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Abstract

In this paper, we consider undirected network design games with fair cost allocation. We introduce two concepts *Potential-Optimal Price of Anarchy* (POPoA) and *Potential-Optimal Price of Stability* (POPoS), where POPoA is the ratio between the worst cost of Nash equilibria with optimal potential and the minimum social cost, and POPoS is the ratio between the best cost of Nash equilibria with optimal potential and the minimum social cost, and show that

- The POPoA and POPoS for undirected broadcast games with n players are $\mathrm{O}(\sqrt{\log n})$.
- The POPoA and POPoS for undirected broadcast games with $|V|$ vertices are $\mathrm{O}(\log |V|)$.
- There exists an undirected broadcast game with n players such that POPoA, POPoS $= \Omega(\sqrt{\log\log n})$.
- There exists an undirected broadcast game with $|V|$ vertices such that $\mathrm{POPoA}, \mathrm{POPoS} = \Omega(\log |V|)$.

1 Introduction

The inefficiency of equilibria in noncooperative games have been extensively investigated in recent years. The *price of anarchy* (PoA) was introduced by Koutsoupias and Papadimitriou [12] as the ratio between the cost of the *worst* Nash equilibria and the optimal social cost, and the *price of stability* (PoS) was introduced by Anshelevich *et al.* [2] as the ratio between the cost of the *best* Nash equilibria and the optimal social cost. Quantifying PoA and PoS is one of the most active research areas in algorithmic game theory.

Network design games introduced by Anshelevich *et al.* [2] are fundamental and well-studied noncooperative games [1–3, 5, 7–9, 11, 13]. In the games, we are given a network and n selfish players. Each player i chooses a path from source vertex s_i to sink vertex t_i. The cost of each edge is shared equally by all players whose paths contain it, where this cost-sharing scheme is referred as the Shapley cost-sharing mechanism. The goal of each player is to minimize the sum of its own costs. A network design game is called a *multicast game* if $t_i = t$ holds for all players i, and a *broadcast game* if in addition, each vertex $v \neq t$ has a player i with $s_i = v$. In the standard setting of broadcast games, each vertex $v \neq t$ is associated with exactly one player, but we here allow more than one player to share the same vertex. It is known [2] that the worst PoA for broadcast, multicast and network design games with n players is equals to n, which also implies that the worst PoA is $+\infty$ in terms of the number of vertices in the network. On the other hand, determining PoS for undirected network design games is a long-standing important open problem. As for PoS, it was shown that it has an upper bound of nth Harmonic number $H(n)$ for network design games [2], $\mathrm{O}(\log n/\log\log n)$ for undirected multicast games [13], and $\mathrm{O}(\log\log n)$ for undirected broadcast games [11]. Bilò *et al.* [5] presented undirected network design, multicast, and broadcast games that have PoS at least 2.245, 1.862, and 1.818, respectively.

Potential games were proposed by Monderer and Shapley [14], which has a property that the incentive of all players to change their strategy can be expressed in one global function, called *potential*. Note that for every potential game, the potential is *unique* up to an additive constant. Potential games contain many important games such as network design games and congestion games. In fact, it is known that finite potential games are isomorphic to congestion games [14]. Various properties of potential games were proven due to the existence of potential functions [2, 4, 10, 11, 13–16]. For example, the $H(n)$-bound for PoS of network design games mentioned above was shown by potential functions.

In potential games, a strategy profile with optimal potential is a Nash equilibrium. More precisely, there exists a one-to-one correspondence between Nash equilibria and local optimizers of the potential function. A strategy profile with the optimal potential has good properties, called *robustness* [17] and *stability* [4]. Furthermore, Blume's logit response dynamics [6] always converges to the set of Nash equilibria with optimum potential. Therefore, it is natural and important to evaluate the inefficiency of those equilibria, i.e., Nash equilibria with optimal potential. We note that Nash equilibria is optimal with respect to some potential if and only if it is optimal with respect to any potential, since the potential is unique up to an additive constant.

In this paper, we define two concepts *Potential-Optimal Price of Anarchy* (POPoA) and *Potential-Optimal Price of Stability* (POPoS), and evaluate them for broadcast games. Here POPoA is the ratio between the worst cost of Nash equilibria with optimal potential and the minimum social cost, and POPoS is the ratio between the best cost of Nash equilibria with optimal potential and the minimum social cost. We note that POPoA has been already studied about some other games under the name of *Inefficiency Ratio of Stable Equilibria* by Asadpour and Saberi [4], whereas POPoS is a new measure of inefficiency of stable equilibria. We further note that several researchers makes use of potential minimizers as evaluations of PoS, e.g., [2, 13]. Namely, they implicitly evaluated POPoA to obtain bounds for PoS.

The results obtained in this paper

In this paper, we consider POPoA and POPoS for undirected broadcast games. We first show that the worst POPoA coincides with the worst POPoS in multicast and broadcast games, and obtain the following upper and lower bounds for POPoA and POPoS:

- The POPoA and POPoS for undirected broadcast games with n players are $\mathrm{O}(\sqrt{\log n})$.
- The POPoA and POPoS for undirected broadcast games with $|V|$ vertices are $\mathrm{O}(\log |V|)$.
- There exists an undirected broadcast game with n players such that POPoA, POPoS $= \Omega(\sqrt{\log \log n})$.
- There exists an undirected broadcast game with $|V|$ vertices such that POPoA, POPoS $= \Omega(\log |V|)$.

Since each vertex without sink allows to have multiple players, the number of players n might be larger than $|V| - 1$. We remark that all the results except for the last one are true in the setting that each vertex without sink is associated with exactly one player. Since the third result is obtained by replacing each vertex with multiple players by a star such that each vertex has exactily one player. We also note that from the second and fourth results, the tight bounds on POPoA and POPoS are obtained for undirected broadcast games with $|V|$ vertices. We summarize the inefficiency of equilibria for undirected broadcast games in Table 1, where our results are written in bold letters.

The rest of the paper is organized as follows. In Section 2, we define network design games and the measures for the inefficiency of equilibria, and show that the worst POPoA coincides with the worst POPoS in multicast and broadcast games. In Section 3, we present upper bounds for POPoS and POPoA of broadcast games, and in Section 4, we present lower bounds for POPoS and POPoA of broadcast games.

Table 1: Inefficiency of equilibria for undirected broadcast games, where our results are written in bold letters.

		PoS	POPoS, POPoA	PoA
n players	Upper Bound	$\mathrm{O}(\log\log n)$ [11]	$\mathbf{O}(\sqrt{\log n})$	n [2]
	Lower Bound	1.818 [5]	$\mathbf{\Omega}(\sqrt{\log\log n})$	n [2]
$\|V\|$ vertices	Upper Bound	$\mathbf{O}(\log \|V\|)$	$\mathbf{O}(\log \|V\|)$	$+\infty$ [2]
	Lower Bound	1.818 [5]	$\mathbf{\Omega}(\log \|V\|)$	$+\infty$ [2]

Due to the space limitation, some of the proofs are omitted, where they can be found in the appendix.

2 Definitions

We define undirected network design games as follows. Consider an undirected graph $G = (V, E)$ with a positive cost function $c : E \to \mathbb{R}_{++}$ on the edges [1]. There exist n players associated with G, where $N = \{1, 2, \ldots, n\}$ denotes the set of n players. Each player $i \in N$ has a source-sink pair $(s_i, t_i) \in V^2$ that it wishes to connect. A strategy of player i consists of a s_i-t_i path $P_i \subseteq E$ in G, where we denote by $\mathcal{P}_i$ the set of strategies of player i. In this paper, we assume that $\mathcal{P}_i \neq \emptyset$ holds for all $i \in N$. Network design games are called *multicast games* if $t_i = t$ holds for all players i, and *broadcast games* if in addition, each vertex $v \neq t$ has a player i with $s_i = v$.

For a strategy profile (or vector) $P = (P_1, P_2, \ldots, P_n) \in \prod_{i \in N} \mathcal{P}_i$, let $\xi_P(e)$ denotes the number of players i that use edge e, i.e., $\xi_P(e) = |\{i \in N \mid P_i \ni e\}|$, and the cost of player i that i minimizes is defined as

$$\mathrm{cost}_i(P) = \sum_{e \in P_i} c(e)/\xi_P(e). \tag{1}$$

The total cost of all players

$$\mathrm{cost}(P) = \sum_{i \in N} \mathrm{cost}_i(P)\ (= \sum_{e \in \bigcup_{i \in N} P_i} c(e)) \tag{2}$$

is called *social cost* (or simply *cost*) of P.

A strategy profile $(P_1, \ldots, P_n)$ is said to be a *Nash equilibrium* if no player has an incentive to change its strategy, assuming that the strategies

[1] In the network design game, the edge cost is usually assumed to be nonnegative, but in this paper, we assume that it is positive, since we can contract all the edges with zero cost, without changing minimum social cost, minimum potential value, and etc. In fact, our main results are applicable to graphs with nonnegative edge cost.

of the other players are fixed, i.e., $\mathrm{cost}_i(P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_n) \leq \mathrm{cost}_i(P_1, \ldots, P_{i-1}, P'_i, P_{i+1}, \ldots, P_n)$ holds for any player $i \in N$ and any strategy $P'_i \in \mathcal{P}_i$. A function $\Phi : \mathcal{P} \to \mathbb{R}$ is called a *potential function* if for any strategy profile $P = (P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_n)$ and any deviation $P' = (P_1, \ldots, P_{i-1}, P'_i, P_{i+1}, \ldots, P_n)$ from P of a single player i, it holds that

$$\Phi(P) - \Phi(P') = \mathrm{cost}_i(P) - \mathrm{cost}_i(P'). \tag{3}$$

It is known [2] that network design games admit potential functions, which can be represented as

$$\Phi(P) = \sum_{e \in E} c(e) \cdot H(\xi_P(e)), \tag{4}$$

where $H(n) = 1 + 1/2 + 1/3 + \cdots + 1/n$ is the nth Harmonic number with $H(0) = 0$. By the definition of potential function, a strategy profile is a Nash equilibrium if and only if it is locally minimal in the potential. In fact, a strategy profile with minimum potential is a Nash equilibrium, which has additional properties, called *robustness* [17] and *stability* [4].

Price of Anarchy (PoA) is defined as the ratio between the cost of the worst Nash equilibrium and the minimum social cost [12], and *Price of Stability* (PoS) is defined as the ratio between the cost of the best Nash equilibrium and the minimum social cost [2]. In this paper, we study *Potential-Optimal Price of Anarchy* (POPoA) and *Potential-Optimal Price of Stability* (POPoS) for broadcast games, in order to evaluate the quality of potential-optimal strategy profiles. POPoA is the ratio between the worst cost of strategy profiles with optimal potential and the minimum social cost, and POPoS is the ratio between the best cost of strategy profiles with optimal potential and the minimum social cost. Namely, we have

$$\mathrm{POPoA} = \frac{\max_{P \in \mathrm{argmin}\,\Phi} \mathrm{cost}(P)}{\min_{P \in \mathcal{P}} \mathrm{cost}(P)}, \tag{5}$$

$$\mathrm{POPoS} = \frac{\min_{P \in \mathrm{argmin}\,\Phi} \mathrm{cost}(P)}{\min_{P \in \mathcal{P}} \mathrm{cost}(P)}. \tag{6}$$

POPoA was studied under the name of *Inefficiency Ratio of Stable Equilibria* by Asadpour and Saberi [4], whereas POPoS is a new measure of inefficiency of stable equilibria. By definitions, we have

$$\mathrm{PoS} \leq \mathrm{POPoS} \leq \mathrm{POPoA} \leq \mathrm{PoA}. \tag{7}$$

The following example shows that POPoS $<$ POPoA for some broadcast game.

Example 1. *Consider a broadcast game with two players* 1 *and* 2 *in* $G = (V = \{s_1, s_2, t\}, E = \{(s_1, s_2), (s_1, t), (s_2, t)\})$ *such that* $c(s_1, s_2) = 1$ *and*

$c(s_i, t) = 2$ *for* $i = 1, 2$ (*see Figure* 1). *In this game, the following three strategy profiles have minimum potential* 4.

$$P = (\{(s_1, t)\}, \{(s_2, t)\}),\ P' = (\{(s_1, t)\}, \{(s_2, s_1), (s_1, t)\}),$$
$$P'' = (\{(s_1, s_2), (s_2, t)\}, \{(s_2, t)\}). \tag{8}$$

Since minimum social cost (*which is attained by* P' *and* P'') *is* 3, *we have* $\mathrm{POPoS} = 1$ *and* $\mathrm{POPoA} = 4/3$.

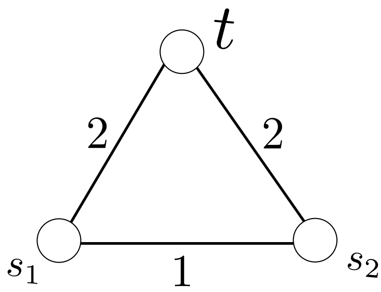

Figure 1: A broadcast game with POPoS $\neq$ POPoA

However, if we deal with the classes of undirected multicast and broadcast games, the supremum of POPoS coincides with the one of POPoA.

Lemma 2. *(1) The supremum of POPoA in undirected multicast (resp., broadcast) games with* n *players coincides with the one of POPoS in undirected multicast (resp., broadcast) games with* n *players.*

(2) The supremum of POPoA in undirected multicast (resp., broadcast) games with $|V|$ *vertices coincides with the one of POPoS in undirected multicast (resp., broadcast) games with* $|V|$ *vertices.*

Proof. We only show broadcast games with n players, since the other cases can be shown similarly. Let α and σ respectively denote the supremum of POPoA and POPoS in undirected broadcast games with n players. Since $\alpha \geq \sigma$ by definition, it is sufficient to prove that $\alpha \leq \sigma$. For any positive ε, there exists a broadcast game $I = (G = (V, E), c, N, \{s_i\}_{i \in N}, t)$ such that $\mathrm{POPoA}(I) \geq \alpha - \varepsilon$, where $\mathrm{POPoA}(I)$ denote POPoA for the game I.

For the game I, let P be a worst potential-optimal strategy profile and P^* be a strategy profile with minimum cost. By definition, we have $\mathrm{POPoA}(I) = \mathrm{cost}_I(P)/\mathrm{cost}_I(P^*)$, where for a strategy profile Q, $\mathrm{cost}_I(Q)$ denotes the social cost of Q in the game I. Let I' denote the broadcast game obtained from I by modifying edge cost c to c':

$$c'(e) = \begin{cases} c(e) & \text{if } e \in \bigcup_i P_i, \\ c(e) + \frac{\mathrm{cost}_I(P^*)}{|V|-1}\varepsilon & \text{otherwise.} \end{cases} \tag{9}$$

We note that the minimum social cost in the modified game I' is at most $(1+\varepsilon)\mathrm{cost}_I(P^*)$, and P is the *unique* strategy profile with minimum potential in the modified game I', since $\bigcup_i P_i$ is a tree in G and any other profile uses at least one edge not in P. Therefore, we have

$$\sigma \geq \frac{\mathrm{cost}_I(P)}{(1+\varepsilon)\cdot \mathrm{cost}_I(P^*)} \geq \frac{\alpha-\varepsilon}{1+\varepsilon}. \tag{10}$$

By taking $\varepsilon \to 0$, we have $\sigma \geq \alpha$, which completes the proof. $\square$

Unfortunately, this proof cannot be applied to general network design games.

3 Upper bounds of POPoA and POPoS for broadcast games

In this section, we show the following theorems.

Theorem 3. *For any undirected broadcast game with n players, we have*

$$\mathrm{POPoA} = \mathrm{O}(\sqrt{\log n}).$$

Theorem 4. *For any undirected broadcast game with $|V|$ vertices, we have*

$$\mathrm{POPoA} = \mathrm{O}(\log |V|).$$

Note that we have no better bounds for POPoS, since by Lemma 2 it immediately implies a better bounds for POPoA.

Before proving these theorems, we note that POPoA $= \mathrm{O}(\log n)$ holds for any undirected broadcast game with n players. This can be shown by the following *potential function method* introduced by Anshelevich *et al.* [2]. Let P and P^* be strategy profiles with minimum potential and minimum cost, respectively. Then by (4) we have

$$\mathrm{cost}(P) \leq \Phi(P) \leq \Phi(P^*) \leq H(n)\cdot \mathrm{cost}(P^*). \tag{11}$$

By $\sqrt{\log n}, \log |V| \leq \log n$, our theorems improve upon this simple result.

In order to obtain our results, we first show properties of POPoA and POPoS for undirected broadcast games with respect to metric closure.

3.1 Metric closure

In this subsection, we show that taking the metric closure does not affect POPoA or POPoS for undirected network design games.

The metric closure of a network $(G = (V,E), c : E \to \mathbb{R}_{++})$ is a network $(\hat{G} = (V, \hat{E} = \binom{V}{2}), \hat{c} : \hat{E} \to \mathbb{R}_{++})$ where $\hat{c}(u,v)$ is defined as the cost of a shortest path from u to v in (G, c). For a network design game $I = (G, c, N, (s_i, t_i)_{i \in N})$, we denote the corresponding game on $(\hat{G}, \hat{c})$ by $\hat{I} = (\hat{G}, \hat{c}, N, (s_i, t_i)_{i \in N})$. For a game I, let cost_I, Φ_I, and $\mathcal{P}_I$ denote the cost, potential and the set of strategy profiles in I, respectively.

We first show that taking the metric closure does not affect the minimum social cost.

Lemma 5.

$$\min_{P \in \mathcal{P}_I} \mathrm{cost}_I(P) = \min_{Q \in \mathcal{P}_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q). \tag{12}$$

Proof. Since $c(e) \geq \hat{c}(e)$ holds for any edge $e \in E$, we have $\min_{P \in \mathcal{P}_I} \mathrm{cost}_I(P) \geq \min_{Q \in \mathcal{P}_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q)$. On the other hand, for any edge (u, v) in $\hat{E}$, (G, c) has a u-v path with cost $\hat{c}(u, v)$. Therefore, for any strategy profile Q in $\mathcal{P}_{\hat{I}}$, replacing all edges in Q by the corresponding shortest path does not increase the cost, which implies $\min_{P \in \mathcal{P}_I} \mathrm{cost}_I(P) \leq \min_{Q \in \mathcal{P}_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q)$. $\square$

Next, we show that taking the metric closure does not affect the best or worst social cost among minimum potential strategies.

Lemma 6.

$$\min_{P \in \operatorname{argmin} \Phi_I} \mathrm{cost}_I(P) = \min_{Q \in \operatorname{argmin} \Phi_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q), \tag{13}$$

$$\max_{P \in \operatorname{argmin} \Phi_I} \mathrm{cost}_I(P) = \max_{Q \in \operatorname{argmin} \Phi_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q). \tag{14}$$

Proof. Let us first show the following equation:

$$\min_{P \in \mathcal{P}_I} \Phi_I(P) = \min_{Q \in \mathcal{P}_{\hat{I}}} \Phi_{\hat{I}}(Q). \tag{15}$$

Since $c(e) \geq \hat{c}(e)$ holds for any edge $e \in E$, we have

$$\Phi_I(P) \geq \Phi_{\hat{I}}(P) \tag{16}$$

for any strategy profile P of I. On the other hand, for any edge $e = (u, v)$ in $\hat{E}$, let $\pi(e)$ denote a u-v path in G with cost $\hat{c}(e)$. Let Q be a strategy profile of $\hat{I}$, and let P be a strategy profile of I which is obtained from Q by replacing all edges e in Q by $\pi(e)$. Then, we have

$$\begin{aligned}
\Phi_{\hat{I}}(Q) &= \sum_{e\in\hat{E}} \hat{c}(e)H(\xi_Q(e)) \\
&= \sum_{e\in\hat{E}} \left(\sum_{f\in\pi(e)} c(f)\right) H(\xi_Q(e)) = \sum_{f\in E} c(f) \left(\sum_{e\in\hat{E}:\pi(e)\ni f} H(\xi_Q(e))\right) \\
&\geq \sum_{f\in E} c(f)H\left(\sum_{e\in\hat{E}:\pi(e)\ni f} \xi_Q(e)\right) \\
&= \sum_{f\in E} c(f)H(\xi_P(f)) = \Phi_I(P), \quad \text{(by concavity of harmonic number)}
\end{aligned} \tag{17}$$

which together with (16) implies (15). Moreover, by (15), inequality (17) holds with equality if Q minimizes the potential function $\Phi_{\hat{I}}$. This implies that for any $f \in E$, at most one edge $e \in \hat{E}$ satisfies both $\pi(e) \ni f$ and $\xi_Q(e) \geq 1$. Hence we have $\mathrm{cost}_{\hat{I}}(Q) = \mathrm{cost}_I(P)$ for $Q \in \mathrm{argmin}\, \Phi_{\hat{I}}$, which implies

$$\min_{P\in\mathrm{argmin}\,\Phi_I} \mathrm{cost}_I(P) \leq \min_{Q\in\mathrm{argmin}\,\Phi_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q), \tag{18}$$

$$\max_{P\in\mathrm{argmin}\,\Phi_I} \mathrm{cost}_I(P) \geq \max_{Q\in\mathrm{argmin}\,\Phi_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q). \tag{19}$$

On the other hand, let P be a strategy profile of I that minimizes the potential function Φ_I. By (15) and (16), $\Phi_I(P) = \Phi_{\hat{I}}(P)$ and P also minimizes $\Phi_{\hat{I}}$. By $\Phi_I(P) = \Phi_{\hat{I}}(P)$, we have $\mathrm{cost}_I(P) = \mathrm{cost}_{\hat{I}}(P)$, which implies

$$\min_{P\in\mathrm{argmin}\,\Phi_I} \mathrm{cost}_I(P) \geq \min_{Q\in\mathrm{argmin}\,\Phi_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q), \tag{20}$$

$$\max_{P\in\mathrm{argmin}\,\Phi_I} \mathrm{cost}_I(P) \leq \max_{Q\in\mathrm{argmin}\,\Phi_{\hat{I}}} \mathrm{cost}_{\hat{I}}(Q). \tag{21}$$

This completes the proof. □

It follows from Lemmas 5 and 6 that metric closure does not affect POPoA or POPoS for undirected network design games.

For a network design game I, let POPoA(I) and POPoS(I) denote POPoA and POPoS for I.

Lemma 7. *For any network design game I, it holds that*

$$\mathrm{POPoA}(I) = \mathrm{POPoA}(\hat{I}), \tag{22}$$

$$\mathrm{POPoS}(I) = \mathrm{POPoS}(\hat{I}). \tag{23}$$

However, there exists a network design game I such that $\mathrm{PoA}(I) \neq \mathrm{PoA}(\hat{I})$ or $\mathrm{PoS}(I) \neq \mathrm{PoS}(\hat{I})$, where $\mathrm{PoA}(I)$ and $\mathrm{PoS}(I)$ denote PoA and PoS for a game I (see Examples 8 and 9).

Example 8. *Consider a broadcast game I in $G = (V = \{v_1, v_2\}, E = \{e_1 = (v_1, v_2), e_2 = (v_1, v_2)\})$ with $c(e_1) = 1$ and $c(e_2) = n$. The game has n players each of which has a source-sink pair (v_1, v_2). See Figure 2. It is not difficult to see that I and $\hat{I}$ both has minimum social cost 1, the worst Nash equilibrium of I is $\{e_2\}$ with cost n, and the worst Nash equilibrium of $\hat{I}$ has cost 1. Therefore, we have $\mathrm{PoA}(I)\,(= n) \neq \mathrm{PoA}(\hat{I})\,(= 1)$.*

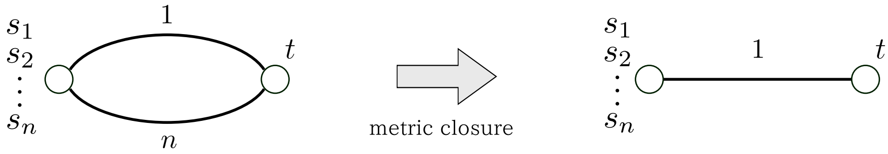

Figure 2: A broadcast game I with $\mathrm{PoA}(I) \neq \mathrm{PoA}(\hat{I})$

Example 9. *Consider a multicast game I shown in Figure 3, where $\varepsilon > 0$ is arbitrarily small. Both games have minimum social cost 24. In the game I, there exists a unique Nash equilibrium that uses $\{(v_1, v_2), (v_2, t), (v_2, v_3), (v_4, t)\}$ and its cost is $25 - \varepsilon$. In the game $\hat{I}$, the best Nash equilibrium uses $\{(v_1, t), (v_3, v_4), (v_4, t)\}$ with cost 24. Therefore, $\mathrm{PoS}(I) = (25 - \varepsilon)/24$ and $\mathrm{PoS}(\hat{I}) = 1$.*

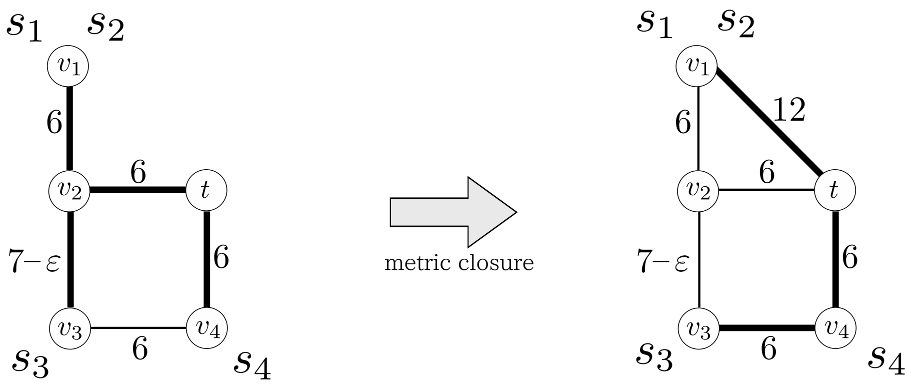

Figure 3: A multicast game I with $\mathrm{PoS}(I) \neq \mathrm{PoS}(\hat{I})$

3.2 The proof of Theorem 4

Let $(G = (V, E), c, N, \{s_i\}_{i \in N}, t)$ be a broadcast game. By Lemma 7, we can assume that (G, c) is a metric network. Let P be a strategy profile with minimum potential, and let $T = \bigcup_{i \in N} P_i$. Since c is positive, T is a spanning tree in G. We regard T as a directed tree of root t, by directing every edge toward t. For $v \in V \setminus \{t\}$, let e_v denotes the edge in T whose tail is v.

For any two vertices $u, v \in V \setminus \{t\}$, we estimate the cost $c(u, v)$ of u-v path in T in terms of $c(e_u)$ and $c(e_v)$.

Lemma 10. *Let $u, v \in V \setminus \{t\}$ be vertices such that u is neither an ancestor nor a descendant of v in T, and let $k = \xi_P(e_u)$, $l = \xi_P(e_v)$. Then we have*

$$c(u,v) \geq \frac{(k+l)H(k) - kH(k+l)}{lH(k) + kH(l)} c(e_u) + \frac{(k+l)H(l) - lH(k+l)}{lH(k) + kH(l)} c(e_v). \tag{24}$$

Proof. Let w be the least common ancestor of u and v in T. Let $P^{u \to v}$ be a strategy profile such that $\bigcup_{i \in N} P_i^{u \to v} = T \setminus \{e_u\} \cup \{(u,v)\}$. Since $T \setminus \{e_u\} \cup \{(u,v)\}$ is a tree, $P^{u \to v}$ is unique. Similarly, let $P^{v \to u}$ be a strategy profile such that $\bigcup_{i \in N} P_i^{v \to u} = T \setminus \{e_v\} \cup \{(u,v)\}$. Since P has a minimum potential, we have that

$$\begin{aligned}
0 &\geq \Phi(P) - \Phi(P^{u \to v}) \\
&= \sum_{e \in T_{u,w}} c(e) H(\xi_P(e), \xi_P(e) - k) \\
&\qquad + \sum_{e \in T_{v,w}} c(e) H(\xi_P(e), \xi_P(e) + k) - c(u,v) H(k),
\end{aligned} \tag{25}$$

where $H(m,n) = H(m) - H(n)$, and for two vertices x and y, $T_{x,y}$ denotes x-y path in T. Similarly, we have

$$\begin{aligned}
0 &\geq \Phi(P) - \Phi(P^{v \to u}) \\
&\geq \sum_{e \in T_{u,w}} c(e) H(\xi_P(e), \xi_P(e) + l) \\
&\qquad + \sum_{e \in T_{v,w}} c(e) H(\xi_P(e), \xi_P(e) - l) - c(u,v) H(l).
\end{aligned} \tag{26}$$

From (25) and (26),

$$\begin{aligned}
&c(u,v)(lH(k) + kH(l)) \\
&\quad \geq \sum_{e \in T_{u,w}} c(e) \left(kH(\xi_P(e), \xi_P(e) + l) + lH(\xi_P(e), \xi_P(e) - k) \right) \\
&\quad + \sum_{e \in T_{v,w}} c(e) \left(kH(\xi_P(e), \xi_P(e) - l) + lH(\xi_P(e), \xi_P(e) + k) \right).
\end{aligned} \tag{27}$$

From Proposition 21,

$$kH(\xi_P(e), \xi_P(e) + l) + lH(\xi_P(e), \xi_P(e) - k) \geq -k \cdot \frac{l}{\xi_P(e) + 1} + l \cdot \frac{k}{\xi_P(e)} > 0,$$

$$kH(\xi_P(e), \xi_P(e) - l) + lH(\xi_P(e), \xi_P(e) + k) \geq k \cdot \frac{l}{\xi_P(e)} - l \cdot \frac{k}{\xi_P(e) + 1} > 0.$$

Therefore, we obtain

$$\begin{aligned}c(u,v) &\geq \frac{kH(k,k+l)+lH(k,k-k)}{lH(k)+kH(l)}c(e_u)+\frac{kH(l,l-l)+lH(l,l+k)}{lH(k)+kH(l)}c(e_v)\\ &= \frac{(k+l)H(k)-kH(k+l)}{lH(k)+kH(l)}c(e_u)+\frac{(k+l)H(l)-lH(k+l)}{lH(k)+kH(l)}c(e_v).\end{aligned} \tag{28}$$

□

We use the following propositions, where the proofs can be found in Appendix

Proposition 11. *Let k, l be positive integers. If $k \leq l$, then we have*

$$\frac{(k+l)H(k)-kH(k+l)}{lH(k)+kH(l)} \geq \frac{1}{4}. \tag{29}$$

Proposition 12. *Let k, l be positive integers. If $l/2 \leq k \leq l$, then*

$$\frac{(k+l)H(l)-lH(k+l)}{lH(k)+kH(l)} \geq \frac{1}{12}. \tag{30}$$

Lemma 13. *If a vertex v is a proper ancestor of a vertex u in T, then $c(u,v) \geq c(e_u)$.*

Proof. Since P is a strategy profile with minimum potential, we have

$$\begin{aligned}0 &\geq \Phi(P)-\Phi(P^{u\to v})\\ &\geq \sum_{e\in T_{u,v}} c(e)\left(H(\xi_P(e))-H(\xi_P(e)-k)\right)-c(u,v)H(k)\\ &\geq \ c(e_u)H(k)-c(u,v)H(k).\end{aligned} \tag{31}$$

Therefore, we obtain $c(u,v) \geq c(e_u)$. □

Lemma 14. *Let $u \in V \setminus \{t\}$ and $v \in V$ with $u \neq v$, $k = \xi_P(e_u)$ and $l = \xi_P(e_v)$. If $k \leq l$, then we have $c(u,v) \geq \frac{c(e_u)}{4}$.*

Proof. Since $k \leq l$, u is not a proper ancestor of v. If v is an ancestor of u, the inequality $c(u,v) \geq c(e_u) \geq c(e_u)/4$ holds by Lemma 13. Otherwise, by Lemma 10, we have

$$\begin{aligned}c(u,v) &\geq \frac{(k+l)H(k)-kH(k+l)}{lH(k)+kH(l)}c(e_u)+\frac{(k+l)H(l)-lH(k+l)}{lH(k)+kH(l)}c(e_v)\\ &\geq \frac{c(e_u)}{4}+\frac{c(e_v)}{12} \geq \frac{c(e_u)}{4} \qquad \text{(by Propositions 11 and 12)}\end{aligned} \tag{32}$$

□

We are now ready to prove Theorem 4.

Proof of Theorem 4. Let P^* be a strategy profile with minimum cost. Since (G, c) is a metric network, there exists a Hamilton path $L = \{(v_1, v_2), (v_2, v_3), \dots, (v_{|V|-1}, v_{|V|})\}$ such that $v_1 = t$ and $c(L) \leq 2 \cdot \mathrm{cost}(P^*)$.

For each $i = 1, 2, \dots, \lfloor |V|/2 \rfloor$, Let x_i and y_i be two vertices such that $\{x_i, y_i\} = \{v_{2i-1}, v_{2i}\}$ and $\xi_P(e_{x_i}) \leq \xi_P(e_{y_i})$, where we assume $\xi_P(e_{v_1}) = +\infty$. By Lemma 14, we have

$$\sum_{i=1}^{\lfloor |V|/2 \rfloor} c(e_{x_i}) \leq \sum_{i=1}^{\lfloor |V|/2 \rfloor} 4c(v_{2i-1}, v_{2i}) \leq 8\mathrm{cost}(P^*). \tag{33}$$

By repeatedly applying the same argument to the remaining vertices, we obtain

$$\frac{\mathrm{cost}(P)}{\mathrm{cost}(P^*)} = \mathrm{O}(\log |V|). \tag{34}$$

□

3.3 The proof of Theorem 3

We need further lemmas to show Theorem 3.

Lemma 15. *Let $u, v \in V \setminus \{t\}$ be vertices such that u is neither an ancestor nor a descendant of v in T, and let $k = \xi_P(e_u)$, $l = \xi_P(e_v)$. If $l/2 \leq k \leq l$, then we have*

$$c(u, v) \geq \frac{1}{12} \left(c(e_u) + c(e_v) \right). \tag{35}$$

Proof. By Lemma 10, we have that

$$\begin{aligned} c(u, v) &\geq \frac{(k+l)H(k) - kH(k+l)}{lH(k) + kH(l)} c(e_u) + \frac{(k+l)H(l) - lH(k+l)}{lH(k) + kH(l)} c(e_v) \\ &\geq \frac{c(e_u)}{4} + \frac{c(e_v)}{12} \qquad \text{(by Propositions 11 and 12)} \\ &\geq \frac{1}{12}(c(e_u) + c(e_v)). \end{aligned} \tag{36}$$

□

For a positive integer α, let $V_\alpha = \{v \in V \mid \alpha \leq \xi_P(e_v) < 2\alpha\}$. We then show that $\sum_{v \in V_\alpha} c(e_v) = O(\mathrm{cost}(P^*))$.

We partition V_α into connected components $W_1, W_2, \dots, W_p$ in T (see Figure 4). Suppose that two vertices u, v are in V_α and u is an ancestor of v. Then any vertex w in the u-v path in T also satisfies $\alpha \leq \xi_P(e_w) < 2\alpha$, and hence $w \in V_\alpha$, which implies that there exists a connected component W_i that contains both u and v. This leads to the following lemma.

Lemma 16. *For $i \neq j$, let W_i and W_j be defined as above, and let u and w be vertices in W_i and W_j, respectively. Then u is neither an ancestor nor a descendant of v.*

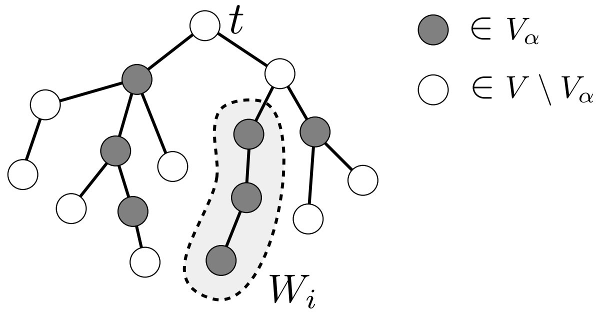

Figure 4: The vertices V_α in the strategy profile P

Lemma 17. *Let V_α be defined as above, and P^* be a strategy profile with minimum cost. Then we have $\sum_{v \in V_\alpha} c(e_v) \leq 27\text{cost}(P^*)$.*

Proof. We first analyze the sum of the costs $c(e_v)$ for $v \in W_i$. Let L be a Hamilton path in G such that t is the initial vertex and $c(L) \leq 2 \cdot \text{cost}(P^*)$. We index vertices in W_i by $v_1, v_2, \ldots, v_{l_i}$ in the increasing order of the distance from t in L, i.e., in L, v_k is closer to t than v_h if $k < h$. We further partition W_i into $\bigcup_k W_{ik}$ such that each W_{ik} is the maximum set that for any two vertices u and v in W_{ik}, the u-v path in L contains no vertex in W_j, $j \neq i$. Let $W_{ik} = \{v_{q_k}, v_{q_k+1} \ldots, v_{q_{k+1}-1}\}$. For each W_{ik}, let v_{r_k} denote the vertex which is the closest to t in T. For $h = q_k, q_k+1, \ldots, q_{k+1}-1$ with $h \neq r_k$, let $e'_{v_h} = (v_h, v_{h+1})$ if $h < r_k$, and (v_h, v_{h-1}) if $h > r_k$. Let

$$T' = \left(T \setminus (\bigcup_k \{e_v \mid v \in W_{ik} \setminus \{v_{r_k}\}\})\right) \cup (\bigcup_k \{e'_v \mid v \in W_{ik} \setminus \{v_{r_k}\}\}), \quad (37)$$

and let P' be the strategy profile that corresponds to T'. Then we have

$$\begin{aligned} 0 &\leq \Phi(P') - \Phi(P) \\ &\leq \sum_k \sum_{h=q_k}^{q_{k+1}-1} [h \neq r_k]\Big(c(e'_{v_h})H(\xi_{P'}(e'_{v_h})) - c(e_{v_h})H(\xi_P(e_{v_h}))\Big), \end{aligned} \quad (38)$$

where $[\cdot]$ is the Iverson bracket that returns one if the condition in the bracket is true, and zero otherwise. By $\xi_{P'}(e'_{v_h}) < 2\alpha$, $\xi_P(e_{v_h}) \geq \alpha$, an upper bound in the right-hand side of (38) is:

$$\begin{aligned}
&\sum_k \sum_{h=q_k}^{q_{k+1}-1} [h \neq r_k]\Big(c(e'_{v_h})H(2\alpha) - c(e_{v_h})H(\alpha)\Big) \\
&\leq \sum_k \sum_{h=q_k}^{q_{k+1}-2} c(v_h, v_{h+1})H(2\alpha) - \sum_k \sum_{h=q_k}^{q_{k+1}-1} [h \neq r_k] c(e_{v_h})H(\alpha) \\
&\leq \sum_k d(v_{q_k}, v_{q_{k+1}-1})H(2\alpha) - \sum_{v \in W_i \setminus U_i} c(e_v)H(\alpha), \qquad (39)
\end{aligned}$$

where $U_i = \bigcup_k \{v_{r_k}\}$, and for two vertices u and w, $d(u, w)$ denotes the distance between u and w in L. The last inequality holds since the network is metric.

Therefore, we have

$$\begin{aligned}
\sum_{v \in W_i \setminus U_i} c(e_v) &\leq \frac{H(2\alpha)}{H(\alpha)} \sum_k d(v_{q_k}, v_{q_{k+1}-1}) \\
&= \left(1 + \frac{H(2\alpha, \alpha)}{H(\alpha)}\right) \sum_k d(v_{q_k}, v_{q_{k+1}-1}) \\
&\leq \left(1 + \max_{1 \leq i \leq \alpha} \left\{\frac{1/(\alpha+i)}{1/i}\right\}\right) \sum_k d(v_{q_k}, v_{q_{k+1}-1}) \\
&= \frac{3}{2} \sum_k d(v_{q_k}, v_{q_{k+1}-1}). \qquad (40)
\end{aligned}$$

Now we evaluate the sum of the costs $c(e_v)$ such that $v \in V_\alpha$. Let $U = \bigcup_{i=1}^p U_i$. For $V_\alpha \setminus U$, we obtain

$$\begin{aligned}
\sum_{v \in V_\alpha \setminus U} c(e_v) &= \sum_{i=1}^p \sum_{v \in W_i \setminus U_i} c(e_v) \\
&\leq \sum_{i=1}^p \frac{3}{2} \sum_k d(v_{q_k}, v_{q_{k+1}-1}) \qquad \text{(by (40))} \\
&\leq \frac{3}{2} c(L) \leq 3\text{cost}(P^*). \qquad (41)
\end{aligned}$$

For U, let us index vertices in U by $u_1, u_2, \ldots$ in the increasing order of the distance from t in L. Then Lemma 16 implies that, for any j, u_j is neither an ancestor nor a descendant of u_{j+1}. Since $\alpha \leq \xi_P(e_{u_j}), \xi_P(e_{u_{j+1}}) < 2\alpha$ yield $\xi_P(e_{u_j})/2 \leq \xi_P(e_{u_{j+1}}) \leq \xi_P(e_{u_j})$ or $\xi_P(e_{u_{j+1}})/2 \leq \xi_P(e_{u_j}) \leq \xi_P(e_{u_{j+1}})$, we have

$$\begin{aligned}
\sum_{u \in U} c(e_u) = \sum_j c(e_{u_j}) &\leq \sum_j 12c(u_j, u_{j+1}) \qquad \text{(by Lemma 15)} \\
&\leq 12c(L) \leq 24\text{cost}(P^*). \qquad (42)
\end{aligned}$$

Combining inequalities (41) and (42), we obtain

$$\sum_{v\in V_\alpha} c(e_v) = \sum_{v\in V_\alpha\setminus U} c(e_v) + \sum_{v\in U} c(e_v) \le 27\mathrm{cost}(P^*). \tag{43}$$

□

Let γ be a positive integer. The total cost of the edges used by at most γ players is

$$\begin{aligned}\sum_{e:1\le\xi_P(e)\le\gamma} c(e) &\le \sum_{k=0}^{\lfloor\lg\gamma\rfloor}\sum_{e:2^k\le\xi_P(e)<2^{k+1}} c(e)\\ &\le \sum_{k=0}^{\lfloor\lg\gamma\rfloor}\sum_{v\in V_{2^k}} c(e_v) = \mathrm{O}(\log\gamma)\mathrm{cost}(P^*). \quad \text{(by Lemma 17)}\end{aligned} \tag{44}$$

On the other hand, the total cost of the edges used by at least γ players is

$$\sum_{e:1\le\gamma\le\xi_P(e)} c(e) = \frac{H(n)}{H(\gamma)}\cdot\mathrm{cost}(P^*) = \mathrm{O}\left(\frac{\log n}{\log\gamma}\right)\mathrm{cost}(P^*) \tag{45}$$

since $\sum_{e:1\le\gamma\le\xi_P(e)} c(e)H(\gamma) \le \Phi(P) \le \Phi(P^*) \le \mathrm{cost}(P^*)H(n)$.

From inequalities (44) and (45), we have that

$$\begin{aligned}\mathrm{cost}(P) &= \sum_{e:1\le\xi_P(e)<\gamma} c(e) + \sum_{e:\gamma\le\xi_P(e)} c(e)\\ &= \mathrm{O}\left(\log\gamma + \frac{\log n}{\log\gamma}\right)\mathrm{cost}(P^*).\end{aligned} \tag{46}$$

By choosing $\gamma = \exp(\sqrt{\log n})$, we obtain

$$\frac{\mathrm{cost}(P)}{\mathrm{cost}(P^*)} = \mathrm{O}(\sqrt{\log n}). \tag{47}$$

4 Lower bounds of POPoA and POPoS for broadcast games

In this section, we show the following theorems.

Theorem 18. *There exists a broadcast game with n players such that*

$$\mathrm{POPoS} = \Omega(\sqrt{\log\log n}).$$

Theorem 19. *There exists a broadcast game with $|V|$ vertices such that*

$$\mathrm{POPoS} = \Omega(\log|V|).$$

For positive integer d, let $\mathrm{lsb}(d)$ denote the maximum number 2^p which divides d, where p denotes a nonnegative integer. For example, $\mathrm{lsb}(12) = 4$, $\mathrm{lsb}(5) = 1$, and $\mathrm{lsb}(8) = 8$.

We construct a family of broadcast games I with $\mathrm{POPoS}(I) = \Omega(\sqrt{\log\log n})$, $\Omega(\log |V|)$.

Our graph $G = (V, E)$ is defined as $V = \{0, 1, 2, 3, \ldots, 2^m\}$ and $E = \bigcup_{i=0}^{3} E_i$, where

$$\begin{aligned}
E_0 &= \{(v-1, v) \mid v = 1, 2, 3, \ldots, 2^m\},\\
E_1 &= \{(0, 2^m)\},\\
E_2 &= \{(v, v - \mathrm{lsb}(v)/2) \mid v = 4\cdot 1, 4\cdot 2, 4\cdot 3, \ldots, 4\cdot 2^{m-2}\},\\
E_3 &= \{(v, v + \mathrm{lsb}(v)/2) \mid v = 4\cdot 1, 4\cdot 2, 4\cdot 3, \ldots, 4\cdot (2^{m-2}-1)\},
\end{aligned}$$

and the edge cost $c : E \to \mathbb{R}_{++}$ is given as

$$c(e) = \begin{cases} 1, & e \in E_0,\\ 2^{m-1}, & e \in E_1,\\ \mathrm{lsb}(v)/4, & e = (v, v - \mathrm{lsb}(v)/2) \in E_2,\\ \mathrm{lsb}(v)/4, & e = (v, v + \mathrm{lsb}(v)/2) \in E_3. \end{cases}$$

An edge in $E_1 \cup E_2 \cup E_3$ with cost 2^{k-1} is called *k-shortcut* (or simply *shortcut*). For a vertex $v \in V$ with $\mathrm{lsb}(v) = 2^k$, let $V_v = \{u \in V \mid v - \mathrm{lsb}(v) < u < v + \mathrm{lsb}(v)\}$. The subgraph induced by V_v (denoted by $G[V_v]$) is called *k-block*, and v is called a *root* of the block.

All the players have a sink $t = 0$, and each vertex $v \in V \setminus \{t\}$ has $f(\mathrm{lsb}(v))$ players as their sources, where $f(k) = 2^{2^{(\lg k)^2}}$. Thus, this game has $n = \sum_{v=1}^{2^m} f(\mathrm{lsb}(v)) = \Theta(2^{2^{m^2}})$ players. We note that $m = \Theta(\sqrt{\log\log n})$. This games are depicted in Figures 5 and 6.

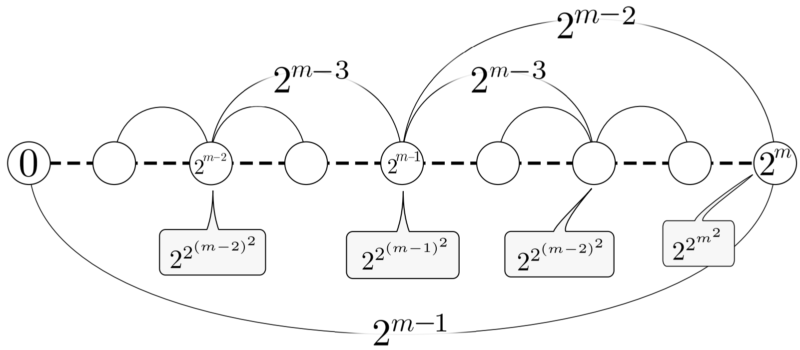

Figure 5: A family of broadcast games to prove Theorems 18 and 19

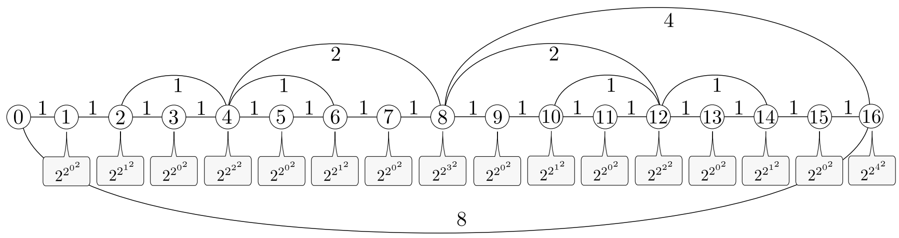

Figure 6: A broadcast game for $m = 4$ in Figure 5

Let P^* and P denote strategy profiles with minimum cost and potential, respectively. Then it is not difficult to see that $\bigcup_{i \in N} P_i^* = E_0$, where N denotes the set of all players. This is because E_0 is a spanning tree of cost 2^m, and any spanning tree has cost at least 2^m. Therefore it is sufficient to show $\mathrm{cost}(P) = \Omega(m \cdot 2^m)$, since it implies

$$\mathrm{POPoS} = \Omega(m) = \Omega(\sqrt{\log \log n}) = \Omega(\log |V|), \tag{48}$$

i.e., Theorems 18 and 19.

To prove our claim, we show that P contains all k-shortcut edges with $k \geq 4$ by using the following propositions on harmonic number, where the proofs can be found in Appendix.

Proposition 20. *For all nonnegative integer m, we have*

$$1 + \frac{m}{2} \leq H(2^m) \leq 1 + m. \tag{49}$$

Proposition 21. *Let m, n be positive integers. If $m \geq n$, then we have*

$$\frac{m-n}{m} \leq H(m, n) \leq \frac{m-n}{n+1}. \tag{50}$$

For a nonnegative integer k, let G_k be a k-block, and let $g(k) = \sum_{l=0}^{k} 2^{2^{l^2}} \cdot 2^{k-l}$ be the number of players whose source vertex is in G_k.

Proposition 22. *$g(k)$ is at most $2f(2^k) = 2 \cdot 2^{2^{k^2}}$.*

Proof. The proof is by induction on k. When $k < 2$, we have $g(0) = 2 \leq 4 = 2 \cdot 2^{2^{0^2}}$, $g(1) = 8 = 2 \cdot 2^{2^{1^2}}$.

Assuming that the induction hypothesis is true for some $k \geq 2$, we consider the case of $k+1$. Since $(k+1)$-block consists of two k-blocks and its root, we have $g(k+1) = 2g(k) + 2^{2^{(k+1)^2}}$.

Then we have:

$$\begin{aligned} g(k+1) &= 2g(k) + 2^{2^{(k+1)^2}} \\ &\leq 2^{2^{k^2}+2} + 2^{2^{(k+1)^2}} && \text{(by induction hypothesis)} \\ &\leq 2^{2^{(k+1)^2}} + 2^{2^{(k+1)^2}} = 2 \cdot 2^{2^{(k+1)^2}}. && \text{(by } k \geq 2\text{)} \end{aligned} \tag{51}$$

This completes the proof. $\square$

Lemma 23. *For $m \geq 3$, any strategy profile with minimum potential contains the m-shortcut edge.*

Proof. Let

$$\varphi_1 = \min\{\Phi(Q) \mid Q : \text{a strategy profile with } m\text{-shortcut edge}\},$$
$$\varphi_2 = \min\{\Phi(Q) \mid Q : \text{a strategy profile without } m\text{-shortcut edge}\}.$$

We claim that $\varphi_1 < \varphi_2$, which proves the lemma.

In order to estimate φ_1, consider a spanning tree $T = (E_0 \setminus \{(2^m - 1, 2^m)\}) \cup E_1$ and the corresponding strategy profile Q. In Q, $e \in E_1$ is used by $f(2^m)$ players, each of which has a source 2^m, and any $e \in E_0 \setminus \{(2^m - 1, 2^m)\})$ is used by at most $g(m-1)$ players. Hence we have

$$\begin{aligned}
\varphi_1 \leq \Phi(Q) &\leq \underbrace{2^{m-1}H(f(2^m))}_{m\text{-shortcut edge}} + \underbrace{(2^m-1)H(g(m-1))}_{\text{edges in } E_0\setminus\{(2^m-1,2^m)\}} \\
&\leq 2^{m-1}H(f(2^m)) + 2^M C(2\cdot 2^{2^{(m-1)^2}}) \qquad \text{(by Proposition 22)} \\
&\leq 2^{m-1}H(f(2^m)) + 2^m(1+1+2^{(m-1)^2}) \qquad \text{(by Proposition 20)} \\
&\leq 2^{m-1}H(f(2^m)) + 2^{m+1} + 2^{m^2-m+1}. \qquad (52)
\end{aligned}$$

We next estimate φ_2. Let T be a spanning tree that does not contain the m-shortcut edge. Then the cost of the 2^m-t path in T is at least $2^{m-1}+1$, which implies

$$\begin{aligned}
\varphi_2 &\geq (2^{m-1}+1)H(2^{2^{m^2}}) \\
&= 2^{m-1}H(2^{2^{m^2}}) + H(2^{2^{m^2}}) \\
&\geq 2^{m-1}H(2^{2^{m^2}}) + 2^{m^2-1}. \qquad \text{(by Proposition 20)} \qquad (53)
\end{aligned}$$

By combining (52) and (53), we obtain

$$\varphi_2 - \varphi_1 \geq 2^{m^2-1} - 2^{m+1} - 2^{m^2-m+1} > 0, \qquad (54)$$

since $m \geq 3$. $\square$

By the following lemma, together with Lemma 23, shows that any strategy profile with minimum potential makes use of all k-shortcut edges with $k \geq 4$.

Lemma 24. *Let k be an integer with $4 \leq k < m$. If any strategy profile with minimum potential contains all the h-shortcut edges of $h \geq k+1$, then it also contains all the k-shortcut edges.*

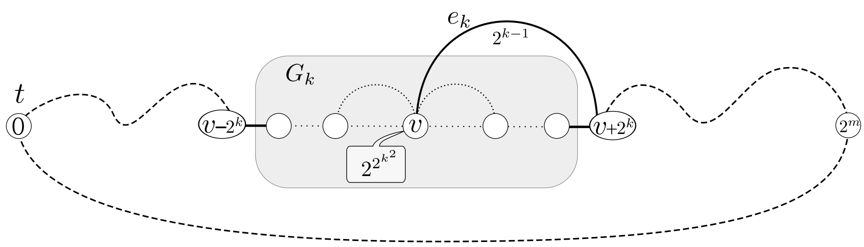

Figure 7: A k-block G_k of root v

Proof. For a k-block G_k, let v be the root of G_k, and let e_k be the k-shortcut edge of incident on v (see Figure 7). Let

$$\varphi_1 = \min\left\{\Phi(Q) \mid Q : \begin{array}{l}\text{a strategy profile with all } h\text{-shortcut}\\ \text{edges of } h \geq k+1 \text{ and } e_k\end{array}\right\}, \tag{55}$$

$$\varphi_2 = \min\left\{\Phi(Q) \mid Q : \begin{array}{l}\text{a strategy profile with all } h\text{-shortcut}\\ \text{edges of } h \geq k+1 \text{ and no } e_k\end{array}\right\}. \tag{56}$$

We claim that $\varphi_1 < \varphi_2$, which proves the lemma.

In order to estimate φ_1, let Q^* be a strategy profile such that $\Phi(Q^*) = \varphi_2$ and it satisfies the conditions in (56). We consider the following spanning tree T in G:

$$\begin{aligned} T =& \Big(E(G_k) \cap E_0\Big) \cup \{e_k\} \\ &\cup \Big(\bigcup_{i\in N} Q_i^* \cap \big(E \setminus (E(G_k) \cup \{(v-2^k, v-2^k+1), (v+2^k-1, v+2^k)\}\big)\Big), \end{aligned}$$

where $E(G_k)$ denotes the edge set of G_k. Let Q be the strategy profile corresponding to T. Note that Q contains all h-shortcut edges of $h \geq k+1$ and e_k, and no player i with source s_i not in G_k uses edge e in G_k. Let

$$\eta(e) = |\{i \in N \mid e \in Q_i^*,\, s_i \notin V(G_k)\}|, \tag{57}$$

where $V(G_k)$ denotes the vertex set of G_k, and let $\varphi = \sum_{e\in E} c(e)H(\eta(e))$. Then

$$\begin{aligned} \varphi_1 &\leq \Phi(Q) = \sum_{e\in E} c(e)H(\xi_Q(e)) \\ &= \varphi + \sum_{e\in E(G_k)} c(e)H(\xi_Q(e)) + \sum_{l=k}^{m} c(e_l)H(\xi_Q(e_l), \eta(e_l)), \end{aligned} \tag{58}$$

where $\{e_k, e_{k+1}, \ldots, e_m\}$ is the v-t path in T. The second term in (58) satisfies

$$
\begin{aligned}
\sum_{e\in E(G_k)} c(e)H(\xi_Q(e)) &< 2^{k+1}H(g(k-1)) \\
&\le 2^{k+1}H(2\cdot 2^{2^{(k-1)^2}}) && \text{(by Proposition 22)} \\
&\le 2^{k+1}(2+2^{(k-1)^2}) && \text{(by Proposition 20)} \\
&= 2^{k+2}+2^{k^2-k+2}. && (59)
\end{aligned}
$$

The third term in (58) satisfies

$$
\sum_{l=k}^{m} c(e_l)H(\xi_Q(e_l),\eta(e_l)) \;\le\; 2^{k-1}H(g(k)) + \sum_{l=k+1}^{m} 2^{l-1}H(f(2^l)+g(k),f(2^l)), \tag{60}
$$

where the first term is bounded as

$$
\begin{aligned}
2^{k-1}H(g(k)) &\le 2^{k-1}H(2\cdot 2^{2^{k^2}}) && \text{(by Proposition 22)} \\
&\le 2^{k-1}(H(2^{2^{k^2}})+1) \\
&= 2^{k-1}H(2^{2^{k^2}})+2^{k-1}, && \text{(by Proposition 21 with } m=2n) \qquad (61)
\end{aligned}
$$

and the second term is bounded as

$$
\begin{aligned}
&\sum_{l=k+1}^{m} 2^{l-1}\cdot H(f(2^l)+g(k),f(2^l)) \\
&\quad\le \sum_{l=k+1}^{m} 2^{l-1}\cdot\frac{g(k)}{f(2^l)} && \text{(by Proposition 21)} \\
&\quad\le \sum_{l=k+1}^{m} 2^{l-1}\cdot\frac{2\cdot 2^{2^{k^2}}}{2^{2^{l^2}}} && \text{(by Proposition 22)} \\
&\quad\le 2. && \text{(by } k\ge 4) \qquad (62)
\end{aligned}
$$

Combining (58), (59), (60), (61), and (62) yields,

$$
\begin{aligned}
\varphi_1 &< \varphi + 2^{k+2}+2^{k^2-k+2}+2^{k-1}H(2^{2^{k^2}})+2^{k-1}+2 \\
&< \varphi + 2^{k-1}H(2^{2^{k^2}})+2^{k^2-k+3}. && \text{(by } k\ge 4) \qquad (63)
\end{aligned}
$$

We next estimate φ_2. Consider the players i with source v. If e_k is not used by i, then any strategy of i uses a path with cost at least $2^{k-1}+1$ to get out of G_k. Therefore, we obtain

$$
\begin{aligned}
\varphi_2 &\ge \varphi + (2^{k-1}+1)H(f(2^k)) \\
&\ge \varphi + 2^{k-1}H(f(2^k))+2^{k^2-1}. && \text{(by Proposition 20)} \qquad (64)
\end{aligned}
$$

It follows from (63) and (64) that

$$\varphi_2 - \varphi_1 > 2^{k^2-1} - 2^{k^2-k+3} \geq 0, \tag{65}$$

since $k \geq 4$. $\square$

By Lemmas 23 and 24, any strategy profile P with minimum potential contains all the k-shortcut edges of $k \geq 4$. Therefore, we have

$$\mathrm{cost}(P) \geq 2^{m-1} + \sum_{l=4}^{m-1} 2^{l-1} \cdot 2^{m-1-l} = (m-2) \cdot 2^{m-2} = \Omega(m \cdot 2^m), \tag{66}$$

which proves Theorems 18 and 19.

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Appendix

Harmonic number

In this section, we present omitted proofs of properties of harmonic number:

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \qquad (n \geq 0). \tag{67}$$

In addition, for nonnegative integers m and n, we define $H(m,n) = H(m) - H(n)$.

Proposition 20. *For all nonnegative integer m, we have*

$$1 + \frac{m}{2} \leq H(2^m) \leq 1 + m. \tag{68}$$

Proof. For $m = 0$, the statement is true with equality. For $m > 0$, we have:

$$\begin{aligned} \frac{1}{2} = 2^{m-1} \cdot \frac{1}{2^m} &\leq H(2^m) - H(2^{m-1}) \\ &= \frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m} \leq 2^{m-1} \cdot \frac{1}{2^{m-1}} = 1. \end{aligned} \tag{69}$$

Since $1/2 \leq H(2^m) - H(2^{m-1})$ and $H(2^0) = 1$, we obtain $H(2^m) \geq 1 + m/2$. Similarly, as $H(2^m) - H(2^{m-1}) \leq 1$ and $H(2^0) = 1$, we obtain $H(2^m) \leq 1 + m$. □

Proposition 25. *Let k, l be positive integers. If $k \leq l$ then,*

$$kH(l) \leq lH(k). \tag{70}$$

Proof. Since $H(n)/n$ is the average of 1, 1/2, ..., $1/n$, it is monotone decreasing in n. Thus we have

$$\frac{H(l)}{l} \leq \frac{H(k)}{k}. \tag{71}$$

□

Proposition 26. *Let k, l be positive integers. If $k \leq l$ then,*

$$\frac{H(k+l,k)}{l} \leq \frac{H(2k,k)}{k} \leq \frac{H(k)}{k}. \tag{72}$$

Proof. It follows from the fact that $H(k+l,k)/l$ is the average of $1/(k+1)$, $1/(k+2)$, ..., $1/(k+l)$. □

Proposition 21. *Let m, n be positive integers. If $m \geq n$, then we have*

$$\frac{m-n}{m} \leq H(m,n) \leq \frac{m-n}{n+1}. \tag{73}$$

Proof. It follows from

$$H(m,n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+(m-n)} \leq \frac{m-n}{n+1}, \tag{74}$$

$$H(m,n) = \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{m-(m-n-1)} \geq \frac{m-n}{m}. \tag{75}$$

□

Proposition 11. *Let k, l be positive integers. If $k \leq l$, then we have*

$$\frac{(k+l)H(k) - kH(k+l)}{lH(k) + kH(l)} \geq \frac{1}{4}. \tag{76}$$

Proof. Since proposition 25 implies $(k+l)H(k) - kH(k+l) \geq 0$, we have

$$\begin{aligned}
\frac{(k+l)H(k) - kH(k+l)}{lH(k) + kH(l)} &\geq \frac{(k+l)H(k) - kH(k+l)}{lH(k) + lH(k)} \quad \text{(by Proposition 25)} \\
&= \frac{1}{2} - \frac{k}{2H(k)} \cdot \frac{H(k+l,k)}{l} \\
&\geq \frac{1}{2} - \frac{k}{2H(k)} \cdot \frac{H(2k,k)}{k} \quad \text{(by Proposition 26)} \\
&= \frac{1}{2} - \frac{1}{2} \cdot \frac{H(2k,k)}{H(k)} \\
&\geq \frac{1}{2} - \frac{1}{2} \cdot \max_{1 \leq i \leq k} \left\{ \frac{1/(k+i)}{1/i} \right\} = \frac{1}{4}.
\end{aligned} \tag{77}$$

□

Proposition 12. *Let k, l be positive integers. If $l/2 \leq k \leq l$, then*

$$\frac{(k+l)H(l) - lH(k+l)}{lH(k) + kH(l)} \geq \frac{1}{12}. \tag{78}$$

Proof. If $l = k = 1$, then we obtain

$$\frac{(k+l)H(l) - lH(k+l)}{lH(k) + kH(l)} = \frac{2H(1) - H(2)}{H(1) + H(1)} = \frac{1}{4} \geq \frac{1}{12}. \tag{79}$$

For $l \geq 2$, we have:

$$
\begin{aligned}
\frac{(k+l)H(l) - lH(k+l)}{lH(k) + kH(l)} &\geq \frac{(k+l)H(l) - lH(k+l)}{lH(k) + lH(k)} && \text{(by Proposition 25)} \\
&= \frac{kH(l) - lH(k+l,l)}{2lH(k)} \\
&\geq \frac{kH(l) - l \cdot (k/l)}{2lH(k)} && \text{(by Proposition 21)} \\
&= \frac{kH(l)}{2lH(k)} \left(1 - \frac{1}{H(l)}\right) \\
&\geq \frac{kH(l)}{2lH(k)} \left(1 - \frac{1}{H(2)}\right) && \text{(by } l \geq 2\text{)} \\
&= \frac{kH(l)}{6lH(k)} \\
&\geq \frac{(l/2) \cdot H(l)}{6lH(l)} = \frac{1}{12}. && (80)
\end{aligned}
$$

□