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Yasushi KAWASE, Yusuke KOBAYASHI
and Yutaro YAMAGUCHI

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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Finding a Path in Group-Labeled Graphs with Two Labels Forbidden

Yasushi Kawase[*] Yusuke Kobayashi[†] Yutaro Yamaguchi[‡]

Abstract

The parity of the length of paths and cycles is a classical and well-studied topic in graph theory and theoretical computer science. A basic problem concerned with parity is to find an odd or even s–t path in an undirected graph. It can be solved in polynomial time easily, whereas its directed version is NP-hard. In this paper, as a generalization of this problem, we focus on the problem of finding an s–t path in a group-labeled graph, which is a directed graph with a group label on each arc. It is not difficult to see that finding an odd or even path in an undirected graph can be formulated as finding a zero path in a $\mathbb{Z}_2$-labeled graph.

For group-labeled graphs, efficient algorithms for finding non-zero paths or cycles with some conditions have been devised recently. On the other hand, the difficulty of finding a zero path is heavily dependent on the group, e.g., it is NP-complete to determine whether there exists an s–t path of label zero (or another specified label) in a $\mathbb{Z}$-labeled graph, but quite easy in a $\mathbb{Z}_2$-labeled graph. It is in fact known that a zero path in a Γ-labeled graph can be found in polynomial time for any constant-size abelian group Γ by getting help of the graph minor theory.

In this paper, we present a solution to finding an s–t path in a group-labeled graph with two labels forbidden. This also leads to an elementary solution to finding a zero path in a $\mathbb{Z}_3$-labeled graph, which is the first nontrivial case of finding a zero path. This case generalizes the 2-disjoint paths problem in undirected graphs, which also motivates us to consider that setting. More precisely, we provide an elementary polynomial-time algorithm for testing whether there are at most two possible labels of s–t paths in a group-labeled graph or not, and finding s–t paths attaining at least three distinct labels if exist. We also give a necessary and sufficient condition for a group-labeled graph to have exactly two possible labels of s–t paths, and our algorithm is based on this characterization.

1 Introduction

1.1 Background

The parity of the length of paths and cycles in a graph is a classical and well-studied topic in graph theory and theoretical computer science. As the simplest example, one can easily check the bipartiteness of a given undirected graph, i.e., we can determine whether it contains a cycle of odd length or not. This can be done in polynomial time also in the directed case by using the ear decomposition. It is also an important problem to test whether a given directed graph contains a directed cycle of even length or not, which is known to be equivalent to Pólya's permanent problem [12] (see, e.g., [11]). A polynomial time algorithm for this problem was devised by Robertson, Seymour, and Thomas [14].

In this paper, we focus on paths connecting two specified vertices s and t. It is easy to test whether a given undirected graph contains an s–t path of odd (or even) length or not,

[*]Tokyo Institute of Technology, Tokyo 152-8550, Japan. E-mail: kawase.y.ab@m.titech.ac.jp

[†]University of Tokyo, Tokyo 113-8656, Japan. Supported by JST, ERATO, Kawarabayashi Large Graph Project, and by KAKENHI Grant Number 24106002, 24700004. E-mail: kobayashi@mist.i.u-tokyo.ac.jp

[‡]University of Tokyo, Tokyo 113-8656, Japan. Supported by JSPS Fellowship for Young Scientists. E-mail: yutaro_yamaguchi@mist.i.u-tokyo.ac.jp

whereas the same problem is NP-complete in the directed case [10] (follows from [5]). A natural generalization of this problem is to consider paths of length p modulo q. One can easily see that, when $q = 2$, both of the following problems generalize the problem of finding an odd (or even) s–t path in an undirected graph:

- finding an s–t path of length p modulo q in an undirected graph, and
- finding an s–t path whose length is NOT p modulo q in an undirected graph, which is equivalent to determining whether all s–t paths are of length p modulo q or not.

Although these two generalizations are similar to each other, they are essentially different in the case of $q \geq 3$. In fact, a linear time algorithm for the second generalization was given by Arkin, Papadimitriou, and Yannakakis [1] for any q, whereas not so much was known about the first generalization.

Recently, as another generalization of the parity constraints, paths and cycles in a group-labeled graph have been investigated, where a group-labeled graph is a directed graph with each arc labeled by a group element. In a group-labeled graph, the label of a walk is defined as the sum (or the ordered product when the underlying group is non-abelian) of the labels of the traversed arcs, where each arc can be traversed in the converse direction and then the label is inversed (see Section 2 for the precise definition). In a similar way to paths of length p modulo q, it is natural to consider the following two problems: for a given element α,

(I) finding an s–t path of label α in a group-labeled graph, and

(II) finding an s–t path whose label is NOT α in a group-labeled graph, which is equivalent to determining whether all s–t paths are of label α or not.

Note that, when we consider Problem (I) or (II), by changing uniformly the labels of the arcs incident to s if necessary, we may assume that α is the identity of the underlying group. Hence, each problem is equivalent to finding a zero path or a non-zero path in a group-labeled graph.

If the underlying group is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = (\{0, 1\}, +)$ and the label of each arc is 1, then the label of a path corresponds to the parity of its length because $-1 = 1$ in $\mathbb{Z}_2$. This shows that both of these two problems generalize the problem of finding an odd (or even) s–t path in an undirected graph. We note that, in a $\mathbb{Z}_2$-labeled graph, finding an s–t path of label $\alpha \in \mathbb{Z}_2$ is equivalent to finding an s–t path whose label is not $\alpha + 1 \in \mathbb{Z}_2$, but such equivalence cannot hold for any other nontrivial group.

Problem (II) can be reduced to testing whether the group-labeled graph (precisely, the 2-connected component of the graph obtained from the input graph by adding an arc from s to t with label α that contains both s and t) contains a non-zero cycle, whose label is not the identity. With this observation, Problem (II) can be easily solved in polynomial time for any underlying group (see, e.g., [18] and Proposition 6). We mention that there are several results for packing non-zero paths [2, 3, 18, 20] and non-zero cycles [9, 19] with some conditions.

On the other hand, the difficulty of Problem (I) is heavily dependent on the underlying group. When the underlying group is isomorphic to $\mathbb{Z}_2$, since Problems (I) and (II) are equivalent as discussed above, it can be easily solved in polynomial time. When the underlying group is $\mathbb{Z}$, Problem (I) is NP-complete since the undirected Hamiltonian path problem reduces to this problem by replacing each edge with a pair of two arcs of opposite directions with label 1 and letting $\alpha := n - 1$, where n denotes the number of vertices. Huynh [8] showed the polynomial-time solvability of Problem (I) for any constant-size abelian group, which is deeply dependent on the graph minor theory.

To investigate the gap between Problems (I) and (II), we make a new approach to these problems by generalizing Problem (II) so that multiple labels are forbidden. In this paper, we provide a solution to the case that two labels are forbidden. Our result also leads to an elementary solution to the first nontrivial case of Problem (I), i.e., when the underlying group is isomorphic to $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} = (\{0, \pm 1\}, +)$.

1.2 Our contribution

Let Γ be an arbitrary group. For a Γ-labeled graph G and two distinct vertices s and t, let $l(G; s, t)$ be the set of all possible labels of s–t paths in G. Our first contribution is to give a characterization of Γ-labeled graphs G with two specified vertices s, t such that $l(G; s, t) = \{\alpha, \beta\}$, where α and β are distinct elements in Γ. Roughly speaking, we show that $l(G; s, t) = \{\alpha, \beta\}$ if and only if G is obtained from planar graphs with some condition by "gluing" them together (see Section 3.2). It is interesting that the planarity, which is a topological condition, appears in the characterization.

Note that there exists an easy characterization of triplets (G, s, t) with $l(G; s, t) = \{\alpha\}$, which is used to solve Problem (II) (see Section 2.2 for details). Hence, our characterization leads to the first nontrivial classification of Γ-labeled graphs in terms of the possible labels of s–t paths, and the classification is complete when $\Gamma \simeq \mathbb{Z}_3$.

We also show an algorithmic result, which is our second contribution. Based on the fact that our characterization can be tested in polynomial time, we present a polynomial-time algorithm for testing whether $|l(G; s, t)| \leq 2$ or not and finding at least three s–t paths whose labels are distinct if exist (see Theorem 11). In particular, our algorithm leads to an elementary solution to Problem (I) when $\Gamma \simeq \mathbb{Z}_3$, i.e., for each $\alpha \in \mathbb{Z}_3$, we can test whether $\alpha \in l(G; s, t)$ or not, and find an s–t path of label α if exists.

Note again that our results are not dependent on Γ, which can be non-abelian or infinite.

1.3 k-disjoint paths problem

Problem (I) in a $\mathbb{Z}_3$-labeled graph in fact generalizes the 2-disjoint paths problem, which also motivates us to consider the situation when two labels are forbidden. The 2-disjoint paths problem is to determine whether there exist two vertex-disjoint paths such that one connects s_1 and t_1 and the other connects s_2 and t_2 for distinct vertices s_1, s_2, t_1, t_2 in a given undirected graph. We can reduce the 2-disjoint paths problem to Problem (I) in a $\mathbb{Z}_3$-labeled graph as follows: let $s := s_1$ and $t := t_2$, replace every edge in the given graph with an arc with label 0, add one arc from t_1 to s_2 with label 1, and ask whether the constructed $\mathbb{Z}_3$-labeled graph contains an s–t path of label 1 or not. If the answer is YES, then there exist desired two disjoint paths, and otherwise there do not.

The 2-disjoint paths problem can be solved in polynomial time [15–17], and the following theorem characterizes the existence of two disjoint paths.

Theorem 1 (Seymour [16])**.** *Let $G = (V, E)$ be an undirected graph and $s_1, t_1, s_2, t_2 \in V$ distinct vertices. Then, there exist two vertex-disjoint paths P_i connecting s_i and t_i $(i = 1, 2)$ if and only if there is no family of disjoint vertex sets $X_1, X_2, \ldots, X_k \subseteq V \setminus \{s_1, t_1, s_2, t_2\}$ such that*

1. *$N(X_i) \cap X_j = \emptyset$ for distinct $i, j \in \{1, 2, \ldots, k\}$,*
2. *$|N(X_i)| \leq 3$ for $i = 1, 2, \ldots, k$, and*
3. *if G' is the graph obtained from G by deleting X_i and adding a new edge joining each pair of distinct vertices in $N(X_i)$ for each $i \in \{1, 2, \ldots, k\}$, then G' can be embedded on a plane so that s_1, s_2, t_1, t_2 are on the outer boundary in this order.*

Our characterization (Theorem 14) for triplets (G, s, t) with $l(G; s, t) = \{\alpha, \beta\}$ is inspired by Theorem 1, and we use this theorem in our proof.

We next mention that the k-disjoint paths problem can also be regarded as a special case of Problem (I) for any fixed integer $k \geq 2$. The k-disjoint paths problem is, for a given undirected graph with $2k$ distinct vertices s_i, t_i $(i = 1, \ldots, k)$, to find k vertex-disjoint paths such that each path connects s_i and t_i. This problem can be formulated as Problem (I) using the alternating group A_{2k-1} (which is indeed isomorphic to $\mathbb{Z}_3$ when $k = 2$) as follows: replace each edge with an arc with label id $\in A_{2k-1}$, add an arc from t_i to s_{i+1} with label $(2i-1\ \ 2i+1\ \ 2i) \in A_{2k-1}$ for each $i = 1, 2, \ldots, k-1$, and ask whether there exists an s–t path of label

$$\sigma := (1\ 3\ 2)(3\ 5\ 4)\cdots(2k-3\ \ 2k-1\ \ 2k-2)$$

or not. It is easy to check that σ is the unique permutation mapping 1 to $2k-1$ which can be constructed in such an A_{2k-1}-labeled graph.

Although the k-disjoint paths problem can be solved in polynomial time for fixed k [13], its solution requires sophisticated arguments based on the graph minor theory. This suggests that Problem (I) is a challenging problem even if the size of the underlying group is bounded.

2 Preliminaries

2.1 Terms and notations

Throughout this paper, let Γ be a group (which can be non-abelian or infinite), for which we usually use multiplicative notation with denoting the identity by 1_Γ (also we sometimes use additive notation with denoting the identity by 0, e.g., for $\Gamma \simeq \mathbb{Z}_3$). We assume that the following operations can be done in constant time for any $\alpha, \beta \in \Gamma$: getting the inverse element $\alpha^{-1} \in \Gamma$, computing the product $\alpha\beta \in \Gamma$, and testing the identification $\alpha = \beta$. A directed graph $G = (V, E)$ with a mapping $\psi_G : E \to \Gamma$ (called a *label function*) is called a Γ-*labeled graph*.

2.1.1 Graphs

Let $G = (V, E)$ be a directed graph. A sequence $W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ is called a *walk* in G if $v_0, v_1, \ldots, v_l \in V$ are vertices, $e_1, e_2, \ldots, e_l \in E$ are arcs, and either $e_i = v_{i-1}v_i$ or $e_i = v_i v_{i-1}$ for each $i = 1, 2, \ldots, l$. A walk W is called a *path* (in particular, a v_0–v_l *path*) if $v_0, v_1, \ldots, v_l$ are distinct, and a *cycle* if $v_0, v_1, \ldots, v_{l-1}$ are distinct and $v_0 = v_l$. We call v_0 and v_l (which may coincide) the *end vertices of* W, and each v_i $(1 \le i \le l-1)$ an *inner vertex on* W. For $0 \le i < j \le l$, let $W[v_i, v_j]$ denote the subwalk $(v_i, e_{i+1}, v_{i+1}, \ldots, e_j, v_j)$ of W. Let $\bar{W}$ denote the reversed walk of W, i.e., $\bar{W} = (v_l, e_l, \ldots, v_1, e_1, v_0)$.

Let $X \subseteq V$ be a vertex set. We denote by $\delta_G(X)$ the set of arcs incident to X in G and by $N_G(X)$ the *neighbor* of X in G, i.e., $\delta_G(X) := \{\, e = xy \in E \mid |\{x, y\} \cap X| = 1 \,\}$ and $N_G(X) := \{\, y \in V \setminus X \mid \delta_G(X) \cap \delta_G(\{y\}) \neq \emptyset \,\}$. We often omit the subscript G and denote a singleton $\{x\}$ by its element x if there is no confusion.

Let $G[X] := (X, E(X))$ denote the subgraph of G *induced by* X, where $E(X) := \{\, e = xy \in E \mid \{x, y\} \subseteq X \,\}$. We denote by $G - X$ the subgraph of G obtained by removing all vertices in X, i.e., $G - X = G[V \setminus X]$. For an arc set $F \subseteq E$, we also denote by $G - F$ the subgraph of G obtained by removing all arcs in F, i.e., $G - F = (V, E \setminus F)$.

For an integer $k \ge 0$ and a vertex set $X \subsetneq V$ with $|X| = k$, we call X a k-cut in G if $G - X$ is not connected. A directed graph is called k-*connected* if it contains at least k vertices[1] and no k'-cut for every $k' < k$. A k-*connected component* of G is a maximal k-connected induced subgraph $G[X]$ $(X \subseteq V$ with $|X| \ge k)$ of G.

Suppose that G is embedded on a plane. We call a unique unbounded face of G the *outer face* of G, and another face an *inner face*. For a face F of G, let $\mathrm{bd}(F)$ denote the cycle obtained by walking the boundary of F in an arbitrary direction from an arbitrary vertex.

2.1.2 Labels

Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G, and $W = (v_0, e_1, v_1, \ldots, e_l, v_l)$ a walk in G. The *label* $\psi_G(W)$ of W is defined as the ordered product $\psi_G(e_l, v_l) \cdots \psi_G(e_2, v_2) \cdot$

[1] Although it may be popular to exclude the case that there are exactly k vertices, we here admit that case.

$\psi_G(e_1, v_1)$, where $\psi_G(e_i, v_i) := \psi_G(e_i)$ if $e_i = v_{i-1}v_i$ and $\psi_G(e_i, v_i) := \psi_G(e_i)^{-1}$ if $e_i = v_iv_{i-1}$. Note that, for the reversed walk $\bar{W}$ of W, we have $\psi_G(\bar{W}) = \psi_G(W)^{-1}$. In particular, since an arc uv with label α and an arc vu with label α^{-1} are equivalent, we identify such two arcs. We say that W is *balanced* (or a *zero walk*) if $\psi_G(W) = 1_\Gamma$ and *unbalanced* (or a *non-zero walk*) otherwise, and also that G is *balanced* if G contains no unbalanced cycle. Note that whether a cycle C is balanced or not does not depend on the choices of the direction and the end vertex, since $\psi_G(\bar{C}) = \psi_G(C)^{-1}$ and $\psi_G(C') = \psi_G(e_1) \cdot \psi_G(C) \cdot \psi_G(e_1)^{-1}$, where $C = W$ and $C' := (v_1, e_2, v_2, \ldots, e_l, v_l = v_0, e_1, v_1)$. Hence, when we consider whether a cycle is balanced or not, we can choose the direction and the end vertex arbitrarily.

For distinct vertices $s, t \in V$, let $l(G; s, t)$ be the set of all possible labels of s–t paths in G. When $l(G; s, t) = \{\alpha\}$ for some $\alpha \in \Gamma$, we also denote the element α itself by $l(G; s, t)$. Without loss of generality, we may assume that there is no vertex $v \in V$ that is not contained in any s–t path, since such a vertex does not make any effect on $l(G; s, t)$. To consider only such cases, let $\mathcal{D}$ be the set of all triplets (G', s, t) such that G' is a Γ-labeled graph with two specified vertices $s, t \in V(G')$ in which every vertex is contained in some s–t path. The following lemma guarantees that one can efficiently compute an induced subgraph G' of G such that $(G', s, t) \in \mathcal{D}$ and $l(G'; s, t) = l(G; s, t)$.

Lemma 2. *For a connected Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$, one can test whether $(G, s, t) \in \mathcal{D}$ or not in polynomial time. Moreover, one can compute, in polynomial time, the set X of vertices that are not contained in any s–t path in G.*

Proof. Add an arc from s to t, and compute the 2-connected component that contains both s and t (e.g., by [6]). Let $Y \subseteq V$ be the vertex set of the 2-connected component (if no 2-connected component contains both s and t, then let $Y := \emptyset$). If $Y = V$, then $(G, s, t) \in \mathcal{D}$. Otherwise, we have $X = V \setminus Y$, and moreover, $(G - X, s, t) \in \mathcal{D}$ unless $Y = \emptyset$. □

2.2 Finding a non-zero path

In this section, we show that a non-zero s–t path can be found (i.e., Problem (II) can be solved) efficiently by using well-known properties of Γ-labeled graphs. The following techniques are often utilized in dealing with Γ-labeled graphs (see, e.g., [2, 3, 18]).

Definition 3 (Shifting)**.** Let $G = (V, E)$ be a Γ-labeled graph. For a vertex $v \in V$ and an element $\alpha \in \Gamma$, *shifting* (*a label function* ψ_G) *by* α *at* v means the following operation: update ψ_G to ψ'_G defined as, for each $e \in E$,

$$\psi'_G(e) := \begin{cases} \psi_G(e) \cdot \alpha^{-1} & (e \in \delta_G(v) \text{ leaves } v) \\ \alpha \cdot \psi_G(e) & (e \in \delta_G(v) \text{ enters } v) \\ \psi_G(e) & (\text{otherwise}). \end{cases}$$

Shifting at $v \in V$ does not change the label of any walk whose end vertices are not v, and neither that of any cycle C whose end vertex is v up to conjugate, i.e., $\psi'_G(C) = \alpha \cdot \psi_G(C) \cdot \alpha^{-1}$. Furthermore, when we apply shifting multiple times, the order of applications does not make any effect on the resulting label function. We say that two Γ-labeled graphs G_1 and G_2 are (s, t)-*equivalent* if G_2 is obtained from G_1 by shifting by some $\alpha_v \in \Gamma$ at each $v \in V \setminus \{s, t\}$ (and then G_1 is obtained from G_2 by shifting by α_v^{-1} at each v). Note that $l(G_1; s, t) = l(G_2; s, t)$ if G_1 and G_2 are (s, t)-equivalent.

Lemma 4. *For a connected and balanced Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$, there exists a Γ-labeled graph G' which is (s, t)-equivalent to G such that*

$$\psi_{G'}(e) = \begin{cases} \alpha & (e \in \delta_G(s) \text{ leaves } s), \\ \alpha^{-1} & (e \in \delta_G(s) \text{ enters } s), \\ 1_\Gamma & (\text{otherwise}), \end{cases}$$

for every arc $e \in E(G') = E$ and for some $\alpha \in \Gamma$ (in fact, $\alpha = l(G; s, t)$).

Proof. Take an arbitrary spanning tree T of G, and assume that all arcs in T are directed toward t. Consider the following procedure. Let $X := \{t\}$. While $X \neq V$, take a neighbor $v \in N_T(X)$, apply shifting the current label function ψ by $\psi(e)$ at v for a unique arc $e \in \delta_T(v) \cap \delta_T(X)$ from v to X (so that $\psi(e) = 1_\Gamma$ after shifting), and update $X := X + v$.

After the procedure, we have $\psi(e) = 1_\Gamma$ for every arc $e \in E(T)$, and also for every arc $e \in E$ since G is balanced. Suppose that we applied shifting by α at s. Then, we obtain desired G' by shifting ψ by α^{-1} at s after the procedure. Note that G' is (s, t)-equivalent to G since the resulting label function does not depends on the order of applications of shifting. □

Lemma 5. *For any $(G, s, t) \in \mathcal{D}$, $|l(G; s, t)| = 1$ if and only if G is balanced.*

Proof. "If" part is obvious from Lemma 4. To prove the converse direction, suppose that G is unbalanced and let C be an unbalanced cycle in G. Since $(G, s, t) \in \mathcal{D}$ implies that $G + st$ is 2-connected (cf. the proof of Lemma 2), for any distinct $x, y \in V(C)$, there exist two disjoint paths (possibly of length 0, i.e., $s = x$ or $y = t$) between $\{s, t\}$ and $\{x, y\}$ in G. Take an s–x path P and a y–t path Q in G so that $V(P) \cap V(C) = \{x\}$, $V(Q) \cap V(C) = \{y\}$, and $V(P) \cap V(Q) = \emptyset$, and choose x as the end vertex of C. Since $\psi_G(\bar{C}[x, y])^{-1} \cdot \psi_G(C[x, y]) = \psi_G(C) \neq 1_\Gamma$, we have $\psi_G(C[x, y]) \neq \psi_G(\bar{C}[x, y])$. Hence, by extending $C[x, y]$ and $\bar{C}[x, y]$ using P and Q, we can construct two s–t paths in G whose labels are distinct, which implies $|l(G; s, t)| \geq 2$. □

Lemmas 2 and 5 lead to the following proposition. Note that G' in Lemma 4 can be found in $O(|E|)$ time, since it is needed just to perform one breadth first search and $|V| - 1$ shiftings.

Proposition 6. *Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then, for any $\alpha \in \Gamma$, one can test whether $l(G; s, t) \subseteq \{\alpha\}$ or not in polynomial time. Furthermore, if $l(G; s, t) \not\subseteq \{\alpha\}$, then one can find an s–t path P with $\psi_G(P) \neq \alpha$ in polynomial time.*

2.3 New operations

For our characterization of triplets $(G, s, t) \in \mathcal{D}$ with $|l(G; s, t)| = 2$, we introduce several new operations which do not make any effect on $l(G; s, t)$. Let $(G = (V, E), s, t) \in \mathcal{D}$.

Definition 7 (3-contraction)**.** For a vertex set $X \subseteq V \setminus \{s, t\}$ such that $|N_G(X)| = 3$ and $G_X := G[X \cup N_G(X)] - E(N_G(X))$ is connected and balanced, the *3-contraction* of X is the following operation (see Fig. 1):

- remove all vertices in X, and
- add an arc from x to y with label $l(G_X; x, y)$ (which consists of a single element by Lemma 5) for each pair of $x, y \in N_G(X)$ if there is no such arc.

The resulting graph is denoted by $G/_3X$.

For vertex sets $X, Y, Z \subseteq V$, we say that X *separates Y and Z in G* if every two vertices $y \in Y \setminus X$ and $z \in Z \setminus X$ are contained in different connected components in $G - X$. In particular, if X separates Y and Z in G and $Y \setminus X \neq \emptyset \neq Z \setminus X$, then X is an $|X|$-cut in G.

Definition 8 (2-contraction)**.** For a vertex set $X \subsetneq V$ with $x \in X$ and $y \in V \setminus X$ such that

- $G[X]$ is a balanced 2-connected component of $G - y$ (hence, $|X| \geq 2$), and
- $\{x, y\}$ separates X and $\{s, t\}$ (hence, $\{x, y\}$ is a 2-cut in G unless $\{x, y\} = \{s, t\}$),

the *2-contraction* of X is the following operation (see Fig. 2):

- apply shifting by $l(G[X]; v, x)$ at each vertex $v \in X - x$, so that the label of every arc in $G[X]$ becomes 1_Γ (which can be checked similarly to Lemma 4),
- merge all vertices in X into a single vertex, which we refer to also as x, and
- identify the parallel arcs with the same label.

The resulting graph is denoted by $G/_2X$.

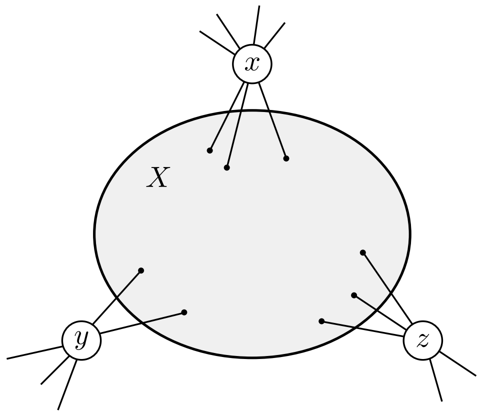

Figure 1: 3-contraction.

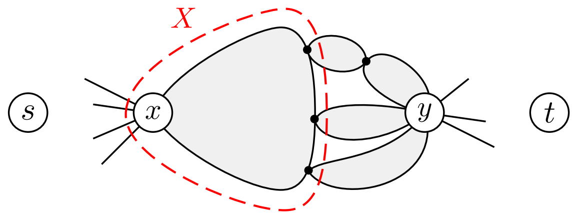

Figure 2: 2-contraction.

Remark. Although the 2-contraction and the 3-contraction are different operations, we use the same term "*contraction*" to refer to one of these two operations because of the following correspondence. Let $X \subseteq V \setminus \{s, t\}$ be a vertex set such that $N_G(X) = \{x, y\}$ for some distinct vertices $x, y \in V$ and $G_X := G[X \cup N_G(X)] - E(N_G(X))$ is balanced. Then, by a sequence of the 2-contractions of some $X' \subseteq X + x$ with $x \in X'$ and $y \notin X'$, we can replace the balanced subgraph G_X with a single arc from x to y with label $l(G_X; x, y)$. This sequential operation and the 3-contraction are analogous to the contractions of X_i in Theorem 1 (which means removing all vertices in X_i and adding an edge between each pair of distinct neighbors of X_i).

We call a sequence of contractions a *contraction sequence*. We also say that a vertex set $X \subsetneq V$ is *2-contractible* (or *3-contractible*) if the 2-contraction (or 3-contraction) of X can be performed in G. Furthermore, X is said to be *contractible* if X is 2-contractible or 3-contractible.

It should be noted that any contraction does not change $l(G; s, t)$, since each s–t path enters the removed vertex set at most once in both cases (2-contraction and 3-contraction).

Definition 9 (Replacing)**.** For a triplet $(H, x', y') \in \mathcal{D}$ with $V(H) \cap V = \emptyset$ and parallel arcs $e_i \in E$ $(i \in I)$ (possibly $|I| = 1$) from $x \in V$ to $y \in V$ with $l(H; x', y') = \{\, \psi_G(e_i) \mid i \in I \,\}$, we say that G' is obtained from G by *replacing* e_i $(i \in I)$ *with* (H, x', y') when G' is obtained from the disjoint union of G and H by removing e_i $(i \in I)$ and by identifying x and y with x' and y', respectively.

The following operation can be regarded as the inverse operation of the replacing.

Definition 10 (Reduction)**.** For a connected induced subgraph H of G with $3 \leq |V(H)| < |V|$ such that $\{x, y\} \subsetneq V(H)$ separates $\{s, t\}$ and $V(H)$, the *reduction* of (H, x, y) is the following operation:

- remove all vertices in $V(H) \setminus \{x, y\}$, and
- add an arc from x to y with label α for each $\alpha \in l(H; x, y)$ if there is no such arc.

The resulting graph is denoted by $G/(H, x, y)$.

It should be remarked again that $l(G; s, t) = l(G'; s, t)$ for a Γ-labeled graph G' obtained from G by any operation shown here, since there exists an s–t path in G of label $\alpha \in \Gamma$ if and only if so does in G'. Moreover, we also have $(G', s, t) \in \mathcal{D}$.

3 Main Results

3.1 Algorithmic results

As described in Section 2.2, Problem (II) can be solved efficiently, i.e., one can find a non-zero s–t path in polynomial time (Proposition 6). The following theorem, one of our main results, is the first nontrivial extension of this property, which claims that not only one label but also another can be forbidden simultaneously.

Theorem 11. *Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then, for any distinct $\alpha, \beta \in \Gamma$, one can test whether $l(G; s, t) \subseteq \{\alpha, \beta\}$ or not in polynomial time. Furthermore, if $l(G; s, t) \not\subseteq \{\alpha, \beta\}$, then one can find an s–t path P with $\psi_G(P) \notin \{\alpha, \beta\}$ in polynomial time.*

Such an algorithm is constructed based on a characterization of Γ-labeled graphs with exactly two possible labels of s–t paths shown in Section 3.2. Our algorithm and a proof of this theorem are presented later in Section 4. It should be mentioned that this theorem leads to a solution to Problem (I) for $\Gamma \simeq \mathbb{Z}_3$.

Corollary 12. *Let $G = (V, E)$ be a $\mathbb{Z}_3$-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then one can compute $l(G; s, t)$ in polynomial time. Furthermore, for each $\alpha \in l(G; s, t)$, one can find an s–t path P with $\psi_G(P) = \alpha$ in polynomial time.*

3.2 Characterizations

Recall that $\mathcal{D}$ denotes the set of all triplets (G, s, t) such that G is a Γ-labeled graph with $s, t \in V(G)$ in which every vertex is contained in some s–t path. In this section, we provide a complete characterization of triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ for some distinct $\alpha, \beta \in \Gamma$. We consider two cases separately: when $\alpha\beta^{-1} = \beta\alpha^{-1}$ and when $\alpha\beta^{-1} \neq \beta\alpha^{-1}$.

First, we give a characterization in the easier case: when $\alpha\beta^{-1} = \beta\alpha^{-1}$. Note that this case does not appear when $\Gamma \simeq \mathbb{Z}_3$. The following proposition holds analogously to Lemmas 4 and 5 in Section 2.2, which characterize triplets $(G, s, t) \in \mathcal{D}$ with $|l(G; s, t)| = 1$.

Proposition 13. *Let α and β be distinct elements in Γ with $\alpha\beta^{-1} = \beta\alpha^{-1}$. For any $(G, s, t) \in \mathcal{D}$, $l(G; s, t) = \{\alpha, \beta\}$ if and only if G is not balanced and there exists a Γ-labeled graph G' which is (s, t)-equivalent to G such that*

$$\psi_{G'}(e) = \begin{cases} \alpha \text{ or } \beta & (e \in \delta_G(s) \text{ leaves } s), \\ \alpha^{-1} \text{ or } \beta^{-1} & (e \in \delta_G(s) \text{ enters } s), \\ 1_\Gamma \text{ or } \alpha\beta^{-1} & (\text{otherwise}), \end{cases} \qquad (*)$$

for every arc $e \in E(G') = E(G)$.

Proof. "If" part is easy to see as follows. Since G is not balanced, $|l(G; s, t)| \geq 2$ by Lemma 5. Furthermore, since $\alpha\beta^{-1} = \beta\alpha^{-1}$, the label of any s–t path in G' is α or β. Hence, the (s, t)-equivalence between G and G' leads to $l(G; s, t) = l(G'; s, t) = \{\alpha, \beta\}$.

The converse direction is rather difficult. Similarly to the proof of Lemma 4, take an arbitrary spanning tree T of G and apply shifting at each $v \in V - t$ so that $\psi(e) = 1_\Gamma$ for every arc $e \in E(T)$, where ψ denotes the resulting label function. Since $l(G; s, t) = \{\alpha, \beta\}$ and $l(T; s, t) = 1_\Gamma$, we applied shifting by α or β at s. Hence, by shifting ψ by α^{-1} or β^{-1}, respectively, at s after the above procedure, we can obtain a Γ-labeled graph G' which is (s, t)-equivalent to G, and this G' is in fact desired one.

To see this, suppose to the contrary that some arc $e' \in E(G')$ does not satisfy $(*)$, and let $E' \subsetneq E(G')$ be the set of arcs satisfying $(*)$. Note that $E(T) \subseteq E'$, and hence $G'[E']$ is connected. Take an s–t path P in G' with $E(P) \setminus E' \neq \emptyset$ so that $|E(P) \setminus E'|$ is minimized.

If $|E(P) \setminus E'| = 1$, then $\psi_{G'}(P) \notin \{\alpha, \beta\}$, which contradicts $l(G'; s, t) = l(G; s, t) = \{\alpha, \beta\}$. Otherwise, we have $|E(P) \setminus E'| \geq 2$. Let $e_1, e_2 \in E(P) \setminus E'$ be the first two arcs traversed in walking along P, and Q be the subpath of P from the head of e_1 to the tail of e_2 (hence, $E(Q) \subseteq E'$). Since $G'[E']$ is connected, there exists a path R from $u \in V(Q)$ to $w \in V(P) \setminus V(Q)$ in $G'[E']$. We can construct an s–t path P' from P by replacing $P[u, w]$ (or $P[w, u]$) with R (or $\bar{R}$) such that $|E(P') \cap \{e_1, e_2\}| = 1$ and $E(P') \setminus E' \subseteq E(P) \setminus E'$. This implies that $1 \leq |E(P') \setminus E'| \leq |E(P) \setminus E'| - 1$, which contradicts the choice of R. □

We next consider the main case, which is much more difficult: when $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. The following theorem, one of our main results, completes a characterization of triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ for some distinct $\alpha, \beta \in \Gamma$. The definition of the set $\mathcal{D}_{\alpha,\beta} \subseteq \mathcal{D}$, which appears in the theorem, is shown later through Definitions 15–17 in Section 3.3. In short, $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ if one can obtain, from G by a contraction sequence, a Γ-labeled graph constructed by "gluing" together planar Γ-labeled graphs with some simple conditions.

Theorem 14. *Let α and β be distinct elements in Γ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. For any $(G, s, t) \in \mathcal{D}$, $l(G; s, t) = \{\alpha, \beta\}$ if and only if $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$.*

Recall that $|l(G; s, t)| = 1$ if and only if G is balanced by Lemma 5, which can be easily tested by shifting along an arbitrary spanning tree of G (cf. the proof of Lemma 4). Hence, these characterizations lead to the first nontrivial classification of Γ-labeled graphs in terms of the number of possible labels of s–t paths, and the classification is also complete when $\Gamma \simeq \mathbb{Z}_3$.

3.3 Definition of $\mathcal{D}_{\alpha,\beta}$

Fix distinct elements $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. In order to characterize triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$, let us define several sets of triplets $(G, s, t) \in \mathcal{D}$ for which it is easy to see that $l(G; s, t) = \{\alpha, \beta\}$. Theorem 14 claims that any triplet $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ is in fact contained in one of them.

Definition 15. For distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, let $\mathcal{D}^0_{\alpha,\beta}$ be the set of all triplets $(G, s, t) \in \mathcal{D}$ such that

- there is no contractible vertex set $X \subsetneq V(G)$ (*well-contracted*), and
- G can be embedded on a plane with the face set $\mathcal{F}$ satisfying the following properties:
  - both s and t are on the boundary of the outer face $F_0 \in \mathcal{F}$,
  - one s–t path along the boundary of F_0 is of label α and the other is of β, and
  - there exists a unique inner face F_1 whose boundary is unbalanced, i.e., $\psi_G(\mathrm{bd}(F)) = 1_\Gamma$ for any $F \in \mathcal{F} \setminus \{F_0, F_1\}$.

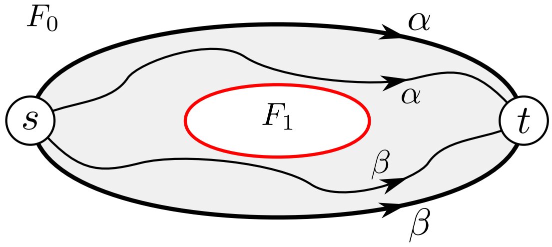

Figure 3: $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$.

It is not difficult to see that $l(G; s, t) = \{\alpha, \beta\}$ for any triplet $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ (see Fig. 3). This also follows from a stronger claim shown later as Lemma 20 in Section 5.2.

Definition 16. For distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, we define $\mathcal{D}^1_{\alpha,\beta}$ as the minimal set of triplets $(G, s, t) \in \mathcal{D}$ satisfying the following conditions:

- $\mathcal{D}^0_{\alpha,\beta} \subseteq \mathcal{D}^1_{\alpha,\beta}$, and
- if $(G/(H, x, y), s, t) \in \mathcal{D}^1_{\alpha,\beta}$ (recall that $G/(H, x, y)$ denotes the Γ-labeled graph obtained from G by the reduction of (H, x, y)) for some triplet $(H, x, y) \in \mathcal{D}^0_{\alpha',\beta'}$, where $\alpha', \beta' \in \Gamma$ satisfy $\alpha'\beta'^{-1} \neq \beta'\alpha'^{-1}$, then $(G, s, t) \in \mathcal{D}^1_{\alpha,\beta}$.

We are now ready to define $\mathcal{D}_{\alpha,\beta}$.

Definition 17. For distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, let $\mathcal{D}_{\alpha,\beta}$ be the set of all triplets $(G, s, t) \in \mathcal{D}$ with the following property: there exists a Γ-labeled graph $\tilde{G}$ obtained from G by a contraction sequence such that $(\tilde{G}, s, t) \in \mathcal{D}^1_{\alpha,\beta}$.

It is also easy to see that $l(G; s, t) = \{\alpha, \beta\}$ for any triplet $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ since contractions and reductions do not change $l(G; s, t)$. A proof of the non-trivial direction ("only if" part of Theorem 14) is presented later in Section 5.

4 Algorithm

In this section, we give a proof of Theorem 11. That is, we present an algorithm to test whether $l(G; s, t) \subseteq \{\alpha, \beta\}$ or not for given distinct $\alpha, \beta \in \Gamma$ and to find an s–t path of label $\gamma \in \Gamma \setminus \{\alpha, \beta\}$ if $l(G; s, t) \not\subseteq \{\alpha, \beta\}$, in a given Γ-labeled graph $G = (V, E)$ with $s, t \in V$. It should be mentioned that, when $\Gamma \simeq \mathbb{Z}_3$, such an algorithm can compute $l(G; s, t)$ itself and find an s–t path of label α for each $\alpha \in l(G; s, t)$. Without loss of generality, we assume that G does not have parallel arcs with the same label.

4.1 Algorithm description

For the simple description, we separate our algorithm into two parts: to test whether $|l(G; s, t)| \leq 2$ or not and return at most two s–t paths which attain all labels in $l(G; s, t)$ when $|l(G; s, t)| \leq 2$, and to find three s–t paths whose labels are distinct when it has turned out that $|l(G; s, t)| \geq 3$ (note that it is easy to find two s–t paths whose labels are distinct when $|l(G; s, t)| \geq 2$).

We first present an algorithm to test whether $|l(G; s, t)| \leq 2$ or not and return at most two s–t paths which attain all labels in $l(G; s, t)$ when $|l(G; s, t)| \leq 2$. It should be noted again that this algorithm can compute $l(G; s, t)$ itself when $\Gamma \simeq \mathbb{Z}_3$. Throughout this algorithm, let $G' = (V', E')$ denote a temporary Γ-labeled graph currently considered.

TestTwoLabels(G, s, t)

Input A Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$.

Output The set $l(G; s, t)$ of all possible labels of s–t paths in G with those which attain the labels if $|l(G; s, t)| \leq 2$, and "$|l(G; s, t)| \geq 3$" otherwise.

Step 0. Compute the set X of vertices which are not contained in any s–t path in G by Lemma 2. If $X = V$, then halt with returning $\emptyset$ since there is no s–t path in G. Otherwise, set $G' \leftarrow G - X$. Note that $(G', s, t) \in \mathcal{D}$ and $l(G'; s, t) = l(G; s, t)$.

Step 1. Test whether G' is balanced or not by Lemma 4 (i.e., take an arbitrary spanning tree, and apply shifting along it). If G' is balanced, then halt with returning the label of an arbitrary s–t path in G with the path. Otherwise, by using an unbalanced cycle, obtain two s–t paths in G whose labels are distinct (cf. the proof of Lemma 5), say $\alpha, \beta \in \Gamma$. In the following steps, we check whether $l(G'; s, t) = \{\alpha, \beta\}$ or not.

Step 2. If $\alpha\beta^{-1} = \beta\alpha^{-1}$, then check the condition in Proposition 13. If it is satisfied, then return $\{\alpha, \beta\}$ with the two s–t paths obtained in Step 1, and "$|l(G; s, t)| \geq 3$" otherwise. Otherwise (i.e., $\alpha\beta^{-1} \neq \beta\alpha^{-1}$), to make G' 2-connected, add to G' an arc from s to t with label α if there is no such arc.

Step 3. While G' is not 3-connected (and $|V'| \geq 3$), do the following procedure. Let $x, y \in V'$ be distinct vertices such that $G' - \{x, y\}$ is not connected. Let X be the vertex set of a connected component of $G' - \{x, y\}$ that contains none of s and t (such X exists, since s and t are adjacent in G'), and $Y := X \cup \{x, y\} \subsetneq V$. Test whether $|l(G'[Y]; x, y)| \leq 2$ or not recursively by TestTwoLabels($G'[Y], x, y$). Update $G' \leftarrow G'/(G'[Y], x, y)$ (reduction) if $|l(G'[Y]; x, y)| \leq 2$, and return "$|l(G; s, t)| \geq 3$" otherwise.

Step 4. While there exists a 3-contractible vertex set $X \subseteq V' \setminus \{s, t\}$, update $G' \leftarrow G'/_3 X$ (3-contraction).

Step 5. Test whether $(G', s, t) \in \mathcal{D}^0_{\alpha,\beta}$ or not by Lemma 18 below (and the case of $|V'| = 2$ is trivial). Return $\{\alpha, \beta\}$ with the s–t paths in G whose labels are α and β which have been obtained in Step 1 if $(G', s, t) \in \mathcal{D}^0_{\alpha,\beta}$, and "$|l(G; s, t)| \geq 3$" otherwise.

Next, we show an algorithm to find three s–t paths whose labels are distinct when it has turned out that $|l(G; s, t)| \geq 3$. Also it should be noted again that this algorithm finds three s–t paths which attain all labels when $\Gamma \simeq \mathbb{Z}_3$.

FindThreePaths(G, s, t)

Input A Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$ such that $|l(G; s, t)| \geq 3$.

Output Three s–t paths in G whose labels are distinct.

Step 0. If $V = \{s, t\}$, then halt with returning three s–t paths each of which consists of a single arc $st \in E$. Note that E consists of at least three parallel arcs st with distinct labels.

Step 1. For each $s' \in N_G(s) - t$, test whether $|l(G - s; s', t)| \leq 2$ or not by TestTwoLabels($G - s, s', t$).

Step 2. If $|l(G - s; s', t)| \leq 2$ for all $s' \in N_G(s) - t$, then we have already obtained s'–t paths which attain all labels in $l(G - s; s', t)$. Choose three s–t paths whose labels are distinct among the s–t paths obtained by extending such s'–t paths using an arc (possibly parallel arcs) $ss' \in E$ for each $s' \in N_G(s) - t$ and the s–t paths each of which consists of a single arc $st \in E$, and halt with returning them.

Step 3. Otherwise, for at least one $\tilde{s} \in N_G(s) - t$, we obtained $|l(G - s; \tilde{s}, t)| \geq 3$. Then, find three $\tilde{s}$–t paths whose labels are distinct by FindThreePaths($G - s, \tilde{s}, t$). Extend the three $\tilde{s}$–t paths using an arc $s\tilde{s} \in E$, and return the extended s–t paths.

4.2 Proof of Theorem 11

To prove Theorem 11, we first show the detailed description of Step 5 in TestTwoLabels.

Lemma 18. *Let $(G, s, t) \in \mathcal{D}$. Suppose that $G = (V, E)$ is 3-connected and contains no 3-contractible vertex set, that s and t are adjacent, and that $\{\alpha, \beta\} \subseteq l(G; s, t)$ for some distinct $\alpha, \beta \in \Gamma$. Then, one can test whether $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ or not in polynomial time.*

Proof. Since s and t are adjacent and G is 3-connected, G contains no 2-contractible vertex set. This implies that G is well-contracted since G contains no 3-contractible vertex set. Hence, it suffices to check the second condition in Definition 15.

First, test the planarity of G. If G is not planar, then we can conclude $(G, s, t) \notin \mathcal{D}^0_{\alpha,\beta}$. Otherwise, compute an embedding of G on a plane in which both s and t are on the outer boundary (because of an arc $st \in E$, there exists a face on whose boundary both s and t are).

Since G is 3-connected, the face set is unique if there are no parallel arcs (see, e.g., [4, Chapter 4]). Although there may be parallel arcs in G, we can say that the number of parallel arcs is bounded as seen below. It should be noted that the planar embeddings can be compted in polynomial time (e.g., by [7]).

Claim. We may assume that there is no parallel arcs from s to t.

Suppose that there exist parallel arcs from s to t, which may be assumed to have distinct labels. Moreover, we may assume that there are exactly two such arcs $e_\alpha, e_\beta \in E$ with labels α, β, respectively, since otherwise, we have $|l(G; s, t)| \geq 3$ and hence $(G, s, t) \notin \mathcal{D}^0_{\alpha,\beta}$. By the 3-connectivity of G, there exists an s–t path in $G - \{e_\alpha, e_\beta\}$, and let γ be its label. If $\alpha \neq \gamma \neq \beta$, then $|l(G; s, t)| \geq 3$. Otherwise, remove e_γ from G. Note that this removal does not violate the hypotheses of this lemma, and does not make an effect on whether $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ or not.

Claim. We may assume that there exists at most one pair of parallel arcs.

Suppose that there exist parallel arcs from x to y with distinct labels, where $\{x, y\} \neq \{s, t\}$. Then, by the 3-connectivity of G, the parallel arcs form an inner face whose boundary is unbalanced. Hence, there is a unique pair of such parallel arcs if $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$, since the existence of at least two pairs of parallel arcs immediately implies $(G, s, t) \notin \mathcal{D}^0_{\alpha,\beta}$.

Recall that we have to test whether there exists an embedding of G such that the outer boundary is unbalanced and there exists a unique inner face whose boundary is unbalanced. Since a pair of parallel arcs is unique if exists, there are at most two possible face sets of G. Furthermore, since there exists exactly one arc from s to t, both of the two faces whose boundaries share the arc $st \in E$ can be the outer face, i.e., there are two choices of the outer face. It can be done in polynomial time to check, in each of the at most four $(= 2 \times 2)$ cases, whether exactly one inner face has an unbalanced boundary or not, and hence one can do the whole procedure in polynomial time. □

We are ready to prove Theorem 11.

Proof of Theorem 11. Recall that our goal is to test whether $|l(G; s, t)| \leq 2$ or not, and to find $\min\{3, |l(G; s, t)|\}$ s–t paths whose labels are distinct. These are achieved as follows. We first test whether $|l(G; s, t)| \leq 2$ or not by TestTwoLabels(G, s, t) for the input triplet (G, s, t) (which may not be in $\mathcal{D}$). If we obtain $|l(G; s, t)| \leq 2$, then we also obtain at most two s–t paths in G which attain all labels in $l(G; s, t)$. Otherwise, we can obtain three s–t paths whose labels are distinct by FindThreePaths(G, s, t). Hence, it suffices to show the correctness and polynomiality of these two algorithms.

The correctness of these two algorithms is almost obvious. It should be noted that we have $l(G'; s, t) = l(G; s, t)$ in any step of TestTwoLabels(G, s, t). This follows from the fact that all of the following operations do not change $l(G'; s, t)$: the removal of X in Step 0, the reductions in Step 2, and the 3-contractions in Step 3.

Let $T_{\text{labels}}(n)$ and $T_{\text{paths}}(n)$ denote the computational time of TestTwoLabels(G, s, t) and FindThreePaths(G, s, t), respectively, where n is the number of vertices in G. TestTwoLabels runs in polynomial time, i.e., $T_{\text{labels}}(n)$ is polynomially bounded, since in the recursion step (Step 2) we just divide the graph G' into two smaller graphs which have $|V'| - |X|$ and $|X| + 2$ vertices. By a reccurence relation

$$T_{\text{paths}}(n) \leq n \cdot T_{\text{labels}}(n-1) + T_{\text{paths}}(n-1),$$

we have $T_{\text{paths}}(n) \leq n^2 \cdot T_{\text{labels}}(n) + \text{const.}$, which is also polynomially bounded. □

5 Proof of Necessity of Theorem 14

In this section, we give a proof of the necessity of Theorem 14, and begin with its outline.

5.1 Proof sketch

To derive a contradiction, assume that there exist distinct $\alpha, \beta \in \Gamma$ and a triplet $(G, s, t) \in \mathcal{D}$ such that $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, $l(G; s, t) = \{\alpha, \beta\}$, and $(G, s, t) \notin \mathcal{D}_{\alpha,\beta}$. We choose such $\alpha, \beta \in \Gamma$ and $(G, s, t) \in \mathcal{D}$ so that G is as small as possible.

Fix an arbitrary arc e_0 in G leaving s, and consider the graph $G' := G - e_0$. By using the minimality of G, we can show that $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$ (see Claims 24 and 25). Furthermore, we can see that a triplet $(\tilde{G}, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ is obtained from $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$ by applying a contraction at most once, which removes the head of e_0 (see Claims 26–28).

By the definition of $\mathcal{D}^0_{\alpha,\beta}$, $\tilde{G}$ can be embedded on a plane satisfying the conditions in Definition 15. By using the fact that G is obtained from $\tilde{G}$ by expanding a vertex set and adding e_0, we try to extend the planar embedding of $\tilde{G}$ to G. Then, we have one of the following cases.

- Such an extension is possible, i.e., G can be embedded on a plane satisfying the conditions in Definition 15. This contradicts that $(G, s, t) \notin \mathcal{D}_{\alpha,\beta}$.
- $\tilde{G}$ contains a contractible vertex set, which contradicts that $(\tilde{G}, s, t) \in \mathcal{D}^0_{\alpha,\beta}$.
- G contains a contractible vertex set or a 2-cut (i.e., $G - \{x, y\}$ is not connected for some distinct $x, y \in V$), which contradicts that G is a minimal counterexample (cf. Claim 23).
- We can construct an s–t path of label $\gamma \in \Gamma \setminus \{\alpha, \beta\}$ in G by using e_0 and some arcs in G', which contradicts that $l(G; s, t) = \{\alpha, \beta\}$.

In each case, we have a contradiction, which completes the proof of the necessity of Theorem 14. We note that Theorem 1 plays an important role in this case analysis.

5.2 Useful lemmas

Before starting the proof, we show several lemmas which are utilized in it.

Lemma 19. *For any $(G = (V, E), s, t) \in \mathcal{D}_{\alpha,\beta}$, we have the following properties.*

(1) *Let G' be the graph obtained from G by shifting by $\gamma \in \Gamma$ at s. Then, $(G', s, t) \in \mathcal{D}_{\alpha',\beta'}$, where $\alpha' := \alpha\gamma^{-1}$ and $\beta' := \beta\gamma^{-1}$.*

(2) *Let $G' := (V + s', E + e')$ be the graph obtained from G by adding a vertex $s' \notin V$ and an arc $e' = s's$ with label $\gamma \in \Gamma$. Then, $(G', s', t) \in \mathcal{D}_{\alpha',\beta'}$, where $\alpha' := \alpha\gamma$ and $\beta' := \beta\gamma$.*

(3) *For an arc $e \in E$ and a triplet $(H, x', y') \in \mathcal{D}$ with $l(H; x', y') = \psi_G(e)$, let G' be the graph obtained from G by replacing e with (H, x', y'). Then, $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$.*

(4) *For parallel arcs $e_1, e_2 \in E$ whose labels are $\alpha', \beta' \in \Gamma$ with $\alpha'\beta'^{-1} \neq \beta'\alpha'^{-1}$, let G' be the graph obtained from G by replacing e_1, e_2 with some $(H, x', y') \in \mathcal{D}_{\alpha',\beta'}$. Then, $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$.*

Proof. (1) Since $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$, there exists a Γ-labeled graph $\tilde{G}$ obtained from G by a contraction sequence such that $(\tilde{G}, s, t) \in \mathcal{D}^1_{\alpha,\beta}$. Note that shifting at any vertex does not make an effect on whether an arbitrarily fixed subgraph is balanced or not, and hence the same contraction sequence can be applied to G'. For the resulting graph $\tilde{G}'$, we show $(\tilde{G}', s, t) \in \mathcal{D}^1_{\alpha',\beta'}$.

Since $(\tilde{G}, s, t) \in \mathcal{D}^1_{\alpha,\beta}$, there exists a sequence of Γ-labeled graphs $G_1, \ldots, G_r = \tilde{G}$ such that $(G_1, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ and $G_i = G_{i+1}/(H_i, x_i, y_i)$ for each $i = 1, \ldots, r-1$, where $(H_i, x_i, y_i) \in \mathcal{D}^0_{\alpha_i,\beta_i}$ and $\alpha_i \neq \beta_i$. Let H'_i be H_i itself if $s \notin \{x_i, y_i\}$, and the Γ-labeled graph obtained from H_i by shifting by γ at x_i if $x_i = s$ (without loss of generality by the symmetry). We then obtain a sequence of Γ-labeled graphs $G'_1, \ldots, G'_r = \tilde{G}'$ such that $(G'_1, s, t) \in \mathcal{D}^0_{\alpha',\beta'}$ and $G'_i = G'_{i+1}/(H'_i, x_i, y_i)$ for each $i = 1, \ldots, r-1$, where $(H'_i, x_i, y_i) \in \mathcal{D}^0_{\alpha'_i,\beta'_i}$, $\alpha'_i = \alpha_i \neq \beta_i = \beta'_i$ if $s \notin \{x_i, y_i\}$, and $\alpha'_i = \alpha_i\gamma^{-1} \neq \beta_i\gamma^{-1} = \beta'_i$ if $x_i = s$. Hence, $(\tilde{G}', s, t) \in \mathcal{D}^1_{\alpha',\beta'}$.

(2) For $X := \{s', s\} \subsetneq V(G')$ with $s' \in X$ and $t \in V(G') \setminus X$, the Γ-labeled graph $G'/_2X$ is obtained from G by shifting by γ^{-1} at s and by identifying $s \in V$ and $s' \in V(G'/_2X)$. This implies $(G'/_2X, s', t) \in \mathcal{D}_{\alpha',\beta'}$ since $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$, and hence we have $(G', s', t) \in \mathcal{D}_{\alpha',\beta'}$.

(3) Suppose that e is from $x \in V$ to $y \in V$, and let $X := (V(H) \setminus \{x', y'\}) + x \subsetneq V(G')$. Then, X can be merged into a single vertex x by a 2-contraction sequence, since $G'[X]$ is balanced (note that H is balanced by Lemma 5) and $\{x, y\}$ separates X and $\{s, t\}$ in G'. The resulting graph coincides with G, which implies $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$.

(4) Suppose that e_1 and e_2 are from $x \in V$ to $y \in V$. Since $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ and $(H, x', y') \in \mathcal{D}_{\alpha',\beta'}$, there exist Γ-labeled graphs $\tilde{G}$ and $\tilde{H}$ obtained from G and H, respectively, by contraction sequences such that $(\tilde{G}, s, t) \in \mathcal{D}^1_{\alpha,\beta}$ and $(\tilde{H}, x', y') \in \mathcal{D}^1_{\alpha',\beta'}$.

By the construction of G', $\{x, y\}$ separates $V(H) \setminus \{x', y'\}$ and V in G'. This implies that the 2-contractions in G and H to construct $\tilde{G}$ and $\tilde{H}$, respectively, can be applied independently also in G', since each 2-contracted vertex set does not contain both x and y. Furthermore, any 3-contraction in G cannot remove x nor y (since, for a vertex set $X \subseteq V \setminus \{s, t\}$ with $X \cap \{x, y\} \neq \emptyset$, G_X cannot be balanced because of the parallel arcs e_1 and e_2), and any 3-contraction in H can remove neither x' nor y'. Hence, also the 3-contractions in G and H to construct $\tilde{G}$ and $\tilde{H}$, respectively, can be applied independently also in G'.

Let $\tilde{G}'$ be the Γ-labeled graph obtained from $\tilde{G}$ by replacing e_1 and e_2 with $(\tilde{H}, x', y')$. By the last paragraph, we can obtain $\tilde{G}'$ also from G' by a contraction sequence, and hence it suffices to show $(\tilde{G}', s, t) \in \mathcal{D}^1_{\alpha,\beta}$. Recall that $(\tilde{G}, s, t) \in \mathcal{D}^1_{\alpha,\beta}$ and $(\tilde{H}, x', y') \in \mathcal{D}^1_{\alpha',\beta'}$. The latter implies that, by a sequence of reductions of triplets $(H_i, x_i, y_i) \in \mathcal{D}^0_{\alpha_i,\beta_i}$, we can obtain from $\tilde{H}$ the Γ-labeled graph H_0 consisting of two parallel arcs from x' to y' with labels α', β' Therefore, by identifying $x, y \in V(\tilde{G}')$ with $x', y' \in V(\tilde{H})$, respectively, and applying the same sequence of reductions to $\tilde{G}'$, we can confirm $(\tilde{G}', s, t) \in \mathcal{D}^1_{\alpha,\beta}$ since $(\tilde{G}, s, t) \in \mathcal{D}^1_{\alpha,\beta}$. □

By Lemma 19-(1), it suffices to consider the case when $\beta = 1_\Gamma$ and $\alpha^{-1} \neq \alpha$ (i.e., $\alpha^2 \neq 1_\Gamma$). The following lemma gives a useful characterization of $\mathcal{D}^0_{1_\Gamma,\alpha}$.

Lemma 20. *For any $(G = (V, E), s, t) \in \mathcal{D}^0_{1_\Gamma,\alpha}$, there exists a Γ-labeled graph G' which is (s, t)-equivalent to G and embeddable with the following conditions.*

1. *The arc set E is partitioned into $\{E^0, E^1\}$, where $E^i := \{\, e \in E \mid \psi_{G'}(e) = \alpha^i \,\}$ $(i = 0, 1)$.*
2. *There exists an s–t path $P = (s = u_0, e_1, u_1, \ldots, e_l, u_l = t)$ along the outer boundary of $G' - E^1$ such that*
   - *every arc in E^1 is embedded on the outer face of $G' - E^1$ and is from $u_i \in V(P)$ to $u_j \in V(P)$ for some $i < j$, and*
   - *for any distinct arcs $e_1 = u_{i_1}u_{j_1}$, $e_2 = u_{i_2}u_{j_2} \in E^1$, one of two paths $P[u_{i_1}, u_{j_1}]$ and $P[u_{i_2}, u_{j_2}]$ is a subpath of the other.*

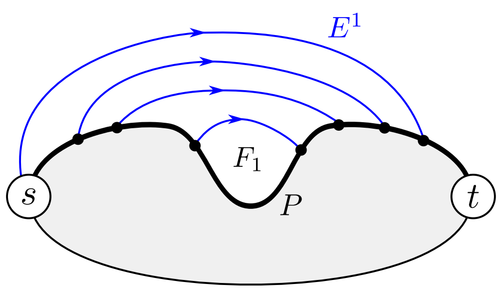

Figure 4: Lemma 20 cliams that $(G', s, t) \in \mathcal{D}^0_{1_\Gamma,\alpha}$ can be embedded as above.

Proof. Fix an embedding of G with the conditions of $\mathcal{D}^0_{1_\Gamma,\alpha}$, and let P_0 and P_1 be the s–t paths along the boundary of the outer face F_0 of G whose labels are 1_Γ and α, respectively.

Let G^* be the dual graph of G (as an undirected graph), i.e., the vertex set of G^* is the face set $\mathcal{F}$ of G, the edge set of G^* coincides with the arc set of G, and each two faces whose boundaries share an arc $e \in E$ in G are connected by the same-named edge e in G^*. Take a shortest F_1–F_0 path Q in $G^* - E(P_0)$. We prove that the second property holds with $E^1 = E(Q)$.

Note that $G' := G - E(Q)$ is connected since Q is a shortest path without the corresponding edge to any arc in $E(P_0)$, and that G' is balanced since F_1 is the unique unbalanced inner face. We then have $l(G'; s, t) = 1_\Gamma$ by Lemma 5. Hence, by Lemma 4, we may assume that $\psi_G(e) = 1_\Gamma$ for any arc $e \in E(G')$ by shifting at some vertices $v \in V \setminus \{s, t\}$. Thus we obtain G with the second property, since $\psi_G(\mathrm{bd}(F)) = 1_\Gamma$ for any $F \in \mathcal{F} \setminus \{F_0, F_1\}$. □

The following two lemmas show properties of triplets $(G, s, t) \in \mathcal{D}$ satisfying $l(G; s, t) = \{\alpha, \beta\}$ for distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$.

Lemma 21. *If a triplet $(G, s, t) \in \mathcal{D}$ satisfies $l(G; s, t) = \{\alpha, \beta\}$ for distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, then G contains no unbalanced cycle C with $\psi_G(\bar{C}) = \psi_G(C)$.*

Proof. We first note that the equality $\psi_G(\bar{C}) = \psi_G(C)$ does not depend on the choices of the direction and the end vertex of the cycle C. Suppose to the contrary that G contains such a cycle C. In the same way as the proof of Lemma 5, take an s–x path P and a y–t path Q in G so that $V(P) \cap V(C) = \{x\}$, $V(Q) \cap V(C) = \{y\}$, $V(P) \cap V(Q) = \emptyset$, and $x, y \in V(C)$ are distinct, and choose y as the end vertex of C.

Let $\alpha' := \psi_G(C[x, y])$ and $\beta' := \psi_G(\bar{C}[x, y])$. We then have $\alpha'\beta'^{-1} = \psi_G(C) = \psi_G(\bar{C}) = \beta'\alpha'^{-1}$. By extending $C[x, y]$ and $\bar{C}[x, y]$ using P and Q, we obtain two s–t paths in G whose labels are $\alpha'' := \psi_G(Q) \cdot \alpha' \cdot \psi_G(P)$ and $\beta'' := \psi_G(Q) \cdot \beta' \cdot \psi_G(P)$, which are distinct. Since $\alpha'\beta'^{-1} = \beta'\alpha'^{-1}$ implies $\alpha''\beta''^{-1} = \beta''\alpha''^{-1}$, we have $\{\alpha'', \beta''\} \not\subseteq \{\alpha, \beta\} = l(G; s, t)$, a contradiction. □

Lemma 22. *Suppose that a triplet $(G, s, t) \in \mathcal{D}$ satisfies $l(G; s, t) = \{\alpha, \beta\}$ for distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. Then, there is no family of two distinct unbalanced cycles C_1, C_2 and three disjoint paths P_1, P_2, P_3 in G (possibly of length 0) such that*

- *$C_1 \cap C_2$ is either empty (Fig. 5) or a path (Fig. 6),*
- *P_1 is from s to $x_1 \in V(C_1) \setminus V(C_2)$, P_3 is from $y_2 \in V(C_2) \setminus V(C_1)$ to t, P_2 is from $y_1 \in V(C_1) - x_1$ to $x_2 \in V(C_2) - y_2$, where possibly $s = x_1$, $y_1 = x_2$, or $y_2 = t$,*
- *$V(P_1) \cap V(C_1) = \{x_1\}$, $V(P_1) \cap V(C_2) = \emptyset$, $V(P_2) \cap V(C_1) = \{y_1\}$, $V(P_2) \cap V(C_2) = \{x_2\}$, $V(P_3) \cap V(C_1) = \emptyset$, and $V(P_3) \cap V(C_2) = \{y_2\}$.*

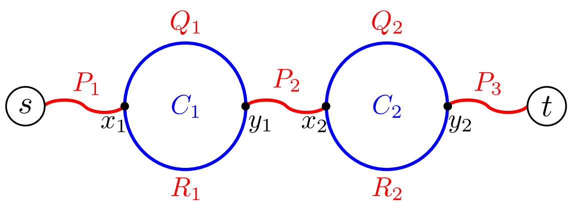

Figure 5: When $C_1 \cap C_2$ is empty.

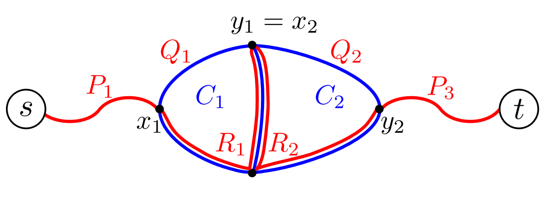

Figure 6: When $C_1 \cap C_2$ is a path.

Proof. Suppose to the contrary that there exist such a family of distinct unbalanced cycles C_1, C_2 and disjoint paths P_1, P_2, P_3 in G. When $C_1 \cap C_2$ is a path, choose one of its end vertices as $x_2 = y_1$. For each $i = 1, 2$, choose x_i as the end vertex of C_i, and let $Q_i := C_i[x_i, y_i]$, $R_i := \bar{C}_i[x_i, y_i]$, $\alpha_i := \psi_G(Q_i)$, and $\beta_i := \psi_G(R_i)$, where the direction of C_i is fixed arbitrarily. Note that, by Claim 21, we have ${\beta_i}^{-1}\alpha_i = \psi_G(C_i) \neq \psi_G(\bar{C}_i) = {\alpha_i}^{-1}\beta_i$ for each $i = 1, 2$.

By concatenating P_1, either Q_1 or R_1, P_2, either Q_2 or R_2, and P_3 (ignoring the path $C_1 \cap C_2$ if it exists and is traversed in both directions), we construct four s–t paths whose labels are

$$\begin{aligned}\gamma_1 &:= \psi_G(P_3) \cdot \alpha_2 \cdot \psi_G(P_2) \cdot \alpha_1 \cdot \psi_G(P_1),\\ \gamma_2 &:= \psi_G(P_3) \cdot \alpha_2 \cdot \psi_G(P_2) \cdot \beta_1 \cdot \psi_G(P_1),\\ \gamma_3 &:= \psi_G(P_3) \cdot \beta_2 \cdot \psi_G(P_2) \cdot \alpha_1 \cdot \psi_G(P_1),\\ \gamma_4 &:= \psi_G(P_3) \cdot \beta_2 \cdot \psi_G(P_2) \cdot \beta_1 \cdot \psi_G(P_1).\end{aligned}$$

Since $l(G; s, t) = \{\alpha, \beta\}$ and $\gamma_1 \neq \gamma_2 \neq \gamma_4 \neq \gamma_3 \neq \gamma_1$, we have $\gamma_1 = \gamma_4$ and $\gamma_2 = \gamma_3$. This implies $\alpha_2 \cdot \psi_G(P_2) \cdot \alpha_1 = \beta_2 \cdot \psi_G(P_2) \cdot \beta_1$ and $\alpha_2 \cdot \psi_G(P_2) \cdot \beta_1 = \beta_2 \cdot \psi_G(P_2) \cdot \alpha_1$. The former leads to $\alpha_1 = \psi_G(P_2)^{-1} \cdot \alpha_2^{-1}\beta_2 \cdot \psi_G(P_2) \cdot \beta_1$, and by substituting this into the latter, we obtain $\alpha_2 \cdot \psi_G(P_2) \cdot \beta_1 = \beta_2\alpha_2^{-1}\beta_2 \cdot \psi_G(P_2) \cdot \beta_1$, which implies $\beta_2^{-1}\alpha_2 = \alpha_2^{-1}\beta_2$, a contradiction. □

5.3 Minimal counterexample

Here we start a proof of "only if" part of Theorem 14. To derive a contradiction, suppose to the contrary that there exist distinct $\alpha, \beta \in \Gamma$ and a triplet $(G, s, t) \in \mathcal{D}$ such that $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, $l(G; s, t) = \{\alpha, \beta\}$, and $(G, s, t) \notin \mathcal{D}_{\alpha,\beta}$. We choose such $\alpha, \beta \in \Gamma$ and $(G = (V, E), s, t) \in \mathcal{D}$ so that the value of $|V| + |E|$ is minimized. Note that we have $|V| \geq 3$ obviously, and we may assume $\beta = 1_\Gamma$ and $\alpha^{-1} \neq \alpha$ (i.e., $\alpha^2 \neq 1_\Gamma$) by Lemma 19-(1). By the minimality, G is well-contracted, and moreover we have the following property.

Claim 23. *There is no 2-cut in G.*

Proof. Suppose to the contrary that there exists a 2-cut $\{x, y\} \subsetneq V$ such that $G[X]$ is one of the connected components of $G - \{x, y\}$ for $X \subseteq V \setminus \{x, y\}$. Since $(G, s, t) \in \mathcal{D}$, we also have $(G[X \cup \{x, y\}], x, y) \in \mathcal{D}$. If $|l(G[X \cup \{x, y\}]; x, y)| \geq 3$, then we also have $|l(G; s, t)| \geq 3$, a contradiction. In the case that $l(G[X \cup \{x, y\}]; x, y) = \{\alpha', \beta'\}$ for distinct $\alpha', \beta' \in \Gamma$ with $\alpha'\beta'^{-1} = \beta'\alpha'^{-1}$, there exists an unbalanced cycle C in $G[X \cup \{x, y\}]$ such that $\psi_G(C)^{-1} = \psi_G(C)$ by Proposition 13, which contradicts Claim 21.

Otherwise, i.e., if $|l(G[X \cup \{x, y\}; x, y)| = 1$ or $l(G[X \cup \{x, y\}]; x, y) = \{\alpha', \beta'\}$ for distinct $\alpha', \beta' \in \Gamma$ with $\alpha'\beta'^{-1} \neq \beta'\alpha'^{-1}$, we can construct a smaller counterexample by the reduction of $(G[X \cup \{x, y\}], x, y)$ (Lemma 19-(3), (4)), a contradiction. Note that, since G is a minimal counterexample, $(G[X \cup \{x, y\}], x, y) \in \mathcal{D}_{\alpha',\beta'}$ if $l(G[X \cup \{x, y\}]; x, y) = \{\alpha', \beta'\}$. □

Fix an arbitrary arc $e_0 = sv_0 \in \delta_G(s)$ leaving s, and let $G' := G - e_0$. We next show the following claims, which lead to $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$.

Claim 24. $(G', s, t) \in \mathcal{D}$.

Proof. Suppose to the contrary that there exists a vertex which is not contained in any s–t path in G'. If G' is not connected, then we have $l(G - e_0; v_0, t) = \{\alpha', \beta'\}$ and $(G - e_0, v_0, t) \notin \mathcal{D}_{\alpha',\beta'}$, where $\alpha' = \alpha \cdot \psi_G(e_0)^{-1}$ and $\beta' = \beta \cdot \psi_G(e_0)^{-1}$ (since otherwise $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ by Lemma 19-(2)). This contradicts the minimality of G.

Otherwise (i.e., if G' is connected and $(G', s, t) \notin \mathcal{D}$), there exists a 1-cut $x \in V - s$ in $G' + st$ (cf. the proof of Lemma 2). This implies that $\{s, x\} \subsetneq V$ is a 2-cut in G as well as in $G + st$, which contradicts Claim 23. □

Claim 25. $l(G'; s, t) = \{\alpha, \beta\}$.

Proof. Since each s–t path in G' is also in G, $l(G'; s, t) \subseteq l(G; s, t) = \{\alpha, \beta\}$. Suppose to the contrary that $|l(G'; s, t)| = 1$. Then, G' is balanced by Lemma 5. Hence, for the vertex set $X := V - s$ with $t \in X$ and $s \in V \setminus X$, we have that $|X| \geq 2$, $G[X]$ ($= G'[X]$) is balanced, and $\{s, t\}$ separates X and $\{s, t\}$. This implies that we can obtain, from G by a 2-contraction sequence, the Γ-labeled graph G_0 consisting of two parallel arcs from s to t whose labels are α and β. Since $(G_0, s, t) \in \mathcal{D}^0_{\alpha,\beta} \subseteq \mathcal{D}^1_{\alpha,\beta}$ is obvious, we have $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$, a contradiction. □

By Claims 24 and 25 and the minimality of G, we have $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$. Hence, by Definitions 16 and 17, there exists a sequence of Γ-labeled graphs $G_0, G_1, \ldots, G_r$ such that

- G_0 consists of two vertices s, t and two parallel arcs from s to t whose labels are α and β,
- for $i = 1, 2, \ldots, r$, we have $G_{i-1} = G_i/(H_i, x_i, y_i)$ for some triplet $(H_i, x_i, y_i) \in \mathcal{D}^0_{\alpha_i,\beta_i}$ with $|V(H_i)| \geq 3$ and $\alpha_i\beta_i^{-1} \neq \beta_i\alpha_i^{-1}$, and
- G_r is obtained from G' by a contraction sequence.

By the minimality of G, we can show the following claims.

Claim 26. *Let $i \in \{2, 3\}$. For any i-contractible vertex set X in $G' = G - e_0$, the head v_0 of e_0 does not remain in the resulting graph $G'/_i X$.*

Proof. Fix an arbitrary contractible vertex set $X \subsetneq V$. If $X \subseteq V \setminus \{s, t\}$ is 3-contractible in G', then we have $v_0 \in X$, since otherwise X is 3-contractible also in G, a contradiction. Hence, $v_0 \notin V(G'/_3 X)$.

Suppose that $X \subsetneq V$ is 2-contractible in G', and to the contrary that $v_0 \in V(G'/_2 X)$. Let $x \in X$ and $y \in V \setminus X$ be the vertices satisfying the conditions in Definition 8, i.e., $\{x, y\}$ separates X and $\{s, t\}$ in G', and $G'[X]$ is a balanced 2-connected component of $G' - y$. Let Y be the vertex set of the connected component of $G' - \{x, y\}$ that contains $X - x$.

If $v_0 \in V \setminus Y$, then X is 2-contractible also in G, a contradiction. Since $v_0 \in V(G'/_2 X) = (V \setminus X) + x$, we have $v_0 \in Y \setminus X =: Z$. Suppose that $G'[Z]$ is not connected. Then, at least one connected component of $G'[Z]$ does not contain v_0, and hence G contains a 2-cut separating such a component and $\{x, y\}$, which contradicts Claim 23. Thus, $G'[Z]$ is connected, and $G'[Y \cup \{x, y\}]$ contains a 2-cut $\{y, z\}$ separating X and Z, where $z \in X - x \subseteq V \setminus \{s, t\}$. This implies $y \neq s$. Since $N_G(Z) = \{s, y, z\}$ and G contains no 3-contractible vertex set, $G_Z := G[Z \cup N_G(Z)] - E(N_G(Z))$ is not balanced, and neither is $G_Z - s$ in particular. Note that $G_Z - \{s, y\}$ is 2-connected, since otherwise some vertex $w \in Z$ separates v_0 and some vertex in $Z - w$ in $G_Z - \{s, y\}$, which implies that $\{w, y\}$ is a 2-cut in G, a contradiction.

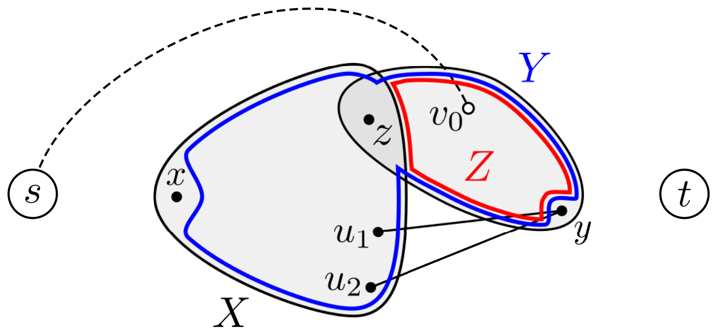

Figure 7: The situation currently considered (possibly $x = s$, $x = t$, or $y = t$).

Recall that $G[X] = G'[X]$ is balanced, and we may assume that the label of every arc in $E(X)$ is 1_Γ by shifting at vertices in $X - x \subseteq V \setminus \{s, t\}$ in advance if necessary. Let $X' := X \setminus \{x, z\}$. Since $N_G(X') = \{x, y, z\}$ and G contains no 3-contractible vertex set, $G_{X'} := G[X' \cup N_G(X')] - E(N_G(X'))$ is not balanced. Hence, at least two arcs $e_1 = u_1 y$ and $e_2 = u_2 y$ in $\delta_G(y)$ with $u_1, u_2 \in X'$ have different labels.

Here we consider the following three cases separately, and derive a contradiction using Claim 22: (a) $x = s$ and $y = t$, (b) $x \neq s$ and $y = t$, and (c) $x \neq s$ and $y \neq t$ (possibly $x = t$). Note that, if $x = s$ and $y \neq t$, then $\{s, y\}$ is a 2-cut also in $G = G' + e_0$, a contradiction.

(a) Suppose that $x = s$ and $y = t$. By Claim 23, $G - t$ is 2-connected. Since G contains no 2-contractible vertex set, $G - t$ contains an unbalanced cycle C_1. Besides, by extending a u_1–u_2

path in $G[X]-s$ using e_1 and e_2, we can construct an unbalanced cycle C_2 in $G[X+y]-s$. Note that $G[X]-s = G'[X]-s$ is conneceted since $G'[X]$ is 2-connected.

If C_1 is contained in $G_Z - \{s,t\}$ (see Fig. 8), then we can take an s–x_1 path P_1 and an y_1–z path P_2' in $G_Z - t$ such that $V(P_1) \cap V(C_1) = \{x_1\}$, $V(P_2') \cap V(C_1) = \{y_1\}$, $V(P_1) \cap V(P_2') = \emptyset$, and $x_1, y_1 \in V(C_1)$ are distinct, since $G_Z - \{s,t\}$ is 2-connected. Besides, we can take an z–x_2 path P_2'' in $G[X]-s$ such that $V(P_2'') \cap V(C_2) = \{x_2\} \subsetneq V(C_2) - t$. Let P_2 be the path obtained by concatenating P_2' and P_2'', and $P_3 := (t)$ a path of length 0. Then, these paths P_i $(i = 1,2,3)$ and cycles C_j $(j = 1,2)$ compose a counterexample for Claim 22, a contradiction.

Otherwise, C_1 traverses $e_0 = sv_0$ and intersects z (see Fig. 9). If C_1 and C_2 are disjoint, then, to derive a contradiction, it suffices just to take a y_1–x_2 path P_2 in $G[X]-s$ such that $y_1 \in V(C_1) - s$ and $x_2 \in V(C_2) - t$. Otherwise, we can choose C_1 and C_2 so that $C_1 \cap C_2$ forms a path, say Q, since the label of every arc in $E(X)$ is 1_Γ and $E(C_2) \setminus \{e_1, e_2\} \subseteq E(X)$. In this case, to derive a contradiction, it suffices just to choose one of the end vertices of Q as $x_1 = y_2$.

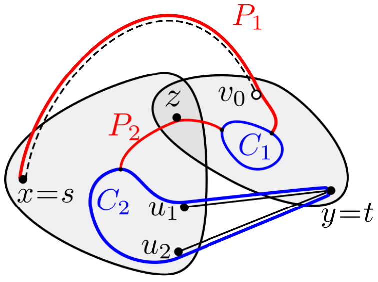

Figure 8: When C_1 is in $G_Z - \{s,t\}$.

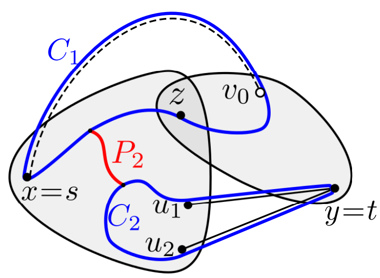

Figure 9: When C_1 traverses $e_0 = sv_0$.

(b) Suppose that $x \neq s$ and $y = t$. Then, since G contains no 2-contractible vertex set, there exists an unbalanced cycle C_1' in the 2-connected component of $G-t$ that contains s and v_0. By taking two disjoint paths between $\{s, v_0\}$ and $\{x_1, y_1\}$ for distinct $x_1, y_1 \in V(C_1')$ and extending them using two x_1–y_1 paths $C_1'[x_1, y_1]$ and $\bar{C}_1'[x_1, y_1]$ and the arc $e_0 = sv_0$ (see Fig. 10), we construct an unbalanced cycle C_1 traversing e_0 (see Fig. 11). Also in this case, we can derive a contradiction in the same way as (a), where it suffices just to take a path P_2 (possibly of length 0) connecting C_1 and C_2.

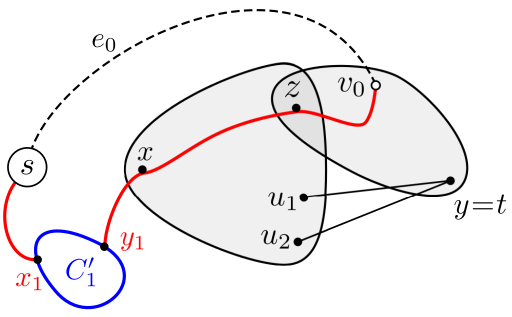

Figure 10: Construction of C_1 from C_1'.

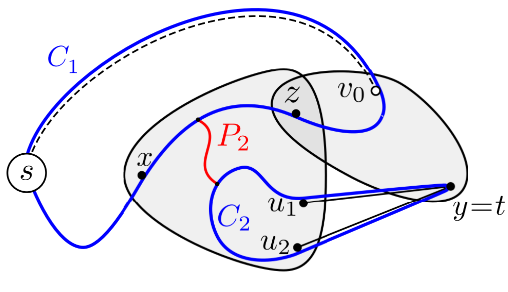

Figure 11: When $x \neq s$ and $y = t$.

(c) Suppose that $x \neq s$ and $y \neq t$ (possibly $x = t$). Let C_2 be an arbitrary unbalanced cycle in $G_{X'}$, which traverses e_1 and e_2. If $G_Z - \{s,y\}$ contains an unbalanced cycle C_1 (see Fig. 12), then, to derive a contradiction, take an s–x_1 path P_1 and a y_1–z path P_2' in $G_Z - y$, a z–x_2 path P_2'' in $G[X]-x$, and a y–t path P_3 in $G-s$ such that $V(P_1) \cap V(C_1) = \{x_1\}$, $V(P_2') \cap V(C_1) = \{y_1\}$, $V(P_1) \cap V(P_2') = \emptyset$, $V(P_2'') \cap V(C_2) = \{x_2\} \subseteq V(C_2) - y$, and $x_1, y_1 \in V(C_1)$ are distinct. Otherwise, there exists an unbalanced cycle C_1 intersecting y in $G_Z - s$ (see Fig. 13). In this case, it suffices just to take an s–x_1 path P_1 in G_Z and a y_2–t path P_3 in $G - \{s,z\}$ such that $x_1 \in V(C_1) - y \subseteq Z$ and $y_2 \in V(C_2) - y \subseteq X'$. □

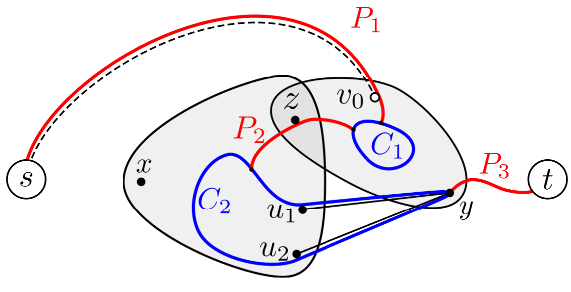

Figure 12: When $y \notin V(C_1)$.

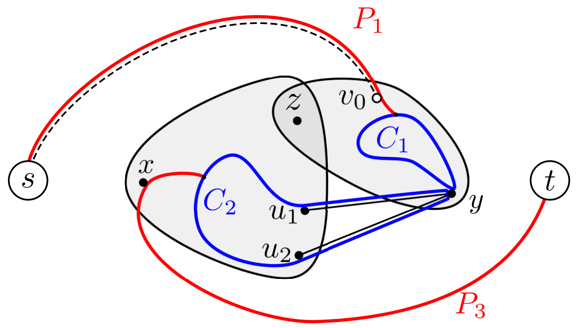

Figure 13: When $y \in V(C_1)$.

Claim 27. *There exists a contraction sequence to construct G_r from G' which consists of at most one contraction.*

Proof. To prove the claim by a contradiction, suppose that any contraction sequence to construct from G' to G_r consists of at least two contractions. Choose one of the shortest such contraction sequences, and let $X, Y \subseteq V$ be the first two contracted vertex sets, i.e., G_r is obtained from $(G' /_i X) /_j Y$ by a contraction sequence (which may be empty) for some $i, j \in \{2, 3\}$. Let $\hat{G} := G' /_i X$. Note that $v_0 \in X$ and $v_0 \notin V(\hat{G})$ by Claim 26

- Suppose that $i = j = 3$. If $Y \cap N_{G'}(X) = \emptyset$, then Y is 3-contractible also in G, which contradicts the minimality of G. Otherwise, $X \cup Y$ is 3-contractible in G', and in particular $G' /_3 (X \cup Y) = \hat{G} /_3 Y$. Hence, there exists a shorter contraction sequence to construct G_r from G', a contradiction.

- Suppose that $i = 3$ and $j = 2$. If $|Y \cap N_{G'}(X)| \neq 2$, then Y is 2-contractible also in G', which contradicts Claim 26. Otherwise, $N_{G'}(X) = \{x_1, x_2, y\}$ for distinct $x_1, x_2 \in Y - x$, where $x \in Y$ and $y \in V(\hat{G}) \setminus Y$ be the vertices satisfying the conditions in Definition 8 (i.e., $\hat{G}[Y]$ is a balanced 2-connected component of $\hat{G} - y$ containing x). Let Y' be the 2-contractible vertex set in G' with $x \in Y'$ and $y \in V \setminus Y'$ such that $Y \subsetneq Y' \subseteq X \cup Y$. If $Y' = X \cup Y$, then we have $G' /_2 Y' = \hat{G} /_2 Y$, which leads to a shorter contraction sequence, a contradiction. Otherwise, by Claim 26, we have $v_0 \in Y'$. In this case, since $G' - y = G'[X \cup Y]$ is not 2-connected, there exists a vertex $z \in X$ separating Y' and some vertex $z' \in X - z$ in $G' - y$. Then, G contains a 2-cut $\{y, z\}$ separating v_0 and z', which contradicts Claim 23.

- Suppose that $i = 2$ and $j = 3$. Let $x \in X$ and $y \in V \setminus X$ be the vertices satisfying the conditions in Definition 8 (hence, x remains in $\hat{G}$ as the merged vertex). Note that $G'[X - x]$ is a connected component of $G' - \{x, y\}$ by Claim 23. We derive a contradiction separately in the following cases.
  - If $Y \cap \{x, y\} = \emptyset$, then Y is 3-contractible also in G.
  - If $x \in Y$, then $X \cup Y$ is also 3-contractible in G', and $G' /_3 (X \cup Y) = \hat{G} /_3 Y$.
  - Otherwise, we have $x \notin Y$ and $y \in Y$. In this case, $(X \cup Y) - x$ is also 3-contractible in G', and the resulting graph coincides with $\hat{G} /_3 Y$.

- Suppose that $i = j = 2$. Let $x \in X$, $y \in V \setminus X$, $x' \in Y$, and $y' \in V(\hat{G}) \setminus Y$ be the vertices satisfying the conditions in Definition 8 with respect to X and Y, respectively. Note again that $G'[X - x]$ is a connected component of $G' - \{x, y\}$ by Claim 23. If $\{x, y\} \subseteq Y$, then $G'[X + y]$ is balanced. Since X is not 2-contractible and $X - x$ is not 3-contractible in G, we have $x = s$, which also implies $x' = s$. In this case, $\{s, y\}$ is a 2-cut in G separating $v_0 \in X - s$ and $y' \neq y$, a contradiction. Otherwise (i.e., if $x \notin Y$ or $y \notin Y$), Y is 2-contractible also in G', a contradiction. □

Claim 28. $(G_r, s, t) \in \mathcal{D}^0_{\alpha,\beta}$.

Proof. We prove $(G_i, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ by induction on i. Suppose that $(G_{i-1}, s, t) \in \mathcal{D}^0_{\alpha,\beta}$, and let F_1 be the unique inner face of G_{i-1} with $\psi_{G_{i-1}}(\mathrm{bd}(F_1)) \neq 1_\Gamma$ as in Definition 15.

We show that the resulting parallel arcs of the reduction of $(H_i, x_i, y_i) \in \mathcal{D}^0_{\alpha_i,\beta_i}$ form the boundary of F_1. Suppose to the contrary that the parallel arcs $x_iy_i \in E(G_{i-1})$ do not form $\mathrm{bd}(F_1)$. Then, there exists a connected subgraph H of G_{i-1} such that $x_i, y_i \in V(H)$, $|V(H)| \geq 3$, and $\{x_i, y_i\}$ separates $V(H)$ and $\{s, t\}$. Let $\tilde{H}$ and $\tilde{H}_i$ be the connected subgraphs of G_r corresponding to H and H_i, respectively (i.e., H and H_i are obtained from $\tilde{H}$ and $\tilde{H}_i$, respectively, by the sequence of reductions to construct G_i from G_r).

By Claim 27, G_r is obtained from G' by at most one contraction. If there is no contraction or the contracted vertex set contains none of $x_i, y_i \in V$, then it is easy to see that $\{x_i, y_i\}$ separates $\{s, t\}$ and each of $V(\tilde{H})$ and $V(\tilde{H}_i)$ in G'. This contradicts Claim 23, since at least one of $\tilde{H}$ and $\tilde{H}_i$ does not contain v_0 and hence $\{x_i, y_i\}$ is a 2-cut in G.

Suppose that $x_i \in V(G_r)$ is the resulting vertex of the 2-contraction of $X \subsetneq V$ with $x_i \in X$ and some $y \in V \setminus X$. If $y \notin V(\tilde{H})$ or $y \notin V(\tilde{H}_i)$, then $\tilde{H}$ or $\tilde{H}_i$, respectively, is a connected subgraph of G' separated from $\{s, t\}$ by $\{x_i, y_i\}$ as well as of G_r and does not contain $v_0 \in X$, which contradicts Claim 23. Otherwise, y must coincide with y_i. In this case, by the definition of the 2-contraction, there exists $x \in X$ (which may coincide with x_i) such that $\{x, y_i\}$ is a 2-cut separating X and $V(\tilde{H})$ in G', which contradicts Claim 23 since $v_0 \notin V(\tilde{H})$.

Then, since $(G_{i-1}, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ and $(H_i, x_i, y_i) \in \mathcal{D}^0_{\alpha_i,\beta_i}$, by combining the embeddings of G_{i-1} and H_i, we obtain an embedding of G_i which satisfies the conditions of $\mathcal{D}^0_{\alpha,\beta}$. □

Let $\tilde{G} := G_r$. Recall that we may assume $\beta = 1_\Gamma$ and $\alpha^{-1} \neq \alpha$ (i.e., $\alpha^2 \neq 1_\Gamma$). By Lemma 20, we may assume also that $\tilde{G} = (\tilde{V}, \tilde{E})$ is embedded on a plane so that the two properties hold (we apply shifting at each vertex $v \in V \setminus \{s, t\}$ to G in advance of the construction of $\tilde{G}$ if necessary). Let $\tilde{E}^i \subseteq \tilde{E}$ be the arc set corresponding to $E^i \subseteq E$ in Lemma 20 for each $i = 0, 1$, and we refer to the path $P = (s = u_0, e_1, u_1, \ldots, e_l, u_l = t)$ along the outer boundary of $\tilde{G} - \tilde{E}^1$ as P itself.

By Claim 27, we consider only three cases:

Case 1. $\tilde{G} = G'$,

Case 2. $\tilde{G} = G'/_2X$ for some vertex set $X \subsetneq V$ containing v_0, and

Case 3. $\tilde{G} = G'/_3X$ for some vertex set $X \subseteq V \setminus \{s, t\}$ containing v_0.

Here, we also assume that, in Cases 2 and 3, the label of every arc in $G[X]$ and $G'_X - y$ for some $y \in N_{G'}(X)$, respectively, is 1_Γ (by shifting in advance of the contractions if necessary), where $G'_X := G'[X \cup N_{G'}(X)] - E(N_{G'}(X))$.

In what follows, we derive a contradiction by showing that $(G, s, t) \in \mathcal{D}_{1_\Gamma,\alpha}$, $\gamma \in l(G; s, t)$ for some $\gamma \in \Gamma \setminus \{1_\Gamma, \alpha\}$ (in particular, $\gamma = \alpha^2$ or α^{-1}), or G contains a 3-contractible vertex set or a 2-cut (which contradicts Claim 23). Recall that $(G', s, t) \in \mathcal{D}$ implies $(\tilde{G}, s, t) \in \mathcal{D}$. Hence, $\tilde{G} - s$ is connected. Since every arc in $\tilde{E}^1$ connects two vertices on the path P in $\tilde{G} - \tilde{E}^1$ defined in Lemma 20, $\tilde{G} - \tilde{E}^1 - s$ is also connected. Thus we have $\psi_G(e_0) \in l(G; s, t) = \{1_\Gamma, \alpha\}$, and consider the following two cases separately: when $\psi_G(e_0) = 1_\Gamma$, and when $\psi_G(e_0) = \alpha$.

Note that we have $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$. To see this, suppose that $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) = \emptyset$. In this case, $\tilde{G} - s$ is balanced, and hence, unless $\tilde{G} = G_0$, there exists a 2-contractible vertex set $X \subseteq \tilde{V} - s$ with $t \in X$ and $s \in \tilde{V} \setminus X$, a contradiction. Hence, we have $\tilde{G} = G_0$, which implies Case 2 and $e_0 \in E(X)$. If $\psi_G(e_0) = 1_\Gamma$, then X is 2-contractible also in G, a contradiction. Otherwise, we have $\psi_G(e_0) = \alpha$, which leads to an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction. We can also see $\tilde{E}^1 \setminus \delta_{\tilde{G}}(t) \neq \emptyset$ in the same way.

Let $\tilde{F}_0$ and $\tilde{F}'_0$ denote the outer faces of $\tilde{G}$ and $\tilde{G} - s$, respectively.

5.4 When $\psi_G(e_0) = 1_\Gamma$

5.4.1 Case 1. $\tilde{G} = G'$

Case 1.1. $v_0 \in V(\mathrm{bd}(\tilde{F}'_0)) \setminus V(P)$

In this case, we can embed $G = \tilde{G} + e_0$ on a plane by adding e_0 on $\tilde{F}_0$ so that (G, s, t) satisfies the conditions of $\mathcal{D}^0_{1_\Gamma,\alpha}$ (see Definition 15), a contradiction.

Case 1.2. $v_0 \in V(\mathrm{bd}(\tilde{F}'_0)) \cap V(P)$

Suppose that $v_0 = u_h \in V(P)$. Take an s–t path P' so that $(P' \cup P) - s$ forms the outer boundary of $\tilde{G} - \tilde{E}^1 - s$. Let j be the minimum index such that $E(P[u_j, t]) \subseteq E(P')$, and i the index such that $P[u_i, u_j] \cup P'[u_i, u_j]$ forms a cycle (i.e., they intersect only at u_i and u_j). Take an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ so that $j' - i'$ is maximized.

Suppose that $j' \leq i$. Then, G contains a 2-cut $\{s, u_i\}$ separating $u_{i-1} \neq s$ and $t \neq u_i$, which contradicts Claim 23. Hence, we have $i < j'$, and consider the following two cases.

Case 1.2.1. Suppose that $v_0 = u_h \in V(P) \cap V(P')$ or $h \leq i'$. In this case, we can embed e_0 without violating the condition of $\mathcal{D}^0_{1_\Gamma,\alpha}$.

Case 1.2.2. Otherwise, we have $j' \leq h < j$ since $u_h = v_0 \in V(\mathrm{bd}(\tilde{F}'_0)) \cap V(P)$. If $0 < i' \leq i$, then we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, $\bar{P}[u_h, u_{j'}]$, $\bar{e}'$, $P[u_{i'}, u_i]$, $P'[u_i, u_j]$, and $P[u_j, t]$. Otherwise, i.e., in the case of $i < i'$, we can derive the same contradiction by replacing $P[u_{i'}, u_i]$ with $\bar{P}[u_{i'}, u_i]$.

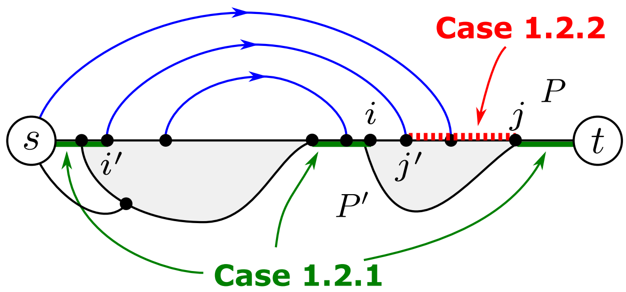

Figure 14: Case 1.2.

Case 1.3. $v_0 \notin V(\mathrm{bd}(\tilde{F}'_0))$

Take a path Q in $\tilde{G} - \tilde{E}^1 - E(P) - s$ from $u_i \in V(P)$ to $u_j \in V(P)$ with $0 < i < j$ such that $Q \cup P[u_i, u_j]$ forms a cycle which encloses v_0 (possibly $v_0 \in V(P)$), i.e., $V(Q \cup P[u_i, u_j])$ separates v_0 and $\{s, t\}$ in $\tilde{G}$. If there are multiple choices of Q, then we choose Q so that the region enclosed by $Q \cup P[u_i, u_j]$ is maximized.

Suppose that $V(Q)$ separates v_0 and $V(P)$ in $\tilde{G}$. Then, there exists a 3-contractible vertex set $X \subseteq V \setminus V(P)$ such that $v_0 \in X$ and $N_G(X) = \{s, w_1, w_2\}$, a contradiction, where $w_1, w_2 \in V(Q)$ are the vertices closest $u_i, u_j \in V(P) \cap V(Q)$, respectively, among those which are reachable from v_0 in $\tilde{G}$ without intersecting Q in between. Thus we can take a v_0–u_h path R in $\tilde{G} - V(Q)$ (possibly of length 0, i.e., $v_0 = u_h$) with $i < h < j$. If there are multiple choices of R, then we choose R so that h is maximized under the condition that $V(R) \cap V(P) = \{u_h\}$.

Case 1.3.1. Suppose that there is no arc in $\tilde{E}^1 \setminus \delta_G(s)$ incident to an inner vertex on $P[u_i, u_j]$. If every arc in $\tilde{E}^1 \cap \delta_{\tilde{G}}(s)$ enters a vertex on $P[s, u_i] \cup P[u_j, t]$, then G contains a 3-contractible vertex set $X \subseteq V \setminus \{s, u_i, u_j\}$ such that $v_0 \in X$, $N_G(X) = \{s, u_i, u_j\}$, and $E(X) \cup \delta_G(X) \subseteq \tilde{E}^0 + e_0$, a contradiction. Otherwise, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$ enters a vertex on $P[s, u_i]$. Then, G contains a 2-cut $\{s, u_i\}$ separating $u_{i-1} \neq s$ and $t \neq u_i$, which contradicts Claim 23.

Case 1.3.2. Suppose that there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ such that $i' < h$ and $i < j' < j$. In this case, similarly to Case 1.2.1, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, R, $P[u_h, u_{j'}]$, $\bar{e}'$, $\bar{P}[u_{i'}, u_i]$, Q, and $P[u_j, t]$ if $i \leq i'$ and $h \leq j'$.

Case 1.3.3. Suppose that every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ connects two vertices on $P[u_h, t]$. In this case, every arc in $\tilde{E}^1 \cap \delta_{\tilde{G}}(s)$ also enters a vertex on $P[u_h, t]$, and $v_0 \neq u_h$ since $v_0 \notin V(\mathrm{bd}(\tilde{F}'_0))$. Let w be the vertex closest to u_j among those on Q which are reachable from v_0 in $G - u_h$ without intersecting Q in between. By the maximality of j and h (i.e., the choice of Q and R), $\{s, u_h, w\}$ separates $v_0 \in V \setminus \{s, u_h, w\}$ and $V(P[u_h, t])$ in G, and hence G contains a 3-contractible vertex set $X \subseteq V \setminus \{s, u_h, w\}$ such that $v_0 \in X$ and $N_G(X) = \{s, u_h, w\}$, a contradiction.

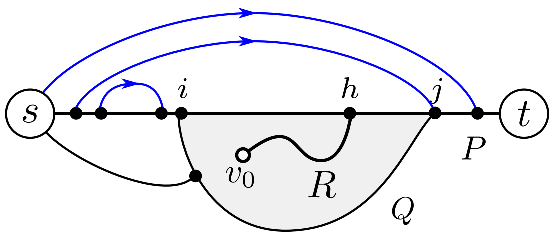

Figure 15: Case 1.3.1.

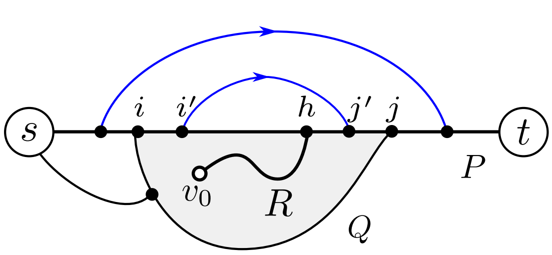

Figure 16: Case 1.3.2.

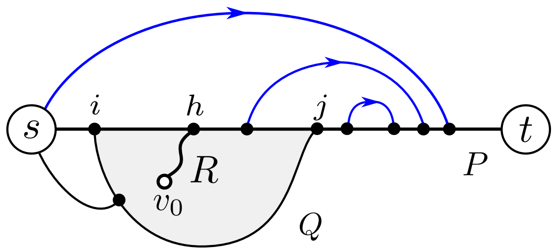

Figure 17: Case 1.3.3.

These three cases imply that there exists an arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ entering a vertex on $P[u_j, t]$. To see this, suppose to the contrary that every such arc enters a vertex on $P[u_1, u_{j-1}]$, and take $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ so that $j' - i'$ is maximized. We may assume $i < j'$ by Case 1.3.1, and hence $h \leq i'$ by Case 1.3.2, which leads to the condition of Case 1.3.3, a contradiction. This implies also that no arc in $\tilde{E}^1 \cap \delta_{\tilde{G}}(s)$ enters a vertex on $P[u_1, u_{j-1}]$.

Case 1.3.4. Suppose that all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ leave the same vertex $u_{i^*} \in V(P)$ with $i^* < h$. In this case, by Case 1.3.2, we may assume that every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enters a vertex on $P[u_j, t]$. Then, since $\{s, u_{i^*}, u_j\}$ separates $v_0 \in V \setminus \{s, u_{i^*}, u_j\}$ and $V(P[u_j, t])$ in G, there exists a 3-contractible vertex set $X \subseteq V \setminus \{s, u_{i^*}, u_j\}$ in G such that $v_0 \in X$ and $N_G(X) = \{s, u_{i^*}, u_j\}$, a contradiction.

Case 1.3.5. Suppose that all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enter the same vertex $u_{j^*} \in V(P)$ with $j \leq j^*$. In this case, $\{s, u_j, u_{j^*}\}$ separates $v_0 \in V \setminus \{s, u_j, u_{j^*}\}$ and $V(P[u_j, t])$ in G. If $j < j^*$, then G contains a 3-contractible vertex set $X \subseteq V \setminus \{s, u_j, u_{j^*}\}$ such that $v_0 \in X$ and $N_G(X) = \{s, u_j, u_{j^*}\}$, a contradiction. Otherwise (i.e., if $j^* = j$), G contains a 2-cut $\{s, u_j\}$ separating v_0 and $t \neq u_j$ (recall that $\tilde{E}^1 \setminus \delta_{\tilde{G}}(t) \neq \emptyset$), a contradiction.

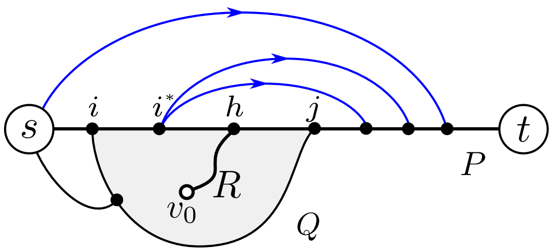

Figure 18: Case 1.3.4.

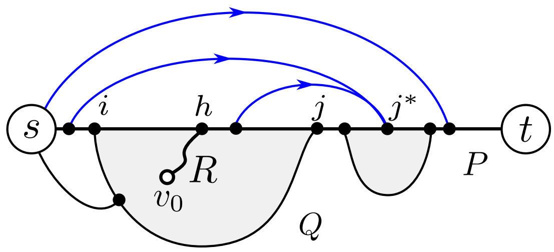

Figure 19: Case 1.3.5.

Otherwise, there exist two arcs $e_1 = u_{i_1}u_{j_1}$ and $e_2 = u_{i_2}u_{j_2}$ in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ such that $i_2 < i_1 < j_1 < j_2$ by Cases 1.3.4 and 1.3.5. We choose e_2 so that $j_2 - i_2$ is maximized. We then have $i_2 < h$ by Case 1.3.3, and $j \leq j_2$ by the argument just after Case 1.3.3. Since there exists an arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ incident to an inner vertex on $P[u_i, u_j]$ by Case 1.3.1, we can choose e_1 so that $i < i_1$ (which is obvious if $i \leq i_2$, and consider Case 1.3.2 otherwise). We then have $h < j_1$, since otherwise we have $i < i_1 < j_1 \leq h < j$, which implies that e_1 satisfies the condition of Case 1.3.2. We choose e_1 so that i_1 is minimized under the condition that $i < i_1$.

Case 1.3.6. Suppose that $j \leq i_1$. In this case, $\{s, u_{i_2}, u_j\}$ separates $v_0 \in V \setminus \{s, u_{i_2}, u_j\}$ and $P[u_j, t]$ in G, and hence G contains a 3-contractible vertex set $X \subseteq V \setminus \{s, u_{i_2}, u_j\}$ such that $v_0 \in X$ and $N_G(X) = \{s, u_{i_2}, u_j\}$, a contradiction.

Case 1.3.7. Suppose that $j_2 = j$. We then have $h \leq i_1$ by $i < i_1 < j_1 < j_2 = j$ and Case 1.3.2. Let h^* be the maximum index such that there exists a w–u_{h^*} path R^* in $\tilde{G} - u_j$ for some $w \in (V(Q) \setminus V(P)) + v_0$ such that $V(R^*) \cap V(Q) \subseteq \{w\}$ and $V(R^*) \cap V(P) = \{u_{h^*}\}$. Note that $h \leq h^*$. If $i_1 < h^*$, then we have $h < h^*$ because of $h \leq i_1$. In this case (see Fig. 22), since R and R^* are disjoint by the maximality of h and h^*, we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, R, $P[u_h, u_{i_1}]$, e_1, $\bar{P}[u_{j_1}, u_{h^*}]$, $\bar{R}^*$, $\bar{Q}[w, u_i]$, $\bar{P}[u_i, u_{i_2}]$, e_2, and $P[u_j, t]$ if $h^* \leq j_1$ and $i_2 \leq i$. Otherwise (i.e., if $h^* \leq i_1$), by the minimality of i_1 and the maximality of h^*, there exists a 2-cut $\{u_{h^*}, u_j\}$ separating u_i and u_{j_1} ($i < h \leq h^* \leq i_1 < j_1 < j_2 = j$) in G (see Fig. 23), a contradiction.

Case 1.3.8. Otherwise, we have $i < i_1 < j < j_2$ (also recall that $i_2 < i_1 < j_1 < j_2$ and $i_2 < h < j_1$). In this case, we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, R, $\bar{P}[u_h, u_{i_1}]$, e_1, $\bar{P}[u_{j_1}, u_j]$, $\bar{Q}$, $P[u_i, u_{i_2}]$, e_2, and $P[u_{j_2}, t]$ if $i_1 \leq h$, $j \leq j_1$, and $i \leq i_2$.

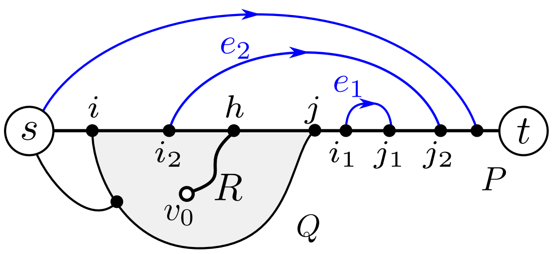

Figure 20: Case 1.3.6.

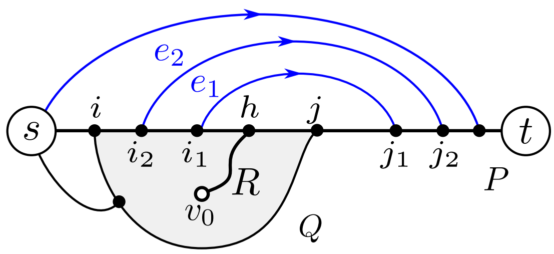

Figure 21: Case 1.3.8.

5.4.2 Case 2. $\tilde{G} = G' /_2 X$ for some $X \subsetneq V$ containing v_0

Let $x \in X$ and $y \in V \setminus X$ be the vertices satisfying the conditions in Definition 8, i.e., $G'[X]$ is a balanced 2-connected component of $G' - y$, and we also refer to the resulting vertex $x \in \tilde{V}$ as v'_0.

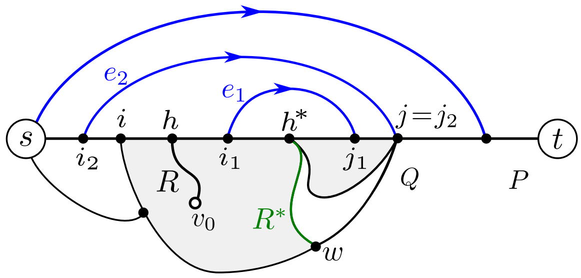

Figure 22: Case 1.3.7 (an s–t path of label α^2).

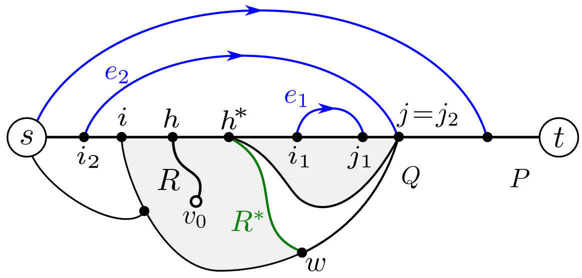

Figure 23: Case 1.3.7 (a 2-cut $\{u_{h^*}, u_j\}$).

Recall that the label of every arc in $G[X]$ is assumed to be 1_Γ by advance shifting. This implies that $s \notin \{x, y\}$, since otherwise X is 2-contractible also in G, a contradiction. By Claim 23, G contains no 2-cut, and hence $G[X - x]$ coincides with the connected component of $G - \{x, y\}$ that contains v_0. Besides, since G contains no 3-contractible vertex set and $G[X]$ is balanced, there exist two arcs x_0y and x_1y in $\delta_G(y)$ with $x_0, x_1 \in X$ (possibly $x_0 = x_1$) whose labels are 1_Γ and either α or α^{-1}, respectively. Since these arcs remain in $\tilde{G}$ as parallel arcs from x to y ($x_0y \in \tilde{E}^0$ and either $x_1y \in \tilde{E}^1$ or $yx_1 \in \tilde{E}^1$), $x = v_0'$ and y are adjacent on P.

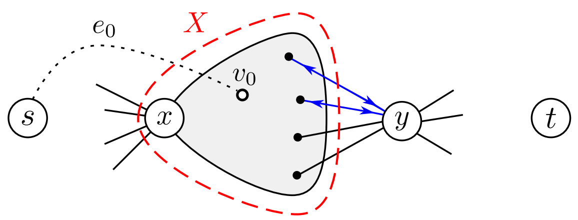

Figure 24: Before the 2-contraction of X.

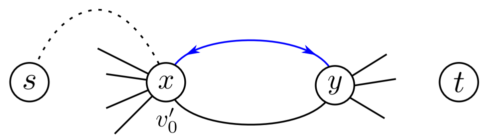

Figure 25: After the 2-contraction of X.

Let $e_0' = sv_0' \notin \tilde{E}$ be an arc with label 1_Γ, and $\tilde{G}^+ := \tilde{G} + e_0'$. If $v_0' \notin V(\mathrm{bd}(\tilde{F}_0'))$, then we can derive a contradiction in the same way as Case 1.3, since any s–t path traversing e_0' in $\tilde{G}^+$ can be expanded into an s–t path of the same label traversing e_0 in G by using some arcs in $E(X) + x_0y$, whose labels are 1_Γ. Hence, we may assume that $v_0' = u_h \in V(\mathrm{bd}(\tilde{F}_0')) \cap V(P)$.

Construct a new Γ-labeled graph H from $G[X+y]$ by splitting the vertex y into two vertices y_0 and y_1 so that every arc entering y in $G[X + y]$ with label $\alpha^{\pm i} \in \{1_\Gamma, \alpha, \alpha^{-1}\}$ enters y_i in H for each $i \in \{0, 1\}$. In particular, $x_i \in X$ is adjacent to y_i for each $i \in \{0, 1\}$. Since $G[X]$ is 2-connected, there exist two disjoint paths between $\{x, v_0\}$ and $\{y_0, y_1\}$ in H. Let Q_1, R_1, Q_2, and R_2 be a v_0–y_0 path, a y_1–x path, a v_0–y_1 path, and a y_0–x path, respectively, in H.

Case 2.1. $y = u_{h+1} \in V(P)$

Note that $y \neq t$, since otherwise $\tilde{E}^1 \setminus \delta_{\tilde{G}}(t) = \emptyset$, a contradiction.

Case 2.1.1. Suppose that one can take the four paths in H so that $V(Q_i) \cap V(R_i) = \emptyset$ for $i = 1, 2$. Then, we can construct two v_0–x paths in $G[X+y]$ whose labels are α^{-1} and α by concatenating Q_i and R_i with identifying $y_0, y_1 \in V(H)$ with $y \in V$ (e.g., see Figs. 26 and 27). Since $l(G; s, t) = \{1_\Gamma, \alpha\}$, for any $j > h+1$, there is no arc $e' = u_iu_j \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$, and no u_i–u_j path P' in $\tilde{G} - \tilde{E}^1 - s$ with $i \leq h$ which does not intersect P in between (otherwise, we can construct an s–t path of label α^2 or α^{-1} not in $\{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, the above v_0–x path of label α or α^{-1} in $G[X + y]$, $\bar{P}[x, u_i]$, e' or P', and $P[u_j, t]$). Hence, G contains a 2-cut $\{s, y = u_{h+1}\}$ separating $x = u_h$ and $t \neq y$, a contradiction.

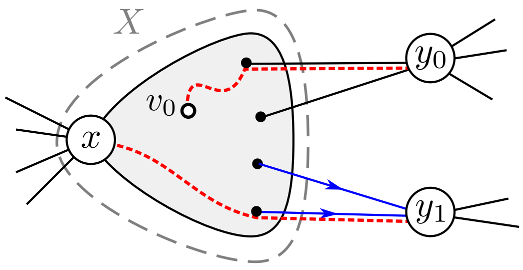

Figure 26: Disjoint paths Q_1 and R_1 in H.

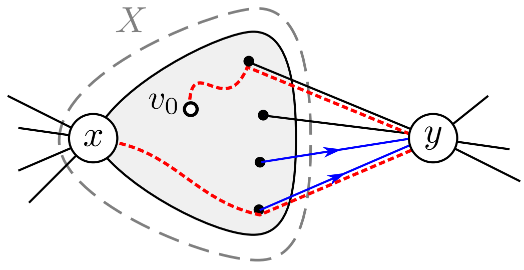

Figure 27: A v_0–x path of label α^{-1} in $G[X+y]$.

Otherwise, exactly one of the two pairs $\{Q_i, R_i\}$ $(i = 1, 2)$ of disjoint paths cannot be taken in H. Then, by Theorem 1, H can be embedded on a plane so that either v_0, y_1, y_0, x or v_0, y_0, y_1, x are on the outer boundary in this order, when either $\{Q_1, R_1\}$ or $\{Q_2, R_2\}$, respectively, cannot be taken in H. Here, we may assume that no contraction (in the sense of Theorem 1) is needed, since if there exists a vertex set $Z \subseteq V(H) \setminus \{x, v_0, y_0, y_1\}$ with $|N_H(Z)| \leq 3$, then G contains a contractible (in the sense of Definitions 7 and 8) vertex set included in Z, a contradiction. These embeddings of H immediately lead to embeddings of $G[X + y]$ by merging $y_0, y_1 \in V(H)$ into $y \in V$. By replacing the parallel arcs xy of $\tilde{G}$ with these embeddings of $G[X + y]$, we can obtain embeddings of $G' = G - e_0$ satisfying the condition of $\mathcal{D}^0_{1_\Gamma,\alpha}$.

Case 2.1.2. Suppose that there is no pair of disjoint paths Q_1, R_1 in H. We then have an embedding of H in which v_0, y_1, y_0, x are on the outer boundary in this order. If there is no arc $u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i' < h < j'$, then we can add $e_0 = sv_0$ to G' without violating the embedding condition of $\mathcal{D}^0_{1_\Gamma,\alpha}$, which contradicts $(G, s, t) \in \mathcal{D}_{1_\Gamma,\alpha}$. Besides, if there exists an arc $u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i' \leq h$ and $h+1 < j'$, then we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G (see Fig. 28), a contradiction, by using the disjoint paths Q_2, R_2 in H. Hence, there exists an arc $u_{i'}u_{h+1} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i' < h$, and all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enter $y = u_{h+1}$.

Recall that $x = u_h \in V(\mathrm{bd}(\tilde{F}_0'))$. This implies that there is no u_i–u_j path in $\tilde{G} - \tilde{E}^1 - s$ with $i < h < j$ which does not intersect P in between. Therefore, $\{s, x, y\}$ separates $u_{i'} \notin \{s, x\}$ and $t \neq y$ in G, and the vertex set of the connected component of $G - \{s, x, y\}$ that contains $u_{i'}$ is 3-contractable in G (see Fig. 29), a contradiction.

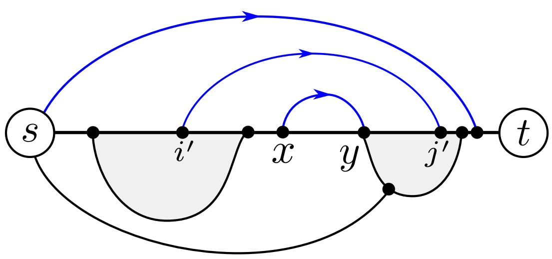

Figure 28: G contains an s–t path of label α^2.

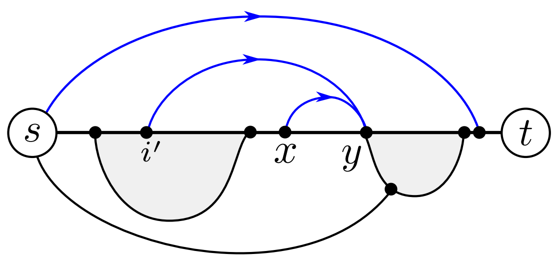

Figure 29: G has a 3-contractible vertex set.

Case 2.1.3. Suppose that there is no pair of disjoint paths Q_2, R_2 in H. We then have an embedding of H in which v_0, y_0, y_1, x are on the outer boundary in this order. If there is no u_i–u_j path in $\tilde{G} - \tilde{E}^1 - s$ with $i < h < j$ which does not intersect P in between, then we can add $e_0 = sv_0$ to G' without violating the embedding condition of $\mathcal{D}^0_{1_\Gamma,\alpha}$, a contradiction. Besides, if there exists a u_i–u_j path in $\tilde{G} - \tilde{E}^1 - s$ with $i \leq h$ and $h + 1 < j$ which does not intersect P in between, then we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G (see Fig. 30), a contradiction, by using the disjoint paths Q_1, R_1 in H. Hence, there exists a u_i–u_{h+1} path in $\tilde{G} - \tilde{E}^1 - s$ with $i < h$ which does not intersect P in between.

The assumption $x = u_h \in V(\mathrm{bd}(\tilde{F}'_0))$ implies that all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ leave $x = u_h$. Therefore, $\{s, x, y\}$ separates $u_i \notin \{s, x\}$ and $t \neq y$ in G, and the vertex set of the connected component of $G - \{s, x, y\}$ that contains u_i is 3-contractable in G (see Fig. 31), a contradiction.

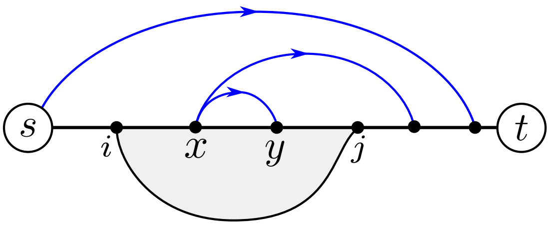

Figure 30: G contains an s–t path of label α^{-1}.

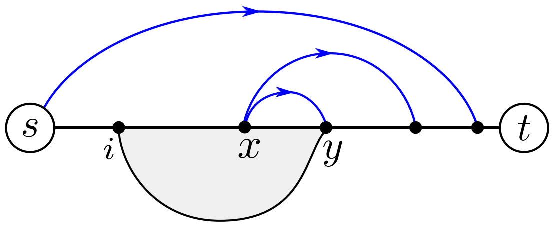

Figure 31: G has a 3-contractible vertex set.

Case 2.2. $y = u_{h-1} \in V(P)$

If there exist a v_0–y_0 path Q_1 and a y_1–x path R_1 such that $V(Q_1) \cap V(R_1) = \emptyset$, then we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, Q_1, R_1, and $P[u_h, t]$ with identifying $y_0, y_1 \in V(H)$ with $y \in V$. Hence, we may assume that there is no pair of such disjoint paths Q_1, R_1 in H. By Theorem 1, we then have an embedding H in which v_0, y_0, y_1, x are on the outer boundary in this order.

If there is no u_i–u_j path in $\tilde{G} - \tilde{E}^1 - s$ with $i < h < j$ which does not intersect P in between, then we can add $e_0 = sv_0$ to G' without violating the condition of $\mathcal{D}^0_{1_\Gamma, \alpha}$ (see Fig. 32), a contradiction. Otherwise, there exists such a u_i–u_j path P'. In this case, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G (see Fig. 33), a contradiction, by concatenating e_0, Q_2, $\bar{P}[u_{h-1}, u_i]$, and $P[u_h, t]$, P', and $P[u_j, t]$ with identifying $y_1 \in V(H)$ with $y \in V$.

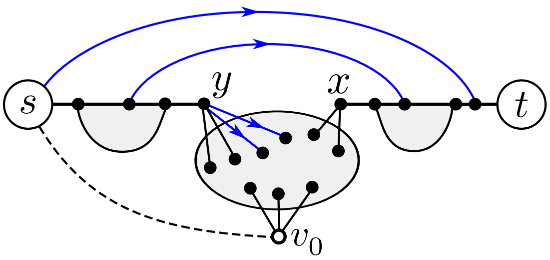

Figure 32: $e_0 = sv_0$ can be embedded.

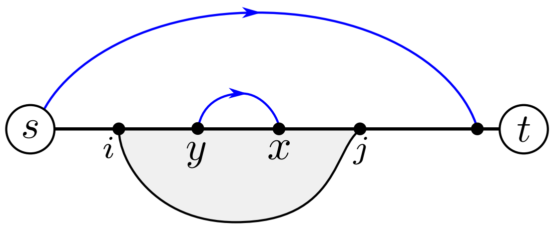

Figure 33: G contains an s–t path of label α^{-1}.

5.4.3 Case 3. $\tilde{G} = G'/_3X$ for some $X \subseteq V \setminus \{s, t\}$ containing v_0

Let $y_1, y_2, y_3 \in \tilde{V}$ be the vertices of the resulting triangle of the 3-contraction of X (i.e., $N_{G'}(X) = \{y_1, y_2, y_3\}$). Consider a plane Γ-labeled graph $\tilde{G}'$ obtained from G' by merging all vertices in X into a single vertex v'_0 and removing parallel arcs with the same label (see Fig. 34). Let $e'_0 = sv'_0 \notin E(\tilde{G}')$ be an arc with label 1_Γ, and $\tilde{G}^+ := \tilde{G}' + e'_0$. If v'_0 is not on the outer boundary of $\tilde{G}' - s$, then we can derive a contradiction in the same way as Case 1.3, since any s–t path traversing e'_0 in $\tilde{G}^+$ can be expanded into an s–t path of the same label traversing e_0 in G by using some arcs in $E(X)$, whose labels are 1_Γ, and any 3-contractible vertex set Y in $\tilde{G}^+$ containing v'_0 can be expanded to a 3-contractible vertex set $X \cup (Y - v'_0)$ in G. Hence, we may assume that v'_0 is on the outer boundary of $\tilde{G}' - s$.

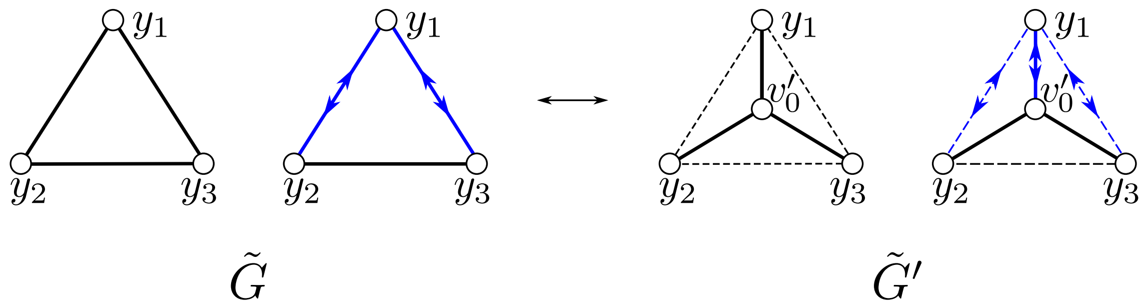

Figure 34: Correspondence between $\tilde{G}$ and $\tilde{G}'$.

Recall that the label of every arc in $G'_X - y$ is assumed to be 1_Γ for some $y \in N_G(X)$, say $y_1 \in N_{G'}(X)$, where $G'_X := G'[X \cup N_{G'}(X)] - E(N_{G'}(X))$. This implies that the label of the arc y_2y_3 in the resulting triangle is 1_Γ. Let $e_1 = y_1y_2$ be the arc in the resulting triangle. We consider the following three cases separately: when $e_1 \in \tilde{E}^1$, when $\bar{e}_1 \in \tilde{E}^1$, and when $e_1 \in \tilde{E}^0$. Note that y_2 and y_3 are symmetric.

Case 3.1. $e_1 = y_1y_2 \in \tilde{E}^1$

We may assume that y_1, y_2, y_3 are on P in this order, and y_2 and y_3 are adjacent. Let h_i be the index such that $y_i = u_{h_i} \in V(P)$ for each $i = 1, 2, 3$, and then we have $h_1 < h_2 = h_3 - 1$.

Case 3.1.1. Suppose that $y_1 = s$. Since $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$ (recall the argument just before Section 5.4), there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$. Take such e' so that $j' - i'$ is maximized. Since $G - \{s, u_h\}$ is connected for any h with $j' \leq h \leq h_2$, there exists an u_i–u_j path P' in $\tilde{G} - \tilde{E}^1 - s$ with $i < j'$ and $h_2 < j$ which does not intersect P in between. Then, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, an arbitrary v_0–y_2 path in $G'_X - \{y_1, y_3\}$, $\bar{P}[y_2, u_{j'}]$, $\bar{e}$, $P[u_{i'}, u_i]$, P', and $P[u_j, t]$ if $i' \leq i$.

Case 3.1.2. Suppose that $y_1 \neq s$ and there exist a v_0–y_1 path Q and a y_2–y_3 path R in G'_X such that $V(Q) \cap V(R) = \emptyset$. Then, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, Q, $P[y_1, y_2]$, R, and $P[y_3, t]$.

Case 3.1.3. Otherwise, there is no such disjoint paths in G'_X. Then, by Theorem 1, G'_X can be embedded on a plane so that v_0, y_2, y_1, y_3 are on the outer boundary in this order. By replacing the triangle $y_1y_2y_3$ in $\tilde{G}$ with this embedding, we obtain an embedding of $G' = G - e_0$ satisfying the condition of $\mathcal{D}^0_{1_\Gamma, \alpha}$. If there is no u_i–u_j path in $\tilde{G} - \tilde{E}^1 - s$ such that $P[u_i, u_j]$ intersects y_2 and y_3, then we can add e_0 to G' without violating the embedding condition of $\mathcal{D}^0_{1_\Gamma, \alpha}$, a contradiction. Otherwise, there exists such a u_i–u_j path P' in $\tilde{G} - \tilde{E}^1 - s$. Then, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, an arbitrary v_0–y_1 path Q in $G'_X - \{y_2, y_3\}$, $P[y_1, u_i]$, P', and $P[u_j, t]$ if $h_1 \leq i$.

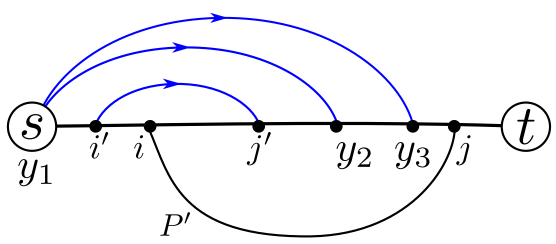

Figure 35: Case 3.1.1.

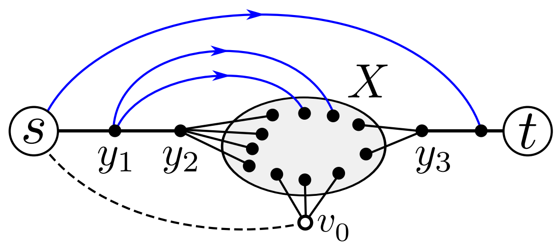

Figure 36: Case 3.1.3 ($e_0 = sv_0$ is embeddable).

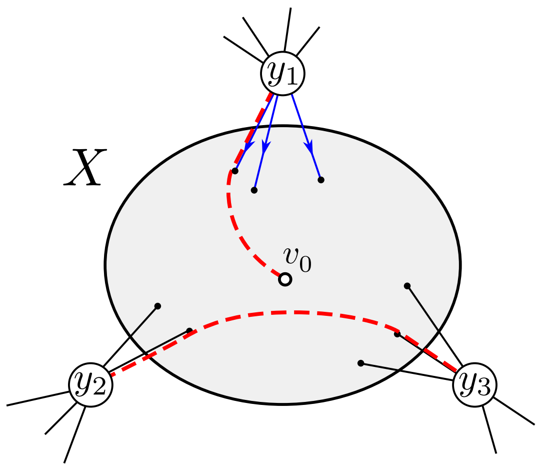

Figure 37: Case 3.1.2 (disjoint paths Q, R in G'_X).

<u>**Case 3.2.** $\bar{e}_1 = y_2y_1 \in \tilde{E}^1$</u>

We may assume that y_1, y_2, y_3 are on P in this order, and y_2 and y_3 are adjacent. Let h_i be the index such that $y_i = u_{h_i} \in V(P)$ for each $i = 1, 2, 3$, and then we have $h_3 + 1 = h_2 < h_1$.

Case 3.2.1. Suppose that there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i' \leq h_3$ and $h_1 < j'$. If there exist a v_0–y_1 path Q and a y_2–y_3 path R in G'_X such that $V(Q) \cap V(R) = \emptyset$, then we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, Q, $\bar{P}[y_1, y_2]$, R, $\bar{P}[y_3, u_{i'}]$, e, and $P[u_{j'}, t]$. Hence, we may assume that there is no such disjoint paths. By Theorem 1, G'_X can be embedded on a plane so that v_0, y_2, y_1, y_3 are on the outer boundary in this order.

By replacing the triangle $y_1y_2y_3$ in $\tilde{G}$ with this embedding of G'_X, we obtain an embedding of $G' = G - e_0$ satisfying the condition of $\mathcal{D}^0_{1_\Gamma,\alpha}$. Recall that the graph $\tilde{G}'$ obetain from G' by merging all vertices in X into a single vertex v'_0 is embedded on a plane so that v'_0 is on the outer boundary of $\tilde{G}' - s$. Hence, there is no u_i–u_j path in $\tilde{G} - \tilde{E}^1 - s$ with $i \leq h_3$ and $h_2 \leq j$ which does not intersect P in between. This implies that we can add $e_0 = sv_0$ to G' without violating the embedding condition of $\mathcal{D}^0_{1_\Gamma,\alpha}$, a contradiction.

Case 3.2.2. Suppose that all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enter y_1. If $y_1 = t$, then $G - t$ is balanced, and hence G contains a 2-contractible vertex set, a contradiction. Otherwise, since G contains no 2-cut, there exists a u_i–u_j path P' in $\tilde{G} - \tilde{E}^1 - s$ with $i < h_1 < j$ which does not intersect P in between. Take P' so that i is minimized. If $h_2 \leq i$, then $\{s, u_i, y_1\}$ separates $y_3 \notin \{s, u_i\}$ and t in G, and the vertex set of the connected component of $G - \{s, u_i, y_1\}$ that contains y_3 is 3-contractible in G, a contradiction.

Otherwise, we have $i < h_2$. If there exist a v_0–y_2 path Q and a y_1–y_3 path R in G'_X such that $V(Q) \cap V(R) = \emptyset$, then we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, Q, $P[y_2, y_1]$, R, $\bar{P}[y_3, u_i]$, P', and $P[u_j, t]$. Hence, we may assume that there is no such disjoint paths. By Theorem 1, G'_X can be embedded on a plane so that v_0, y_1, y_2, y_3 are on the outer boundary in this order. Since $\tilde{G}'$ is embedded on a plane so that v'_0 is on the outer boundary of $\tilde{G}' - s$, there is no arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ leaving an inner vertex on $P[s, y_3]$. Also in this case, we can add $e_0 = sv_0$ to G' without violating the embedding condition of $\mathcal{D}^0_{1_\Gamma,\alpha}$, a contradiction.

Case 3.2.3. Otherwise, there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $h_2 \leq i' < j' < h_1$. Take such e so that i' is minimized, and take a u_i–u_j path P' in $\tilde{G} - \tilde{E}^1 - s$ with $i < h_2$

and $j \neq h_1$ which does not intersect P in between so that j is maximized (possibly a y_3–y_2 path consisting of a single arc $y_3y_2 \in \tilde{E}^0$). If $j \leq i'$, then $\{s, u_j, y_1\}$ separates $y_3 \notin \{s, u_j\}$ and t in G, and the vertex set of the connected component of $G - \{s, u_j, y_1\}$ that contains y_3 is 3-contractible in G, a contradiction. Thus we have $i' < j$.

If there is no pair of a v_0–y_2 path Q and a y_3–y_1 path R in G'_X such that $V(Q) \cap V(R) = \emptyset$, then, by Theorem 1, G'_X can be embedded on a plane so that v_0, y_1, y_2, y_3 are on the outer boundary in this order. Since $\tilde{G}'$ is embedded on a plane so that v'_0 is on the outer boundary of $\tilde{G}' - s$, there is no arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ leaving an inner vertex on $P[s, y_3]$, which implies that we can add $e_0 = sv_0$ to G' without violating the embedding condition of $\mathcal{D}^0_{1_\Gamma, \alpha}$, a contradiction. Hence, we may assume that there exist a v_0–y_2 path Q and a y_3–y_1 path R in G'_X such that $V(Q) \cap V(R) = \emptyset$.

If $i' < j < h_1$, then we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, Q, $P[y_2, u_{i'}]$, e, $\bar{P}[u_{j'}, u_j]$, $\bar{P}'$, $P[u_i, y_3]$, R, and $P[y_1, t]$ if $j \leq j'$. Otherwise, we have $h_1 < j$. Then, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G by concatenating e_0, Q, $P[y_2, y_1]$, $\bar{R}$, $\bar{P}[y_3, u_i]$, P', and $P[u_j, t]$.

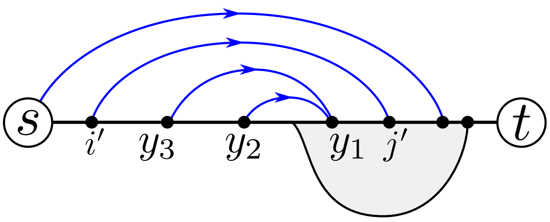

Figure 38: Case 3.2.1.

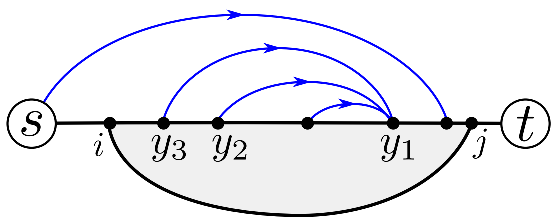

Figure 39: Case 3.2.2.

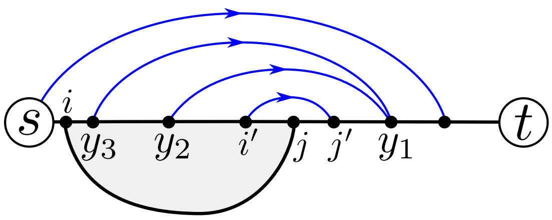

Figure 40: Case 3.2.3 (an s–t path of label α^2).

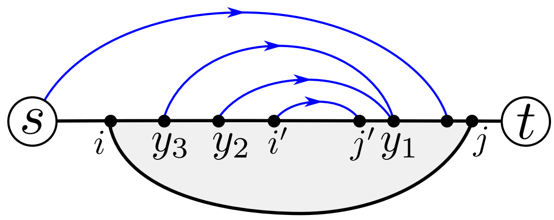

Figure 41: Case 3.2.3 (an s–t path of label α^{-1}).

Case 3.3. $e_1 = y_1y_2 \in \tilde{E}^0$

In this case, y_1, y_2, y_3 are all symmetric, and $s \notin \{y_1, y_2, y_3\}$ since $G = G' + e_0$ contains no 3-contractible vertex set. Recall that the Γ-labeled graph $\tilde{G}'$ obtained from G' by merging all vertices in X into a single vertex v'_0 and removing parallel arcs with the same label is embedded on a plane so that v'_0 is on the outer boundary of $\tilde{G}' - s$. This implies that at least one arc $y_iy_j \in \tilde{E}^0$ $(1 \leq i < j \leq 3)$ is in $\mathrm{bd}(\tilde{F}'_0)$. Take an s–t path P' so that $(P' \cup P) - s$ forms the outer boundary of $\tilde{G} - \tilde{E}^1 - s$.

Case 3.3.1. Suppose that some arc $y_iy_j \in \tilde{E}^0$ is in $E(\mathrm{bd}(\tilde{F}'_0)) \setminus E(P)$. Without loss of generality, take such an arc y_1y_3 so that y_3 is closest to t along P' among y_1, y_2, y_3. There exist a v_0–y_2 path Q and a y_1–y_3 path R in G'_X such that $V(Q) \cap V(P) = \emptyset$, since otherwise, by Theorem 1, we can embed $G = G' + e_0$ on a plane so that (G, s, t) satisfies the conditions of $\mathcal{D}^0_{1_\Gamma, \alpha}$, a contradiction.

If $y_2 \notin V(\mathrm{bd}(\tilde{F}'_0))$, then we can derive a contradiction in the same way as Case 1.3. To see this, let $e'_0 = sy_2 \notin \tilde{E}$ be an arc with label 1_Γ, and consider the Γ-labeled graph

$\tilde{G}^+ := \tilde{G} + e_0'$. Then, any s–t path in $\tilde{G}^+$ traversing e_0' can be expanded to an s–t path of the same label in G traversing e_0 by using Q and R (if it traverses $y_1y_3 \in \tilde{E}^0$), and any 3-contractible vertex set Y in $\tilde{G}^+$ containing y_2 can be expanded to a 3-contractible vertex set $X \cup Y$ in G. Hence, we may assume that $y_2 \in V(\mathrm{bd}(\tilde{F}_0'))$.

If $y_2 \in V(\mathrm{bd}(\tilde{F}_0')) \cap V(P')$, then, by the choice of y_1y_3, we have $N_{\tilde{G}}(y_1) = \{y_2, y_3\}$, which implies that $X + y_1$ is 3-contractible in G, a contradiction. Thus we have $y_2 \in V(\mathrm{bd}(\tilde{F}_0')) \setminus V(P') \subseteq (V(\mathrm{bd}(\tilde{F}_0')) \cap V(P)) - t$, and hence all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ either enter leave those on $P[y_2, t]$ or vertices on $P[s, y_2]$. In the former case, $\{s, y_2, y_3\}$ separates y_1 and t in G, and the vertex set of the connected component of $G - \{s, y_2, y_3\}$ containing y_1 is 3-contractible in G, a contradiction.

In the latter case, take an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ so that j' is maximized, and let i and j be the indices, respectively, such that $P[u_i, u_j] \cup P'[u_i, u_j]$ forms a cycle and u_i, y_1, y_3, u_j are on P' in this order (possibly $u_i = y_1$ or $y_3 = u_j$). Note that u_j is strictly closer to t along P than y_2 since $y_2 \in V(\mathrm{bd}(\tilde{F}_0')) \setminus V(P')$. If $j' \leq i$, then G contains a 2-cut $\{s, u_i\}$ separating $u_{i'}$ and t, a contradiction. Otherwise, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, Q, $\bar{P}[y_2, u_{j'}]$, $\bar{e}$, $P[u_{i'}, u_i]$, $P'[u_i, u_j]$, and $P[u_j, t]$ if $i' \leq i$.

Case 3.3.2. Otherwise, take an arc $y_1y_3 \in E(\mathrm{bd}(\tilde{F}_0')) \cap E(P)$ so that y_3 is closest to t along P among y_1, y_2, y_3. Let i and j be the indices, respectively, such that $P[u_i, u_j] \cup P'[u_i, u_j]$ forms a cycle and u_i, y_1, y_3, u_j are on P in this order, and h the index such that $y_1 = u_h$. Then, since the arcs y_1y_2 and y_2y_3 in $\tilde{E}^0$ are not in $E(\mathrm{bd}(\tilde{F}_0')) \setminus E(P)$, we have $i \leq h < j$ and P' does not traverse these arcs y_1y_2, $y_2y_3 \in \tilde{E}^0$, and hence there is no arc $u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i' \leq h < j$ (recall that $\tilde{G}'$ is embedded on a plane so that v_0' is on the outer boundary of $\tilde{G}' - s$). Take an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ so that $j' - i'$ is maximized.

Suppose that every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enters a vertex on $P[s, y_1]$, i.e., $j' \leq h$. If $j' \leq i$, then G contains a 2-cut $\{s, u_i\}$ separating $u_{i'}$ and t, a contradiction. Otherwise, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, an arbitrary v_0–y_1 path in $G_X' - \{y_2, y_3\}$, $\bar{P}[y_1, u_{j'}]$, $\bar{e}$, $P[u_{i'}, u_i]$, $P'[u_i, u_j]$, and $P[u_j, t]$ if $i' \leq i$.

Otherwise, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ leaves a vertex on $P[y_3, t]$, i.e., $h < i'$. In this case, there exist a v_0–y_2 path Q and a y_1–y_3 path R in G_X' such that $V(Q) \cap V(P) = \emptyset$, since otherwise, by Theorem 1, we can embed $G = G' + e_0$ on a plane so that (G, s, t) satisfies the conditions of $\mathcal{D}^0_{1_\Gamma, \alpha}$, a contradiction. We then have $y_2 \in V(\mathrm{bd}(\tilde{F}_0'))$, since otherwise we can derive a contradiction in the same way as Case 1.3 (see Case 3.3.1).

If $y_2 \in V(\mathrm{bd}(\tilde{F}_0')) \setminus V(P)$, then $\{s, y_2, y_3\}$ separates y_1 and t in G, and the vertex set of the connected component of $G - \{s, y_2, y_3\}$ containing y_1 is 3-contractible in G, a contradiction. Otherwise, $y_2 \in V(\mathrm{bd}(\tilde{F}_0')) \cap V(P)$. Also in this case, $X + y_1$ is 3-contractible in G ($N_G(X + y_1) = \{s, y_2, y_3\}$), a contradiction.

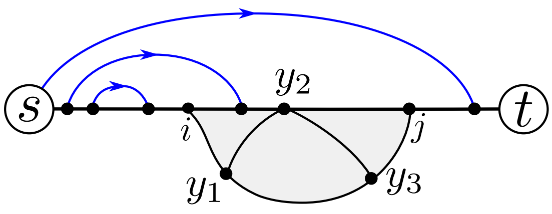

Figure 42: Case 3.3.1.

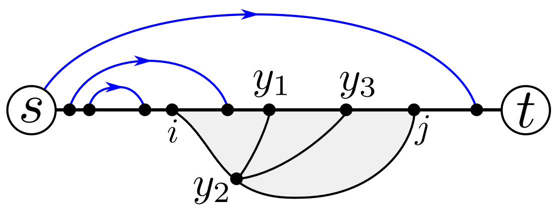

Figure 43: Case 3.3.2.

5.5 When $\psi_G(e_0) = \alpha$

This case is rather easier than when $\psi_G(e_0) = 1_\Gamma$. When $\tilde{G} = G'/_3 X$ for some $X \subseteq V \setminus \{s,t\}$ containing v_0, we redefine $\tilde{G}$ as a plane Γ-labeled graph obtained from G' by merging all vertices in X into a single vertex v_0' and removing parallel arcs with the same label (cf. $\tilde{G}'$ in Section 5.4.3). This $\tilde{G}$ also satisfies $(\tilde{G}, s, t) \in \mathcal{D}^0_{1_\Gamma,\alpha}$ with a natural embedding obtained from $G'/_3 X$ (put v_0' in the resulting triangle), and redefine $\tilde{E}^0, \tilde{E}^1, P, \tilde{F}_0, \tilde{F}_0'$ for this $\tilde{G}$ in the same way. In the other cases, let v_0' be the corresponding vertex in $\tilde{G}$, i.e., $v_0' = v_0$ when $\tilde{G} = G'$, and $v_0' = x$ when $\tilde{G} = G'/_2 X$ for some $X \subsetneq V$ with $x \in X$ and $y \in V \setminus X$ containing v_0.

If there exists a v_0'–t path of label α in $\tilde{G}$ then we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by expanding it to a v_0–t path of the same label in G' and extending it using e_0. Hence, we may assume that there is no such v_0'–t path in $\tilde{G}$. In what follows, to distinguish from Section 5.4, we use Roman numerals for the case numbers.

Case i. $v_0' \in V(\mathrm{bd}(\tilde{F}_0')) \setminus V(P)$

Let i be the minimum index such that there exists a v_0'–u_i path Q in $\tilde{G} - \tilde{E}^1 - s$ which does not intersect P in between. If there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i < j'$, then we can construct a v_0'–t path of label α, a contradiction, e.g., by concatenating Q, $\bar{P}[u_i, u_{i'}]$, e', and $P[u_{j'}, t]$ if $i' \leq i$. Hence, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ connects two vertices on $P[u_1, u_i]$. This implies that G contains a 2-cut $\{s, u_i\}$ separating u_1 and v_0, a contradiction.

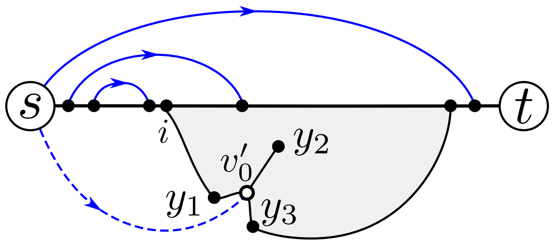

Figure 44: Case i.

Case ii. $v_0' \in V(\mathrm{bd}(\tilde{F}_0')) \cap V(P)$

Suppose that $v_0' = u_h \in V(P)$. If there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $h < j'$, then we can construct a v_0'–t path of label α in $\tilde{G}$, a contradiction, e.g., by concatenating $P[u_h, u_{i'}]$, e, and $P[u_{j'}, t]$ if $h \leq i'$. Hence, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ connects two vertices on $P[u_1, u_h]$. Then, we can embed an arc $e_0' = sv_0'$ on some face of $\tilde{G}$ without violating the embedding conditions in Definition 15.

Case ii.1. If $v_0' = v_0$, then this immediately contradicts that $(G, s, t) \notin \mathcal{D}_{1_\Gamma,\alpha}$.

Case ii.2. Suppose that v_0' is the resulting vertex of the 2-contraction of some $X \subsetneq V$ with $x \in X$ and $y \in V \setminus X$. By the same argument as Case 2 in Section 5.4, either $y = u_{h+1}$ or $y = u_{h-1}$. Since there must be an arc connecting x and y in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$, we have $y = u_{h-1}$. Construct a new Γ-labeled graph H from $G[X + y]$ in the same way as Case 2 in Section 5.4, i.e., by splitting the vertex y into two vertices y_0 and y_1 so that every arc entering y in $G[X + y]$ with label $\alpha^i \in \{1_\Gamma, \alpha\}$ leaves y_i in H for each $i \in \{0, 1\}$. If there exist a v_0–y_0 path Q and a y_1–x path R in H such that $V(Q) \cap V(R) = \emptyset$, then we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, Q, R, and $P[x, t]$, with identifying $y_0, y_1 \in V(H)$ with $y \in V$. Otherwise, by Theorem 1, H can be embedded on a plane so that v_0, y_1, y_0, x are on the outer boundary in this order, and hence we can add e_0 to G' without violating the embedding conditions of $\mathcal{D}^0_{1_\Gamma,\alpha}$, a contradiction.

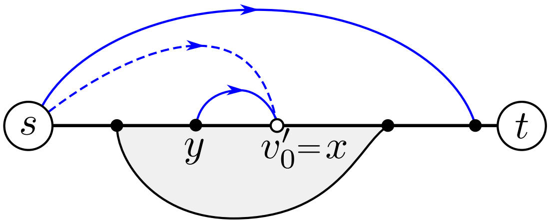

Figure 45: Case ii.2.

Case ii.3. Otherwise, v_0' is the new vertex obtained by merging some $X \subseteq V \setminus \{s,t\}$. In the same way as Case 3 in Section 5.4, let $y_1, y_2, y_3 \in \tilde{V}$ be the vertices of the resulting triangle, and assume that the label of every arc in $G'_X - y_1$ is 1_Γ.

Without loss of generality, assume that the arc y_1y_3 is in the outer boundary of $G'/_3X$, and y_3 is closest to t along P among y_1, y_2, y_3. If there is no pair of a v_0–y_2 path Q and a y_1–y_3 path R in $G'_X := G'[X \cup N_{G'}(X)] - E(N_{G'}(X))$ such that $V(Q) \cap V(R) = \emptyset$, then, by Theorem 1, G'_X can be embedded on a plane so that v_0, y_1, y_2, y_3 are on the outer boundary in this order, and hence we can add e_0 to G' without violating the embedding conditions of $\mathcal{D}^0_{1_\Gamma,\alpha}$, a contradiction. Hence, there exist such disjoint paths Q, R in G'_X.

Suppose that $y_1v_0' \in \tilde{E}^1$. In this case, $y_1 \neq s$ since otherwise X is 3-contractible also in G, a contradiction. Besides, $y_2 \in V(\mathrm{bd}(\tilde{F}_0')) \cap V(P)$ since there exists an arc y_1y_2 with label α in $G'/_3X$. Then, we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by concatenating e_0, Q, $\bar{P}[y_2, y_1]$, R, and $P[y_3, t]$.

Otherwise, the label of every arc in G'_X is 1_Γ. If $y_2 \notin V(\mathrm{bd}(\tilde{F}_0'))$, then we can reduce this case to Case iii below by regarding y_2 as v_0'. Otherwise, $y_2 \in V(\mathrm{bd}(\tilde{F}_0')) \setminus V(P)$. If $y_1 = s$, then $G - s$ is balanced, and hence G contains 2-contractible vertex set, a contradiction. Hence, by the same argument as Case i, in which we regard y_2 as v_0' and $G'/_3X$ as $\tilde{G}$ (i.e., take an y_2–u_i path in $G'/_3X$ so that i is minimized, and so on), we can derive a contradiction. Note that we can use the arc $y_1y_3 \in E(P)$ because of the two disjoint paths Q, R in G'_X.

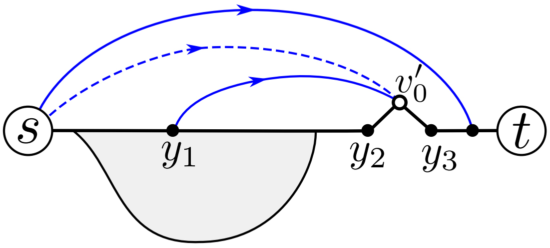

Figure 46: Case ii.3 ($y_1v_0' \in \tilde{E}^1$).

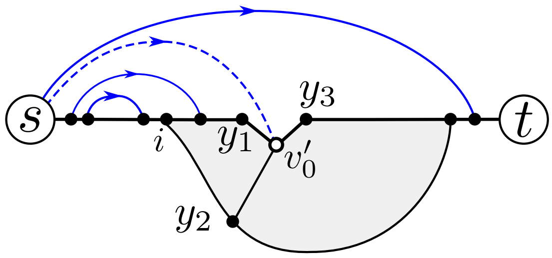

Figure 47: Case ii.3 ($y_1v_0' \in \tilde{E}^0$).

<u>**Case iii.** $v_0' \notin V(\mathrm{bd}(\tilde{F}_0'))$</u>

Let i and j be the minimum and maximum indices, respectively, such that there exist a v_0'–u_i path Q and a v_0'–u_j path R in $\tilde{G} - \tilde{E}^1 - s$ which do not intersect P in between. If there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i < j'$, then we can construct a v_0'–t path of label α in $\tilde{G}$, a contradiction, e.g., by concatenating Q, $\bar{P}[u_i, u_{i'}]$, e', and $P[u_{j'}, t]$ if $i' \leq i$.

Otherwise, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enters a vertex on $P[u_1, u_i]$, and at least one such arc $u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ exists. Since G contains no 3-contractible vertex set, there exists an arc from s to the connected component of $\tilde{G} - \{s, u_i, u_j\}$ containing v_0' with label 1_Γ in $\tilde{G}$. Hence, there is no path from an inner vertex on $P[s, u_i]$ to a vertex on $P[u_j, t]$ in $\tilde{G} - \tilde{E}^1 - s$ which does not intersect P in between. This implies that G contains a 2-cut $\{s, u_i\}$ separating $u_{i'}$ and t, a contradiction.

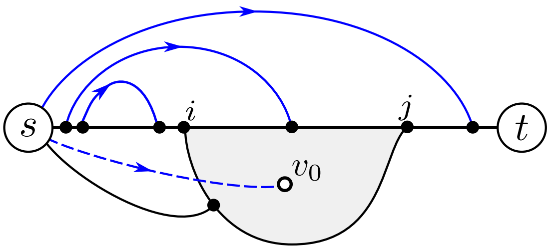

Figure 48: Case iii.

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