\beta\in\mathcal{UZ}_K} \log|\beta| + \log|D_K| + \log|D_P|. \tag{19}$$

Proof. The relations (16) and (19) are known as Bode's integral formula and have been shown in [11] and [12], respectively. The relations (17) and (18) can be easily derived using Jensen's formula [10]. □

Now, we have independently obtained two theorems similar to each other, Theorems 3 and 4. Note that Theorem 3, our main result, shows inequality constraints, while in Theorem 4 the constraints are in equalities. From our analysis, it is unclear when the equalities hold and moreover whether we can show a condition for the equality to hold by the information theoretic approach. However, as we described in Section 4.1, Theorem 3 can deal with more general systems than Theorem 4.

Also note that in Theorem 3, the unstable poles and zeros of the controller K do not appear in the constraints. This is due to the assumption that the initial state $\mathbf{z}(0)$ of K is deterministic. Although, this assumption is the reason for the inequalities as we mentioned above, it enables us to deal with the nonlinear controller K in (8).

To summarize the results obtained so far, we list the differences between Theorems 3 and 4 as follows:

- Theorem 3
  - Analysis of the ratios of power spectral densities
  - Time-domain
  - Square MIMO system
  - A class of nonlinear controllers
  - Constraints in inequalities
- Theorem 4
  - Analysis of the transfer functions
  - Frequency-domain
  - SISO system
  - Linear controllers
  - Constraints in equalities

5 The entropy rates of input and output signals

In this section, we give some results which are required to derive our main result.

We note that because of the relation given by Lemma 1, the ratio $\overline{T}_{\mathbf{ts}}$ of power spectral densities can be expressed as the difference in the entropy rates. Hence, in this section, we analyze the entropy rates of the input signal $\mathbf{s}$ and the output signal $\mathbf{t}$ to evaluate the ratio $\overline{T}_{\mathbf{ts}}$ of power spectral densities.

The following proposition gives the relation between the entropy rates $h_\infty(\mathbf{s})$ and $h_\infty(\mathbf{t})$.

Proposition 1. *Consider the system depicted in Fig. 2. The following inequalities hold:*

$$\text{i)}\quad h_\infty(\mathbf{u}) - h_\infty(\mathbf{w}) \geq \liminf_{k\to\infty} \frac{I(\mathbf{u}^k; \mathbf{x}(0))}{k}, \tag{20}$$

$$\text{ii)}\quad h_\infty(\mathbf{y}) - h_\infty(\mathbf{w}) \geq \liminf_{k\to\infty} \frac{I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0))}{k} + \log|\det D_P|, \tag{21}$$

$$\text{iii)}\quad h_\infty(\mathbf{u}) - h_\infty(\mathbf{d}) \geq \liminf_{k\to\infty} \frac{I(\mathbf{u}^k; \mathbf{x}(0))}{k} + \mathcal{D}_K, \tag{22}$$

iv) $h_\infty(\mathbf{y}) - h_\infty(\mathbf{d}) \geq \liminf_{k\to\infty} \frac{I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0))}{k} + \log|\det D_P| + \mathcal{D}_K.$ (23)

This proposition gives the relation between the entropy rates of the input signal $\mathbf{s}$ and the output signal $\mathbf{t}$. It is noted that no assumption is made on the stability of the feedback system.

Proposition 1 can be shown by a conservation law between the entropies of $\mathbf{s}$ and $\mathbf{t}$. We describe this in the following as a lemma.

Lemma 2. *Consider the system depicted in Fig. 2. The following relations holds:*

$$\text{i)}\quad h(\mathbf{u}^k) = h(\mathbf{w}^k) + I(\mathbf{u}^k; \mathbf{x}(0)) + \sum_{i=0}^{k} I(\mathbf{u}(i); \mathbf{d}^{i-1}|\mathbf{x}(0), \mathbf{u}^{i-1}), \tag{24}$$

$$\begin{aligned}\text{ii)}\quad h(\mathbf{y}_\nu^{k+\nu}) &= h(\mathbf{w}^k) + I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0)) + (k+1)\log|\det D_P| \\ &\quad + \sum_{i=0}^{k} I(\mathbf{u}(i); \mathbf{d}^{i-1}|\mathbf{x}(0), \mathbf{u}^{i-1}),\end{aligned} \tag{25}$$

$$\begin{aligned}\text{iii)}\quad h(\mathbf{u}^k) &= h(\mathbf{d}^k) + I(\mathbf{u}^k; \mathbf{x}(0)) + \sum_{i=0}^{k} \mathrm{E}\left[\log|J_\phi(\mathbf{e}(i))|\right] \\ &\quad + \sum_{i=0}^{k} I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{x}(0), \mathbf{u}^{i-1}),\end{aligned} \tag{26}$$

$$\begin{aligned}\text{iv)}\quad h(\mathbf{y}_\nu^{k+\nu}) &= h(\mathbf{d}^k) + I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0)) + (k+1)\log|\det D_P| \\ &\quad + \sum_{i=0}^{k} \mathrm{E}\left[\log|J_\phi(\mathbf{e}(i))|\right] + \sum_{i=0}^{k} I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{x}(0), \mathbf{u}^{i-1}).\end{aligned} \tag{27}$$

Proof. Here, we give the proof for (27). The equation (27) describes the relation between the entropy of the input signal at time k, $h(\mathbf{d}(k))$, and that of the output signal, $h(\mathbf{y}(k))$. Hence, we should study how $h(\mathbf{d}(k))$ affects $h(\mathbf{y}(k))$. However, because of the time delay of ν steps within P, $\mathbf{d}(k)$ has an influence on the output $\mathbf{y}$ only after time $k + \nu$.

To deal with this problem, we define the auxiliary system P_0 and the signal $\mathbf{y}^+$ as

$$\begin{aligned} P_0(z) &:= z^\nu P(z), \\ \mathbf{y}^+(k) &:= \mathbf{y}(k+\nu). \end{aligned}$$

The state-space representation of P_0 is given by

$$P_0: \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{y}^+(k) \end{bmatrix} = \begin{bmatrix} A_P & B_P \\ C_P A_P^\nu & D_P \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{bmatrix}.$$

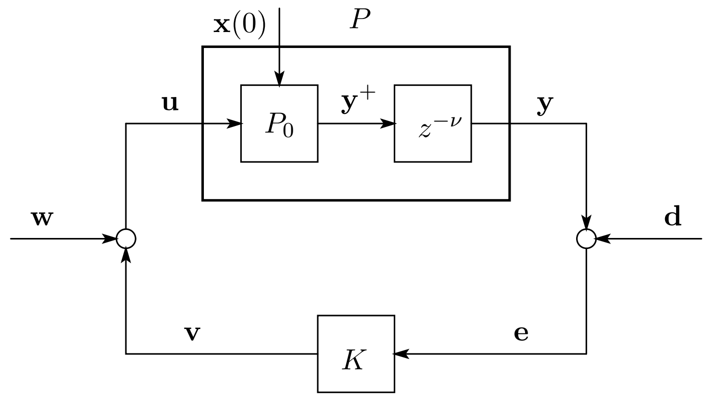

Figure 3: Equivalent system with P_0.

We have $\det D_P = \det C_P A_P^{\nu-1} B_P \neq 0$ because of (7), and hence P_0 is a biproper system. By using P_0 and $\mathbf{y}^+$, the system in Fig. 2 can be expressed as that in Fig. 3.

We now consider a conservation law between the entropies of $\mathbf{d}$ and $\mathbf{y}^+$. It follows that

$$\begin{aligned} h(\mathbf{y}^+(i)|(\mathbf{y}^+)^{i-1}) &= h(\mathbf{y}^+(i)|(\mathbf{y}^+)^{i-1}, \mathbf{x}(0)) + I(\mathbf{y}^+(i); \mathbf{x}(0)|(\mathbf{y}^+)^{i-1}) \\ &= h(\mathbf{u}(i)|\mathbf{u}^{i-1}, \mathbf{x}(0)) + \log|\det D_P| \\ &\qquad + I(\mathbf{y}^+(i); \mathbf{x}(0)|(\mathbf{y}^+)^{i-1}), \end{aligned} \tag{28}$$

where the first equality follows by (1) and the second one follows by (4), (5), and (6). Moreover, the first term on the right-hand side of (28) can be expressed as

$$\begin{aligned} h(\mathbf{u}(i)|\mathbf{u}^{i-1}, \mathbf{x}(0)) &= h(\mathbf{u}(i)|\mathbf{u}^{i-1}, \mathbf{x}(0), \mathbf{w}^{i-1}) + I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{u}^{i-1}, \mathbf{x}(0)) \\ &= h(\mathbf{d}(i)|\mathbf{d}^{i-1}, \mathbf{x}(0), \mathbf{w}^{i-1}) + \mathrm{E}[\log|J_\phi(\mathbf{e}(i))|] \\ &\quad + I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{u}^{i-1}, \mathbf{x}(0)). \end{aligned}$$

Since $\mathbf{x}(0)$, $\mathbf{w}$, and $\mathbf{d}$ are mutually independent, $\mathbf{x}(0)$ and $\mathbf{w}^{i-1}$ vanish in the first term on the right-hand side of the last equation. Thus, we have

$$\begin{aligned} h(\mathbf{u}(i)|\mathbf{u}^{i-1}, \mathbf{x}(0)) &= h(\mathbf{d}(i)|\mathbf{d}^{i-1}) + \mathrm{E}[\log|J_\phi(\mathbf{e}(i))|] \\ &\qquad + I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{u}^{i-1}, \mathbf{x}(0)). \end{aligned} \tag{29}$$

We substitute (29) into (28) and obtain

$$\begin{aligned} h(\mathbf{y}^+(i)|(\mathbf{y}^+)^{i-1}) &= h(\mathbf{d}(i)|\mathbf{d}^{i-1}) + I(\mathbf{y}^+(i); \mathbf{x}(0)|(\mathbf{y}^+)^{i-1}) \\ &\quad + \log|\det D_P| + \mathrm{E}[\log|J_\phi(\mathbf{e}(i))|] \\ &\quad + I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{u}^{i-1}, \mathbf{x}(0)). \end{aligned}$$

Now, by summing both sides of the above equation for $i = 0, 1, ..., k$, we have (27).

The other equations (24)–(26) can be proven in a similar way. □

Lemma 2 shows that a conservation law of entropy holds in the feedback system in Fig. 2. For example, (27) shows such a law between the entropies of $\mathbf{d}$ and $\mathbf{y}$. Other terms in (27) can be explained as follows. The terms $\log|\det D_P|$ and $\mathrm{E}[\log|J_\phi|]$ reflect the scaling caused by the direct feedthrough terms of P and K (see (5)). The terms of mutual information $I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0))$ and $I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{u}^{i-1}, \mathbf{x}(0))$ show the effects of $\mathbf{x}(0)$ and $\mathbf{w}$; these can be viewed as external inputs to the feedback system other than $\mathbf{d}$, which is the input signal we focus on.

Proof of Proposition 1. We give the proof of (23). In (27), we have

$$\sum_{i=0}^{k} I(\mathbf{u}(i); \mathbf{w}^{i-1}|\mathbf{x}(0), \mathbf{u}^{i-1}) \geq 0$$

from the property (1). Thus, it follows that

$$\begin{aligned} h(\mathbf{y}_\nu^{k+\nu}) \geq h(\mathbf{d}^k) + I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0)) + (k+1)\log|\det D_P| \\ + \sum_{i=0}^{k} \mathrm{E}\left[\log|J_\phi(\mathbf{e}(i))|\right]. \end{aligned}$$

We obtain (23) by dividing the above inequality by k and taking the limsup as $k \to \infty$ on both sides. □

We next show as a lemma that the lower bounds in Proposition 1 can be further bounded by the unstable poles or the unstable zeros of the plant P.

Lemma 3 ([5, 7])**.** *Consider the system depicted in Fig. 2. If the signals* $\mathbf{u}$ *and* $\mathbf{y}$ *are weakly stationary and* $\sup_k \mathrm{E}[\mathbf{x}(k)^\top \mathbf{x}(k)] < \infty$*, then the following inequalities hold:*

$$\liminf_{k\to\infty} \frac{I(\mathbf{u}^k; \mathbf{x}(0))}{k} \geq \sum_{\lambda \in \mathcal{UP}_P} \log|\lambda|, \tag{30}$$

$$\liminf_{k\to\infty} \frac{I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0))}{k} \geq \sum_{\beta \in \mathcal{UZ}_P} \log|\beta|. \tag{31}$$

We have some remarks regarding this result. In general, from the viewpoint of the open-loop system, when the system is unstable, the system amplifies the initial state at a level depending on the size of the unstable poles. Hence, we can say that in systems having more unstable dynamics, the signals contain more information about the initial state. Therefore, in Fig. 2, we can expect the mutual information between the input $\mathbf{u}$ and $\mathbf{x}(0)$ to be a function of the unstable poles. The relation (30) corresponds to this observation. Also, (31) corresponds to the same observation about the inverse system of P.

Table 1: Integrals of ratios of power spectral densities versus their theoretical lower bounds.

$K = K_1$	$\frac{1}{2\pi}\int \log \lvert\bar{T}_{\mathbf{ts}}(\omega)\rvert \mathrm{d}\omega$	$\mathrm{LB}_{\mathrm{Bode}}$	$\mathrm{LB}_{\mathrm{Th.3}}$
$\bar{T}_{\mathbf{uw}}$	1.3794	0.1553	0.0721
$\bar{T}_{\mathbf{yw}}$	−5.2786	−6.5243	−6.6075
$\bar{T}_{\mathbf{ud}}$	1.4182	0.7487	0.7487
$\bar{T}_{\mathbf{yd}}$	−5.2398	−5.9309	−5.9309

$K = K_2$	$\frac{1}{2\pi}\int \log \lvert\bar{T}_{\mathbf{ts}}(\omega)\rvert \mathrm{d}\omega$	$\mathrm{LB}_{\mathrm{Bode}}$	$\mathrm{LB}_{\mathrm{Th.3}}$
$\bar{T}_{\mathbf{uw}}$	6.2312	0.5929	0.0721
$\bar{T}_{\mathbf{yw}}$	−0.4484	−6.0867	−6.6075
$\bar{T}_{\mathbf{ud}}$	6.2319	6.2305	6.2305
$\bar{T}_{\mathbf{yd}}$	−0.4477	−0.4491	−0.4491

6 Numerical examples

In this section, we illustrate the results presented in Section 4 through numerical examples.

6.1 Feedback systems with two additive noise channels

Consider the system shown in Fig. 2. Suppose that $m = 1$ and P is an unstable non-minimum phase system given by the transfer function

$$P(z) = \frac{0.01005z - 0.01026}{z^2 - 2.031z + 1.03}. \tag{32}$$

We consider two systems K_1, K_2 as stabilizing controllers for the plant P. The transfer functions of K_1 and K_2 are given as follows:

$$K_1(z) = \frac{-1.598z^3 + 1.414z^2 + 1.595z - 1.418}{z^3 - 2.72z^2 + 2.444z - 0.7248}, \tag{33}$$

$$K_2(z) = \frac{-71.43z^3 + 212z^2 - 209.7z + 69.16}{z^3 - 3.423z^2 + 3.842z - 1.418}. \tag{34}$$

We take $\mathbf{x}(0)$, $\mathbf{w}(k)$, and $\mathbf{d}(k)$ as Gaussian (pseudo) random variables with mean 0 and variance 1, and compute $\mathbf{u}$ and $\mathbf{y}$ from time 0.00 to 100.00 [sec] with sampling period 0.01 [sec]. Then, we calculate the ratios $\bar{T}_{\mathbf{ts}}$ of power spectral densities and examine the inequalities in Theorem 3.

In Table 1, we show the result of the simulations. Here, $\mathrm{LB}_{\mathrm{Th.3}}$ and $\mathrm{LB}_{\mathrm{Bode}}$ denote the lower bounds given by Theorems 3 and 4, respectively. Note that although Table 1 shows the data for certain samples of random

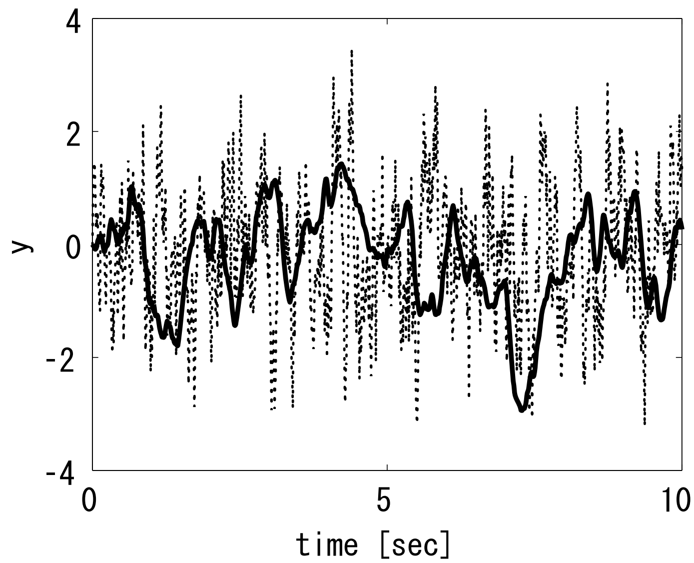

Figure 4: Sampling paths of $\mathbf{y}$ for K_1 and K_2 (solid line: $K = K_1$, dotted line: $K = K_2$).

variables $\mathbf{x}(0)$, $\mathbf{w}$, and $\mathbf{d}$, we have repeated the simulation for various samples and have obtained similar data.

In Table 1, we can observe that all inequalities in Theorems 3 and 4 hold for both cases $K = K_1$ and $K = K_2$. Moreover, the lower bounds given by these theorems are fairly close to the values obtained by the simulations. This shows that the results are not conservative for these systems.

We note that the properties of K_1 and K_2 are significantly different. The controller K_1 aims at achieving high stability with small gain, while K_2 aims at assuring not stability but agility. Fig. 4 shows the difference between K_1 and K_2 in the sample paths of the output signal $\mathbf{y}$. Here, the solid line is the response of $\mathbf{y}$ for the case $K = K_1$ and the dotted line is for the case $K = K_2$. We can see that the signal changes rapidly when $K = K_2$. As we have just described, although properties of the feedback system are different depending on the controller, Table 1 shows that inequalities in Theorem 3 hold for both controllers.

6.2 Feedback systems with quantizers

In this subsection, we consider networked control systems including digital channels. For communication through a digital channel, we have to convert analog signals to digital signals by a quantizer and to represent them in a finite number of bits. In such feedback systems, we should consider effects of errors between analog signals and quantized digital signals. The channels

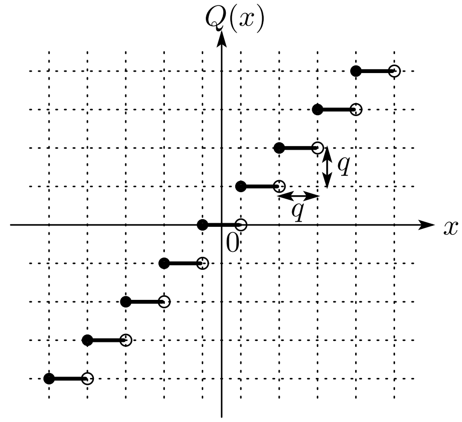

Figure 5: The input and the output of is uniform quantizer Q.

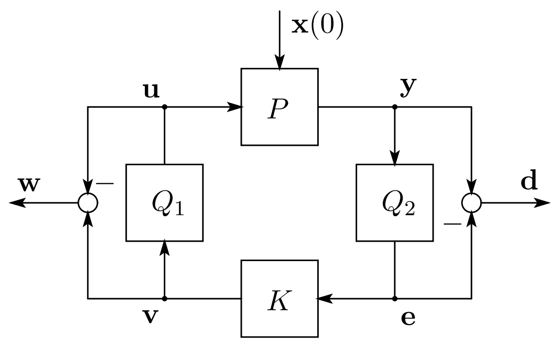

Figure 6: Model of a networked control system including quantizers.

with additive noises depicted in Fig. 2 are often used as a model of these errors due to quantization, especially for uniform quantizers. The uniform quantizer Q with quantization step size q is shown in Fig. 5.

Consider the feedback system in Fig. 6, where P is the plant and its initial state $\mathbf{x}(0)$ is a random variable, K is a stabilizing controller, and Q_1 and Q_2 are uniform quantizers. Here, we compute the ratio $\overline{T}_{\mathbf{ts}}$ of power spectral densities of the quantization error $\mathbf{s} \in \{\mathbf{w}, \mathbf{d}\}$ and $\mathbf{t} \in \{\mathbf{u}, \mathbf{y}\}$, and examine the relations in Theorem 3.

Suppose that the plant P and the controller K are given as in (32) and (33). Also, the quantizers Q_1 and Q_2 have a common quantization step size denoted by q. We consider two quantization steps of $q = 0.1$ and $q = 0.5$. Other settings of the simulation are the same as those in the previous subsection.

We show the result of the simulation and the difference in $\mathbf{y}$ due to the

Table 2: Integrals of ratios of power spectral densities versus their theoretical lower bounds.

$q = 0.1$	$\frac{1}{2\pi}\int \log\lvert\bar{T}_{\mathbf{ts}}(\omega)\rvert \mathrm{d}\omega$	$\mathrm{LB}_{\mathrm{Bode}}$	$\mathrm{LB}_{\mathrm{Th.3}}$
$\bar{T}_{\mathbf{uw}}$	1.5719	0.1553	0.0721
$\bar{T}_{\mathbf{yw}}$	−4.8736	−6.5243	−6.6075
$\bar{T}_{\mathbf{ud}}$	1.6543	0.7487	0.7487
$\bar{T}_{\mathbf{yd}}$	−4.7912	−5.9309	−5.9309

$q = 0.5$	$\frac{1}{2\pi}\int \log\lvert\bar{T}_{\mathbf{ts}}(\omega)\rvert \mathrm{d}\omega$	$\mathrm{LB}_{\mathrm{Bode}}$	$\mathrm{LB}_{\mathrm{Th.3}}$
$\bar{T}_{\mathbf{uw}}$	1.5677	0.1553	0.0721
$\bar{T}_{\mathbf{yw}}$	−4.8664	−6.5243	−6.6075
$\bar{T}_{\mathbf{ud}}$	1.6577	0.7487	0.7487
$\bar{T}_{\mathbf{yd}}$	−4.7764	−5.9309	−5.9309

quantization steps in Table 2 and Fig. 7, respectively. In Table 2, we confirm that all inequalities in Theorems 3 and 4 hold for both quantization steps. However, we note that in this setting, signals $\mathbf{w}$ and $\mathbf{d}$ are not stochastic processes, but sequences of numbers which are deterministically calculated resulting from the random variable $\mathbf{x}(0)$. This implies that the assumption on $\mathbf{w}$ and $\mathbf{d}$ being weakly stationary stochastic processes, presented in Section 3, might not be satisfied. Even though this means that Theorem 3 does not assure the properties of the system in Fig. 6, Table 2 shows that the theorem gives good approximate lower bounds for $\bar{T}_{\mathbf{ts}}$.

7 Conclusion

In this paper, we have considered a class of networked control systems and have analyzed the effects of channel noises on the input and the output of the plants by evaluating the entropy of the signals. In particular, we have uniformly derived constraints on the sensitivity-like functions expressed by the unstable poles and zeros of the plant and direct feedthrough terms. These constraints extend Bode's integral formula to a more general system. Future research will deal with finding conditions for the equalities to hold in Theorem 3.

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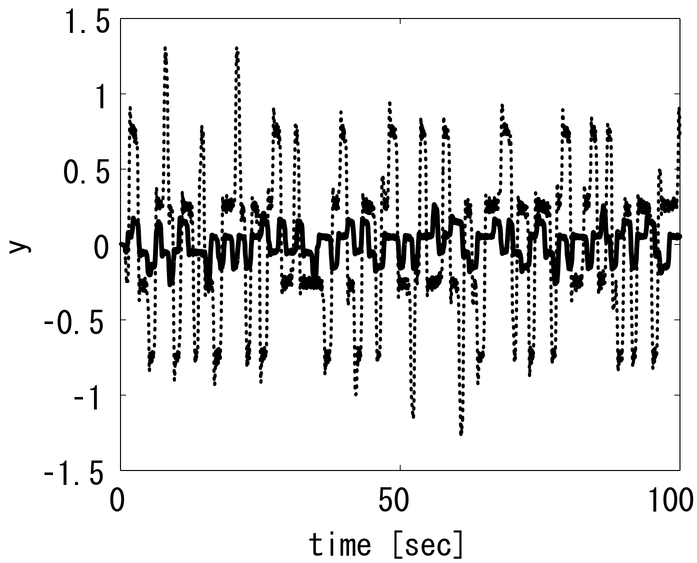

Figure 7: Sample paths of $\mathbf{y}$ with the quantization steps q (solid line: $q = 0.1$, dotted line: $q = 0.5$).

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