MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

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(Communicated by Kazuo MUROTA)

METR 2008–25 May 2008

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Sum-of-Squares Decomposition via Generalized KYP Lemma

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May 27th, 2008

Abstract

The Kalman-Yakubovich-Popov (KYP) lemma establishes the equivalence between a frequency domain inequality (FDI) of a proper rational function and a linear matrix inequality (LMI). A recent result generalized the KYP lemma to characterize an FDI of a possibly nonproper rational function on a portion of a curve on the complex plane. This note examines implications of the generalized KYP result to sum-of-squares (SOS) decompositions of matrix-valued nonnegative polynomials of a single complex variable on a curve in the complex plane. Our result generalizes and unifies some existing SOS results, and also establishes equivalences among FDI, LMI, and SOS.

1 Introduction

A fundamental problem in systems theory is to determine whether a Hermitian-valued proper transfer function $\mathcal{H}(\lambda)$ satisfies $\mathcal{H}(\lambda) \succeq 0$, or it is positive semidefinite, for all frequencies $\lambda \in \mathbf{\Lambda}_o$, where $\mathbf{\Lambda}_o$ is either the imaginary axis or the unit circle on the complex plane. The Kalman-Yakubovich-Popov (KYP) lemma [1],[2] states that such frequency domain inequality (FDI) $\mathcal{H}(\lambda) \succeq 0$ holds for all $\lambda \in \mathbf{\Lambda}_o$ if and only if there exists a Hermitian matrix P satisfying a linear matrix inequality (LMI) $M(P) \succeq 0$, thus reducing the problem to a tractable semidefinite programming (SDP) problem.

Recent results [3]–[5] generalized the standard KYP lemma so that $\mathcal{H}(\lambda)$ is possibly a nonproper rational function and λ varies over a prescribed frequency range $\mathbf{\Lambda} \subseteq \mathbf{\Lambda}_o$. The result states that $\mathcal{H}(\lambda) \succeq 0$ holds for $\lambda \in \mathbf{\Lambda}$ if and only if there exist Hermitian matrices P and Q such that LMIs $M(P, Q) \succeq 0$ and $Q \succeq 0$ hold. This extension has been shown to have a significant impact on control applications including digital filters [5], PID control [6], and robustness analysis [7].

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	matrix-valued polynomial	multiple variables	complex variables	restricted range
[12]		$\star$		$\star$
[13]	$\star$	$\star$		$\star$
[14]			$\star$	$\star$
[15]	$\star$		$\star$	
GKYP	$\star$		$\star$	$\star$

In this note, we will establish a connection between the generalized KYP lemma and the sum-of-squares (SOS) decomposition of nonnegative polynomials. The SOS representation can be viewed as an algebraic tool for systems analysis that provides a general framework to reduce important engineering problems to SDPs [8], [9]. It is well known that, if $\mathcal{H}(\lambda)$ is a scalar-valued polynomial of a real variable λ, inequality $\mathcal{H}(\lambda) \succeq 0$ holds for all real λ if and only if it can be expressed as an SOS of polynomials. We will show, using the generalized KYP lemma, that a matrix-valued polynomial $\mathcal{H}(\lambda)$ of complex variable λ is nonnegative on a prescribed set $\mathbf{\Lambda}$ if and only if $\mathcal{H}(\lambda)$ can be expressed as a weighted SOS.

There are many robust control design specifications which can be recast as positive polynomial constraints (see e.g. [10], [11]). The problem of describing a nonnegative polynomial as an SOS, and its application to structured SDPs, have been thus extensively addressed in the systems and controls literature [12]–[15]. These existing results characterized nonnegative polynomials in different settings — matrix/scalar-valued polynomials of single/multiple variable(s) that is real/complex and varies in a restricted/unrestricted range. Table 1 summarizes the results, where $\star$ indicates that the corresponding item is addressed in a more general manner. For instance, [13] treats matrix-valued polynomials of multiple real variables in restricted ranges, while [14] treats scalar-valued polynomials of a single complex variable in a restricted range (on the imaginary axis or unit circle). The last row labeled as "GKYP" indicates the case considered in this note. Thus, the main contribution of this note is to show that the generalized KYP lemma provides a complete characterization of matrix-valued nonnegative polynomials of a single complex variable on a restricted range in the complex plane, unifying the previous results in [14], [15].

We use the following notation. The set of nonnegative integers up to q is denoted by $\mathbb{Z}_q$, i.e., $\mathbb{Z}_q := \{0, 1, \ldots, q\}$. For a matrix M, its transpose, complex conjugate, and conjugate transpose are denoted by M^{T}, $\bar{M}$, and M^*, respectively. The set of $n \times n$ Hermitian matrices is denoted by $\mathbb{H}^n$. For $X \in \mathbb{H}^n$, the notation $X \succeq 0$ means that X is positive semidefinite, and the set of such matrices is denoted by $\mathbb{H}^n_+$.

2 Main Result

Consider a Hermitian-valued rational function $\mathcal{H}(\lambda)$ of complex variable $\lambda \in \mathbb{C}$. We are interested in characterizing the class of such functions that are nonnegative on $\mathbf{\Lambda} \subseteq \mathbf{\Lambda}_o$, a portion of a curve on the complex plane. In particular, the sets $\mathbf{\Lambda}_o$ and $\mathbf{\Lambda}$ are characterized by

$$\begin{array}{lll} \mathbf{\Lambda}_o & := & \{\, \lambda \in \mathbb{C} \mid \sigma(\lambda, \Phi) = 0 \,\}, \\ \mathbf{\Lambda} & := & \{\, \lambda \in \mathbb{C} \mid \sigma(\lambda, \Phi) = 0,\ \sigma(\lambda, \Psi) \geq 0 \,\}, \end{array} \tag{1}$$

where

$$\sigma(x, S) := \begin{bmatrix} x \\ 1 \end{bmatrix}^* S \begin{bmatrix} x \\ 1 \end{bmatrix},$$

and $\Phi, \Psi \in \mathbb{H}^2$ are given matrices such that $\sigma(\lambda, \Phi) = 0$ defines a curve on the complex plane and the additional constraint $\sigma(\lambda, \Psi) \geq 0$ specifies a portion of the curve. For a technical reason, the set $\mathbf{\Lambda}$ is defined to include infinity if it is unbounded.

A state space version of the generalized KYP lemma [5] can be interpreted in terms of SOS of rational functions as follows.

Theorem 1 *Let matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $\Theta \in \mathbb{H}^{n+m}$, and $\Phi, \Psi \in \mathbb{H}^2$ be given and define $\mathbf{\Lambda}$ by (1). Suppose that $\mathbf{\Lambda}$ represents curves on the complex plane, and the pair (A, B) is controllable. Let $\mathbf{\Omega}$ be the set of eigenvalues of A in $\mathbf{\Lambda}$. Consider the rational function*

$$\mathcal{H}(\lambda) := \begin{bmatrix} \mathcal{F}(\lambda) \\ I \end{bmatrix}^* \Theta \begin{bmatrix} \mathcal{F}(\lambda) \\ I \end{bmatrix}, \quad \mathcal{F}(\lambda) := (\lambda I - A)^{-1} B,$$

$$F := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}.$$

The following statements are equivalent.

(i) The function $\mathcal{H}(\lambda)$ is nonnegative on $\mathbf{\Lambda}$, that is, $\mathcal{H}(\lambda) \succeq 0$ holds for all $\lambda \in \mathbf{\Lambda}\backslash\mathbf{\Omega}$.

(ii) There exist $P \in \mathbb{H}^n$ and $Q \in \mathbb{H}^n_+$ such that

$$\Theta \succeq F^*(\Phi \otimes P + \Psi \otimes Q)F. \tag{2}$$

(iii) There exist $P \in \mathbb{H}^n$, $Q \in \mathbb{H}^n_+$ and $X \in \mathbb{H}^{n+m}_+$ such that

$$\Theta = F^*(\Phi \otimes P + \Psi \otimes Q)F + X. \tag{3}$$

(iv) There exist $P \in \mathbb{H}^n$, $Q \in \mathbb{H}^n_+$ and $X \in \mathbb{H}^{n+m}_+$ such that

$$\begin{aligned}
&\mathcal{H}(\lambda) = \sigma(\lambda,\Phi)\mathsf{P}(\lambda) + \sigma(\lambda,\Psi)\mathsf{Q}(\lambda) + \mathsf{X}(\lambda),\\
&\mathsf{P}(\lambda) := \mathcal{F}(\lambda)^* P \mathcal{F}(\lambda), \quad \mathsf{Q}(\lambda) := \mathcal{F}(\lambda)^* Q \mathcal{F}(\lambda),\\
&\mathsf{X}(\lambda) := \begin{bmatrix} \mathcal{F}(\lambda) \\ I \end{bmatrix}^* X \begin{bmatrix} \mathcal{F}(\lambda) \\ I \end{bmatrix}.
\end{aligned} \tag{4}$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has been proved in [5]. The equivalence (ii) $\Leftrightarrow$ (iii) simply follows by introducing a slack variable X. Statement (iii) implies (iv) by multiplying equation (3) by $[\ \mathcal{F}(\lambda)^* \ \ I\ ]$ from the left and its complex conjugate transpose from the right. Finally, (iv) clearly implies (i). ∎

Nonnegativity of the rational function $\mathcal{H}(\lambda)$ on $\mathbf{\Lambda}$ is equivalent to feasibility of the LMI (2) in statement (ii). This is the generalized KYP result in [5]. Theorem 1 further shows that the coefficients of every rational function nonnegative on $\mathbf{\Lambda}$ can be linearly parametrized by positive semidefinite matrices Q and X as in (3). Consequently, such rational function admits a weighted SOS decomposition as in (4), where we note that $\mathsf{Q}(\lambda)$ and $\mathsf{X}(\lambda)$ are SOSs of rational functions but $\mathsf{P}(\lambda)$ is not, and that the term containing $\mathsf{P}(\lambda)$ vanishes on $\mathbf{\Lambda}$ because $\sigma(\lambda,\Phi) = 0$.

We now specialize Theorem 1 to characterize nonnegativity of the following pseudo-polynomial:

$$\mathsf{G}(\lambda) := \mathsf{L}_q(\lambda)^* \Theta \mathsf{L}_q(\lambda) = \sum_{i=0}^{q}\sum_{j=0}^{q} \Theta_{ij}\bar{\lambda}^i \lambda^j, \tag{5}$$

where $\Theta_{ij} \in \mathbb{C}^{m\times m}$ are the partitioned block matrices of $\Theta \in \mathbb{H}^{n+m}$ and

$$\mathsf{L}_q(\lambda) := \begin{bmatrix} I_m & \lambda I_m & \cdots & \lambda^q I_m \end{bmatrix}^{\mathsf{T}}.$$

For a general Φ, we have

$$\sigma(\lambda,\Phi) = 0 \quad \Rightarrow \quad \bar{\lambda} = -\frac{\lambda\phi_{21} + \phi_{22}}{\lambda\phi_{11} + \phi_{12}}$$

and hence $\mathsf{G}(\lambda)$ on $\mathbf{\Lambda}_o$ defined by (1) is given by

$$\mathsf{G}(\lambda) = \sum_{i=0}^{q}\sum_{j=0}^{q} \left(-\frac{\lambda\phi_{21} + \phi_{22}}{\lambda\phi_{11} + \phi_{12}}\right)^i \lambda^j.$$

Thus $\mathsf{G}(\lambda)$ is a rational function in general. If $\phi_{11} = 0$, then $\mathbf{\Lambda}_o$ is a straight line on the complex plane and $\mathsf{G}(\lambda)$ is a polynomial of λ. If $\phi_{12} = 0$, then $\mathbf{\Lambda}_o$ represents a circle and $\mathsf{G}(\lambda)$ is a quasi-polynomial, i.e., a linear combination of λ^ℓ and $\lambda^{-\ell}$ with $\ell \in \mathbb{Z}_q$. We have the following result for $\mathsf{G}(\lambda)$.

Corollary 1 (Nonnegative (pseudo-)polynomial on $\mathbf{\Lambda}$) *Let positive integers m and q be given and define $n := mq$ and*

$$U := \begin{bmatrix} 0 & I_n \end{bmatrix}, \quad V := \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad F := \begin{bmatrix} U \\ V \end{bmatrix} \tag{6}$$

where $U, V \in \mathbb{R}^{n\times(n+m)}$. Let matrices $\Theta \in \mathbb{H}^{n+m}$, and $\Phi, \Psi \in \mathbb{H}^2$ be given and define $\mathbf{\Lambda}$ by (1). Suppose that $\mathbf{\Lambda}$ represents curves on the complex plane. Consider the pseudo-polynomial $\mathsf{G}(\lambda)$ in (5). The following statements are equivalent.

(i) $\mathsf{G}(\lambda) \succeq 0$ *holds for all* $\lambda \in \mathbf{\Lambda}$.

(ii) There exist $P \in \mathbb{H}^n$ and $Q \in \mathbb{H}^n_+$ such that

$$\Theta \succeq F^*(\Phi \otimes P + \Psi \otimes Q)F. \tag{7}$$

(iii) There exist $P \in \mathbb{H}^n$, $Q \in \mathbb{H}^n_+$ and $X \in \mathbb{H}^{n+m}_+$ such that

$$\Theta = F^*(\Phi \otimes P + \Psi \otimes Q)F + X. \tag{8}$$

(iv) There exist $P \in \mathbb{H}^n$, $Q \in \mathbb{H}^n_+$ and $X \in \mathbb{H}^{n+m}_+$ such that

$$\mathsf{G}(\lambda) = \sigma(\lambda, \Phi)\mathsf{P}(\lambda) + \sigma(\lambda, \Psi)\mathsf{Q}(\lambda) + \mathsf{X}(\lambda), \tag{9}$$

$$\mathsf{P}(\lambda) := \mathsf{L}_{q-1}(\lambda)^* P \mathsf{L}_{q-1}(\lambda), \quad \mathsf{Q}(\lambda) := \mathsf{L}_{q-1}(\lambda)^* Q \mathsf{L}_{q-1}(\lambda),$$

$$\mathsf{X}(\lambda) := \mathsf{L}_q(\lambda)^* X \mathsf{L}_q(\lambda).$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows as a special case of Theorem 1 by noting that

$$\mathsf{G}(\lambda) = |\lambda^q|^2 \mathcal{H}(\lambda), \quad \mathcal{F}(\lambda) := \mathsf{L}_{q-1}(\lambda)/\lambda^q$$

and a realization of $\mathcal{F}(\lambda)$ is given by

$$\begin{bmatrix} A & B \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & I & 0 \\ 0 & \cdots & \cdots & 0 & I \end{array}\right].$$

Statement (iii) follows from (ii) by introducing the slack variable X. Statement (iv) follows from (iii) by multiplying $\mathsf{L}_q(\lambda)$ and $\mathsf{L}_q(\lambda)^*$ from the right and left, respectively, and noting that $U\mathsf{L}_q(\lambda) = \lambda \mathsf{L}_{q-1}(\lambda)$ and $V\mathsf{L}_q(\lambda) = \mathsf{L}_{q-1}(\lambda)$. ■

Statement (ii) provides an LMI condition for checking positivity of the Hermitian-valued polynomial $\mathsf{G}(\lambda)$ on $\mathbf{\Lambda}$. Once the LMI in (7) is solved for P and Q, equation (9) in statement (iv) gives a weighted SOS representation of $\mathsf{G}(\lambda)$ where X is calculated from (8). The first term involving $\mathsf{P}(\lambda)$ vanishes on $\mathbf{\Lambda}$ as noted earlier, and the second term involving $\mathsf{Q}(\lambda)$ is positive semidefinite on $\mathbf{\Lambda}$, but can have negative eigenvalue(s) elsewhere. The pseudo-polynomial $\mathsf{G}(\lambda)$ is not (necessarily) positive semidefinite for all $\lambda \in \mathbf{\Lambda}_o$, but the weighted pseudo-polynomial $\mathsf{G}(\lambda) - \sigma(\lambda, \Psi)\mathsf{Q}(\lambda)$ is. Thus, the $\mathsf{Q}(\lambda)$ term has the effect of raising eigenvalues when $\lambda \in \mathbf{\Lambda}_o$ is outside of $\mathbf{\Lambda}$.

3 Special Cases

Next, we will further specializes Corollary 1 in the previous section for the case where $\boldsymbol{\Lambda}_o$ is either a circle or a straight line on the complex plane. Corollary 1 shows the equivalence of the original FDI condition in (i) and the weighted SOS decomposition expressed as (9) in (iv). Our purpose in this section is to derive the simpler weighted SOS decompositions by eliminating a variable P in (8) or a positive polynomial $\mathrm{P}(\lambda)$ in (9) for the special two cases. The derived weighted SOS decompositions with less variables may have an advantage for numerical computations, and it means that Corollary 1 actually generalizes and unifies some of the existing results on SOS decomposition in [14], [15].

Corollary 2 (Nonnegative quasi-polynomial on a circle) *Let scalars* $r, \theta_1, \theta_2 \in \mathbb{R}$ *and matrices* $G_\ell \in \mathbb{C}^{m\times m}$ ($|\ell| \in \mathbb{Z}_q$) *be given such that* $r > 0$, $\theta_1 < \theta_2$, *and* $G_{-\ell} = G_\ell^*$ *for all* $\ell \in \mathbb{Z}_q$. *Let* $\boldsymbol{\Lambda}$ *be the arc on the circle of radius* r *with center at the origin, with its angle ranging from* θ_1 *to* θ_2, *that is,*

$$\boldsymbol{\Lambda} := \{\, re^{\mathrm{j}\theta} \in \mathbb{C} : \ \theta_1 \le \theta \le \theta_2 \,\}.$$

Define F *by (6) with* $n := mq$, *and*

$$\Psi := \begin{bmatrix} 0 & e^{\mathrm{j}\theta_c} \\ e^{-\mathrm{j}\theta_c} & -2r\cos\theta_w \end{bmatrix}, \quad \theta_c := (\theta_1+\theta_2)/2, \quad \theta_w := (\theta_1-\theta_2)/2.$$

The following statements are equivalent.

(i) $\displaystyle\sum_{\ell=-q}^{q} \lambda^\ell G_\ell \succeq 0$ *holds for all* $\lambda \in \boldsymbol{\Lambda}$.

(ii) There exist $X \in \mathbb{H}_+^{n+m}$ *and* $Q \in \mathbb{H}_+^n$ *such that, for each* $\ell \in \mathbb{Z}_q$,

$$G_\ell = \sum_{j-i=\ell} r^i Y_{ij}, \quad Y := F^*(\Psi\otimes Q)F + X,$$

where $Y_{ij} \in \mathbb{C}^{m\times m}$ ($i, j \in \mathbb{Z}_q$) *are obtained by equal partitioning of matrix* Y, *and the summation is over all pairs* (i, j) *such that* $j - i = \ell$ *and* $i, j \in \mathbb{Z}_q$.

Proof. The result is a special case of Theorem 1 where Ψ is as indicated above and

$$\Phi := \begin{bmatrix} 1 & 0 \\ 0 & -r^2 \end{bmatrix}.$$

In this case, $\sigma(\lambda, \Phi) = 0$ defines a circle of radius r on the complex plane, and $\sigma(\lambda, \Psi) \ge 0$ defines the half plane containing $e^{\mathrm{j}\theta_c}$ with boundary being the straight line passing through $re^{\mathrm{j}\theta_1}$ and $re^{\mathrm{j}\theta_2}$. Hence $\boldsymbol{\Lambda}$ is an arc of the circle.

For $\lambda \in \mathbf{\Lambda}$, we have $\lambda^* = r/\lambda$ and hence

$$\mathsf{G}(\lambda) = \sum_{i=0}^{q}\sum_{j=0}^{q} \Theta_{ij}(r/\lambda)^i \lambda^j = \sum_{i=0}^{q}\sum_{j=0}^{q} \Theta_{ij} r^i \lambda^{j-i} = \sum_{\ell=-q}^{q} \lambda^\ell G_\ell,$$

$$G_\ell := \sum_{j-i=\ell} r^i \Theta_{ij}.$$

The result follows from (8) if we show that

$$\sum_{j-i=\ell} r^i Z_{ij} = 0, \quad Z := F^*(\Phi \otimes P)F.$$

This fact can readily be seen for the case $q = 3$:

$$P := \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ & P_{11} & P_{12} \\ & & P_{22} \end{bmatrix},$$

$$Z = \begin{bmatrix} -rP_{00} & -rP_{01} & -rP_{02} & 0 \\ & P_{00} - rP_{11} & P_{01} - rP_{12} & P_{02} \\ & & P_{11} - rP_{22} & P_{12} \\ & & & P_{22} \end{bmatrix}$$

where the entries in the blank space are implied by the symmetry. It should be obvious how to extend the idea for the general case. ∎

Corollary 3 (Nonnegative polynomial on a straight line) *Let scalars $\theta, r_1, r_2 \in \mathbb{R}$ and matrices $G_\ell \in \mathbb{C}^{m\times m}$ ($\ell \in \mathbb{Z}_{2q}$) be given such that $r_1 < r_2$. Let $\mathbf{\Lambda}$ be part of the straight line on the complex plane that passes through the origin and $e^{\mathrm{j}\theta}$, that is,*

$$\mathbf{\Lambda} := \{\, re^{\mathrm{j}\theta} \in \mathbb{C} : \ r_1 \le r \le r_2 \,\} \quad \textit{or} \quad \mathbf{\Lambda} := \{\, re^{\mathrm{j}\theta} \in \mathbb{C} : \ r \le r_1 \ \textit{or} \ r_2 \le r \,\}.$$

Define $F \in \mathbb{R}^{2n\times(n+m)}$ by (6) with $n := mq$ and

$$\Psi := \pm \begin{bmatrix} -1 & r_c e^{\mathrm{j}\theta} \\ r_c e^{-\mathrm{j}\theta} & -r_1 r_2 \end{bmatrix}, \quad r_c := (r_1 + r_2)/2, \tag{10}$$

where the plus (minus) sign is taken if $\mathbf{\Lambda}$ is an interval (semi-infinite intervals). The following statements are equivalent.

(i) $\sum_{\ell=0}^{2q} \lambda^\ell G_\ell \succeq 0$ *holds for all* $\lambda \in \mathbf{\Lambda}$.

(ii) There exist $X \in \mathbb{H}_+^{n+m}$ and $Q \in \mathbb{H}_+^n$ such that

$$G_\ell = \sum_{i+j=\ell} (e^{j\theta})^{-2i} Y_{ij}, \quad Y := F^*(\Psi \otimes Q)F + X,$$

where $Y_{ij} \in \mathbb{C}^{m\times m}$ $(i, j \in \mathbb{Z}_q)$ are the partitioned block matrices of Y, and the summation is taken over all (i, j) such that $i + j = \ell$ and $i, j \in \mathbb{Z}_q$.

Proof. The line on the complex plane passing through the origin and $e^{j\theta}$ is expressed by $\sigma(\lambda, \Phi) = 0$ with

$$\Phi = \mathrm{j} \begin{bmatrix} 0 & z \\ -1/z & 0 \end{bmatrix}, \quad z := e^{j\theta}. \tag{11}$$

The condition $\sigma(\lambda, \Psi) \geq 0$ defines the region inside or outside of the circle of radius $(r_2 - r_1)/2$ with center at $r_c e^{j\theta}$. Thus $\mathbf{\Lambda}$ is a portion of the straight line.

For $\lambda \in \mathbf{\Lambda}$, we have $\lambda^* = \lambda/z^2$ and hence

$$\mathsf{G}(\lambda) = \sum_{i=0}^{q}\sum_{j=0}^{q} \Theta_{ij}(\lambda/z^2)^i \lambda^j = \sum_{i=0}^{q}\sum_{j=0}^{q} \Theta_{ij} z^{-2i}\lambda^{i+j} = \sum_{\ell=0}^{2q} \lambda^\ell G_\ell,$$

$$G_\ell := \sum_{i+j=\ell} z^{-2i}\Theta_{ij}.$$

Now, it can be verified that

$$\sum_{i+j=\ell} z^{-2i} Z_{ij} = 0, \quad Z := F^*(\Phi \otimes P)F \tag{12}$$

holds. For instance, for the case $q = 3$, we have

$$P := \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ & P_{11} & P_{12} \\ & & P_{22} \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & -P_{00}/z & -P_{01}/z & -P_{02}/z \\ zP_{00} & zP_{01} - P_{10}/z & zP_{02} - P_{11}/z & -P_{12}/z \\ zP_{10} & zP_{11} - P_{20}/z & zP_{12} - P_{21}/z & -P_{22}/z \\ zP_{20} & zP_{21} & zP_{22} & 0 \end{bmatrix},$$

from which we see that the claim is valid. ■

Choosing $r = 1$ in Corollary 2, we have a characterization of nonnegative trigonometric quasi-polynomial matrices of degree q. It implies that such a trigonometric quasi-polynomial matrix can be represented by a weighted sum of two nonnegative quasi-polynomial matrices on the unit circle of the form

$$\mathbf{G}(\theta) := \sum_{\ell=-q}^{q} e^{j\ell\theta} G_\ell = \mathbf{X}(\theta) + w(\theta) \cdot \mathbf{Q}(\theta),$$

where $\mathbf{X}(\theta)$ and $\mathbf{Q}(\theta)$ are nonnegative trigonometric quasi-polynomial matrices with degrees q and $q-1$, respectively, and $w(\theta) = 2\{\cos(\theta-\theta_c) - \cos\theta_w\}$, which takes nonnegative value for any $\theta \in [\theta_1, \theta_2]$.

On the other hand, choices $\theta = 0$ and $\theta = \pi/2$ in Corollary 3 give parametrizations of nonnegative polynomial matrices on the real and imaginary axes, respectively. The case of $\theta = 0$ for example implies that real polynomial matrix with degree $2q$ can be represented by a weighted sum of two nonnegative real polynomial matrices of the form

$$\mathbf{G}(x) := \sum_{\ell=0}^{2q} G_\ell = \mathbf{X}(x) + w(x)\cdot\mathbf{Q}(x) \ ,$$

where $\mathbf{X}(x)$ and $\mathbf{Q}(x)$ are nonnegative polynomial matrices with degrees $2q$ and $2(q-1)$, respectively, and $w(x) = -(x-r_1)(x-r_2)$, which takes nonnegative value for any $x \in [r_1, r_2]$. It should be noted that X and Q in statement (ii) of Corollaries 2 and 3 can be restricted to real matrices without loss of generality if G_ℓ and Ψ are real. However, when $\mathbf{\Lambda}$ is asymmetric with respect to the real axis so that Ψ has complex entries, X and Q are still complex in general even if G_ℓ are real.

The above remarks clearly show that our results generalize and unify the previous results for scalar polynomials on a restricted range [14] and for matrix-valued polynomials on an unrestricted range [15].

4 Design Example

Consider the mechanical system consisting of three masses connected in series by springs and dashpots as shown in Fig. 1. Two actuators are placed at masses m_1 and m_2 to apply forces u_1 and u_2, and two sensors are placed at masses m_1 and m_3 to measure the velocities y_1 and y_2. All the parameter values except for k_2 are assumed given as $m_1 = m_2 = m_3 = k_1 = c_1 = c_2 = 1$ for simplicity. Recall from [4] that the mechanical system is easily controllable within the frequency range $|\omega| \leq \varpi$ if the transfer function $H(s)$ from u to y is positive real in the frequency range. Let us call the largest of such ϖ the band-width of $H(s)$ and denote it by ϖ_o. We will consider the problem of designing the spring constant $k := k_2$ so that $H(s)$ has a prescribed band-width ϖ_o.

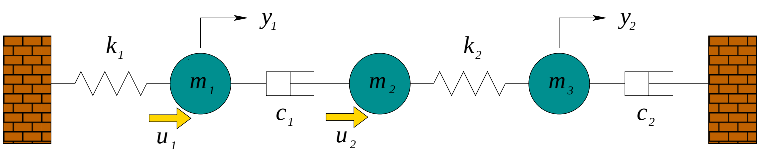

Figure 1: Mechanical system

The transfer function from u to y is given by

$$H(s) := sC(Ms^2 + Ds + K)^{-1}B,$$

$$M = I, \quad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & -k \\ 0 & -k & k \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recall that the transfer function is positive real in the range $|\omega| \le \varpi$ if $H(s) + H(-s)^{\mathsf{T}} \succeq 0$ holds for $s \in \mathbf{\Lambda} := \{\ \mathrm{j}\omega :\ |\omega| \le \varpi\ \}$. With $y = H(s)u$, this condition is equivalent to

$$u^*y + y^*u = \xi^* W(s)\xi \ge 0 \quad \forall\, \xi \text{ such that } L(s)\xi = 0,$$

$$W(s) := \begin{bmatrix} 0 & -sC^{\mathsf{T}} \\ sC & 0 \end{bmatrix}, \quad L(s) := \begin{bmatrix} Ms^2 + Ds + K & -B \end{bmatrix}.$$

Let $N(s)$ be a polynomial matrix whose columns form a basis for the null space of $L(s)$. Then, the condition reduces to $N(-s)^{\mathsf{T}} W(s) N(s) \succeq 0$, or equivalently,

$$\mathsf{G}(s) := \sum_{\ell=0}^{2} s^\ell G_\ell \succeq 0, \quad \forall\, \lambda \in \mathbf{\Lambda}, \tag{13}$$

$$G_0 = 2k \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix}.$$

Note that the frequency set $\mathbf{\Lambda}$ can be characterized by (1) with Ψ and Φ specified by (10) and (11), respectively, where $\theta = -\pi/2$ and $r_2 = -r_1 = \varpi$.

The finite frequency positive real (FFPR) condition (13) is a special case of statement (i) of Corollary 3, where $q = 1$, $m = 2$, $n = 2$, and $\lambda := s$. We see that condition (13) is satisfied if and only if there exist positive semidefinite matrices $Q \in \mathbb{H}_+^2$ and $X \in \mathbb{H}_+^4$ such that

$$G_0 = -\varpi^2 Q + X_{11}, \quad G_1 = X_{12} - X_{21}, \quad G_2 = Q - X_{22}, \tag{14}$$

where X_{ij} are 2×2 partitioned block matrices of X. On the other hand, Corollary 1 provides an LMI characterization of the FFPR condition. In particular, (7) reduces to

$$\begin{bmatrix} G_0 & G_1/2 \\ G_1^{\mathsf{T}}/2 & G_2 \end{bmatrix} \succeq \begin{bmatrix} \varpi^2 Q & P \\ P & -Q \end{bmatrix}, \tag{15}$$

where $Q \in \mathbb{H}_+^2$ and $P \in \mathbb{H}^2$. When deriving the above condition, $\mathsf{G}(s)$ has been expressed as in (5) where Θ is equal to the left hand side of (15). Conditions (14) and (15) are related to each other as follows. Let X be the positive semidefinite matrix obtained by subtracting the right hand side of (15) from the left hand side. Then (14) is obtained by solving the defining equation for G_0, G_1, and G_2.

Finally, a value of the spring constant k to achieve a prescribed band-width ϖ_o can be found by solving the LMI in (15), with $\varpi := \varpi_o$, for the variables k, $Q \in \mathbb{H}^2_+$, and $P \in \mathbb{H}^2$. Note that if (15) is satisfied for $k = k_o$, then it is also satisfied for all $k \geq k_o$ because G_o is linear in k with a positive definite coefficient matrix. The smallest stiffness required to achieve the band-width ϖ_o can be obtained by minimizing k subject to (15), which is an eigenvalue problem. Conversely, for a given spring constant k, one can determine ϖ for which the transfer function $H(s)$ is positive real in $s \in \mathbf{\Lambda}$. The largest value of such ϖ is the frequency band-width ϖ_o, and can be computed by maximizing ϖ subject to the LMI constraint (15) with variables $Q \in \mathbb{H}^2_+$ and $P \in \mathbb{H}^2$. This is a generalized eigenvalue problem. In fact, it is possible for this simple example to find an analytical expression for the maximum ϖ as follows: $\varpi_o = \sqrt{(\sqrt{32k^2 + 8k + 1} - 1)/2 - 2k}$.

Figure 2 shows the plot of the minimum eigenvalue of $H(\mathrm{j}\omega) + H(\mathrm{j}\omega)^*$ as a function of ω for two cases $k = 1$ and $k = 10$. The above formula for ϖ_o gives the lowest frequency at which the plot goes negative as indicated by circles. Note that the band-width ϖ_o for positive realness increases with the stiffness k. This makes sense because $H(s)$ is positive real in the entire frequency range in the limit $k \to \infty$ due to the merging of m_2 and m_3 that makes actuator u_2 and sensor y_2 collocated.

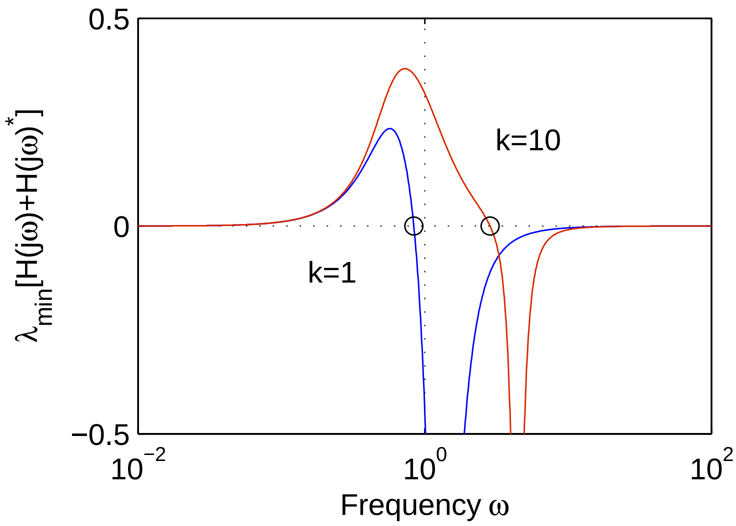

Figure 2: Finite frequency positive real property

5 Conclusion

We have shown that the generalized KYP lemma is directly useful for characterizing polynomial matrices, that are positive semidefinite on a portion of a line on the complex plane, in terms of a weighted SOS. Our result extends and unifies existing results on nonnegative polynomials of a single variable. While the generalized KYP lemma previously stated the equivalence between FDI and LMI, our result adds SOS to this equivalence list. With another result [16] that treated time domain inequality (TDI), the body of generalized KYP theory has now established the equivalence among FDI, TDI, LMI, and SOS, in the context of matrix-valued, rational or polynomial, functions of a single but restricted variable.

Acknowledgements: The authors would like to thank Dr. M. Kanno for deriving the analytical expression in the design example.

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