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# Semidefinite Programming for Dynamic Steady-State Analysis of Structures under Uncertain Harmonic Loads

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## Abstract

This paper presents semidefinite programming approaches for computing confidential bounds for the dynamic steady-state responses of a damped structure subjected to uncertain driving loads. We assume that amplitudes of harmonic driving loads obey a non-probabilistic uncertainty model. The semidefinite programming problems are formulated for finding confidential bounds for various characteristic amounts of dynamic steady-state response, including the modulus and phase angle of the complex amplitude of the displacement and stress. Numerical examples demonstrate that sufficiently tight bounds can be obtained by solving the presented semidefinite programming problems.

## Keywords

Semidefinite program; Data uncertainty; Uncertain linear equations; Forced oscillation; Structural dynamics.

## 1 Introduction

Methodologies as well as numerical techniques for robustness and uncertainty analyses have received increasing attention in structural and mechanical design. Since structures built in the real-world always have various uncertainties caused by manufacture errors, limitation of knowledge of input disturbance, observation errors, etc., various approaches have been developed to estimate responses of structures including uncertainties [5, 16, 27–29, 31, 35].

In this paper, we propose a numerical approach to estimate the dynamic structural response under uncertain harmonic loads. There exist two different frameworks with which we treat the

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uncertain property of a structural system. The one is probabilistic uncertainty modeling and the other non-probabilistic uncertainty modeling. By using probabilistic uncertainty models, various uncertainty analysis methods have been developed; see, e.g., [35, 39], and the references therein. However, it is often difficult to accurately estimate stochastic parameters which are used in probabilistic structural analyses, especially when the number of samples of the uncertain parameters is limited.

In contrast to probabilistic approaches, the non-probabilistic uncertainty framework treats the uncertain parameters as the so-called *unknown-but-bounded* parameters, which does not require to estimate the probability distribution of the uncertain parameters. One of well-known approaches with a non-probabilistic uncertainty model is the so-called *convex model* approach [3, 5, 31]. Note that the conventional convex model approaches are valid only if the magnitude of uncertainty is small enough, because it is essentially based on the first-order approximation of the structural response with respect to the uncertain parameters.

As a non-probabilistic decision theory applicable to uncertainties with arbitrary large magnitude, Ben-Haim [4] proposed the *info-gap decision theory*, and its applications have been found in various fields. In the info-gap decision theory, the *robustness function* plays a key role which represents the greatest level of uncertainty at which any failure cannot occur. The authors proposed numerical methods for computing the robustness function of structures considering the static response [19] and plastic limit-load factor [24].

For computing the exact bounds of structural responses of a truss including unknown-but-bounded uncertainties, mixed integer programming approaches have been developed for the member stress [16] and for the plastic limit-load factor [20].

Concerning uncertainty analysis in structural dynamics, the fuzzy theory has been applied to uncertainty analysis [12, 26], which are also regarded as a non-probabilistic approach. For *uncertain linear equations* (ULE) the interval linear algebra has been well developed [1], which gives a conservative bound for the solution set of the given ULE. The interval algebra has been applied to structural analyses considering uncertainties [28, 29]. Particularly, there have been numerous studies of interval analysis for eigenvalue problems including uncertain parameters [10, 11, 13–15, 23, 25, 32, 33, 36]. These studies are aimed at finding bounds for eigenvalues of structures [10, 11, 13, 15, 23, 25, 32, 33], for eigenmodes [14], modal parameters including the frequency response function [36]. A comparison between the interval analysis and a probabilistic approach for dynamic uncertainty analysis was performed by Qiu and Wang [34].

For convex optimization problems, including the linear program, semidefinite program (SDP) [17], and second-order cone program (SOCP) [2], etc., a unified methodology of *robust optimization* was established by Ben-Tal and Nemirovski [7], in which given data in optimization problems are assumed to have non-probabilistic uncertainties. Calafiore and El Ghaoui [9] proposed a method for finding an ellipsoidal bound for the solution set of ULE by using the SDP relaxation. The authors formulated an SDP problem which provides a confidential ellipsoidal bound for static response of uncertain structural systems, particularly of a truss [18] and a braced frame [21].

In this paper we consider the dynamic responses of a damped structure under uncertain driving load. We are particularly interested in the steady state of a forced oscillation induced by harmonic driving loads, where the amplitudes of driving loads have non-probabilistic uncertainties. We at-

tempt to find various bounds for characteristic parameters, e.g. lower and upper bounds for the modulus and phase angle of the complex amplitude of nodal displacements. It is shown that each of these bound detection problems is formulated as an optimization problem with infinitely many constraints, or as the form of robust optimization. We next present a sufficient condition, with which the (infinitely many) constraints in the optimization problem stated above are satisfied, by embedding the governing equations with uncertain parameters into quadratic inequalities and by applying the so-called  $\mathcal{S}$ -lemma [8, section 2.6.3]. Based on the sufficient condition presented, we construct an approximation problem which is guaranteed to provide a conservative approximation of the exact optimal solution to the bound detection problem. It should be emphasized that our goal formulations for conservative bounds are either an SDP [17] or an SOCP [2] problem, both of which are convex optimization problems and can be solved efficiently by using the primal-dual interior-point method.

It is known that SDP and SOCP problems can be solved by using the primal-dual interior-point method in polynomial time [2, 17]. Hence, our method can find a bound for the structural response in polynomial time, in contrast to the fact that most of numerical methods based on the interval algebra have in general exponential complexity [6, section 6.5.3].

This paper is organized as follows. Section 2 introduces the governing equation of our problem, as well as a non-stochastic uncertainty model of the dynamic load. In section 3, we formulate SDP problems which give conservative bounds for the modulus and phase angle of the displacement amplitude. In section 4, we particularly consider a nodal displacement vector, and propose further bounds for the norm and ratio of the moduli of the nodal displacement amplitudes. Numerical examples are presented in section 5 for the bound detection problems investigated in the two previous sections. In section 6, we consider a slightly generalized case of our uncertainty model, and demonstrate numerical experiments. Some conclusions are drawn in section 7. For readability, all proofs of the technical results are collected in appendix A.

We present a few words regarding our notation at first. All vectors are assumed to be column vectors. For  $\mathcal{X} \subseteq \mathbb{R}^n$  and  $\mathcal{Y} \subseteq \mathbb{R}^m$ , their Cartesian product is defined by  $\mathcal{X} \times \mathcal{Y} = \{(\mathbf{x}^T, \mathbf{y}^T)^T \in \mathbb{R}^{n+m} \mid \mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}\}$ , where  $^T$  denotes the transpose of a vector or a matrix. Particularly, we write  $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ . The  $(n+m)$ -dimensional column vector  $(\mathbf{x}^T, \mathbf{y}^T)^T$  consisting of  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{y} \in \mathbb{R}^m$  is often written simply as  $(\mathbf{x}, \mathbf{y})$ . We denote by  $\mathbb{R}_+^n$  the non-negative orthant defined by  $\mathbb{R}_+^n = \{\mathbf{x} = (x_i) \in \mathbb{R}^n \mid x_i \geq 0 (i = 1, \dots, n)\}$ . We write  $\mathbf{x} \geq \mathbf{0}$  if  $\mathbf{x} \in \mathbb{R}_+^n$ . We denote by  $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$  the set of all  $n \times n$  real symmetric matrices. We write  $A \succeq O$  if  $A \in \mathcal{S}^n$  is positive semidefinite. It should be clear that  $\mathbf{x} \geq \mathbf{y}$  and  $A \succeq B$  mean  $\mathbf{x} - \mathbf{y} \geq \mathbf{0}$  and  $A - B \succeq O$ , respectively. For a vector  $\mathbf{x} \in \mathbb{R}^n$  we denote by  $\|\mathbf{x}\|$  the standard Euclidean norm, i.e.  $\|\mathbf{x}\| = (\mathbf{x}^T \mathbf{x})^{1/2}$ . We denote by  $I_n$  the  $n \times n$  identity matrix. We also write  $I$  instead of  $I_n$ , unless its size is not clear from the context. Let  $\text{Re } z$  and  $\text{Im } z$  denote the real and imaginary parts of  $z \in \mathbb{C}$ , respectively. We denote by  $|z|$  the modulus, or the absolute value, of  $z \in \mathbb{C}$ , i.e.  $|z| = [(\text{Re } z)^2 + (\text{Im } z)^2]^{1/2}$ . We denote by  $\text{Arg } z$  the principal value of the argument of  $z \in \mathbb{C}$ , which satisfies  $-\pi < \text{Arg } z \leq \pi$ .

## 2 Uncertain equations for steady state

Consider a finite-dimensional linear elastic structure. Small displacements and small strains are assumed.

### 2.1 Governing equations

Let  $d$  denote the number of degrees of freedom of displacements. In this paper we consider the harmonic driving load defined by  $\mathbf{f}e^{i\omega t}$ , where  $\mathbf{f} \in \mathbb{R}^d$  and  $\omega \in \mathbb{R}$ . We denote by  $K \in \mathcal{S}^d$ ,  $M \in \mathcal{S}^d$ , and  $C \in \mathcal{S}^d$  the stiffness, mass and damping matrices, respectively. The d'Alembert principle yields

$$M\ddot{\hat{\mathbf{u}}} + C\dot{\hat{\mathbf{u}}} + K\hat{\mathbf{u}} = \mathbf{f}e^{i\omega t}, \quad (1)$$

where  $\hat{\mathbf{u}} \in \mathbb{C}^d$  is the displacement vector. By substituting  $\hat{\mathbf{u}} = \mathbf{u}e^{i\omega t}$  into (1), we find the steady state oscillation,  $\mathbf{u} \in \mathbb{C}^d$ , driven by  $\mathbf{f}e^{i\omega t}$  as the solution to

$$(-\omega^2 M + i\omega C + K)\mathbf{u} = \mathbf{f}. \quad (2)$$

### 2.2 Uncertainty model

In this section we define a non-probabilistic uncertainty model of the harmonic driving load.

Suppose that the amplitude vector of the load,  $\mathbf{f}$ , in (2) is not known precisely, or is uncertain. Throughout the paper we assume that the uncertainty exists only in  $\mathbf{f}$ , and that  $K$ ,  $M$ ,  $C$ , and  $\omega$  in (2) are known precisely.

Let  $\tilde{\mathbf{f}} \in \mathbb{R}^d$  denote the nominal value, or the best estimate, of  $\mathbf{f}$ . We describe the uncertainty of  $\mathbf{f}$  by using an unknown vector  $\boldsymbol{\zeta}$ . Assume that  $\mathbf{f}$  depends on  $\boldsymbol{\zeta} \in \mathbb{R}^k$  affinely as

$$\mathbf{f} = \tilde{\mathbf{f}} + F_0\boldsymbol{\zeta}, \quad (3)$$

where  $F_0 \in \mathbb{R}^{d \times k}$  is a constant matrix. Note that the matrix  $F_0$  represents the relative magnitude of the uncertainty of  $f_j$  ( $j = 1, \dots, d$ ) and the relationship of the uncertainties among  $f_1, \dots, f_d$ . See, for an illustrative example, Example 2.1.

Suppose that  $\boldsymbol{\zeta}$  satisfies

$$\alpha \geq \|T_j\boldsymbol{\zeta}\|, \quad j = 1, \dots, \ell, \quad (4)$$

where  $T_j \in \mathbb{R}^{m_j \times k}$  ( $j = 1, \dots, \ell$ ) are constant matrices, and  $\alpha \in \mathbb{R}_+$  is constant. Throughout the paper we assume that the set  $\{\boldsymbol{\zeta} \mid \alpha \geq \|T_j\boldsymbol{\zeta}\| \ (j = 1, \dots, \ell)\}$  is bounded for any given  $\alpha \in \mathbb{R}_+$ . This assumption is satisfied if  $T_1, \dots, T_\ell$  satisfy

$$\text{rank} \begin{bmatrix} T_1 \\ \vdots \\ T_\ell \end{bmatrix} = k.$$

We call  $\boldsymbol{\zeta}$  the vector of *uncertain parameters*, or *unknown-but-bounded parameters*.

It follows (3) and (4) that the amplitude of driving load,  $\mathbf{f}$ , is running through the set

$$\mathcal{F}(\alpha) = \left\{ \tilde{\mathbf{f}} + F_0\boldsymbol{\zeta} \mid \alpha \geq \|T_j\boldsymbol{\zeta}\| \ (j = 1, \dots, \ell) \right\}. \quad (5)$$

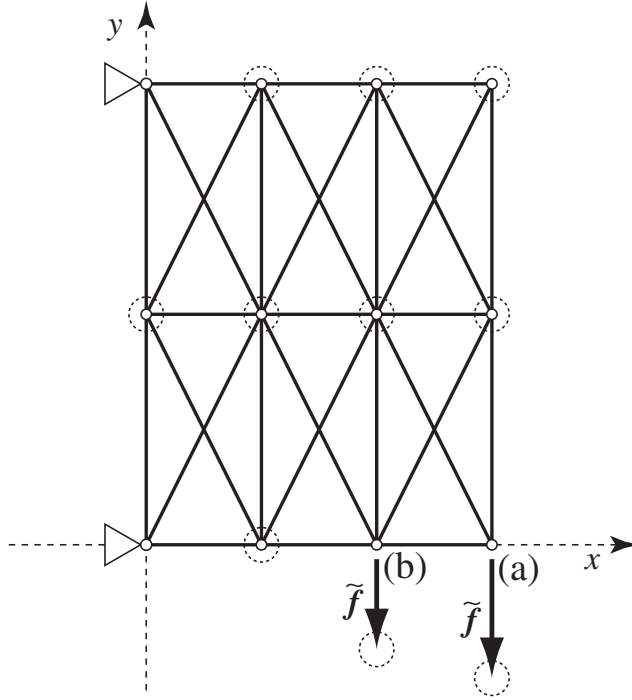


Figure 1: A 29-bar truss.

We call  $\mathcal{F}(\alpha)$  the *uncertainty set* of  $\mathbf{f}$ . Recall (2), and then the *uncertain linear equations* (ULE) that we are interested in are written as

$$(-\omega^2 M + i\omega C + K)\mathbf{u} = \mathbf{f}, \quad \mathbf{f} \in \mathcal{F}(\alpha). \quad (6)$$

Throughout the paper we assume that the equations in (6) have a solution for any  $\mathbf{f} \in \mathcal{F}$ .

In the uncertainty model defined by (5), we see that the greater the value of  $\alpha$ , the greater the range of possible variation of  $\mathbf{f}$ , and hence we call  $\alpha$  the *uncertainty parameter*. More precisely,  $0 \leq \alpha_1 < \alpha_2$  implies  $\mathcal{F}(\alpha_1) \subset \mathcal{F}(\alpha_2)$ , which meets the nesting axiom in the info-gap theory [4]. As another important property it should be mentioned that our model in (5) satisfies  $\mathcal{F}(0) = \{\tilde{\mathbf{f}}\}$ , which implies that the estimate  $\tilde{\mathbf{f}}$  is correct at  $\alpha = 0$ . This property is called the contraction axiom in the info-gap theory [4]. In the remainder of the paper we often write  $\mathcal{F}$  instead of  $\mathcal{F}(\alpha)$ , unless the meaning is not clear from the context.

For simple presentation we assume that  $F_0$  in (5) is nonsingular in sections 3–5. This assumption implies that all elements,  $f_1, \dots, f_d$ , of  $\mathbf{f}$  have uncertainties, and that  $d = k$ . See section 6 for the case in which  $F_0$  is singular.

**Example 2.1.** Consider a plane truss shown in Figure 1, where  $d = 20$ . As the nominal load  $\tilde{\mathbf{f}}$ , relatively large forces are applied at the nodes (a) and (b). Suppose that uncertain loads may possibly exist at all free nodes, and then the external forces are independently running through the circles depicted with the dotted lines in Figure 1. Such an uncertainty model can be represented by (5) as follows.

Note that  $k = 20$ , because  $F_0$  is assumed to be regular. Suppose that all circles in Figure 1 have common range. Then we may put  $F_0 = \tilde{f}_0 I_{20}$ . Since we consider 10 independent circles in Figure 1,

put  $\ell = 10$  and define  $T_j \in \mathbb{R}^{2 \times 20}$  ( $j = 1, \dots, \ell$ ) as

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \end{bmatrix}, \dots, T_{10} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Consequently, (5) with these  $T_j$  represents the uncertainty model illustrated in Figure 1, where the nodal force at each node may perturb in the circle with the radius  $\alpha \bar{f}_0$ . ■

**Example 2.2.** Consider the uncertainty model investigated in Example 2.1. Now we suppose that the uncertain forces are running through squares instead of circles illustrated in Figure 1. Such an uncertainty model is called the interval uncertainty model and widely used in uncertainty analysis of structures [10, 11, 28, 29]. Let  $F_0$  in (5) be a diagonal matrix, where  $d = k = 20$ . Since we consider 20 independent intervals, put  $\ell = 20$  and define  $T_1, \dots, T_\ell \in \mathbb{R}^{1 \times 20}$  by  $T_j = \mathbf{e}_j^T$ , where  $\mathbf{e}^j$  denote the  $j$ th column vector of the  $20 \times 20$  identity matrix. Thus (5) includes the conventional interval uncertainty model as a particular case. ■

Note again that uncertain forces can exist at all free nodes in the uncertainty models studied in Example 2.1 and Example 2.2, because  $F_0$  is nonsingular. In section 6 we suppose that uncertain forces can exist only at specified nodes, in which  $F_0$  is singular.

### 3 Bounds for complex amplitude

For the given  $\mathbf{f} \in \mathcal{F}$ , let  $\mathbf{u}(\mathbf{f}) \in \mathbb{C}^d$  denote a solution (complex number) to (2). We denote by  $u_q(\mathbf{f})$  the  $q$ th element of  $\mathbf{u}(\mathbf{f})$ . In this section, we attempt to find the bounds for  $|u_q|$  and  $\text{Arg } u_q$  when  $\mathbf{f} \in \mathcal{F}$ . More precisely, the problem in this study is stated as follows: given the ULE (6), find  $\bar{r}$ ,  $\underline{r}$ ,  $\bar{\theta}$ , and  $\underline{\theta} \in \mathbb{R}$  satisfying

$$\begin{aligned} \underline{r} &\leq |u_q(\mathbf{f})| \leq \bar{r}, & \forall \mathbf{f} \in \mathcal{F}(\alpha), \\ \underline{\theta} &\leq \text{Arg } u_q(\mathbf{f}) \leq \bar{\theta}, & \forall \mathbf{f} \in \mathcal{F}(\alpha). \end{aligned}$$

Such a set of bounds is not unique in general. Naturally we are interested in a set of bounds which is as tight as possible.

In this section we assume that  $F_0$  in (5) is regular, and hence  $k = d$ . A formulation that works without this assumption will be given in section 6.

#### 3.1 Upper bound for modulus of displacement amplitude

The maximum value,  $r_{\max}$ , of  $|u_q|$  is defined by

$$r_{\max} = \max_{\mathbf{f}} \{|u_q(\mathbf{f})| : \mathbf{f} \in \mathcal{F}\}. \quad (7)$$

In this section we formulate an optimization problem in order to obtain an upper bound of  $r_{\max}$ , i.e.  $\bar{r} \in \mathbb{R}$  satisfying  $\bar{r} \geq r_{\max}$ . Certainly we are interested in finding a small upper bound  $\bar{r}$  as far as possible.

Observe that (2) is rewritten as

$$\begin{bmatrix} -\omega^2 M + K & -\omega C \\ \omega C & -\omega^2 M + K \end{bmatrix} \begin{bmatrix} \text{Re } \mathbf{u} \\ \text{Im } \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}. \quad (8)$$



For  $\mathbf{u} \in \mathbb{C}^d$ , define  $\mathbf{v} \in \mathbb{R}^{2d}$  by

$$\mathbf{v} = \begin{bmatrix} \operatorname{Re} \mathbf{u} \\ \operatorname{Im} \mathbf{u} \end{bmatrix}. \quad (9)$$

Then (8) is rewritten simply as

$$S_1 \mathbf{v} = \mathbf{f}, \quad (10)$$

$$S_2 \mathbf{v} = \mathbf{0}, \quad (11)$$

where  $S_1, S_2 \in \mathbb{R}^{d \times 2d}$  are defined by

$$S_1 = \begin{bmatrix} K - \Omega_e M & -2\beta K \\ 2\beta K & K - \Omega_e M \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 2\beta K & K - \Omega_e M \end{bmatrix}.$$

Define the set  $\mathcal{V} \subseteq \mathbb{R}^{2d}$  by

$$\mathcal{V} = \{\mathbf{v} \mid S_1 \mathbf{v} = \mathbf{f}, S_2 \mathbf{v} = \mathbf{0}, \mathbf{f} \in \mathcal{F}\}, \quad (12)$$

i.e.  $\mathcal{V}$  is the set of all possible solutions to the system of equations, (10) and (11), when  $\mathbf{f}$  is running through the uncertainty set  $\mathcal{F}$ . Note that  $\mathcal{V}$  is related to the solution set to ULE (6) as

$$\mathcal{V} = \left\{ \begin{bmatrix} \operatorname{Re} \mathbf{u} \\ \operatorname{Im} \mathbf{u} \end{bmatrix} \mid (-\omega^2 M + i\omega C + K)\mathbf{u} = \mathbf{f}, \mathbf{f} \in \mathcal{F} \right\}.$$

Let  $G \in \mathbb{R}^{2 \times 2d}$  denote a constant matrix satisfying

$$G\mathbf{v} = \begin{bmatrix} \operatorname{Re} u_q \\ \operatorname{Im} u_q \end{bmatrix}, \quad (13)$$

which yields

$$|u_q| = \|G\mathbf{v}\|.$$

It should be clear that  $\|G\mathbf{v}\|$  denotes the standard Euclidean norm of the real vector  $G\mathbf{v} \in \mathbb{R}^2$ .

We first reformulate the optimization problem (7) into the form similar to the robust optimization [7]. Our reformulation is based on the following idea. Assume that  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is bounded above on  $\mathcal{X} \subseteq \mathbb{R}^n$ . Then we can show the relation

$$\max_{\mathbf{x}} \{g(\mathbf{x}) : \mathbf{x} \in \mathcal{X}\} = \min_t \{t : t \geq g(\mathbf{x}) (\forall \mathbf{x} \in \mathcal{X})\},$$

because  $t$  is regarded as an upper bound of  $g(\mathbf{x})$  if  $t$  satisfies  $t \geq g(\mathbf{x}) (\forall \mathbf{x} \in \mathcal{X})$ . By applying this idea to (7) we obtain the following proposition.

**Proposition 3.1.**  *$r_{\max}$  defined in (7) satisfies*

$$r_{\max} = \min_r \{r : r \geq \|G\mathbf{v}\| (\forall \mathbf{v} \in \mathcal{V})\}. \quad (14)$$

Note that all proofs in this section appear in appendix A.1.

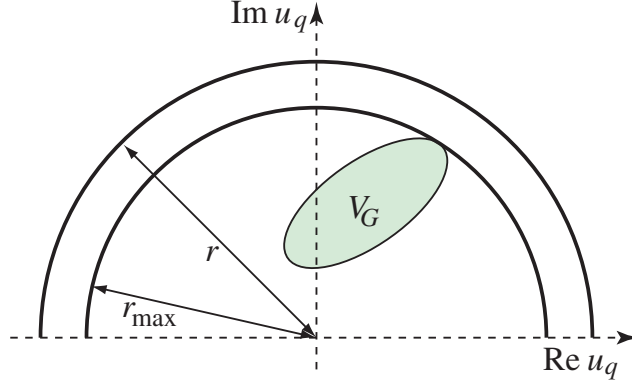


Figure 2: A circle providing an upper bound of  $|u_q|$ .

*Remark 3.2.* The assertion of Proposition 3.1 is interpreted geometrically as follows. In Figure 2, we define the set  $\mathcal{V}_G \subset \mathbb{R}^2$  by  $\mathcal{V}_G = \{G\mathbf{v} \mid \mathbf{v} \in \mathcal{V}\}$  for exposition. In the problem (14), we attempt to find a point  $v_G \in \mathcal{V}_G$  which attains the maximum Euclidean distance from the origin. In contrast, in the problem (14), we consider a circle which includes  $\mathcal{V}_G$  and is centered at the origin, and attempt to minimize the radius of such a circle. Then the minimum radius becomes equivalent to  $r_{\max}$ . ■

In this paper we are particularly interested in dealing with the optimization problem in the form of (14). It is difficult to solve the problem (14) directly, since it has an infinite number of constraint conditions. Hence, we attempt to replace such an intractable constraint condition with a tractable one which is guaranteed to be conservative. The basic idea with which we can address our tractable and conservative reformulation is stated in the following proposition.

**Proposition 3.3.** *Consider two conditions:*

$$\begin{aligned}
 \text{(a) : } & \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^T Q_i \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} \geq 0 \quad (i = 1, \dots, m), \quad \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^T P_j \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = 0 \quad (j = 1, \dots, k); \\
 \text{(b) : } & \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^T Q_0 \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} \geq 0.
 \end{aligned}$$

*The implication*

$$(a) \Rightarrow (b)$$

*holds if there exist  $\tau_1, \dots, \tau_m$  and  $\sigma_1, \dots, \sigma_k$  satisfying*

$$\begin{aligned}
 Q_0 & \succeq \sum_{i=1}^m \tau_i Q_i + \sum_{j=1}^k \sigma_j P_j, \\
 \tau_1, \dots, \tau_m & \geq 0.
 \end{aligned}$$

Observe that the conditions (a) and (b) in Proposition 3.3 are represented by some quadratic inequalities. In the following proposition, we eliminate the uncertain parameters  $\zeta$  from the condition  $\mathbf{v} \in \mathcal{V}$  which appears in (14), and obtain a finite number of quadratic inequalities.

**Proposition 3.4.** *The set  $\mathcal{V}$  defined in (12) can be represented by some quadratic inequalities of  $\mathbf{v}$ . That is,  $\mathbf{v} \in \mathcal{V}$  if and only if  $\mathbf{v}$  satisfies*

$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \Psi_j \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \geq 0, \quad j = 1, \dots, \ell, \quad (15)$$

$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \Theta \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = 0, \quad (16)$$

where  $\Psi_j$  ( $j = 1, \dots, \ell$ ) and  $\Theta$  are constant matrices defined by

$$\Psi_j = \text{Diag}(\mathbf{0}, \alpha^2) - \begin{bmatrix} S_1^T \\ -\tilde{\mathbf{f}}^T \end{bmatrix} (T_j F_0^{-1})^T T_j F_0^{-1} \begin{bmatrix} S_1 & -\tilde{\mathbf{f}} \end{bmatrix}, \quad j = 1, \dots, \ell,$$

$$\Theta = - \begin{bmatrix} S_2^T \\ O \end{bmatrix} \begin{bmatrix} S_2 & O \end{bmatrix}.$$

Proposition 3.4 implies that the condition  $\mathbf{v} \in \mathcal{V}$  can be rewritten as a finite number of quadratic inequalities in terms of  $\mathbf{v}$ . By applying Proposition 3.3, we construct a sufficient condition for the constraint condition of the problem (14) in the following proposition.

**Proposition 3.5.**  *$r \in \mathbb{R}_+$  satisfies the condition*

$$\forall \mathbf{v} \in \mathcal{V} : \quad r \geq \|G\mathbf{v}\| \quad (17)$$

if there exists a vector  $(\mathbf{w}, s) \in \mathbb{R}_+^\ell \times \mathbb{R}$  satisfying

$$\begin{bmatrix} -G^T G & \mathbf{0} \\ \mathbf{0}^T & r^2 \end{bmatrix} \succeq \sum_{j=1}^{\ell} w_j \Psi_j + s\Theta. \quad (18)$$

Proposition 3.5 presents a sufficient condition for the constraint condition of the problem (14). This naturally leads us to construct the following problem which yields a conservative bound for  $r_{\max}$ .

**Proposition 3.6.** *Define  $\rho^*$  by*

$$\rho^* = \min_{\rho, \mathbf{w}, s} \left\{ \rho : \begin{bmatrix} -G^T G & \mathbf{0} \\ \mathbf{0}^T & \rho \end{bmatrix} - \sum_{j=1}^{\ell} w_j \Psi_j - s\Theta \succeq O, \quad \mathbf{w} \geq \mathbf{0} \right\}. \quad (19)$$

Then  $r_{\max}^2 \leq \rho^*$  holds for  $r_{\max}$  defined in (7), i.e.  $\sqrt{\rho^*}$  corresponds to an upper bound of  $|u_q|$ .

Since (19) is an SDP problem [17] in the variables  $\rho \in \mathbb{R}$ ,  $\mathbf{w} \in \mathbb{R}^\ell$ , and  $s \in \mathbb{R}$ , we can obtain  $\rho^*$  efficiently by using the primal-dual interior-point method.

### 3.2 Lower bound for modulus of displacement amplitude

In a manner similar to (7), the minimum value,  $r_{\min}$ , of  $|u_q|$  is defined by

$$r_{\min} = \min_{\mathbf{f}} \{|u_q(\mathbf{f})| : \mathbf{f} \in \mathcal{F}(\alpha)\}. \quad (20)$$

Figure 3 depicts the definition of  $r_{\min}$ . In contrast to the case of  $r_{\max}$  investigated in section 3.1, the following proposition shows that we can reformulate the problem (20) directly as a convex optimization problem.

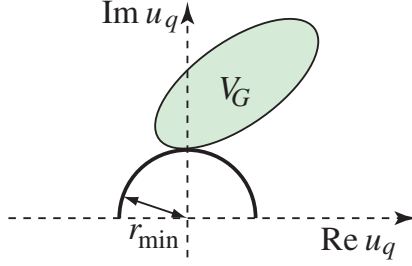


Figure 3: A circle providing the minimum value of  $|u_q|$ .

**Proposition 3.7.** Define  $t^*$  by

$$t^* = \min_{t, \mathbf{v}, \boldsymbol{\zeta}} \left\{ t : r \geq \|G\mathbf{v}\|, S_1\mathbf{v} = \tilde{\mathbf{f}} + F_0\boldsymbol{\zeta}, S_2\mathbf{v} = \mathbf{0}, \alpha \geq \|T_j\boldsymbol{\zeta}\| (j = 1, \dots, \ell) \right\}. \quad (21)$$

Then  $t^* = r_{\min}$  holds for  $r_{\min}$  defined in (20), i.e.  $t^*$  corresponds to the exact minimum value of  $|u_q|$ .

The proof appears in appendix A.2 in order to improve the readability.

It should be emphasized that the problem (21) is an SOCP problem [2] in the variables  $t \in \mathbb{R}$ ,  $\mathbf{v} \in \mathbb{R}^{2d}$ , and  $\boldsymbol{\zeta} \in \mathbb{R}^k$ . Hence, we can compute the global optimal solution of (21) easily by using the primal-dual interior-point method.

*Remark 3.8.* In a manner similar to Proposition 3.7, we can show that  $r_{\max}$  is equal to the optimal value of the following problem:

$$\left. \begin{array}{l} \max_{t, \mathbf{v}, \boldsymbol{\zeta}} t \\ \text{s.t.} \quad t \leq \|G\mathbf{v}\|, \\ S_1\mathbf{v} = \tilde{\mathbf{f}} + F_0\boldsymbol{\zeta}, \\ S_2\mathbf{v} = \mathbf{0}, \\ \alpha \geq \|T_j\boldsymbol{\zeta}\|, \quad j = 1, \dots, \ell. \end{array} \right\} \quad (22)$$

In contrast to (21), the problem (22) is a nonconvex optimization because the constraint condition  $t \leq \|G\mathbf{v}\|$  is nonconvex. Consequently, standard nonlinear programming approaches are likely to fail for finding the global optimal solution of (22): they converge to a local solution in general. At least, standard nonlinear programming approaches cannot give a proof of the global optimality if they are applied to (22). Certainly the objective value of (22) at a local solution, that is not globally optimal, is smaller than  $r_{\max}$ . It should be emphasized that, for the purpose of robustness analysis, the proof of global optimality is strongly desired, because it guarantees that the modulus of amplitude cannot be larger than the computed value. This motivates us to solve the convex optimization problem (19): its global optimal solution can be computed easily by using an existing algorithm, i.e. the primal-dual interior-point method, and it is guaranteed that  $|u_q|$  cannot be larger than the computed value of  $\sqrt{\rho^*}$ . ■

### 3.3 Bounds for phase angle

In this section, we consider the distribution of the phase angle of  $u_q$  ( $q = 1, \dots, d$ ).

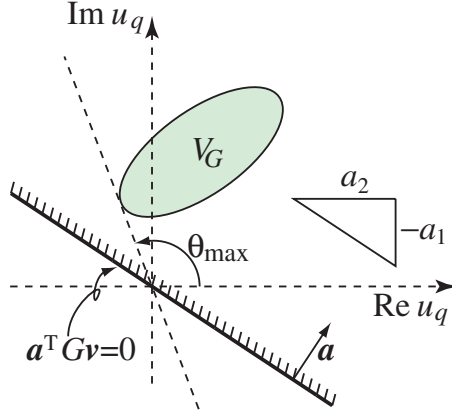


Figure 4: A bounding line which provides an upper bound of  $\text{Arg } u_q$ .

In a manner similar to  $|u_q|$ , we define the maximum value,  $\theta_{\max}$ , and the minimum value,  $\theta_{\min}$ , of the phase angle of  $u_q$  by

$$\theta_{\max} := \max_{\mathbf{f}} \{\text{Arg } u_q(\mathbf{f}) : \mathbf{f} \in \mathcal{F}\}, \quad (23)$$

$$\theta_{\min} := \min_{\mathbf{f}} \{\text{Arg } u_q(\mathbf{f}) : \mathbf{f} \in \mathcal{F}\}. \quad (24)$$

Throughout this section we assume that  $-\pi < \theta_{\min} \leq \theta_{\max} < \pi$ .

*Remark 3.9.* The conditions  $-\pi < \theta_{\min}$  and  $\theta_{\max} < \pi$  are not always satisfied. For example, if  $r_{\min} = 0$ , then  $\theta_{\max} = \pi$ ,  $\inf_{\mathbf{f}} \{\text{Arg } u_q(\mathbf{f}) : \mathbf{f} \in \mathcal{F}(\alpha)\} = -\pi$ , and  $\theta_{\min}$  is not defined consistently in (24). Such a case can be detected by solving (21), because it provides the exact value of  $r_{\min}$ ; see, e.g., Figure 18 (i) in section 5.2 for an example of  $r_{\min} = 0$ . ■

We first consider the case in which  $0 < \theta_{\min} \leq \theta_{\max} < \pi$  is satisfied. The results for the other cases are summarized in Propositions 3.15 and 3.16 below.

In a manner similar to section 3.1, we reformulate the optimization problem (23) into a form of the robust optimization [7]. A key observation is stated as follows: if  $u_q \in \mathbb{C}$  and  $\mathbf{a} = (a_i) \in \mathbb{R}^2$  satisfy

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \text{Re } u_q \\ \text{Im } u_q \end{bmatrix} = 0, \quad (25)$$

then we obtain

$$\tan \text{Arg } u_q = \frac{\text{Im } u_q}{\text{Re } u_q} = -\frac{a_1}{a_2}.$$

Here, we assume  $\text{Re } u_q \neq 0$  and  $a_2 \neq 0$  for simplicity. Consequently, it is sufficient to find a bound for  $\mathbf{a}$  satisfying (25), instead of a bound for  $\text{Arg } u_q$ .

In order to simplify the notation, let  $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$  denote

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

**Proposition 3.10.** *Assume  $0 < \theta_{\max} < \pi$ . Then  $\theta_{\max}$  defined in (23) satisfies*

$$-\frac{1}{\tan \theta_{\max}} = \min_{\mathbf{a}} \left\{ \mathbf{e}_2^T \mathbf{a} : \mathbf{a}^T G \mathbf{v} \geq 0 \ (\forall \mathbf{v} \in \mathcal{V}), \ \mathbf{e}_1^T \mathbf{a} = 1 \right\}. \quad (26)$$

Note that all proofs in this section are collected in appendix A.3 for readability.

A geometrical interpretation of the problem (26) is given in Figure 4. Recall that  $\mathcal{V}_G = \{G\mathbf{v} \mid \mathbf{v} \in \mathcal{V}\}$  and  $G\mathbf{v} = (\operatorname{Re} u_q, \operatorname{Im} u_q)^T$ . It is observed from Figure 4 that any  $\mathbf{v}_G \in \mathcal{V}_G$  satisfies  $\mathbf{a}^T \mathbf{v}_G \geq 0$ , or, equivalently,  $\mathbf{v} \in \mathcal{V}$  satisfies  $\mathbf{a}^T G \mathbf{v} \geq 0$ . Thus, the normal vector,  $\mathbf{a}$ , of the line in Figure 4 satisfies the constraint condition of the problem (26). It is also observed from Figure 4 that  $-a_1/a_2$ , the incline of the line, satisfies  $-a_1/a_2 \geq \tan \theta_{\max}$ , and hence it gives an upper bound of  $\theta_{\max}$ . Since the normal vector  $\mathbf{a}$  is normalized by  $\mathbf{e}_1^T \mathbf{a} = a_1 = 1$  in the problem (26), the upper bound mentioned above becomes more tight by minimizing  $a_2$ .

The following proposition gives a sufficient condition for the constraint condition of (26).

**Proposition 3.11.** *The condition*

$$\forall \mathbf{v} \in \mathcal{V} : \quad \mathbf{a}^T G \mathbf{v} \geq 0 \quad (27)$$

holds if there exists a vector  $(\mathbf{w}, s) \in \mathbb{R}_+^\ell \times \mathbb{R}$  satisfying

$$\begin{bmatrix} O & G^T \mathbf{a} \\ \mathbf{a}^T G & 0 \end{bmatrix} \succeq \sum_{j=1}^{\ell} w_j \Psi_j + s \Theta. \quad (28)$$

Based on the sufficient condition provided by Proposition 3.11, we formulate the following problem, which is a robust counterpart of (26), and gives a conservative bound for  $\theta_{\max}$ .

**Proposition 3.12.** *Assume  $0 < \theta_{\max} < \pi$ . Define  $\varphi^*$  by*

$$a_2^* = \min_{\mathbf{a}, \mathbf{w}, s} \left\{ \mathbf{e}_2^T \mathbf{a} : \begin{bmatrix} O & G^T \mathbf{a} \\ \mathbf{a}^T G & 0 \end{bmatrix} - \sum_{j=1}^{\ell} w_j \Psi_j - s \Theta \succeq O, \ \mathbf{w} \geq \mathbf{0}, \ \mathbf{e}_1^T \mathbf{a} = 1 \right\}, \quad (29)$$

$$\varphi^* = \begin{cases} \operatorname{Arctan}(-1/a_2^*), & (a_2^* \leq 0), \\ \operatorname{Arctan}(-1/a_2^*) + \pi, & (a_2^* > 0). \end{cases} \quad (30)$$

Then  $\theta_{\max} \leq \varphi^*$  holds for  $\theta_{\max}$  defined in (23), i.e.  $\varphi^*$  corresponds to an upper bound for  $\operatorname{Arg} u_q$ .

The proof appears in appendix A.3.3.

Note that (29) is an SDP problem in the variables  $\mathbf{a} \in \mathbb{R}^2$ ,  $\mathbf{w} \in \mathbb{R}^\ell$ , and  $s \in \mathbb{R}$ , and hence it can be solved easily.

We next consider a lower bound for  $\theta_{\min}$ . Analogous to Proposition 3.10, the definition, (24), of  $\theta_{\min}$  is rewritten into the form of robust optimization as

$$-\frac{1}{\tan \theta_{\min}} = \max_{\mathbf{a}} \left\{ \mathbf{e}_2^T \mathbf{a} : \mathbf{a}^T G \mathbf{v} \leq 0 \ (\forall \mathbf{v} \in \mathcal{V}), \ \mathbf{e}_1^T \mathbf{a} = 1 \right\}. \quad (31)$$

Figure 5 shows an example of  $\mathbf{a}$  satisfying the constraint condition  $\mathbf{a}^T G \mathbf{v} \leq 0 \ (\forall \mathbf{v} \in \mathcal{V})$  in (31). A sufficient condition for this constraint condition is given in the following proposition.

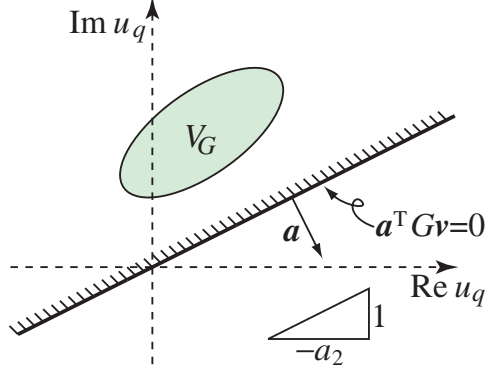


Figure 5: A bounding line which provides a lower bound of  $\text{Arg } u_q$ .

**Proposition 3.13.** *The condition*

$$\forall \mathbf{v} \in \mathcal{V}: \quad \mathbf{a}^T G \mathbf{v} \leq 0$$

holds if there exists a vector  $(\mathbf{w}, s) \in \mathbb{R}_+^\ell \times \mathbb{R}$  satisfying

$$\begin{bmatrix} O & -G^T \mathbf{a} \\ -\mathbf{a}^T G & 0 \end{bmatrix} \succeq \sum_{j=1}^{\ell} w_j \Psi_j + s \Theta.$$

The proof is analogous to Proposition 3.11, and hence is omitted.

We construct a robust counterpart to the problem (31) by replacing the constraint condition  $\mathbf{a}^T G \mathbf{v} \leq 0$  ( $\forall \mathbf{v} \in \mathcal{V}$ ) with its sufficient condition provided by Proposition 3.13. Thus we obtain an SDP problem which gives a lower bound of  $\text{Arg } u_q$  as follows.

**Proposition 3.14.** *Assume  $0 < \theta_{\min} < \pi$ . Put  $a_2^*$  as*

$$a_2^* = \max_{\mathbf{a}, \mathbf{w}, s} \left\{ \mathbf{e}_2^T \mathbf{a} : \begin{bmatrix} O & -G^T \mathbf{a} \\ -\mathbf{a}^T G & 0 \end{bmatrix} - \sum_{j=1}^{\ell} w_j \Psi_j - s \Theta \succeq O, \mathbf{w} \geq \mathbf{0}, \mathbf{e}_1^T \mathbf{a} = 1 \right\}, \quad (32)$$

and define  $\varphi^*$  by (30). Then  $\varphi^* \leq \theta_{\min}$  holds for  $\theta_{\min}$  defined in (24), i.e.  $\varphi^*$  corresponds to a lower bound for  $\text{Arg } u_q$ .

The proof is analogous to Proposition 3.12, and hence is omitted.

For the cases of  $-\pi < \theta_{\max} < 0$  and  $-\pi < \theta_{\min} < 0$ , we show only results in the following two propositions.

**Proposition 3.15.** *Assume  $-\pi < \theta_{\max} < 0$ . Define  $\varphi^*$  by*

$$a_2^* = \min_{\mathbf{a}, \mathbf{w}, s} \left\{ \mathbf{e}_2^T \mathbf{a} : \begin{bmatrix} O & -G^T \mathbf{a} \\ -\mathbf{a}^T G & 0 \end{bmatrix} - \sum_{j=1}^{\ell} w_j \Psi_j - s \Theta \succeq O, \mathbf{e}_1^T \mathbf{a} = 1, \mathbf{w} \geq \mathbf{0}, \mathbf{e}_1^T \mathbf{a} = 1 \right\}, \quad (33)$$

$$\varphi^* = \begin{cases} \text{Arctan}(-1/a_2^*), & (a_2^* \leq 0), \\ \text{Arctan}(-1/a_2^*) - \pi, & (a_2^* > 0). \end{cases} \quad (34)$$

Then  $\theta_{\max} \leq \varphi^*$  holds for  $\theta_{\max}$  defined in (23), i.e.  $\varphi^*$  corresponds to an upper bound for  $\text{Arg } u_q$ .

**Proposition 3.16.** Assume  $-\pi < \theta_{\min} < 0$ . Put  $a_2^*$  as

$$a_2^* = \max_{\mathbf{a}, \mathbf{w}, s} \left\{ \mathbf{e}_2^T \mathbf{a} : \begin{bmatrix} O & G^T \mathbf{a} \\ \mathbf{a}^T G & 0 \end{bmatrix} - \sum_{j=1}^{\ell} w_j \Psi_j - s \Theta \succeq O, \mathbf{e}_1^T \mathbf{a} = 1, \mathbf{w} \geq \mathbf{0}, \mathbf{e}_1^T \mathbf{a} = 1 \right\}, \quad (35)$$

and define  $\varphi^*$  (34). Then  $\varphi^* \leq \theta_{\min}$  holds for  $\theta_{\min}$  defined in (24), i.e.  $\varphi^*$  corresponds to a lower bound for  $\text{Arg } u_q$ .

Consequently, an upper bound for  $\text{Arg } u_q$  can be obtained by solving either (29) or (33); a lower bound for  $\text{Arg } u_q$  can be obtained by solving either (32) or (35).

*Remark 3.17.* In this section 3 we have investigated the distribution of the amplitude of the displacement,  $u_q$ . For this purpose we define the constant matrix  $G$  by (13). Certainly the SDP and SOCP formulations presented are irrespective to the definition of  $G$ , which implies that those formulations are valid for computing bounds for any linear function of the displacement vector. For example, the member stress of a truss is written as a linear function of the displacement vector, and hence we can compute bounds for the amplitude of member stress by using the formulations presented in section 3; see section 5.1.3 for numerical examples of bounds for member stresses. ■

## 4 Bounds for nodal oscillation

Consider a structure in the two-dimensional space. For the  $p$ th node we denote by  $u_{px} \in \mathbb{C}$  and  $u_{py} \in \mathbb{C}$  the nodal displacement in the  $x$ - and  $y$ -th directions, respectively. We have investigated bounds for  $|u_{px}|$  and  $\text{Arg } u_{px}$  in section 3. In this section we consider the distribution of  $\boldsymbol{\nu}_p = (\nu_{px}, \nu_{py}) \in \mathbb{R}^2$  defined by

$$\boldsymbol{\nu}_p = \begin{bmatrix} |u_{px}| \\ |u_{py}| \end{bmatrix}.$$

More precisely, we propose optimization problems in order to find bounds for  $\|\boldsymbol{\nu}_p\|$  and  $\nu_{py}/\nu_{px}$ .

*Remark 4.1.* In this section we consider bounds for  $\sqrt{|u_{px}|^2 + |u_{py}|^2}$  and  $|u_{py}|/|u_{px}|$  in order to estimate the distribution of  $\boldsymbol{\nu}_p = (|u_{px}|, |u_{py}|)$ . Note that further information of the distribution has been obtained from the results in sections 3.1 and 3.2, where we have investigated upper and lower bounds for  $|u_q|$ . Consequently, we can compute upper and lower bounds for  $\sqrt{|u_{px}|^2 + |u_{py}|^2}$ ,  $|u_{py}|/|u_{px}|$ ,  $|u_{px}|$ , and  $|u_{py}|$ . The vector  $(|u_{px}|, |u_{py}|)$  can exist in the intersection of those bounds; see Figure 11 in section 5.1.2 for illustrative examples. ■

### 4.1 Bounds for norm of moduli

In this section we consider bounds for  $\|\boldsymbol{\nu}_p\|$ . Recall that  $\boldsymbol{\nu}_p$  depends on  $\mathbf{f}$ . We denote by  $r_{\max}^{\nu}$  and  $r_{\min}^{\nu}$  the maximum and minimum values of  $\|\boldsymbol{\nu}_p\|$ , which are defined by

$$r_{\max}^{\nu} = \max_{\mathbf{f}} \{\|\boldsymbol{\nu}_p(\mathbf{f})\| : \mathbf{f} \in \mathcal{F}\}, \quad (36)$$

$$r_{\min}^{\nu} = \min_{\mathbf{f}} \{\|\boldsymbol{\nu}_p(\mathbf{f})\| : \mathbf{f} \in \mathcal{F}\}. \quad (37)$$



We can obtain bounds for  $\|\boldsymbol{\nu}_p\|$  in similar manners to sections 3.1 and 3.2. From the definition of  $\boldsymbol{\nu}_p$ , we obtain

$$\|\boldsymbol{\nu}_p\| = \left\| \begin{bmatrix} |u_{px}| \\ |u_{py}| \end{bmatrix} \right\| = \left\| \begin{bmatrix} \text{Re } u_{px} \\ \text{Im } u_{px} \\ \text{Re } u_{py} \\ \text{Im } u_{py} \end{bmatrix} \right\|.$$

Let  $\hat{G} \in \mathbb{R}^{4 \times 2d}$  denote a constant matrix satisfying

$$\hat{G}\boldsymbol{v} = \begin{bmatrix} \text{Re } u_{px} \\ \text{Im } u_{px} \\ \text{Re } u_{py} \\ \text{Im } u_{py} \end{bmatrix}.$$

Then the problem (36) is reduced to

$$r_{\max}^\nu = \max_{\boldsymbol{v}} \left\{ \|\hat{G}\boldsymbol{v}\| : \boldsymbol{v} \in \mathcal{V} \right\},$$

which is mathematically equivalent to the problem investigated in section 3.1. Consequently, we obtain an SDP problem giving an upper bound for  $r_{\max}^\nu$  from (19) in Proposition 3.6, by replacing  $G$  with  $\hat{G}$ . Similarly, an SOCP problem for computing  $r_{\min}^\nu$  can be obtained from (21) in Proposition 3.7 by replacing  $G$  with  $\hat{G}$ . These observations are summarized in the following two propositions.

**Proposition 4.2.** Define  $\rho^*$  by

$$\rho^* = \min_{\rho, \boldsymbol{w}, s} \left\{ \rho : \begin{bmatrix} -\hat{G}^T \hat{G} & \mathbf{0} \\ \mathbf{0}^T & \rho \end{bmatrix} - \sum_{j=1}^{\ell} w_j \Psi_j - s\Theta \succeq O, \boldsymbol{w} \geq \mathbf{0} \right\}. \quad (38)$$

Then  $(r_{\max}^\nu)^2 \leq \rho^*$  holds for  $r_{\max}^\nu$  defined in (36), i.e.  $\sqrt{\rho^*}$  corresponds to an upper bound of  $\|\boldsymbol{\nu}_p\|$ .

**Proposition 4.3.**  $r_{\min}^\nu$  defined in (37) satisfies

$$t^* = \min_{t, \boldsymbol{v}, \boldsymbol{\zeta}} \left\{ t : r \geq \|\hat{G}\boldsymbol{v}\|, S_1 \boldsymbol{v} = \tilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}, S_2 \boldsymbol{v} = \mathbf{0}, \alpha \geq \|T_j \boldsymbol{\zeta}\| (j = 1, \dots, \ell) \right\}, \quad (39)$$

i.e. the minimum value of  $\|\boldsymbol{\nu}_p\|$  can be obtained by solving the problem (39).

## 4.2 Bounds for ratio of moduli

We consider bounds for ratio  $\nu_{py}/\nu_{px}$ . For this purpose, we attempt to compute bounds for  $\nu_{py}^2/\nu_{px}^2$  instead of  $\nu_{py}/\nu_{px}$ . Introduce auxiliary variables  $\mu_x = (\text{Re } u_{px})^2 + (\text{Im } u_{px})^2 = \nu_{px}^2$  and  $\mu_y = (\text{Re } u_{py})^2 + (\text{Im } u_{py})^2 = \nu_{py}^2$ , and we consider bounds for  $\mu_y/\mu_x$ . Note that  $\mu_y \geq 0$  and  $\mu_x \geq 0$ . We assume  $\mu_x \neq 0$ .

The maximum value,  $t_{\max}$ , of  $|u_{py}|/|u_{px}|$  is defined by

$$t_{\max} = \max_{\boldsymbol{f}} \{ \nu_{py}/\nu_{px} : \nu_{px} = |u_{px}(\boldsymbol{f})|, \nu_{py} = |u_{py}(\boldsymbol{f})|, \boldsymbol{f} \in \mathcal{F} \}. \quad (40)$$

We attempt to find an upper bound for  $t_{\max}$ .

Define  $G_{px}, G_{py} \in \mathbb{R}^{2 \times 2d}$  as constant matrices satisfying

$$G_x \mathbf{v} = \begin{bmatrix} \operatorname{Re} u_{px} \\ \operatorname{Im} u_{px} \end{bmatrix}, \quad G_y \mathbf{v} = \begin{bmatrix} \operatorname{Re} u_{py} \\ \operatorname{Im} u_{py} \end{bmatrix}.$$

By using  $\mathcal{V}$  defined in (12), (40) is reduced to

$$t_{\max} = \max_{\mathbf{v}} \{ \nu_y / \nu_x : \nu_x = \|G_x \mathbf{v}\|, \nu_y = \|G_y \mathbf{v}\|, \mathbf{v} \in \mathcal{V} \}. \quad (41)$$

However, instead of solving this problem directly, we treat

$$(t_{\max})^2 = \max_{\mathbf{v}} \{ \mu_y / \mu_x : \mu_x = \|G_x \mathbf{v}\|^2, \mu_y = \|G_y \mathbf{v}\|^2, \mathbf{v} \in \mathcal{V} \}. \quad (42)$$

In a manner similar to Proposition 3.10, we reformulate the problem (42) into the form of robust optimization [7] as follows. Notice here that  $\mu_x \geq 0$  and  $\mu_y \geq 0$  are satisfied in (42), which corresponds to the assumption in Proposition 3.10.

**Proposition 4.4.**  $t_{\max}$  defined in (40) satisfies

$$-\frac{1}{(t_{\max})^2} = \min_{\mathbf{a}} \{ \mathbf{e}_2^T \mathbf{a} : \mathbf{a}^T \boldsymbol{\mu} \geq 0 \ (\forall \boldsymbol{\mu} \in \{ (\|G_x \mathbf{v}\|^2, \|G_y \mathbf{v}\|^2)^T \mid \mathbf{v} \in \mathcal{V} \}), \mathbf{e}_1^T \mathbf{a} = 1 \}. \quad (43)$$

The proof is analogous to Proposition 3.10, and hence is omitted.

Let  $\hat{\Theta}_1, \hat{\Theta}_2 \in \mathcal{S}^{2d+3}$  be

$$\hat{\Theta}_1 = \begin{bmatrix} O & O & \begin{bmatrix} 1/2 \\ 0 \\ \mathbf{0} \\ 0 \end{bmatrix} \\ O & -G_{px} G_{px}^T & \mathbf{0} \\ \begin{bmatrix} 1/2 & 0 \end{bmatrix} & \mathbf{0}^T & 0 \end{bmatrix}, \quad \hat{\Theta}_2 = \begin{bmatrix} O & O & \begin{bmatrix} 0 \\ 1/2 \\ \mathbf{0} \\ 0 \end{bmatrix} \\ O & -G_{py} G_{py}^T & \mathbf{0} \\ \begin{bmatrix} 0 & 1/2 \end{bmatrix} & \mathbf{0}^T & 0 \end{bmatrix},$$

By using  $\Psi_j$  ( $j = 1, \dots, \ell$ ) and  $\Theta$  defined in Proposition 3.4, define  $\hat{\Psi}_1, \dots, \hat{\Psi}_\ell \in \mathcal{S}^{2d+3}$  and  $\hat{\Theta}_3 \in \mathcal{S}^{2d+3}$  by

$$\hat{\Psi}_j = \begin{bmatrix} O & O \\ O & \Psi_j \end{bmatrix}, \quad j = 1, \dots, \ell,$$

$$\hat{\Theta}_3 = \begin{bmatrix} O & O \\ O & \Theta \end{bmatrix}.$$

As a conservative approximation of the problem (43), we obtain the following SDP problem which provides an upper bound for  $t_{\max}$ .

**Proposition 4.5.** Define  $\varphi^*$  by

$$a_2^* = \min_{\mathbf{a}, \mathbf{w}, \mathbf{s}} \left\{ \mathbf{e}_2^T \mathbf{a} : \begin{bmatrix} O & O & \mathbf{a} \\ O & O & \mathbf{0} \\ \mathbf{a}^T & \mathbf{0}^T & 0 \end{bmatrix} - \sum_{j=1}^{\ell} w_j \hat{\Psi}_j - \sum_{i=1}^3 s_i \hat{\Theta}_i \succeq O, \mathbf{w} \geq \mathbf{0}, \mathbf{e}_1^T \mathbf{a} = 1 \right\} \quad (44)$$

and  $\varphi^* = \sqrt{-1/a_2^*}$ . Then  $t_{\max} \leq \varphi^*$  holds for  $t_{\max}$  defined in (40), i.e.  $\varphi^*$  corresponds to an upper bound for  $\mu_y / \mu_x$ .

The proof appears in appendix A.4.

We next consider the minimum value,  $t_{\min}$ , of  $|u_{py}|/|u_{px}|$  which is defined by

$$t_{\min} = \min_{\mathbf{f}} \{ \nu_{py}/\nu_{px} : \nu_{px} = |u_{px}(\mathbf{f})|, \nu_{py} = |u_{py}(\mathbf{f})|, \mathbf{f} \in \mathcal{F} \}. \quad (45)$$

A robust optimization reformulation of (45) is obtained as

$$-\frac{1}{(t_{\min})^2} = \max_{\mathbf{a}} \{ \mathbf{e}_2^T \mathbf{a} : \mathbf{a}^T \boldsymbol{\mu} \leq 0 \ (\forall \boldsymbol{\mu} \in \{ (\|G_{px}\mathbf{v}\|^2, \|G_{py}\mathbf{v}\|^2)^T \mid \mathbf{v} \in \mathcal{V} \}), \mathbf{e}_1^T \mathbf{a} = 1 \}. \quad (46)$$

In a manner similar to Proposition 4.5, we can show that a lower bound for  $t_{\min}$  is obtained by solving the following SDP problem.

**Proposition 4.6.** Define  $\varphi^*$  by

$$\max_{\mathbf{a}, \mathbf{w}, \mathbf{s}} \left\{ \mathbf{e}_2^T \mathbf{a} : \begin{bmatrix} O & O & -\mathbf{a} \\ O & O & \mathbf{0} \\ -\mathbf{a}^T & \mathbf{0}^T & 0 \end{bmatrix} - \sum_{j=1}^{\ell} w_j \hat{\Psi}_j - \sum_{i=1}^3 s_i \hat{\Theta}_i \succeq O, \mathbf{w} \geq \mathbf{0}, \mathbf{e}_1^T \mathbf{a} = 1 \right\} \quad (47)$$

and  $\varphi^* = \sqrt{-1/a_2^*}$ . Then  $\varphi^* \leq t_{\min}$  holds for  $t_{\min}$  defined in (45), i.e.  $\varphi^*$  corresponds to a lower bound for  $\mu_y/\mu_x$ .

## 5 Numerical experiments

Conservative bounds for dynamic responses under the load uncertainties are found for various structures by solving SDP problems and SOCP problems. Computation has been carried out on Core 2 Duo (1.2 GHz with 2.0 GB memory) with MATLAB Ver. 7.5.0 [40]. We solve the SDP and SOCP problems by using SeDuMi Ver. 1.05 [37], which implements the primal-dual interior-point method for the linear programming problems over symmetric cones.

In the following examples, the elastic modulus is 200 GPa, and the mass density of members is  $7.86 \times 10^3$  kg/m<sup>3</sup>. In (8), we assume the complex damping, or the linear hysteretic damping, i.e. the damping matrix  $C$  is given as

$$\omega C = 2\beta K,$$

where we put  $\beta = 0.02$  in the following examples.

### 5.1 2-bar truss

Consider a two-bar truss illustrated in Figure 6. The nodes (b) and (c) are pin-supported at  $(x, y) = (0, 100)$  and  $(0, 0)$  in cm, respectively, while the node (a) is free, i.e.  $d = 2$ . The initial lengths of members (1) and (2) are 100 cm and  $100\sqrt{2}$  cm, respectively.

As the nominal load,  $\tilde{\mathbf{f}}$ , the external force 10 kN is applied in the positive direction of the  $x$ -axis at the node (a). In accordance with (5), the uncertainty model of the load is defined as

$$\mathbf{f} = \tilde{\mathbf{f}} + \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad \alpha \geq \|\boldsymbol{\zeta}\|, \quad j = 1, 2, \quad (48)$$

where we put  $F^0 = 1.0$  (kN)  $\times I_2$ ,  $\ell = 1$ , and  $T_1 = I_2$ . Consequently, the uncertain load  $\mathbf{f}$  is running through the circle depicted with the dashed lines in Figure 6.

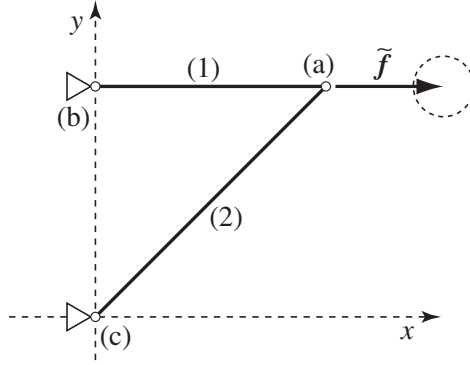


Figure 6: A 2-bar truss.

### 5.1.1 Bounds for complex amplitude

The bounds obtained are illustrated in Figure 7. The complex amplitude corresponding to the nominal load,  $\tilde{\mathbf{f}}$ , is

$$\tilde{\mathbf{u}} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0.141 \\ 0.033 \end{bmatrix} + i \begin{bmatrix} -1.158 \\ 4.877 \end{bmatrix},$$

which is depicted in Figure 7 with  $*$ .

We first choose  $\omega = \omega_1^0$ , where  $\omega_1^0$  is the undamped fundamental natural circular frequency, and put  $\alpha = 1$ . For the displacement in the  $x$ -direction illustrated in Figure 7 (i), the upper bound of the modulus of the complex amplitude is 1.665 cm, which is obtained by solving the SDP problem (19); the minimum modulus is 0.672 cm, which is obtained by solving the SOCP problem (21); the upper and lower bounds for the argument are  $-1.358$  rad and  $-1.488$  rad, respectively, which are obtained by solving the SDP problems (33) and (35). Note that we solve (33) and (35) because  $\tilde{u}_x$  exists in the fourth quadrant. For the displacement in the  $y$ -direction (Figure 7 (ii)), upper and lower bounds for the argument are 1.566 rad and 1.559 rad, respectively, which are obtained by solving the SDP problems (29) and (32). In summary, we obtain the bounds for  $\mathbf{u} = (u_x, u_y)^T$  as

$$\begin{aligned} 0.672 \text{ cm} &\leq |u_x| \leq 1.665 \text{ cm}, & -1.488 \text{ rad} &\leq \text{Arg } u_x \leq -1.358 \text{ rad}, \\ 2.754 \text{ cm} &\leq |u_y| \leq 7.000 \text{ cm}, & 1.559 \text{ rad} &\leq \text{Arg } u_y \leq 1.566 \text{ rad}. \end{aligned}$$

The system is almost in the quadrature with the driving load, but  $u_x$  and  $u_y$  are in anti-phase with each other, because  $\text{Arg } u_x \simeq -\pi/2$  and  $\text{Arg } u_y \simeq \pi/2$ .

In order to verify these results, we randomly generate a number of loads,  $\mathbf{f}$ , satisfying (48), and compute the corresponding amplitudes. The obtained displacements are shown by a lot of points in Figure 8 with the bounds obtained. It is observed in Figure 8 that all generated amplitudes are included in the bounds, which confirms that the obtained bounds correspond to confidential, or outer, approximations of the sets of complex amplitudes. In addition, we can also see that all bounds are sufficiently tight.

We next investigate the dependence of the distribution of  $\mathbf{u}$  on the system parameters. Figure 9 depicts the variation of  $|u_x|$  and  $\text{Arg } u_x$  with respect to the magnitude of uncertainty,  $\alpha$ . In Figure 10, we illustrate the bounds of  $u_x$  and  $u_y$  for various  $\omega$ , as well as the samples generated randomly. From

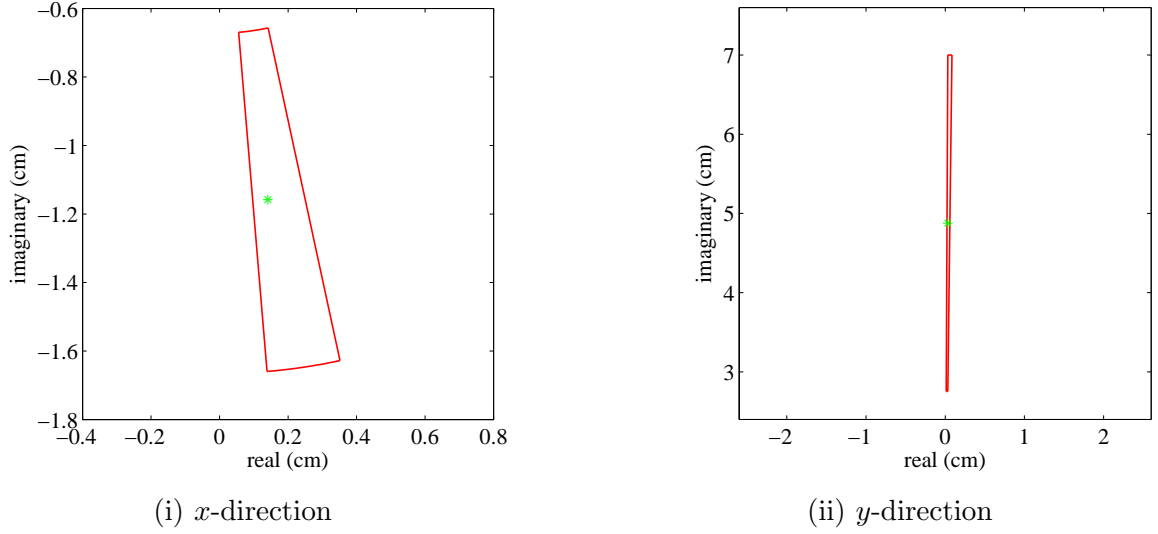


Figure 7: Bounds for the complex amplitudes of the displacements of the 2-bar truss ( $\omega = \omega_1^0$ ). \*: nominal values; (i) displacement in the  $x$ -direction; (ii) displacement in the  $y$ -direction.

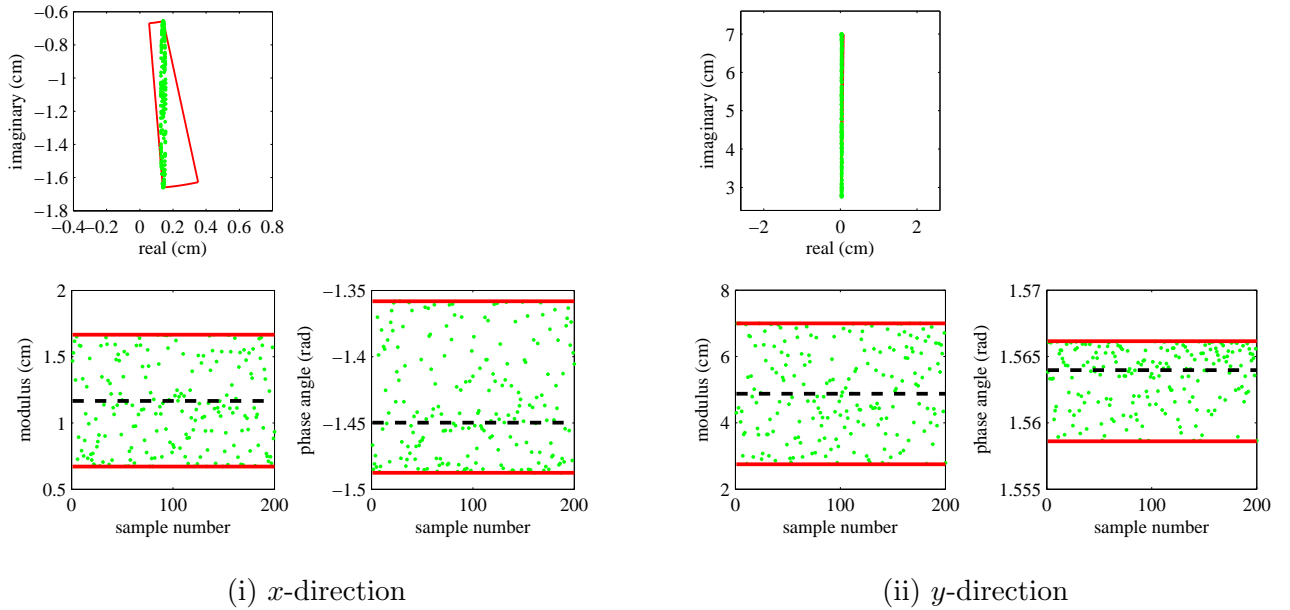


Figure 8: Bounds and randomly generated samples for the member stresses of the 2-bar truss ( $\omega = \omega_1^0$ ). ‘—’: bounds found in Figure 7; ‘- -’: nominal values.

Figure 10 we can observe the resonant behavior at  $\omega = \omega_2^0$ , where  $u_x$  and  $u_y$  are in the same phase. In Figure 10 (iii) and Figure 10 (iv) we see that the system is in phase with the driving load, i.e.  $\text{Arg } u_x \simeq 0$  and  $\text{Arg } u_y \simeq 0$ .

### 5.1.2 Bounds for nodal oscillation

We next consider the distribution of the vector  $\boldsymbol{\nu} = (|u_x|, |u_y|)^T$ , as is investigated in section 4. Upper and lower bounds for  $\sqrt{|u_x|^2 + |u_y|^2}$  are obtained by solving (38) and (39), respectively; upper and

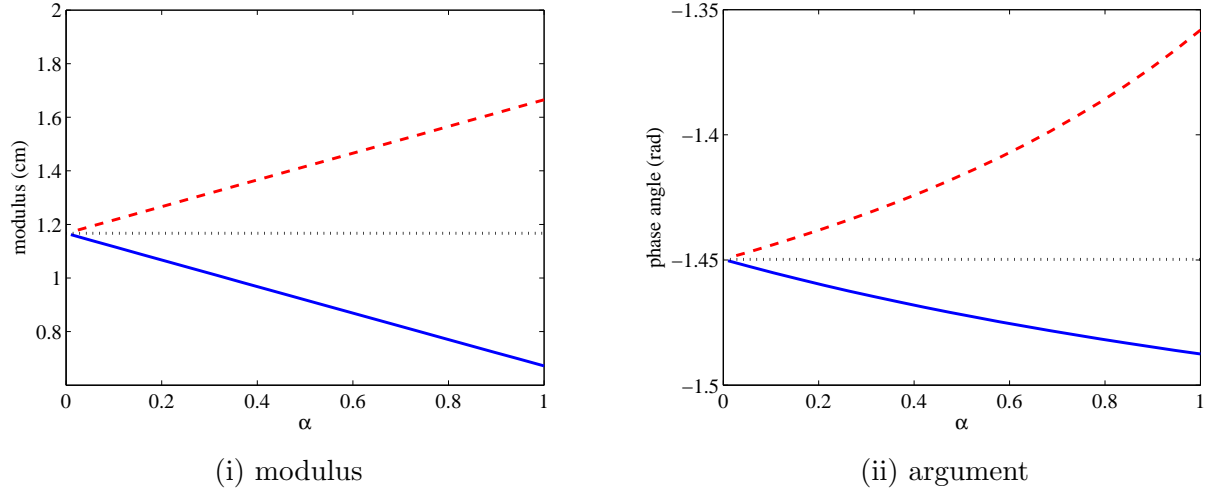


Figure 9: Variations of bounds for the complex amplitudes of the displacements in the  $x$ -direction of the 2-bar truss with respect to the magnitude of uncertainty  $\alpha$ . (i) modulus of the amplitude; (ii) phase angle. ‘...’: nominal value; ‘—’: lower bounds; ‘- -’: upper bounds.

lower bounds for  $|u_y|/|u_x|$  are obtained by solving (44) and (47), respectively. The obtained bounds are shown in Figure 11 for various  $\omega$ . Here, the vector  $(|u_x|, |u_y|)$  exists in the intersection of the two sets shown with the solid and dashed lines, as discussed in Remark 4.1. Note that we plot  $-|u_x|$  and  $|u_y|$  in Figures 11 (i) and (ii), because we can see from Figure 7 and Figure 10 that  $u_x$  is in anti-phase with  $u_y$ . In contrast, we plot  $|u_x|$  and  $|u_y|$  in Figures 11 (iii) and (iv).

The directions of two eigenmodes are also depicted in Figure 11. We can see the resonances in Figure 11 (i) and (iv). From Figure 12 we can confirm that the bounds shown in Figure 11 are confidential and sufficiently tight.

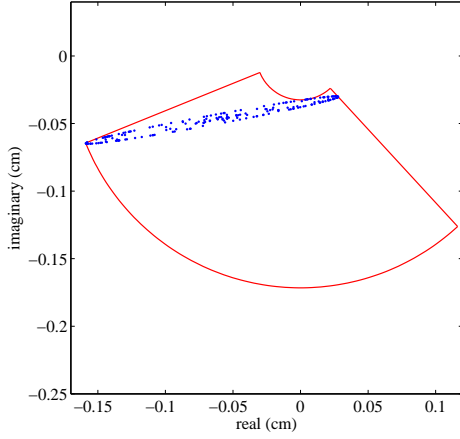
### 5.1.3 Bounds for member stress

As discussed in Remark 3.17, the distribution of (the complex amplitudes of) the member stresses is investigated in Figure 13 and Figure 14 by using the SDP and SOCP formulations presented in section 3. It is observed in Figure 13 that the stresses of both members are in anti-phase with the driving load of  $\omega = \omega_1^0$ . From Figure 14 we see that the stress of the member (2) is in phase with the driving load of  $\omega = 1.1\omega_1^0$ , while there exists a case in which the stress of the member (1) is in anti-phase with the driving load, i.e. the phase angle of the member (1) has very large uncertainty.

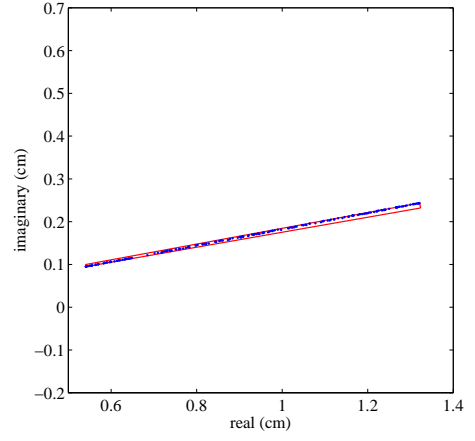
## 5.2 29-bar truss

Next we consider a 29-bar truss shown in Figure 1. The lengths of members in the directions of the  $x$ - and  $y$ -axes are 50 cm and 100 cm, respectively.

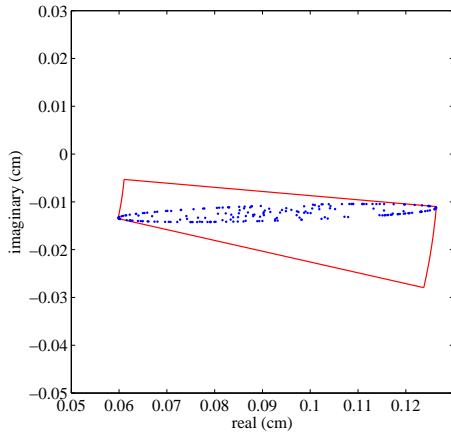
As the nominal load,  $\tilde{\mathbf{f}}$ , the external forces 12 kN and 8 kN are applied at the nodes (a) and (b), respectively, in the negative direction of  $y$ -axis. The driving frequency is  $\omega = \omega_1^0$ . The uncertainty model of  $\mathbf{f}$ , (5), is defined as discussed in Example 2.1 with  $\bar{f}_0 = 1$  kN. Consequently, the external forces are running through the circles depicted with the dotted lines in Figure 1. Note that uncertain forces may possibly exist at all free nodes, and any two sets of the uncertain loads have no relation.



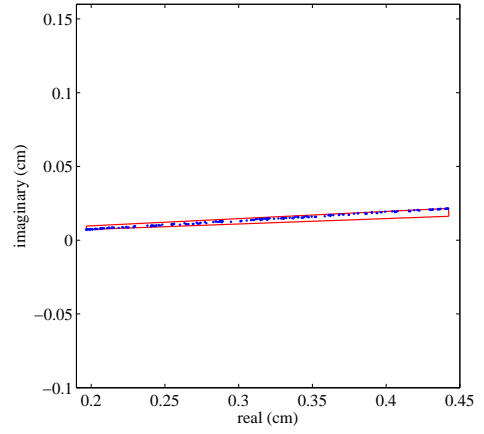
(i)  $x$ -direction ( $\omega = 1.1\omega_1^0$ )



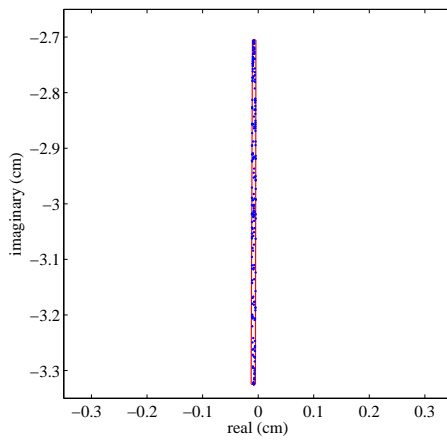
(ii)  $y$ -direction ( $\omega = 1.1\omega_1^0$ )



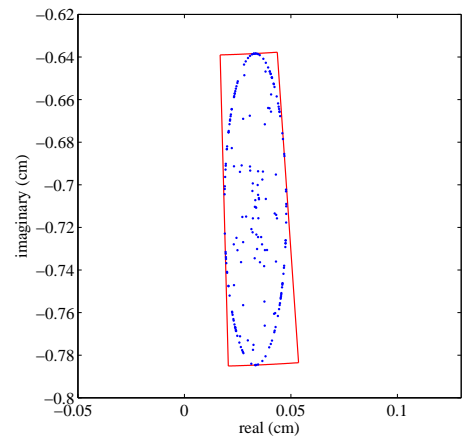
(iii)  $x$ -direction ( $\omega = 1.3\omega_1^0$ )



(iv)  $y$ -direction ( $\omega = 1.3\omega_1^0$ )



(v)  $x$ -direction ( $\omega = \omega_2^0$ )



(vi)  $y$ -direction ( $\omega = \omega_2^0$ )

Figure 10: Bounds and randomly generated samples for the complex amplitudes of the displacements of the 2-bar truss.

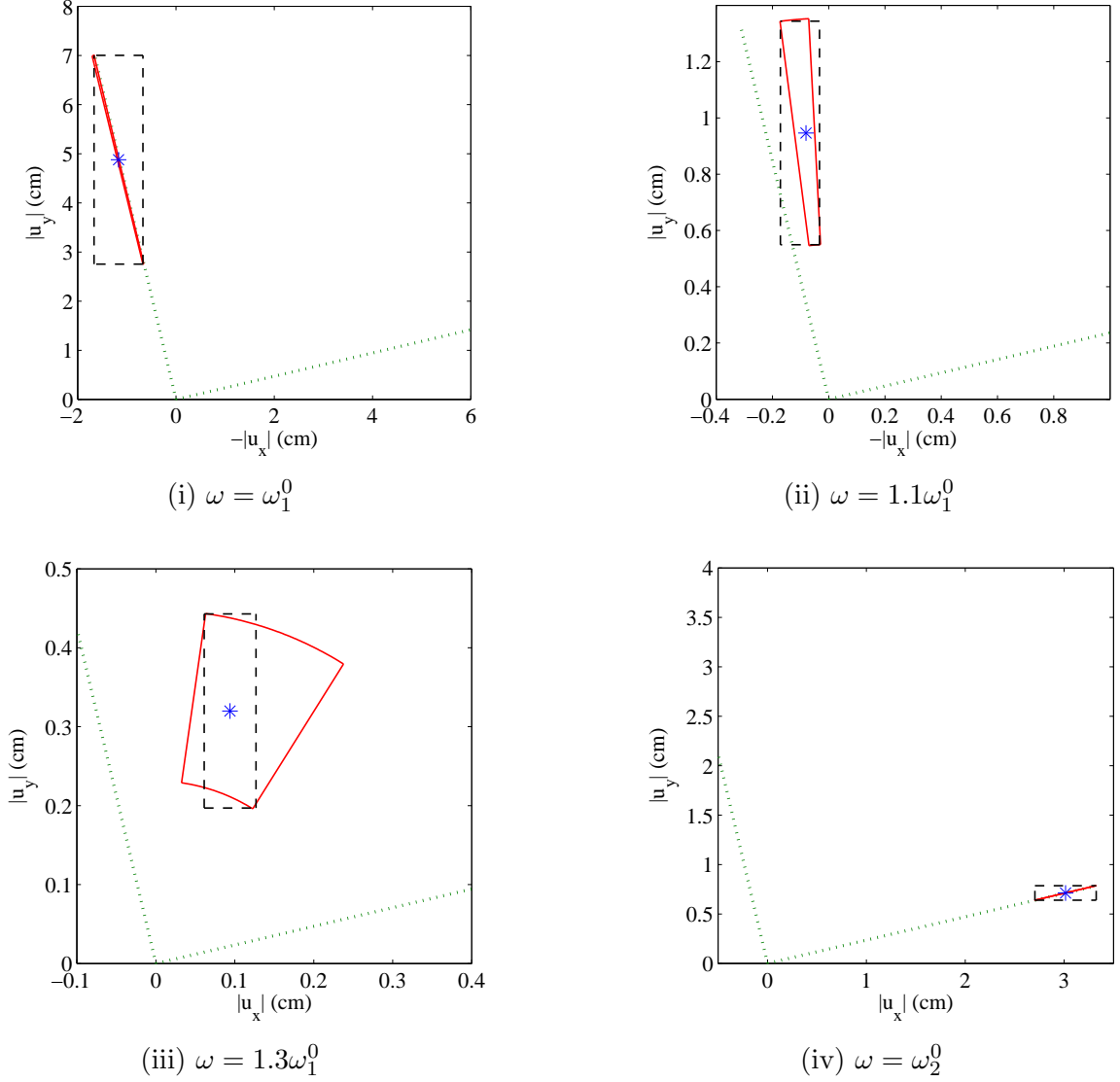


Figure 11: Bounds for the nodal oscillation of the 2-bar truss. \*: nominal values; ‘—’: bounds obtained by solving (38), (39), (44), and (47); ‘- -’: bounds obtained by solving (19) and (21); ‘...’: directions of eigenmodes without damping.

We investigate the displacement of the node (a), which is denoted by  $(u_x, u_y)$ . We put  $\alpha = 1.0$  in (5). For the amplitudes in the  $x$ - and  $y$ -th directions, we obtain the bounds of the moduli and phase angles, which are shown in Figure 15. The obtained bounds are confirmed in Figure 16 to be conservative and sufficiently tight. Variations of  $|u_x|$ ,  $|u_y|$ ,  $\text{Arg } u_x$ , and  $\text{Arg } u_y$  with respect to the magnitude of uncertainty,  $\alpha$ , are depicted in Figure 17. Figure 18 illustrates the variations of  $|u_x|$  and  $|u_y|$  with respect to the frequency of the driving load,  $\omega$ , in the range of  $\omega \in [\omega_1^0, \omega_2^0]$ , where  $\omega_1^0 = 1.248$  rad/s and  $\omega_2^0 = 1.872$  rad/s. It is observed from Figure 18 (i) that the minimum value,  $r_{\min}$ , of  $|u_x|$  is  $r_{\min} = 0$  for  $\omega \in [1.76, \omega_2^0]$  as discussed in Remark 3.9.

For the node (a), Figure 19 depicts the bounds for the distribution of the vector  $(|u_x|, |u_y|)$ , which are confirmed in Figure 20. It is observed from Figure 19 (i) that the driving force with  $\omega = \omega_1^0$  yields the resonance, where the oscillation is in the direction of the first eigenmode. In contrast,



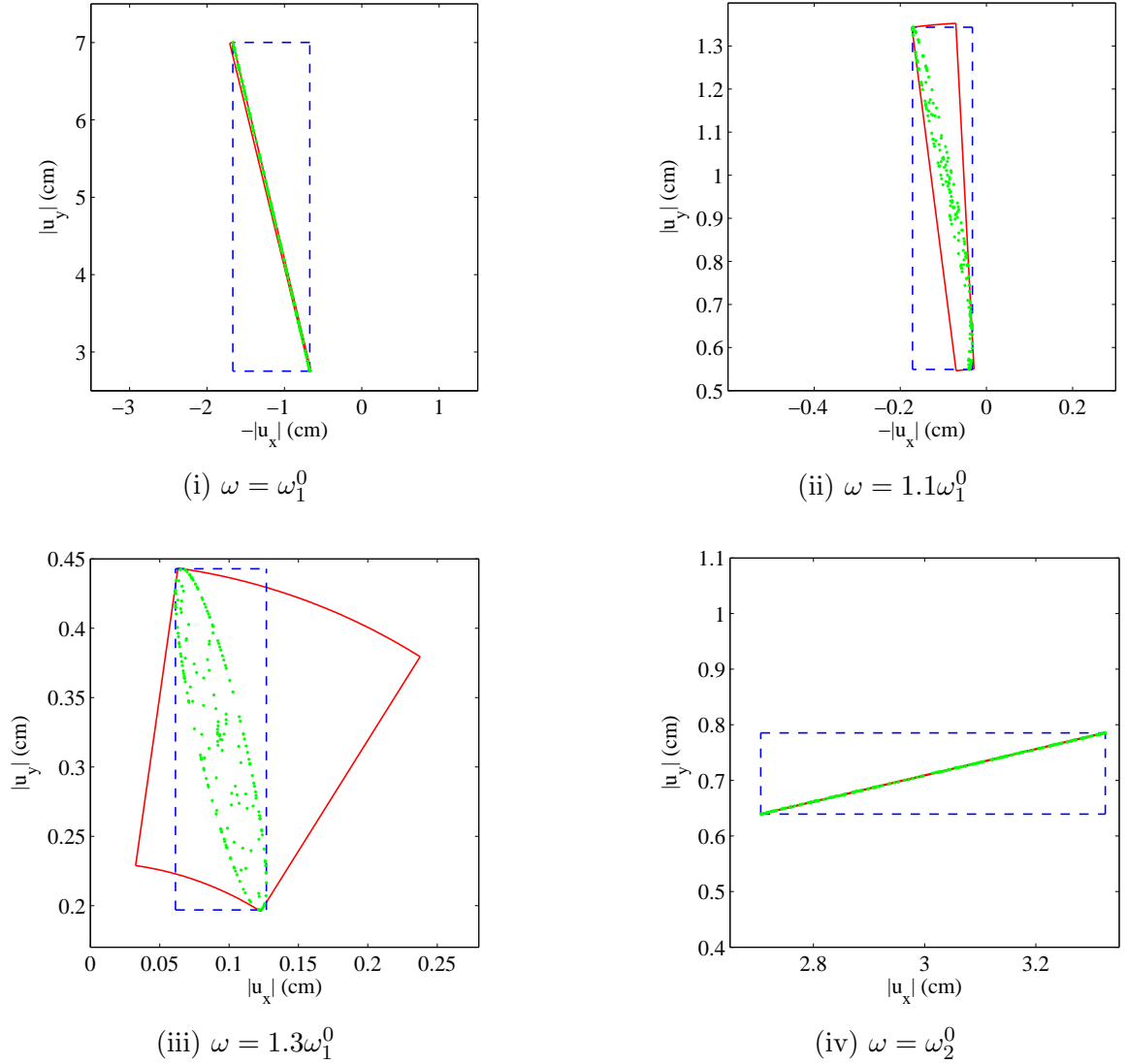


Figure 12: Bounds and randomly generated samples for the nodal oscillation of the 2-bar truss.

in Figure 19 (iv) with  $\omega = \omega_2^0$ , the uncertainty of the oscillation direction,  $|u_y|/|u_x|$ , is very large, although there exists a case in which the oscillation is approximately in the direction of the second eigenmode. It should be emphasized that in such a case the obtained bounds are very tight, which can be seen in Figure 20 (iv). Figure 21 illustrates the variations of  $\sqrt{|u_x|^2 + |u_y|^2}$  with respect to  $\alpha$  and  $\omega$ .

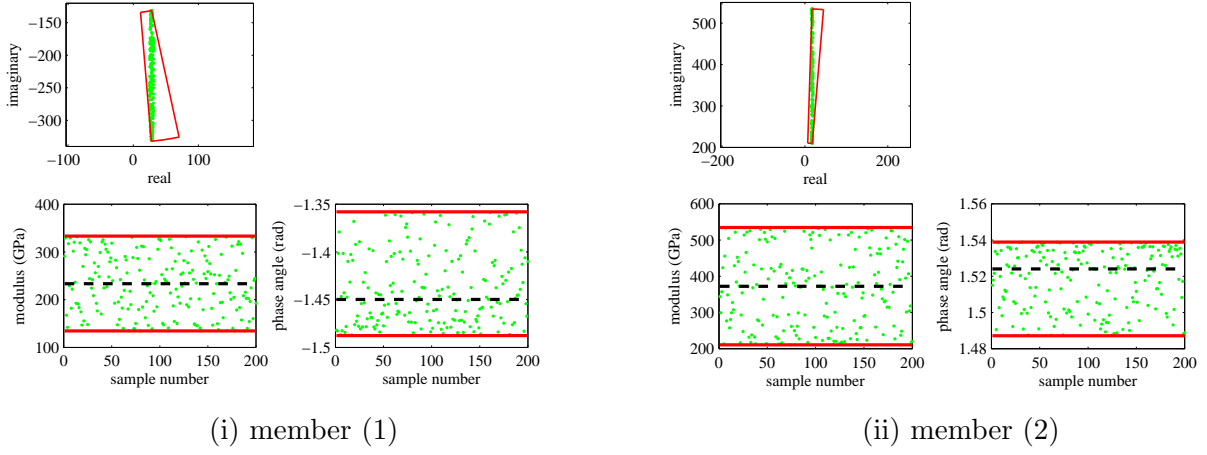


Figure 13: Bounds and randomly generated samples for the member stresses of the 2-bar truss ( $\omega = \omega_1^0$ ). ‘—’: bounds; ‘- -’: nominal values.

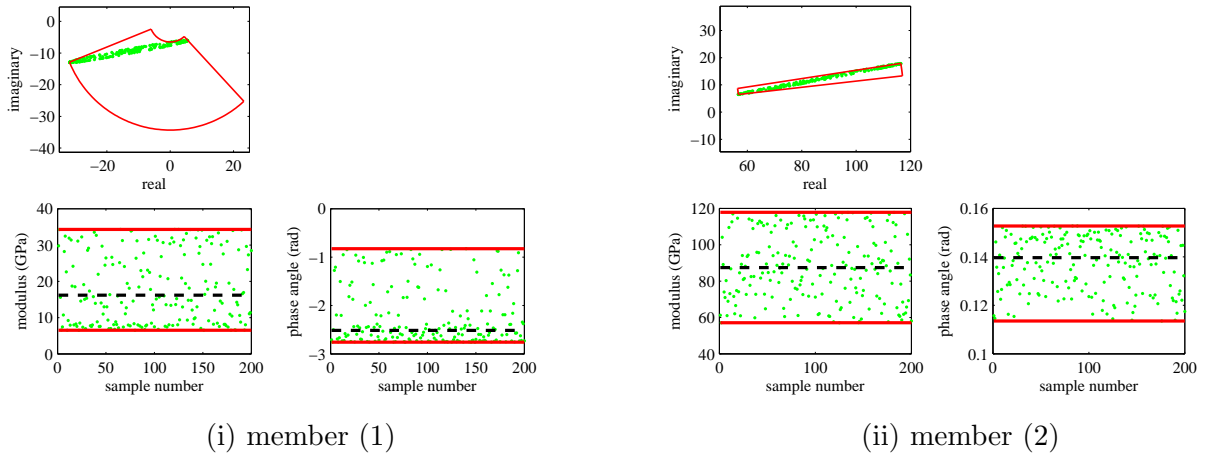
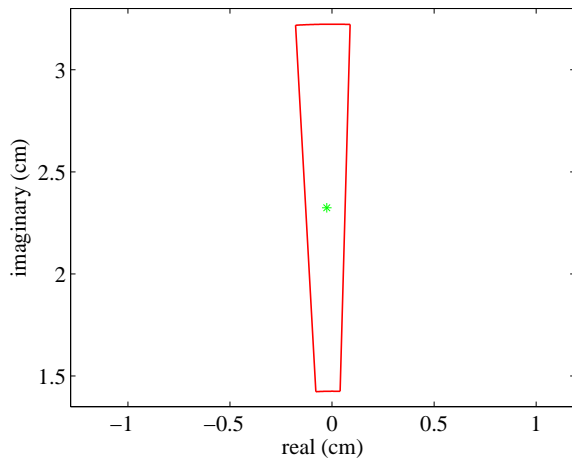
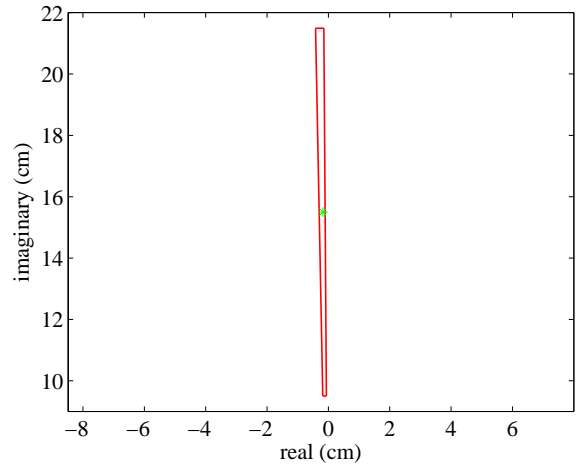


Figure 14: Bounds and randomly generated samples for the member stresses of the 2-bar truss ( $\omega = 1.1\omega_1^0$ ). ‘—’: bounds; ‘- -’: nominal values.

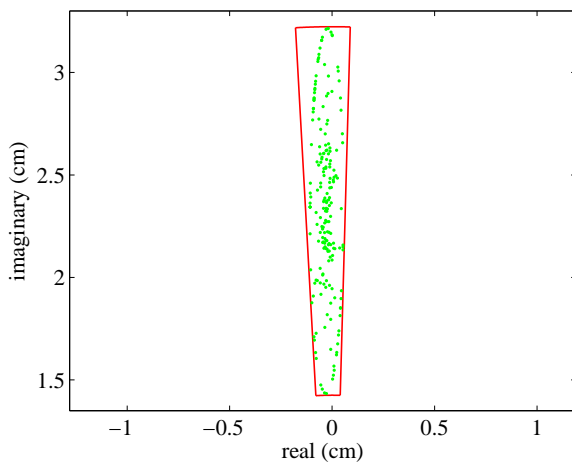


(i)  $x$ -direction

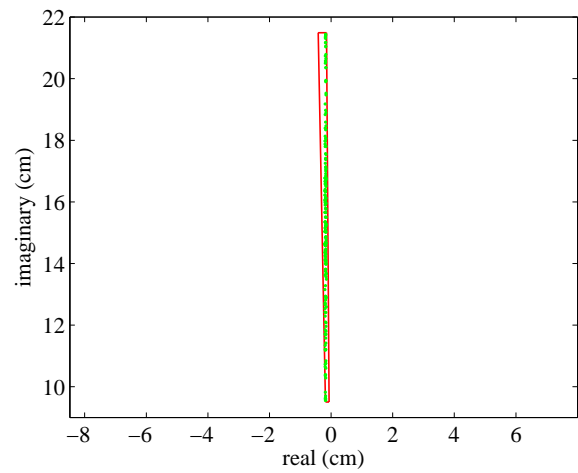


(ii)  $y$ -direction

Figure 15: Bounds for the complex amplitudes of the displacements of the 29-bar truss ( $\omega = \omega_1^0$ ). \*: nominal values; (i) displacement in the  $x$ -direction; (ii) displacement in the  $y$ -direction.

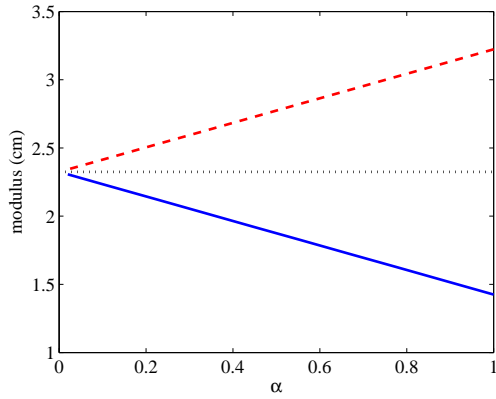


(i)  $x$ -direction

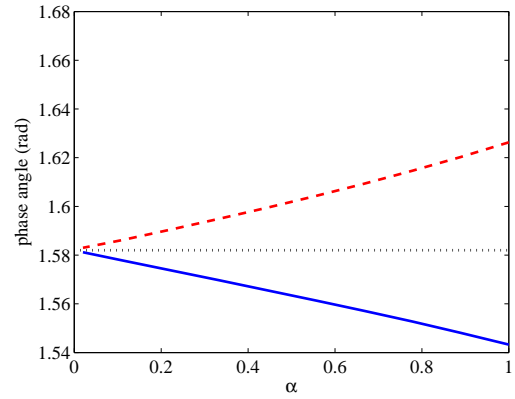


(ii)  $y$ -direction

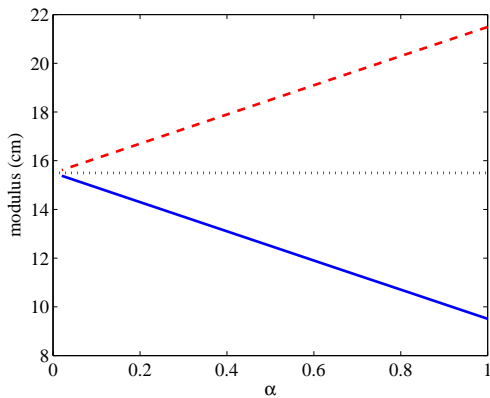
Figure 16: Bounds and randomly generated samples for the complex amplitudes of the displacements of the 29-bar truss ( $\omega = \omega_1^0$ ). ‘—’: bounds found in Figure 15; ‘ $\cdot\cdot\cdot$ ’: nominal values.



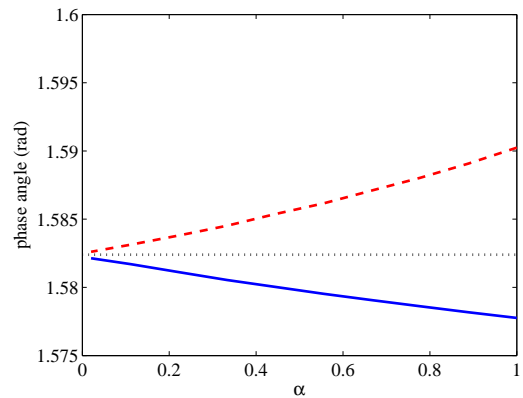
(i) modulus of the  $x$ -direction



(ii) argument of the  $x$ -direction

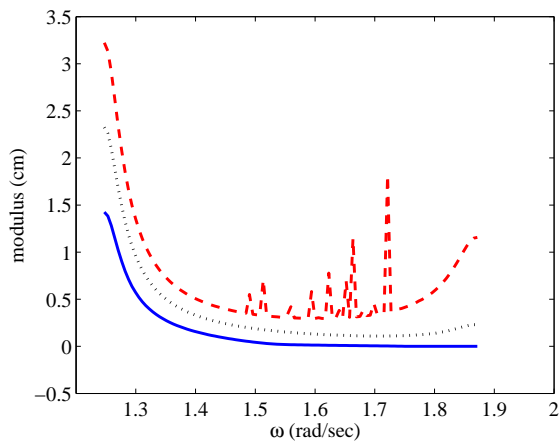


(iii) modulus of the  $y$ -direction

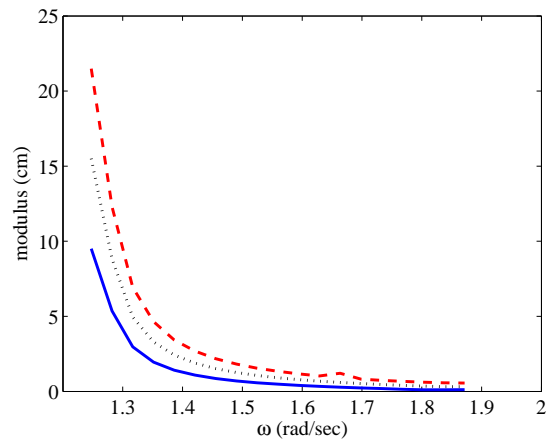


(iv) argument of the  $y$ -direction

Figure 17: Variations of bounds for the complex amplitudes of the displacements in the  $x$ -direction of the 29-bar truss with respect to the magnitude of uncertainty  $\alpha$ . ‘ $\cdots$ ’: nominal value; ‘—’: lower bounds; ‘- -’: upper bounds.



(i)  $x$ -direction



(ii)  $y$ -direction

Figure 18: Variations of bounds for the complex amplitudes of the displacements in the  $x$ -direction of the 29-bar truss with respect to the frequency  $\omega$  of the driving force. ‘ $\cdots$ ’: nominal value; ‘—’: lower bounds; ‘- -’: upper bounds.

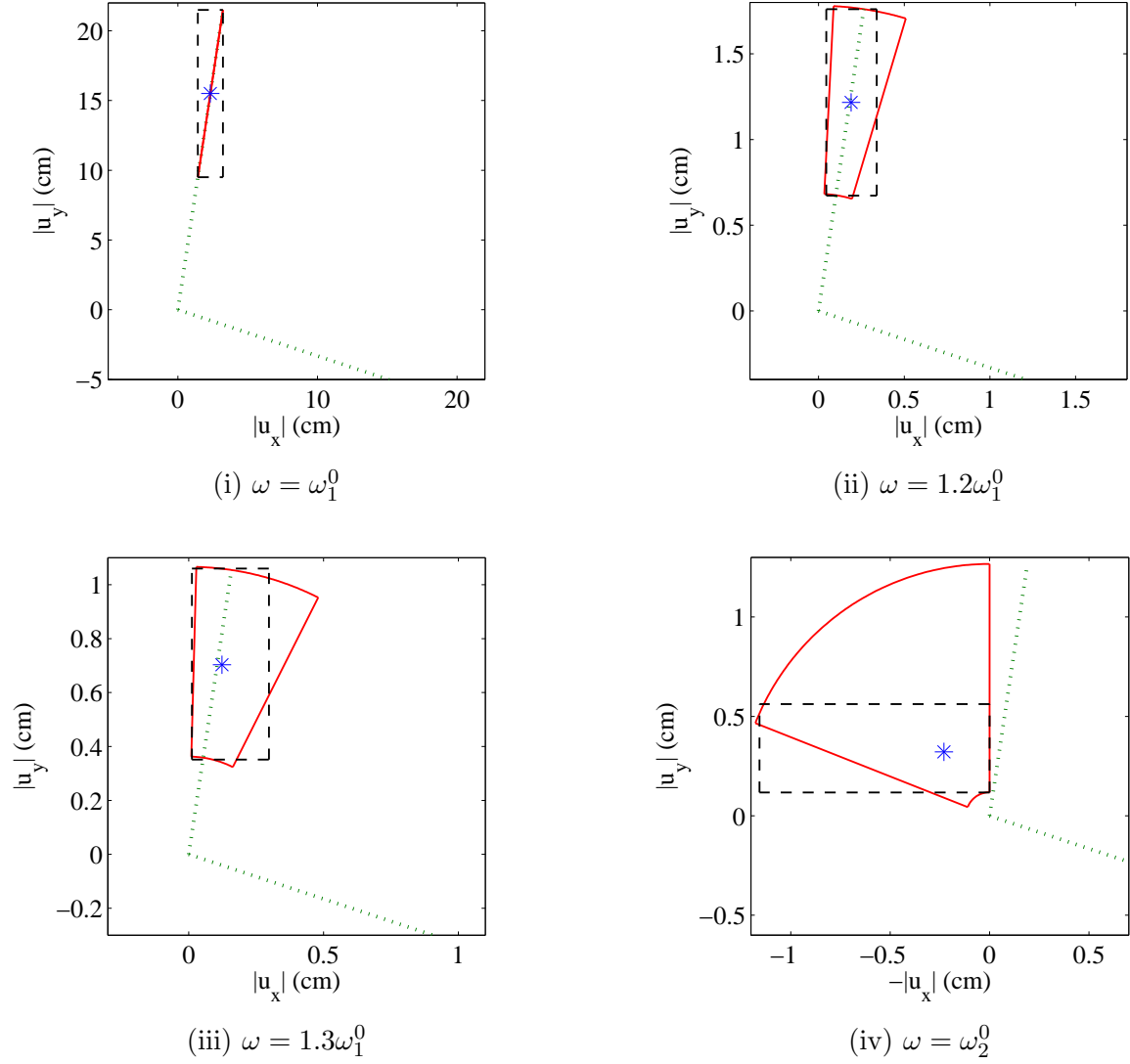


Figure 19: Bounds for the nodal oscillation of the 29-bar truss. \*: nominal values; ‘—’: bounds obtained by solving (38), (39), (44), and (47); ‘- -’: bounds obtained by solving (19) and (21); ‘...’: directions of eigenmodes without damping.

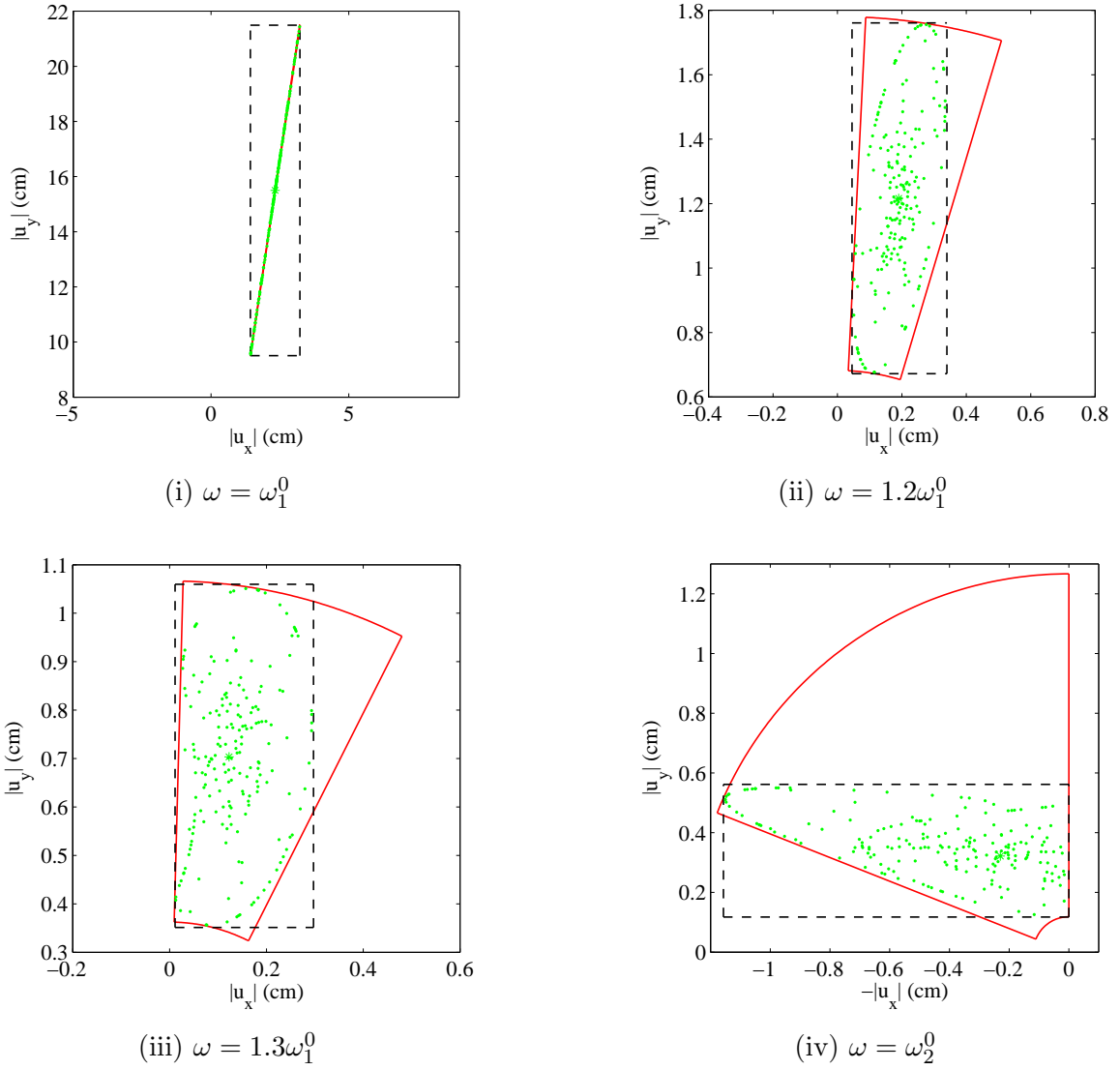


Figure 20: Bounds and randomly generated samples for the nodal oscillation of the 29-bar truss.

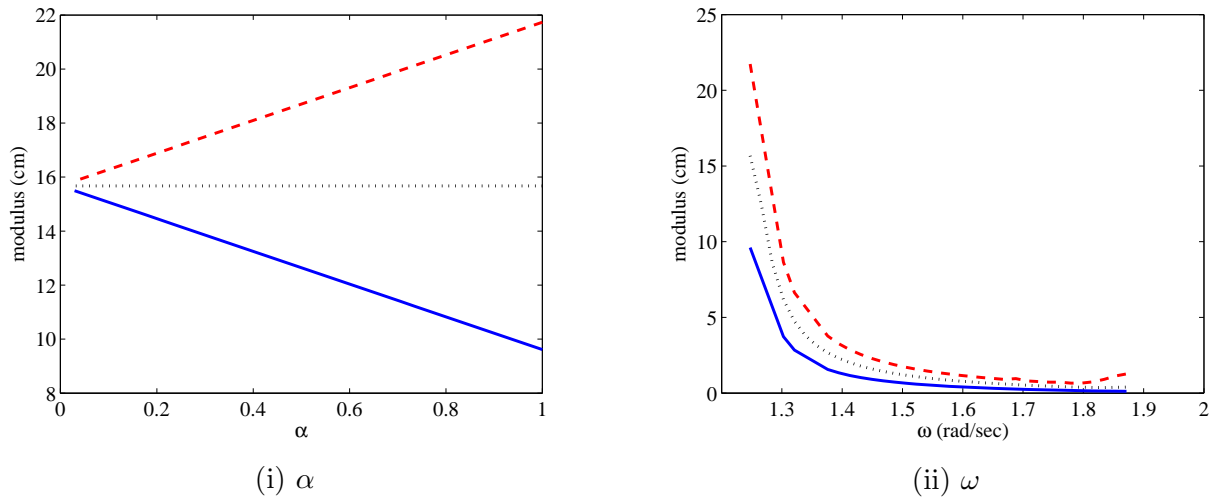


Figure 21: Variations of bounds for the nodal oscillation of the 29-bar truss with respect to  $\alpha$  and  $\omega$ . ‘ $\cdots$ ’: nominal value; ‘—’: lower bounds; ‘- -’: upper bounds.

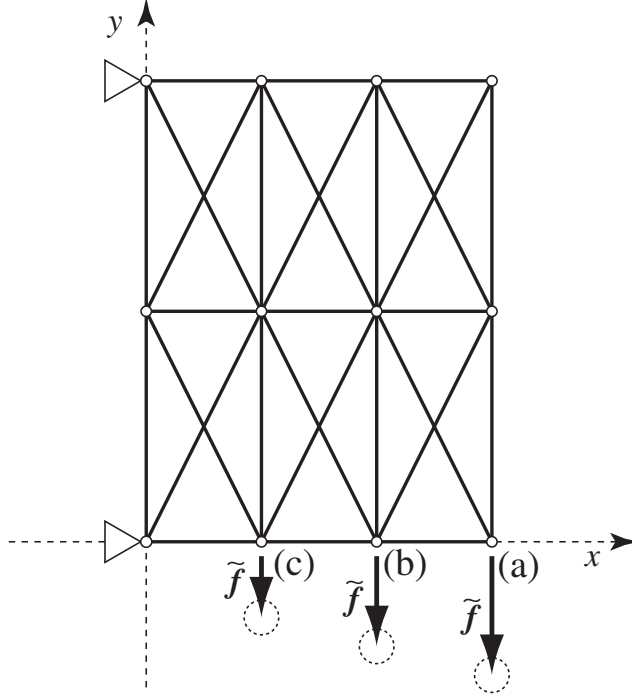


Figure 22: 29-bar truss under the uncertain load defined with a singular  $F_0$  in (5).

## 6 Further generalization of uncertainty model

### 6.1 Formulations for generalized uncertainty model

Recall that we have assumed in sections 3 and 4 that the matrix  $F_0 \in \mathbb{R}^{d \times k}$  defining the uncertainty set, (5), is nonsingular. In this section we consider formulations for confidence bounds without this assumption. Note that the matrices  $\Psi_1, \dots, \Psi_\ell$  defined in Proposition 3.4 includes  $F_0^{-1}$ , hence formulations presented in sections 3.1, 3.3, 4.1, and 4.2 are not valid for a singular  $F_0$ . In contrast, the SOCP formulation in section 3.2 can be applied to a singular  $F_0$  in order to find a lower bound for the modulus of the displacement amplitude.

As investigated in Example 2.1 and Example 2.2, if  $F_0$  is nonsingular, then uncertain forces can exist at all free nodes. By using a singular  $F_0$ , it is possible to consider the uncertainty model where uncertain forces can exist only at specified nodes.

**Example 6.1.** Consider a plane truss illustrated in Figure 22. The nominal harmonic forces are applied at the nodes (a)–(c). In contrast to Figure 1, suppose that uncertain forces can exist only at the nodes (a)–(c). Such an uncertainty model can be represented as (5) by putting

$$F_0 = \bar{f}_0 \begin{bmatrix} I_6 \\ O \end{bmatrix} \in \mathbb{R}^{20 \times 6},$$

$$T_1 = \begin{bmatrix} I_2 & O & O \end{bmatrix}, \quad T_2 = \begin{bmatrix} O & I_2 & O \end{bmatrix}, \quad T_3 = \begin{bmatrix} O & O & I_2 \end{bmatrix} \in \mathbb{R}^{2 \times 6},$$

where  $k = 6$  and  $\ell = 3$ . Thus we see that  $F_0$  is singular. ■

**Proposition 6.2.** *Suppose that  $F_0$  in (5) is singular. For  $\mathcal{V}$  defined in (12),  $\mathbf{v} \in \mathcal{V}$  if and only if  $\mathbf{v}$  satisfies*

$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \check{\Psi}_j \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \geq 0, \quad j = 1, \dots, \ell, \quad (49)$$

$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \check{\Theta} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = 0, \quad (50)$$

where  $\check{\Psi}_j$  ( $j = 1, \dots, \ell$ ) and  $\check{\Theta}$  are constant matrices defined by

$$\begin{aligned} \check{\Psi}_j &= \text{Diag}(\mathbf{0}, \alpha^2) - \begin{bmatrix} S_1^T \\ -\tilde{\mathbf{f}}^T \end{bmatrix} (T_j F_0^\dagger)^T T_j F_0^\dagger \begin{bmatrix} S_1 & -\tilde{\mathbf{f}} \end{bmatrix}, \quad j = 1, \dots, \ell, \\ \check{\Theta} &= - \begin{bmatrix} ((F_0^\perp)^T S_1)^T & S_2^T \\ -((F_0^\perp)^T \tilde{\mathbf{f}})^T & O \end{bmatrix} \begin{bmatrix} (F_0^\perp)^T S_1 & -(F_0^\perp)^T \tilde{\mathbf{f}} \\ S_2 & O \end{bmatrix}, \end{aligned}$$

where  $F_0^\dagger \in \mathbb{R}^{k \times d}$  is the Moore–Penrose pseudoinverse of  $F_0$ ,  $F_0^\perp \in \mathbb{R}^{d \times (d - \text{rank } F_0)}$  the basis for the left null space of  $F_0$ , and  $\text{rank } F_0$  the row rank of  $F_0$ .

The proof appears in appendix A.5.

**Example 6.3.** In the case of Example 6.1, the matrices in Proposition 6.2 are obtained as

$$F_0^\dagger = \frac{1}{\bar{f}_0} \begin{bmatrix} I_6 & O \end{bmatrix} \in \mathbb{R}^{6 \times 20}, \quad F_0^\perp = \begin{bmatrix} O \\ I_{14} \end{bmatrix} \in \mathbb{R}^{20 \times 14}.$$

where  $\text{rank } F_0 = 6$ . ■

It follows from Proposition 6.2 that the SDP formulations presented in sections 3.1, 3.3, 4.1, and 4.2 are generalized for a singular  $F_0$  by replacing  $\Psi_1, \dots, \Psi_\ell$  and  $\Theta$  with  $\check{\Psi}_1, \dots, \check{\Psi}_\ell$  and  $\check{\Theta}$ , respectively.

## 6.2 Numerical example for generalized uncertainty model

Consider a 29-bar truss shown in Figure 22. As the nominal load,  $\tilde{\mathbf{f}}$ , the external forces 7.5 kN, 5 kN and 2.5 kN are applied at the nodes (a), (b), and (c), respectively, in the negative direction of  $y$ -axis. The uncertainty model of  $\mathbf{f}$ , (5), is defined as discussed in Example 6.3 with  $\bar{f}_0 = 1$  kN. Consequently, the external forces are running through the circles depicted with the dotted lines in Figure 22. Note that uncertain forces may possibly exist only at the nodes (a)–(c), and any two sets of the uncertain loads have no relation.

We investigate the displacement of the node (a), which is denoted by  $(u_x, u_y)$ . We put  $\alpha = 1.0$  in (5). For the amplitudes in the  $x$ - and  $y$ -th directions, we obtain the bounds of the moduli and phase angles, which are shown in Figure 23 for various frequency of the driving load. Figure 24 depicts the bounds for the distribution of the vector  $(|u_x|, |u_y|)$ . It is observed from Figure 23 and Figure 25 that the obtained bounds are confirmed to be conservative and very tight.



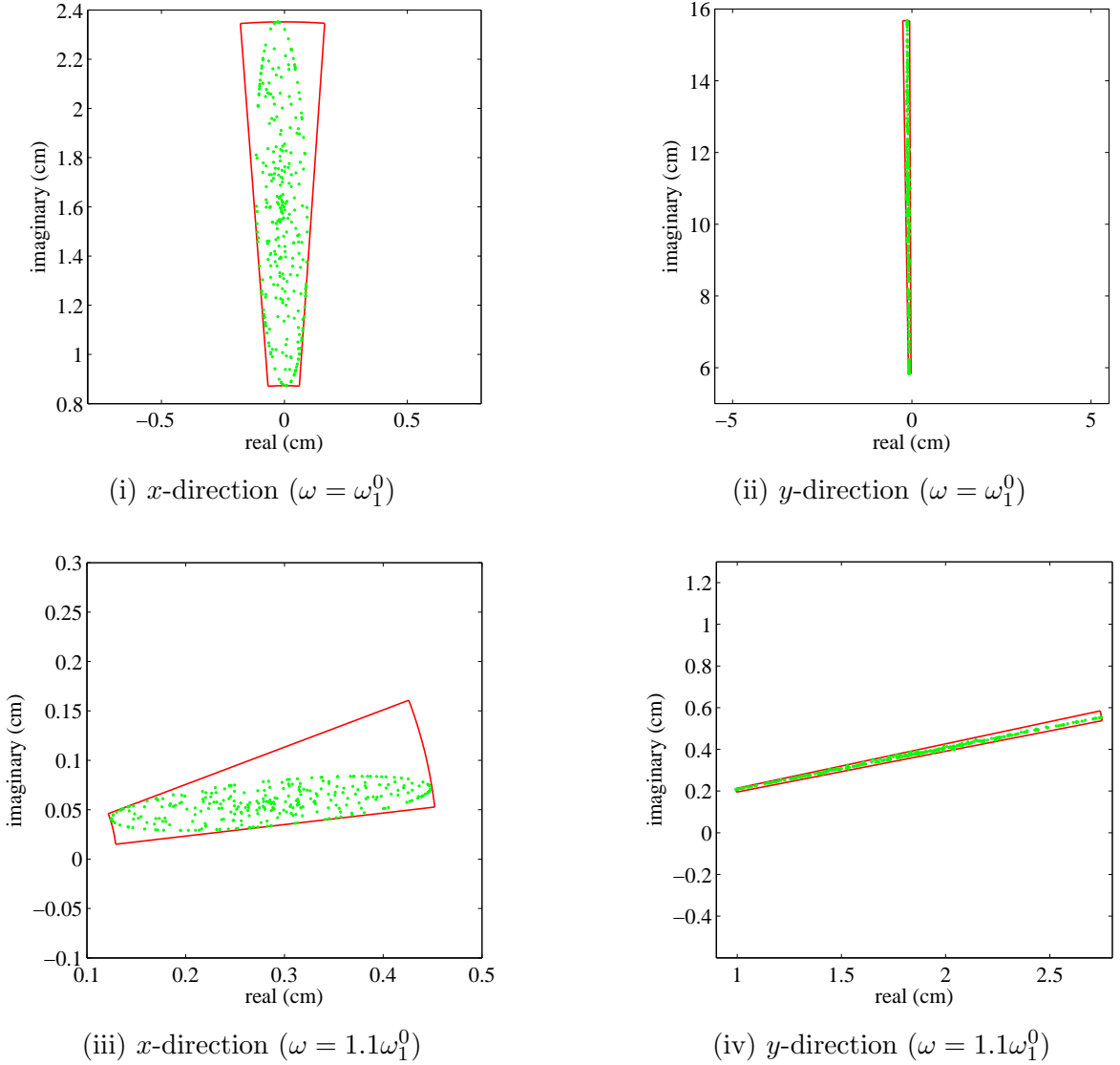


Figure 23: Bounds and random samples for the complex amplitudes of the displacements at the node (a) of the model defined in Figure 22.

## 7 Conclusions

We have proposed tractable formulations for computing confidential bounds for the dynamic steady-state behaviors of a structure subjected to uncertain driving loads. We considered non-stochastic uncertainties of amplitudes of harmonic driving loads, and estimate conservative bounds for structural responses at the steady-state.

We have formulated the bound detection problems as optimization problems for finding the minimal bounds of various characteristic amounts of dynamic response, including the modulus and phase angle of the complex amplitude of the displacement and stress. By using the quadratic-embedding of the uncertain parameters, and applying the  $\mathcal{S}$ -lemma, we constructed a numerically tractable problem which approximates the bound detection problem, and provides a conservative bound for the dynamic response. The obtained conservative problems are shown to be either an SDP (semidefinite programming) problem or an SOCP (second-order cone programming) problem,

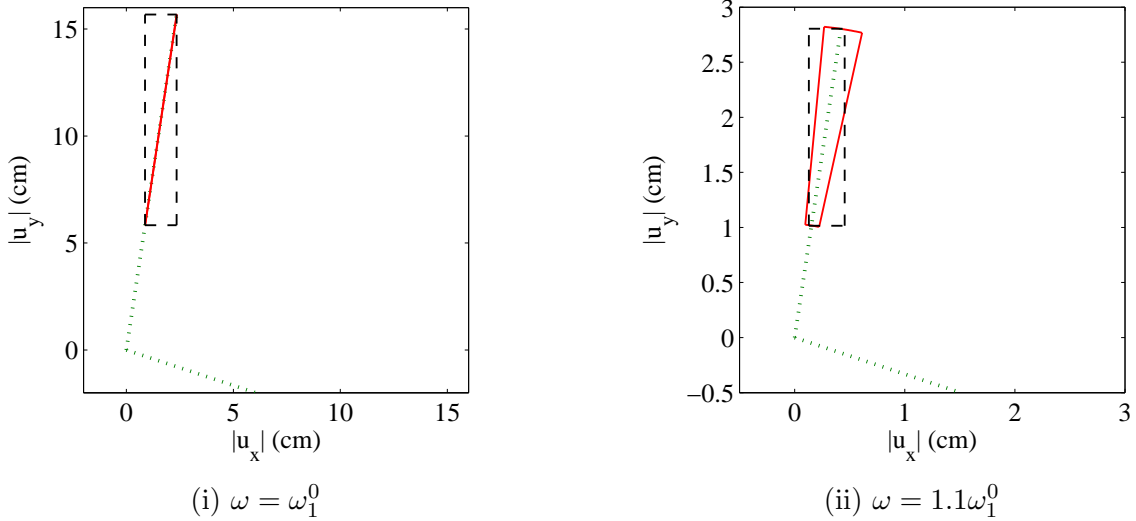


Figure 24: Bounds for the oscillation of the node (a) of the 29-bar truss defined in Figure 22. ‘—’: bounds obtained by solving (38), (39), (44), and (47); ‘- -’: bounds obtained by solving (19) and (21); ‘...’: directions of eigenmodes without damping.

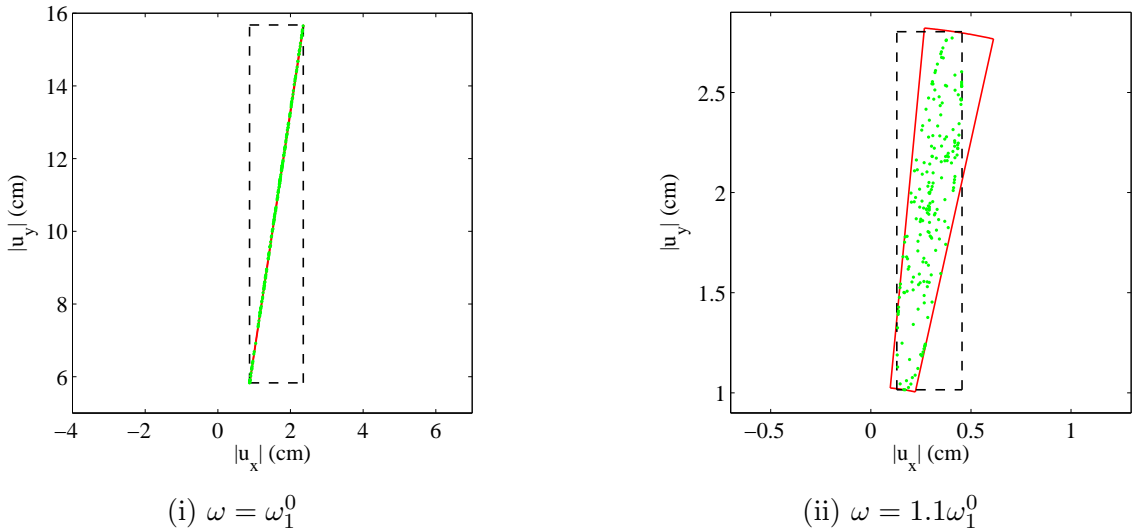


Figure 25: Random samples for the oscillation of the node (a) of the 29-bar truss defined in Figure 22.

both of which are convex optimization problems, and can be solved very effectively by using existing well-developed software based on the primal-dual interior-point method.

In the numerical examples, it has been shown that confidence bounds for dynamic responses can be obtained effectively by using the primal-dual interior-point method. We have also illustrated through numerical examples that the bounds provided by the proposed SDP and SOCP problems are sufficiently tight even for a moderately large magnitude of uncertainty.

## Appendix A. Proofs of technical results

### A.1 For section 3.1

#### A.1.1 Proof of Proposition 3.1

By using  $G$  defined in (13), the problem (7) is equivalently rewritten as

$$r_{\max} = \max_{\mathbf{v}} \{\|G\mathbf{v}\| : \mathbf{v} \in \mathcal{V}\}. \quad (51)$$

Since we assume that  $\mathcal{F}$  defined in (5) is bounded,  $\mathcal{V}$  is also bounded. Hence,  $\|G\mathbf{v}\|$  is bounded above, which implies that there exists an  $r' \in \mathbb{R}$  satisfying  $r' \geq \|G\mathbf{v}\|$  ( $\forall \mathbf{v} \in \mathcal{V}$ ). Here,  $r'$  is regarded as an upper bound of  $r_{\max}$  in (51), and hence  $r' \geq r_{\max}$ . Moreover, it is easy to see that  $r_{\max}$  satisfies  $r_{\max} \geq \|G\mathbf{v}\|$  ( $\forall \mathbf{v} \in \mathcal{V}$ ), i.e.  $r_{\max}$  is a feasible solution of (14). Consequently,  $r_{\max}$  is the minimum value of  $r'$  when  $r'$  is an upper bound of  $\|G\mathbf{v}\|$  over  $\mathcal{V}$ . This is (14).  $\square$

#### A.1.2 Proof of Proposition 3.3

The following two lemmas are used for the proof of Proposition 3.3.

**Lemma A.1.** *Let  $Q \in \mathcal{S}^n$ ,  $\mathbf{p} \in \mathbb{R}$ ,  $\gamma \in \mathbb{R}$ , and  $\mathbf{x} \in \mathbb{R}^n$ . Then the following two conditions are equivalent:*

$$\begin{aligned} \text{(a)} : \quad & \forall \mathbf{x} \in \mathbb{R}^n : \quad \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^T \begin{bmatrix} Q & \mathbf{p} \\ \mathbf{p}^T & \gamma \end{bmatrix} Q \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} \geq 0; \\ \text{(b)} : \quad & \begin{bmatrix} Q & \mathbf{p} \\ \mathbf{p}^T & \gamma \end{bmatrix} \succeq O. \end{aligned}$$

*Proof.* The implication from (b) to (a) is trivial. We show that (a) implies (b) by the contradiction. Suppose that (b) does not hold, i.e. we assume that there exist  $\mathbf{x}' \in \mathbb{R}^n$  and  $\xi \in \mathbb{R}$  satisfying

$$\begin{bmatrix} \mathbf{x}' \\ \xi \end{bmatrix}^T \begin{bmatrix} Q & \mathbf{p} \\ \mathbf{p}^T & \gamma \end{bmatrix} \begin{bmatrix} \mathbf{x}' \\ \xi \end{bmatrix} < 0. \quad (52)$$

If  $\xi \neq 0$ , then (52) is reduced to

$$\begin{bmatrix} \mathbf{x}'/\xi \\ 1 \end{bmatrix}^T \begin{bmatrix} Q & \mathbf{p} \\ \mathbf{p}^T & \gamma \end{bmatrix} \begin{bmatrix} \mathbf{x}'/\xi \\ 1 \end{bmatrix} < 0,$$

which contradicts the condition (a). Alternatively, if  $\xi = 0$ , then (52) is reduced to

$$\mathbf{x}'^T Q \mathbf{x}' < 0. \quad (53)$$

Put  $\mathbf{x} = \eta \mathbf{x}'$ , and the left-hand side of (a) is reduced to

$$(\mathbf{x}^T Q \mathbf{x}') \eta^2 + 2(\mathbf{p}^T \mathbf{x}') \eta + \gamma, \quad (54)$$

which is regarded as a function of  $\eta$ . The condition (53) implies that (54) is not bounded below, from which it follows that there exists  $\eta$  such that (54) becomes negative. Thus, we see the contradiction to (a).  $\square$

**Lemma A.2** (*S*-lemma). Let  $f_0, f_1, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$  be quadratic functions. The implication

$$f_1(\mathbf{x}) \geq 0, \dots, f_m(\mathbf{x}) \geq 0 \quad \Rightarrow \quad f_0(\mathbf{x}) \geq 0$$

holds if there exist a vector  $\boldsymbol{\tau} = (\tau_i) \in \mathbb{R}^m$  satisfying

$$\begin{aligned} \forall \mathbf{x} \in \mathbb{R}^n : \quad f_0(\mathbf{x}) &\geq \sum_{i=1}^m \tau_i f_i(\mathbf{x}), \\ \tau_1, \dots, \tau_m &\geq 0. \end{aligned}$$

*Proof.* See Boyd *et al.* [8, section 2.6.3] and the references therein.  $\square$

**Proof of Proposition 3.3** For simplicity we write

$$\begin{aligned} q_i(\mathbf{x}) &= \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^\top Q_i \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}, \quad i = 0, 1, \dots, m, \\ p_j(\mathbf{x}) &= \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^\top P_j \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}, \quad j = 1, \dots, k. \end{aligned}$$

Observe that the quadratic equation  $p_j(\mathbf{x}) = 0$  in (a) is equivalent to quadratic inequalities

$$p_j(\mathbf{x}) \geq 0, \quad -p_j(\mathbf{x}) \geq 0.$$

It follows from Lemma A.2 that the implication (a)  $\Rightarrow$  (b) holds if there exist  $\tau_1, \dots, \tau_m, \rho_{m+1}^+, \dots, \rho_{m+k}^+$ , and  $\rho_{m+1}^-, \dots, \rho_{m+k}^-$  satisfying

$$\begin{aligned} \forall \mathbf{x} \in \mathbb{R}^n : \quad q_0(\mathbf{x}) &\geq \sum_{i=1}^m \tau_i q_i(\mathbf{x}) + \sum_{j=1}^k \rho_j^+ p_j(\mathbf{x}) + \sum_{j=1}^k \rho_j^- (-p_j(\mathbf{x})), \\ \tau_1, \dots, \tau_m &\geq 0, \quad \rho_1^+, \dots, \rho_k^+ \geq 0, \quad \rho_1^-, \dots, \rho_k^- \geq 0. \end{aligned}$$

By putting  $\sigma_j = \rho_j^+ - \rho_j^-$  ( $j = 1, \dots, k$ ), this condition is equivalently rewritten as

$$\begin{aligned} \forall \mathbf{x} \in \mathbb{R}^n : \quad q_0(\mathbf{x}) &\geq \sum_{i=1}^m \tau_i q_i(\mathbf{x}) + \sum_{j=1}^k \sigma_j p_j(\mathbf{x}), \\ \tau_1, \dots, \tau_m &\geq 0. \end{aligned} \tag{55}$$

The assertion of this proposition is obtained by applying Lemma A.1 to (55).  $\square$

### A.1.3 Proof of Proposition 3.4

It follows from (5) and (12) that  $\mathbf{v} \in \mathcal{V}$  if and only if

$$S_1 \mathbf{v} = \tilde{\mathbf{f}} + F_0 \boldsymbol{\zeta}, \quad \alpha \geq \|T_j \boldsymbol{\zeta}\|, \quad j = 1, \dots, \ell, \tag{56}$$

$$S_2 \mathbf{v} = \mathbf{0}. \tag{57}$$

Since  $F_0$  is regular, (56) is rewritten as

$$T_j \boldsymbol{\zeta} = T_j F_0^{-1} (S_1 \mathbf{v} - \tilde{\mathbf{f}}), \quad \alpha \geq \|T_j \boldsymbol{\zeta}\|, \quad j = 1, \dots, \ell. \tag{58}$$

For each  $j = 1, \dots, \ell$ , we see that

$$\alpha \geq \|T_j \boldsymbol{\zeta}\| \Leftrightarrow \alpha^2 - \|T_j \boldsymbol{\zeta}\|^2 \geq 0$$

holds. From this observation it follows that (58) is equivalent to the quadratic inequalities

$$\alpha^2 - \left\| T_j F_0^{-1}(S_1 \mathbf{v} - \tilde{\mathbf{f}}) \right\|^2 \geq 0, \quad j = 1, \dots, \ell,$$

which corresponds to (15). The equation (57) is equivalent to the quadratic equality

$$\|S_2 \mathbf{v}\|^2 = 0,$$

which corresponds to (16). □

#### A.1.4 Proof of Proposition 3.5

We start with observing that the condition (17) is equivalent to the implication

$$\mathbf{v} \in \mathcal{V} \Rightarrow r \geq \|G\mathbf{v}\|. \quad (59)$$

Here, the inequality on the right-hand side of (59) is reduced to

$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \begin{bmatrix} -G^T G & \mathbf{0} \\ \mathbf{0}^T & r^2 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \geq 0. \quad (60)$$

Note that the right-hand side of (59) is equivalent to (60) and  $r \geq 0$ .

Recall that, in Proposition 3.4, the left-hand side of (59) is reduced to quadratic inequalities. Consequently, the implication (59) can be rewritten equivalently as the following implication:

$$\begin{aligned} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \Psi_j \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \geq 0 \quad (j = 1, \dots, \ell), \quad \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \Theta \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = 0 \\ \Rightarrow \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \begin{bmatrix} -G^T G & \mathbf{0} \\ \mathbf{0}^T & r^2 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \geq 0, \end{aligned}$$

where  $r \geq 0$ . Then the assertion of Proposition 3.5 follows from Proposition 3.3. □

#### A.1.5 Proof of Proposition 3.6

Recall Proposition 3.1, i.e.  $r_{\max}$  is equivalent to the optimal value of the problem (14). Now we observe that the problem (19) is obtained by replacing the constraint condition, (17), of the problem (14) with (18). Hence, the assertion of Proposition 3.6 follows immediately from Proposition 3.5. □

## A.2 For section 3.2

### A.2.1 Proof of Proposition 3.7

By using  $G$  defined by (13), we see that (20) is equivalently rewritten as

$$r_{\min} = \min_{\mathbf{v}} \{ \|G\mathbf{v}\| : \mathbf{v} \in \mathcal{V} \}. \quad (61)$$

By introducing an auxiliary variable  $t$  which is equivalent to  $\|G\mathbf{v}\|$ , and by substituting the definition (12) of  $\mathcal{V}$ , the problem (61) is reduced to

$$\left. \begin{aligned} \min_{t, \mathbf{v}, \boldsymbol{\zeta}} \quad & t \\ \text{s.t.} \quad & t = \|G\mathbf{v}\|, \\ & S_1\mathbf{v} = \tilde{\mathbf{f}} + F_0\boldsymbol{\zeta}, \\ & S_2\mathbf{v} = \mathbf{0}, \\ & \alpha \geq \|T_j\boldsymbol{\zeta}\|, \quad j = 1, \dots, \ell, \end{aligned} \right\} \quad (62)$$

without changing the optimal value. Since the problem (62) is the minimization of  $t$ , we can replace the constraint condition  $r = \|G\mathbf{v}\|$  with  $r \geq \|G\mathbf{v}\|$ , which results in (21).  $\square$

### A.3 For section 3.3

#### A.3.1 Proof of Proposition 3.10

Since  $0 < \theta_{\max} < \pi$ , we may restrict ourselves to  $u_q$  satisfying  $\text{Im } u_q \neq 0$  without loss of generality. It is easy to see that if  $a_2 \in \mathbb{R}$  and  $u_q \in \mathbb{C}$  satisfy

$$\begin{bmatrix} 1 & a_2 \end{bmatrix} \begin{bmatrix} \text{Re } u_q \\ \text{Im } u_q \end{bmatrix} = 0,$$

then the relation

$$a_2 = -\frac{\text{Re } u_q}{\text{Im } u_q} = -\frac{1}{\tan \text{Arg } u_q}$$

holds. Hence, (23) is reduced to

$$-\frac{1}{\tan \theta_{\max}} = \max_{\mathbf{a}} \{ \mathbf{e}_2^T \mathbf{a} : \mathbf{a}^T G\mathbf{v} = 0, \mathbf{v} \in \mathcal{V}, \mathbf{e}_1^T \mathbf{a} = 1 \}, \quad (63)$$

Moreover, if  $a_2$  and  $u_q$  satisfy

$$\begin{bmatrix} 1 & a_2 \end{bmatrix} \begin{bmatrix} \text{Re } u_q \\ \text{Im } u_q \end{bmatrix} \geq 0, \quad (64)$$

then the inequality

$$a_2 \geq -\frac{1}{\tan \text{Arg } u_q}$$

holds. Hence, (64) implies that  $a_2$  is an upper bound for  $-1/\tan \text{Arg } u_q$ . From this observation it follows that if  $a_2$  satisfies

$$\begin{bmatrix} 1 & a_2 \end{bmatrix} G\mathbf{v} \geq 0, \quad \forall \mathbf{v} \in \mathcal{V},$$

then  $a_2$  corresponds to an upper bound for  $-1/\tan \text{Arg } u_q(\mathbf{f})$  when  $\mathbf{f}$  is running through  $\mathcal{F}$ . Consequently, in a manner similar to Proposition 3.1, the maximization problem (63) of  $-1/\tan \text{Arg } u_q$  is equivalently rewritten as the minimization problem of the upper bound for  $-1/\tan \text{Arg } u_q$ , which concludes the proof.  $\square$

### A.3.2 Proof of Proposition 3.11

We first observe that the condition (27) is equivalent to the following implication:

$$\mathbf{v} \in \mathcal{V} \quad \Rightarrow \quad \mathbf{a}^T G \mathbf{v} \geq 0.$$

Recall that the condition  $\mathbf{v} \in \mathcal{V}$  has been embedded into some quadratic inequalities in Proposition 3.4. We easily see that

$$\mathbf{a}^T G \mathbf{v} \geq 0$$

is equivalent to

$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \begin{bmatrix} O & G^T \mathbf{a} \\ \mathbf{a}^T G & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \geq 0.$$

Then the assertion follows from Proposition 3.3.  $\square$

### A.3.3 Proof of Proposition 3.12

Recall Proposition 3.10, i.e.  $\theta_{\max}$  is equivalent to the optimal value of the problem (26). Observe that the problem (29) is constructed from the problem (26) by replacing the constraint condition (27) with its sufficient condition (28) provided by Proposition 3.11. Let  $(\mathbf{a}', \mathbf{w}', s')$  denote a feasible solution of the problem (29). From the construction of (29) it follows that  $\mathbf{a}'$  satisfies (27). Hence, by using Proposition 3.10, we obtain the inequality

$$-\frac{1}{\tan \theta_{\max}} \leq a_2^*,$$

which concludes the proof.  $\square$

## A.4 For section 4.2

### A.4.1 Proof of Proposition 4.5

We investigate the constraint condition

$$\forall \boldsymbol{\mu} \in \{(\|G_x \mathbf{v}\|^2, \|G_y \mathbf{v}\|^2)^T \mid \mathbf{v} \in \mathcal{V}\} : \quad \mathbf{a}^T \boldsymbol{\mu} \geq 0 \quad (65)$$

in the problem (43). It follows from Proposition 3.4 that the condition  $\boldsymbol{\mu} \in \{(\|G_x \mathbf{v}\|^2, \|G_y \mathbf{v}\|^2)^T \mid \mathbf{v} \in \mathcal{V}\}$  in (65) can be reduced to quadratic inequalities in terms of  $\mathbf{u}$  and  $\boldsymbol{\mu}$  as

$$\begin{aligned} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \boldsymbol{\Psi}_j \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} &\geq 0, & j = 1, \dots, \ell, \\ \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}^T \boldsymbol{\Theta} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} &= 0, \\ \mu_x^2 - \|G_x \mathbf{v}\|^2 &= 0, & \mu_y^2 - \|G_y \mathbf{v}\|^2 = 0. \end{aligned}$$

which are rewritten as

$$\begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{v} \\ 1 \end{bmatrix}^T \hat{\Psi}_j \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{v} \\ 1 \end{bmatrix} \geq 0, \quad j = 1, \dots, \ell, \quad (66)$$

$$\begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{v} \\ 1 \end{bmatrix}^T \hat{\Theta}_i \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{v} \\ 1 \end{bmatrix} = 0, \quad i = 1, 2, 3. \quad (67)$$

The condition  $\mathbf{a}^T \boldsymbol{\mu} \geq 0$  in (65) is equivalently rewritten as

$$\begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{v} \\ 1 \end{bmatrix}^T \begin{bmatrix} O & O & \mathbf{a} \\ O & O & \mathbf{0} \\ \mathbf{a}^T & \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{v} \\ 1 \end{bmatrix} \geq 0. \quad (68)$$

Consequently, the condition (65) is equivalent to the implication

$$(66) \ \& \ (67) \ \Rightarrow \ (68). \quad (69)$$

By applying Proposition 3.3 to (69), we see that the implication (69) holds if there exist  $\mathbf{w} \in \mathbb{R}_+^\ell$  and  $\mathbf{s} \in \mathbb{R}^3$  satisfying

$$\begin{bmatrix} O & O & \mathbf{a} \\ O & O & \mathbf{0} \\ \mathbf{a}^T & \mathbf{0}^T & 0 \end{bmatrix} \succeq \sum_{j=1}^{\ell} w_j \hat{\Psi}_j + \sum_{i=1}^3 s_i \hat{\Theta}_i. \quad (70)$$

Note that the problem (44) is obtained from the problem (43) by replacing the constraint condition (65) with its sufficient condition (70). Hence,  $a_2^*$  defined in (44) satisfies  $a_2^* \geq -1/(t_{\max})^2$ , from which we obtain Proposition 4.5.  $\square$

## A.5 For section 6.1

### A.5.1 Proof of Proposition 6.2

From (5) and (12), and the definitions of  $F_0^\dagger$  and  $F_0^\perp$ , we see that  $\mathbf{v} \in \mathcal{V}$  if and only if  $\mathbf{v}$  satisfies

$$F_0^\dagger(S_1 \mathbf{v} - \tilde{\mathbf{f}}) = \boldsymbol{\zeta}, \quad \alpha \geq \|T_j \boldsymbol{\zeta}\|, \quad j = 1, \dots, \ell, \quad (71)$$

$$(F_0^\perp)^T(S_1 \mathbf{v} - \tilde{\mathbf{f}}) = \mathbf{0}, \quad (72)$$

$$S_2 \mathbf{v} = \mathbf{0}. \quad (73)$$

The condition (71) is reduced to

$$\alpha \geq \left\| T_j F_0^\dagger(S_1 \mathbf{v} - \tilde{\mathbf{f}}) \right\|, \quad j = 1, \dots, \ell,$$

which is rewritten equivalently as (49). Moreover, the linear equations (72) and (73) are equivalently rewritten as (50), which concludes the proof.  $\square$



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