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Abstract

The matroid parity (or matroid matching) problem, introduced as a common generalization of matching and matroid intersection problems, is so general that it requires an exponential number of oracle calls. Lovász (1980) has shown that this problem admits a min-max formula and a polynomial algorithm for linearly represented matroids. Since then efficient algorithms have been developed for the linear matroid parity problem.

In this paper, we present a combinatorial, deterministic, polynomial-time algorithm for the weighted linear matroid parity problem. The algorithm builds on a polynomial matrix formulation using Pfaffian and adopts a primal-dual approach with the aid of the augmenting path algorithm of Gabow and Stallmann (1986) for the unweighted problem.

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1 Introduction

The matroid parity problem [16] (also known as the matchoid problem [15] or the matroid matching problem [17]) was introduced as a common generalization of matching and matroid intersection problems. In the worst case, it requires an exponential number of independence oracle calls [14, 19]. Nevertheless, Lovász [17, 19, 20] has shown that the problem admits a min-max theorem for linear matroids and presented a polynomial algorithm that is applicable if the matroid in question is represented by a matrix.

Since then, efficient combinatorial algorithms have been developed for this linear matroid parity problem [8, 26, 27]. Gabow and Stallmann [8] developed an augmenting path algorithm with the aid of a linear algebraic trick, which was later extended to the linear delta-matroid parity problem [10]. Orlin and Vande Vate [27] provided an algorithm that solves this problem by repeatedly solving matroid intersection problems coming from the min-max theorem. Later, Orlin [26] improved the running time bound of this algorithm. The current best deterministic running time bound due to [8, 26] is $O(nm^{\omega})$, where n is the cardinality of the ground set, m is the rank of the linear matroid, and ω is the matrix multiplication exponent, which is at most 2.38. These combinatorial algorithms, however, tend to be complicated.

An alternative approach that leads to simpler randomized algorithms is based on an algebraic method. This is originated by Lovász [18], who formulated the linear matroid parity problem as rank computation of a skew-symmetric matrix that contains independent parameters. Substituting randomly generated numbers to these parameters enables us to compute the optimal value with high probability. A straightforward adaptation of this approach requires iterations to find an optimal solution. Cheung, Lau, and Leung [3] have improved this algorithm to run in $O(nm^{\omega-1})$ time, extending the techniques of Harvey [12] developed for matching and matroid intersection.

While matching and matroid intersection algorithms have been successfully extended to their weighted version, no polynomial algorithms have been known for the weighted linear matroid parity problem for more than three decades. Camerini, Galbiati, and Maffioli [2] developed a random pseudopolynomial algorithm for the weighted linear matroid parity problem by introducing a polynomial matrix formulation that extends the matrix formulation of Lovász [18]. This algorithm was later improved by Cheung, Lau, and Leung [3]. The resulting complexity, however, remained pseudopolynomial. Tong, Lawler, and Vazirani [32] observed that the weighted matroid parity problem on gammoids can be solved in polynomial time by reduction to the weighted matching problem. As a relaxation of the matroid matching polytope, Vande Vate [33] introduced the fractional matroid matching polytope. Gijswijt and Pap [11] devised a polynomial algorithm for optimizing linear functions over this polytope. The polytope was shown to be half-integral, and the algorithm does not necessarily yield an integral solution.

This paper presents a combinatorial, deterministic, polynomial-time algorithm for the weighted linear matroid parity problem. To do so, we combine algebraic approach and augmenting path technique together with the use of node potentials. The algorithm builds on a polynomial matrix formulation, which naturally extends the one discussed in [9] for the unweighted problem. The algorithm employs a modification of the augmenting path search procedure for the unweighted problem by Gabow and Stallmann [8]. It adopts a primal-dual approach without writing an explicit LP description. The correctness proof for the optimality is based on the idea of combinatorial relaxation for polynomial matrices due to Murota [24]. The algorithm is shown to require $O(n^3m)$ arithmetic operations. This leads to a strongly polynomial algorithm for linear matroids represented over a finite field. For linear matroids represented over the rational field, one can exploit our algorithm to solve the problem in polynomial time.

Independently of the present work, Gyula Pap has obtained another combinatorial, deterministic, polynomial-time algorithm for the weighted linear matroid parity problem based on a different approach.

The matroid matching theory of Lovász [20] in fact deals with more general class of matroids that enjoy the double circuit property. Dress and Lovász [6] showed that algebraic matroids satisfy this property. Subsequently, Hochstättler and Kern [13] showed the same phenomenon for pseudomodular matroids. The min-max theorem follows for this class of matroids. To design a polynomial algorithm, however, one has to establish how to represent those matroids in a compact manner. Extending this approach to the weighted problem is left for possible future investigation.

The linear matroid parity problem finds various applications: structural solvability analysis of passive electric networks [23], pinning down planar skeleton structures [21], and maximum genus cellular embedding of graphs [7]. We describe below two interesting applications of the weighted matroid parity problem in combinatorial optimization.

A T-path in a graph is a path between two distinct vertices in the terminal set T. Mader [22] showed a min-max characterization of the maximum number of openly disjoint T-paths. The problem can be equivalently formulated in terms of $\mathcal{S}$-paths, where $\mathcal{S}$ is a partition of T and an $\mathcal{S}$-path is a T-path between two different components of $\mathcal{S}$. Lovász [20] formulated the problem as a matroid matching problem and showed that one can find a maximum number of disjoint $\mathcal{S}$-paths in polynomial time. Schrijver [30] has described a more direct reduction to the linear matroid parity problem.

The disjoint $\mathcal{S}$-paths problem has been extended to path packing problems in group-labeled graphs [4, 5, 28]. Tanigawa and Yamaguchi [31] have shown that these problems also reduce to a matroid matching problem with double circuit property. Yamaguchi [34] clarifies a characterization of the groups for which those problems reduce to the linear matroid parity problem.

As a weighted version of the disjoint $\mathcal{S}$-paths problem, it is quite natural to think of finding disjoint $\mathcal{S}$-paths of minimum total length. It is not immediately clear that this problem reduces to the weighted linear matroid parity problem. A recent paper of Yamaguchi [35] clarifies that this is indeed the case. He also shows that the reduction results on the path packing problems on group-labeled graphs also extend to the weighted version.

The weighted linear matroid parity is also useful in the design of approximation algorithms. Prömel and Steger [29] provided an approximation algorithm for the Steiner tree problem. Given an instance of the Steiner tree problem, construct a hypergraph on the terminal set such that each hyperedge corresponds to a terminal subset of cardinality at most three and regard the shortest length of a Steiner tree for the terminal subset as the cost of the hyperedge. The problem of finding a minimum cost spanning hypertree in the resulting hypergraph can be converted to the problem of finding minimum spanning tree in a 3-uniform hypergraph, which is a special case of the weighted parity problem for graphic matroids. The minimum spanning hypertree thus obtained costs at most 5/3 of the optimal value of the original Steiner tree problem, and one can construct a Steiner tree from the spanning hypertree without increasing the cost. Thus they gave a 5/3-approximation algorithm for the Steiner tree problem via weighted linear matroid parity. This is a very interesting approach that suggests further use of weighted linear matroid parity in the design of approximation algorithms, even though the performance ratio is larger than the current best one for the Steiner tree problem [1].

2 The Minimum-Weight Parity Base Problem

Let A be a matrix of row-full rank over an arbitrary field $\mathbf{K}$ with row set U and column set V. Assume that both $m = |U|$ and $n = |V|$ are even. The column set V is partitioned into pairs, called *lines*. Each $v \in V$ has its *mate* $\bar{v}$ such that $\{v, \bar{v}\}$ is a line. We denote by L the set of lines, and suppose that each line $\ell \in L$ has a weight $w_\ell \in \mathbb{R}$.

The linear dependence of the column vectors naturally defines a matroid $\mathbf{M}(A)$ on V. Let $\mathcal{B}$ denote its base family. A base $B \in \mathcal{B}$ is called a *parity base* if it consists of lines. As a weighted version of the linear matroid parity problem, we will consider the problem of finding a parity base of minimum weight, where the weight of a parity base is the sum of the weights of lines in it. We denote the optimal value by $\zeta(A, L, w)$. This problem generalizes finding a minimum-weight perfect matching in graphs and a minimum-weight common base of a pair of linear matroids on the same ground set.

As another weighted version of the matroid parity problem, one can think of finding a matching (independent parity set) of maximum weight. This problem can be easily reduced to the minimum-weight parity base problem.

Associated with the minimum-weight parity base problem, we consider a skew-symmetric polynomial matrix $\Phi_A(\theta)$ in variable θ defined by

$$\Phi_A(\theta) = \begin{pmatrix} O & A \\ -A^\top & D(\theta) \end{pmatrix},$$

where $D(\theta)$ is a block-diagonal matrix in which each block is a 2×2 skew-symmetric polynomial matrix $D_\ell(\theta) = \begin{pmatrix} 0 & -\tau_\ell \theta^{w_\ell} \\ \tau_\ell \theta^{w_\ell} & 0 \end{pmatrix}$ corresponding to a line $\ell \in L$. Assume that the coefficients τ_ℓ are independent parameters (or indeterminates). For a skew-symmetric matrix Φ whose rows and columns are indexed by W, the *support graph* of Φ is the graph $\Gamma = (W, E)$ with edge set $E = \{(u, v) \mid \Phi_{uv} \neq 0\}$. We denote by $\mathrm{Pf}\,\Phi$ the *Pfaffian* of Φ, which is defined as follows:

$$\mathrm{Pf}\,\Phi = \sum_M \sigma_M \prod_{(u,v)\in M} \Phi_{uv},$$

where the sum is taken over all perfect matchings M in Γ and σ_M takes ± 1 in a suitable manner, see [21]. It is well-known that $\det \Phi = (\mathrm{Pf}\,\Phi)^2$ and $\mathrm{Pf}\,(S\Phi S^\top) = \mathrm{Pf}\,\Phi \cdot \det S$ for any square matrix S.

We have the following lemma that characterizes the optimal value of the minimum-weight parity base problem.

Lemma 2.1. *The optimal value of the minimum-weight parity base problem is given by*

$$\zeta(A, L, w) = \sum_{\ell \in L} w_\ell - \deg_\theta \mathrm{Pf}\,\Phi_A(\theta).$$

In particular, if $\mathrm{Pf}\,\Phi_A(\theta) = 0$*, then there is no parity base.*

Proof. We split $\Phi_A(\theta)$ into Ψ_A and $\Delta(\theta)$ such that

$$\Phi_A(\theta) = \Psi_A + \Delta(\theta), \qquad \Psi_A = \begin{pmatrix} O & A \\ -A^\top & O \end{pmatrix}, \qquad \Delta(\theta) = \begin{pmatrix} O & O \\ O & D(\theta) \end{pmatrix}.$$

The row and column sets of these skew-symmetric matrices are indexed by $W := U \cup V$. By [25, Lemma 7.3.20], we have

$$\mathrm{Pf}\,\Phi_A(\theta) = \sum_{X \subseteq W} \pm \mathrm{Pf}\,\Psi_A[W \setminus X] \cdot \mathrm{Pf}\,\Delta(\theta)[X],$$

where each sign is determined by the choice of X, $\Delta(\theta)[X]$ is the principal submatrix of $\Delta(\theta)$ whose rows and columns are both indexed by X, and $\Psi_A[W \setminus X]$ is defined in a similar way. One can see that $\mathrm{Pf}\,\Delta(\theta)[X] \neq 0$ if and only if $X \subseteq V$ (or, equivalently $B := V \setminus X$) is a union of lines. One can also see for $X \subseteq V$ that $\mathrm{Pf}\,\Psi_A[W \setminus X] \neq 0$ if and only if $A[U, V \setminus X]$ is nonsingular, which means that B is a base of $\mathbf{M}(A)$. Thus, we have

$$\mathrm{Pf}\,\Phi_A(\theta) = \sum_B \pm \mathrm{Pf}\,\Psi_A[U \cup B] \cdot \mathrm{Pf}\,\Delta(\theta)[V \setminus B],$$

where the sum is taken over all parity bases B. Note that no term is canceled out in the summation, because each term contains a distinct set of independent parameters. For a parity base B, we have

$$\deg_\theta(\mathrm{Pf}\,\Psi_A[U \cup B] \cdot \mathrm{Pf}\,\Delta(\theta)[V \setminus B]) = \sum_{\ell \in V \setminus B} w_\ell = \sum_{\ell \in L} w_\ell - \sum_{\ell \in B} w_\ell,$$

which implies that the minimum weight of a parity base is $\sum_{\ell \in L} w_\ell - \deg_\theta \mathrm{Pf}\,\Phi_A(\theta)$. □

3 Algorithm Outline

In this section, we describe the outline of our algorithm for solving the minimum-weight parity base problem.

The algorithm works on a vertex set $V^* \supseteq V$ that includes some new vertices generated during the execution. The algorithm keeps a nested (laminar) collection $\Lambda = \{H_1, \dots, H_{|\Lambda|}\}$ of vertex subsets of V^* such that $H_i \cap V$ is a set of lines for each i. The indices satisfy that, for any two members $H_i, H_j \in \Lambda$ with $i < j$, either $H_i \cap H_j = \emptyset$ or $H_i \subsetneq H_j$ holds. Each member of Λ is called a *blossom*. The algorithm maintains a potential $p : V^* \to \mathbb{R}$ and a nonnegative variable $q : \Lambda \to \mathbb{R}_+$, which are collectively called *dual variables*. It also keeps a subset $B^* \subseteq V^*$ such that $B := B^* \cap V \in \mathcal{B}$.

The algorithm starts with splitting the weight w_ℓ into $p(v)$ and $p(\bar{v})$ for each line $\ell = \{v, \bar{v}\} \in L$, i.e., $p(v) + p(\bar{v}) = w_\ell$. Then it executes the greedy algorithm for finding a base $B \in \mathcal{B}$ with minimum value of $p(B) = \sum_{u \in B} p(u)$. If B is a parity base, then B is obviously a minimum-weight parity base. Otherwise, there exists a line $\ell = \{v, \bar{v}\}$ in which exactly one of its two vertices belongs to B. Such a line is called a *source line* and each vertex in a source line is called a *source vertex*. A line that is not a source line is called a *normal line*.

The algorithm initializes $\Lambda := \emptyset$ and proceeds iterations of primal and dual updates, keeping dual feasibility. In each iteration, the algorithm applies the breadth-first search to find an augmenting path. In the meantime, the algorithm sometimes detects a new blossom and adds it to Λ. If an augmenting path P is found, the algorithm updates B along P. This will reduce the number of source lines by two. If the search procedure terminates without finding an augmenting path, the algorithm updates the dual variables to create new tight edges. The algorithm repeats this process until B becomes a parity base. Then B is a minimum-weight parity base.

The rest of this paper is organized as follows. In Section 4, we introduce new notions attached to blossoms. The feasibility of the dual variables is defined in Section 5. In Section 6, we show that a parity base that admits feasible dual variables attains the minimum weight. In Section 7, we describe a search procedure for an augmenting path. The validity of the procedure is shown in Section 8. In Section 9, we describe how to update the dual variables when the search procedure terminates without finding an augmenting path. If the search procedure succeeds in finding an augmenting path P, the algorithm updates the base B along P. The details of this process is presented in Section 10. Finally, in Section 11, we describe the entire algorithm and analyze its running time.

4 Blossoms

In this section, we introduce buds and tips attached to blossoms and construct auxiliary matrices that will be used in the definition of dual feasibility.

Each blossom contains at most one source line, and a blossom that contains a source line is called a *source blossom*. A blossom with no source line is called a *normal blossom*. Let $\Lambda_{\rm s}$ and $\Lambda_{\rm n}$ denote the sets of source blossoms and normal blossoms, respectively. Each normal blossom $H_i \in \Lambda_{\rm n}$ contains mutually disjoint vertices b_i, t_i, and $\bar{t}_i$ outside V, where b_i, t_i, and $\bar{t}_i$ are called the *bud* of H_i, the *tip* of H_i, and the *mate* of t_i, respectively. The vertex set V^* is defined to be $V^* := V \cup \{b_i, t_i, \bar{t}_i \mid H_i \in \Lambda_{\rm n}\}$, For every i, j with $H_j \in \Lambda_{\rm n}$, they satisfy $b_j, t_j, \bar{t}_j \in H_i$ if and only if $H_j \subseteq H_i$ (see Fig. 1). Although $\bar{t}_i$ is called the mate of t_i, we call $\{t_i, \bar{t}_i\}$ a *dummy line* instead of a line. If $H_i \in \Lambda_{\rm s}$, we regard $\{b_i\}$, $\{t_i\}$, and $\{\bar{t}_i\}$ as $\emptyset$. The algorithm keeps a subset $B^* \subseteq V^*$ such that $B := B^* \cap V \in \mathcal{B}$, $|B^* \cap \{b_i, t_i\}| = 1$, and $|B^* \cap \{t_i, \bar{t}_i\}| = 1$ for each i with $H_i \in \Lambda_{\rm n}$. It also keeps $H_i \cap V \neq H_j \cap V$ for distinct $H_i, H_j \in \Lambda$. This implies that $|\Lambda| = O(n)$, where $n = |V|$, and hence $|V^*| = O(n)$.

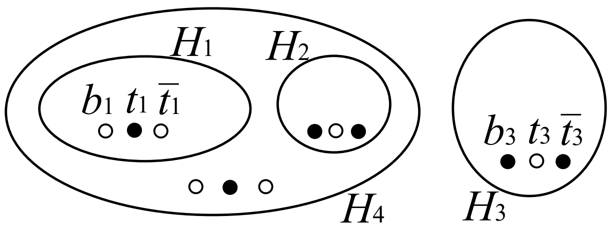

Figure 1: Illustration of blossoms. Black nodes are in B^* and white nodes are in $V^* \setminus B^*$.

The *fundamental circuit matrix* C with respect to a base B is a matrix with row set B and column set $V \setminus B$ obtained by $C = A[U, B]^{-1} A[U, V \setminus B]$. In other words, $[I\ C]$ is obtained from A by identifying B and U, applying row transformations, and changing the ordering of columns. We keep a matrix C^* whose row and column sets are B^* and $V^* \setminus B^*$, respectively, such that the restriction of C^* to V is the fundamental circuit matrix C with respect to B, that is, $C = C^*[V \cap B^*, V \setminus B^*]$. If the row and column sets of C^* are clear, for a vertex set $X \subseteq V^*$, we denote $C^*[X \cap B^*, X \setminus B^*]$ by $C^*[X]$. For each i with $H_i \in \Lambda_{\rm n}$, the matrix C^* satisfies the following properties.

(BT)
- If $b_i, \bar{t}_i \in B^*$ and $t_i \in V^* \setminus B^*$, then $C^*_{b_i t_i} \neq 0$, $C^*_{\bar{t}_i t_i} \neq 0$, $C^*_{b_i v} = 0$ for any $v \in (V^* \setminus B^*) \setminus H_i$, and $C^*_{\bar{t}_i v} = 0$ for any $v \in (V^* \setminus B^*) \setminus \{t_i\}$.
- If $b_i, \bar{t}_i \in V^* \setminus B^*$ and $t_i \in B^*$, then $C^*_{t_i b_i} \neq 0$, $C^*_{t_i \bar{t}_i} \neq 0$, $C^*_{u b_i} = 0$ for any $u \in B^* \setminus H_i$, and $C^*_{u \bar{t}_i} = 0$ for any $u \in B^* \setminus \{t_i\}$.

We henceforth denote by λ the current number of blossoms, i.e., $\lambda := |\Lambda|$. For $i = 0, 1, \ldots, \lambda$, we recursively define a matrix C^i with row set B^* and column set $V^* \setminus B^*$ as follows. Set $C^0 := C^*$. For $i \geq 1$, if $H_i \in \Lambda_{\rm s}$, then define $C^i := C^{i-1}$. Otherwise, define C^i as follows.

- If $b_i \in B^*$ and $t_i \in V^* \setminus B^*$, then C^i is defined to be the matrix obtained from C^{i-1} by a column transformation eliminating $C^{i-1}_{b_i v}$ with $C^{i-1}_{b_i t_i}$ for every $v \in (V^* \setminus B^*) \setminus \{t_i\}$. That is,

$$C^i_{uv} := \begin{cases} C^{i-1}_{uv} - (C^{i-1}_{u t_i} \cdot C^{i-1}_{b_i v} / C^{i-1}_{b_i t_i}) & \text{if } v \in (V^* \setminus B^*) \setminus \{t_i\}, \\ C^{i-1}_{uv} & \text{if } v = t_i. \end{cases} \tag{1}$$

- If $b_i \in V^* \setminus B^*$ and $t_i \in B^*$, then C^i is defined to be the matrix obtained from C^{i-1} by a row transformation eliminating $C^{i-1}_{u b_i}$ with $C^{i-1}_{t_i b_i}$ for every $u \in B^* \setminus \{t_i\}$. That is,

$$C^i_{uv} := \begin{cases} C^{i-1}_{uv} - (C^{i-1}_{t_i v} \cdot C^{i-1}_{u b_i} / C^{i-1}_{t_i b_i}) & \text{if } u \in B^* \setminus \{t_i\}, \\ C^{i-1}_{uv} & \text{if } u = t_i. \end{cases} \tag{2}$$

In the definition of C^i, we use the fact that $C^{i-1}_{b_i t_i} \neq 0$ or $C^{i-1}_{t_i b_i} \neq 0$, which is guaranteed by the following lemma.

Lemma 4.1. *For any $j \in \{0, 1, \ldots, \lambda\}$ and $i \in \{1, \ldots, \lambda\}$ with $H_i \in \Lambda_{\rm n}$, the following statements hold.*

(1) *If $b_i, \bar{t}_i \in B^*$ and $t_i \in V^* \setminus B^*$, then we have the following.*

  (1-1) *$C^j_{b_i v} = 0$ for any $v \in (V^* \setminus B^*) \setminus H_i$ and $C^j_{b_i t_i} = C^*_{b_i t_i} \neq 0$.*

  (1-2) *$C^j_{\bar{t}_i v} = 0$ for any $v \in (V^* \setminus B^*) \setminus H_i$.*

  (1-3) *Suppose that a vertex $u \in B^*$ satisfies that $C^i_{uv} = 0$ for any $v \in (V^* \setminus B^*) \setminus H_i$. If $j \geq i$, then $C^j_{uv} = C^i_{uv}$ for any $v \in V^* \setminus B^*$ and $C^j_{b_i v} = 0$ for any $v \in (V^* \setminus B^*) \setminus \{t_i\}$.*

(2) *If* $b_i, \bar{t}_i \in V^* \setminus B^*$ *and* $t_i \in B^*$*, then we have the following.*

(2-1) $C^j_{ub_i} = 0$ *for any* $u \in B^* \setminus H_i$ *and* $C^j_{t_i b_i} = C^*_{t_i b_i} \neq 0$.

(2-2) $C^j_{u\bar{t}_i} = 0$ *for any* $u \in B^* \setminus H_i$.

(2-3) *Suppose that a vertex* $v \in V^* \setminus B^*$ *satisfies that* $C^i_{uv} = 0$ *for any* $u \in B^* \setminus H_i$. *If* $j \geq i$*, then* $C^j_{uv} = C^i_{uv}$ *for any* $u \in B^*$ *and* $C^j_{ub_i} = 0$ *for any* $u \in B^* \setminus \{t_i\}$.

Proof. We show the claims by induction on j. Suppose that $b_i, \bar{t}_i \in B^*$ and $t_i \in V^* \setminus B^*$.

We first show (1-1). When $j = 0$, the claim is obvious by (BT). For $j \geq 1$ and for $v \in (V^* \setminus B^*) \setminus H_i$, we have the following by induction hypothesis.

- Suppose that $b_j, \bar{t}_j \in B^*$ and $t_j \in V^* \setminus B^*$.
  - If $H_i \cap H_j = \emptyset$ or $H_i \subsetneq H_j$, then $C^{j-1}_{b_i t_j} = 0$ by induction hypothesis (1-1).
  - If $H_j \subseteq H_i$, then $C^{j-1}_{b_j v} = 0$ by induction hypothesis (1-1).
- Suppose that $b_j, \bar{t}_j \in V^* \setminus B^*$ and $t_j \in B^*$. Then, $C^{j-1}_{b_i b_j} = 0$ by induction hypothesis (1-1) or (2-1).

In each case, by the definition of C^j, we have $C^j_{b_i v} = C^{j-1}_{b_i v} = 0$. Similarly, we also obtain that $C^j_{b_i t_i} = C^{j-1}_{b_i t_i}$, which is not zero by induction hypothesis.

When $j = 0$, (1-2) is obvious by (BT). In the same way as (1-1), since $C^{j-1}_{\bar{t}_i t_j} = 0$, $C^{j-1}_{b_j v} = 0$, or $C^{j-1}_{\bar{t}_i b_j} = 0$ by induction hypothesis, we have $C^j_{\bar{t}_i v} = C^{j-1}_{\bar{t}_i v} = 0$, which shows (1-2).

When $j = i$, it is obvious that $C^j_{uv} = C^i_{uv}$ for any $v \in V^* \setminus B^*$. For $j \geq i + 1$, since $C^{j-1}_{ut_j} = 0$ or $C^{j-1}_{ub_j} = 0$ by induction hypothesis, we have that $C^j_{uv} = C^{j-1}_{uv}$ for any $v \in V^* \setminus B^*$, which shows the first half of (1-3). Since $C^i_{b_i v} = 0$ for any $v \in (V^* \setminus B^*) \setminus \{t_i\}$ by (1-1) and by the definition of C^i, we have the second half of (1-3).

The case when $b_i, \bar{t}_i \in V^* \setminus B^*$ and $t_i \in B^*$ can be dealt with in the same way. □

5 Dual Feasibility

In this section, we define feasibility of the dual variables and show their properties. Our algorithm for the minimum-weight parity base problem is designed so that it keeps the dual feasibility.

Recall that a potential $p : V^* \to \mathbb{R}$, and a nonnegative variable $q : \Lambda \to \mathbb{R}_+$ are called dual variables. A blossom H_i is said to be *positive* if $q(H_i) > 0$. For distinct vertices $u, v \in V^*$ and for $H_i \in \Lambda$, we say that a pair (u, v) *crosses* H_i if $|\{u, v\} \cap H_i| = 1$. For distinct $u, v \in V^*$, we denote by I_{uv} the set of indices $i \in \{1, \ldots, \lambda\}$ such that (u, v) crosses H_i. The maximum element of I_{uv} is denoted by i_{uv}. We also denote by J_{uv} the set of indices $i \in \{1, \ldots, \lambda\}$ such that $t_i \in \{u, v\}$. We introduce the set F_Λ of ordered vertex pairs defined by

$$F_\Lambda := \{(u, v) \mid u \in B^*,\ v \in V^* \setminus B^*,\ C^{i_{uv}}_{uv} \neq 0\}.$$

Note that F_Λ is closely related to the nonzero entries in C^λ as we will see later in Observation 7.1. For $u, v \in V^*$, we define

$$Q_{uv} := \sum_{i \in I_{uv} \setminus J_{uv}} q(H_i) - \sum_{i \in I_{uv} \cap J_{uv}} q(H_i).$$

The dual variables are called *feasible with respect to* C^* *and* Λ if they satisfy the following.

(DF1) $p(v) + p(\bar{v}) = w_\ell$ for every line $\ell = \{v, \bar{v}\} \in L$.

(DF2) $p(v) - p(u) \geq Q_{uv}$ for every $(u, v) \in F_\Lambda$.

(DF3) $p(b_j) = p(\bar{t}_j) = p(t_j)$ for every $H_j \in \Lambda_{\mathrm{n}}$.

If no confusion may arise, we omit C^* and Λ when we discuss dual feasibility.

Note that if p satisfies (DF1), $\Lambda = \emptyset$, and $B \in \mathcal{B}$ minimizes $p(B) = \sum_{u \in B} p(u)$ in $\mathcal{B}$, then p and q are feasible. This ensures that the initial setting of the algorithm satisfies the dual feasibility.

We now show some properties of feasible dual variables.

Lemma 5.1. *For distinct vertices $u \in B^*$ and $v \in V^* \setminus B^*$, we have $(u, v) \in F_\Lambda$ if and only if $C^*[X]$ is nonsingular, where $X := \{u, v\} \cup \bigcup \{\{b_i, t_i\} \mid i \in I_{uv} \setminus J_{uv},\ H_i \in \Lambda_{\mathrm{n}}\}$.*

Proof. By (1-3) and (2-3) of Lemma 4.1, if $b_i \in B^*$ for $i \in I_{uv} \setminus J_{uv}$, then $C^{i_{uv}}_{b_i v'} = 0$ for any $v' \in (V^* \setminus B^*) \setminus \{t_i\}$ and $C^{i_{uv}}_{b_i t_i} \neq 0$, and if $b_i \in V^* \setminus B^*$ for $i \in I_{uv} \setminus J_{uv}$, then $C^{i_{uv}}_{u' b_i} = 0$ for any $u' \in B^* \setminus \{t_i\}$ and $C^{i_{uv}}_{t_i b_i} \neq 0$. This implies that $C^{i_{uv}}_{uv} \neq 0$ is equivalent to that $C^{i_{uv}}[X]$ is nonsingular.

For any $j \in \{1, \ldots, i_{uv}\}$, either $X \cap H_j = \emptyset$ or $\{b_j, t_j\} \subseteq X$ holds. By (1-1) and (2-1) of Lemma 4.1, this shows that either $C^j[X] = C^{j-1}[X]$ or $C^j[X]$ is obtained from $C^{j-1}[X]$ by applying elementary operations. Therefore, $C^{i_{uv}}[X]$ is obtained from $C^*[X]$ by applying elementary operations, and hence the nonsingularity of $C^{i_{uv}}[X]$ is equivalent to that of $C^*[X]$. □

When we are given a set of blossoms Λ, there may be more than one way of indexing the blossoms so that for any two members $H_i, H_j \in \Lambda$ with $i < j$, either $H_i \cap H_j = \emptyset$ or $H_i \subsetneq H_j$ holds. Lemma 5.1 guarantees that this flexibility does not affect the definition of the dual feasibility. Thus, we can renumber the indices of the blossoms if necessary.

The following lemma guarantees that we can remove (or add) a blossom H with $q(H) = 0$ from (or to) Λ. The proof is given in Appendix A.

Lemma 5.2. *Suppose that $p : V^* \to \mathbb{R}$ and $q : \Lambda \to \mathbb{R}_+$ are dual variables, and let $i \in \{1, 2, \dots, \lambda\}$ be an index such that $q(H_i) = 0$. Suppose that $p(b_i) = p(t_i) = p(\bar{t}_i)$ if $H_i \in \Lambda_{\rm n}$. Let q' be the restriction of q to $\Lambda' := \Lambda \setminus \{H_i\}$. Then, p and q are feasible with respect to Λ if and only if p and q' are feasible with respect to Λ'. Here, we do not remove $\{b_i, t_i, \bar{t}_i\}$ from V^* even when we consider dual feasibility with respect to Λ'.*

The next lemma shows that $p(v) - p(u) \geq Q_{uv}$ holds if u and v satisfy a certain condition. The proof is given in Appendix A.

Lemma 5.3. *Let p and q be feasible dual variables and let $k \in \{0, 1, \dots, \lambda\}$. For any $u \in B^*$ and $v \in V^* \setminus B^*$ with $i_{uv} \leq k$ and $C^k_{uv} \neq 0$, it holds that*

$$p(v) - p(u) \geq Q_{uv} = \sum_{i \in I_{uv} \setminus J_{uv}} q(H_i) - \sum_{i \in I_{uv} \cap J_{uv}} q(H_i). \tag{3}$$

By using Lemma 5.3, we have the following lemma.

Lemma 5.4. *Suppose that p and q are feasible dual variables. Let k be an integer and let $X \subseteq V^*$ be a vertex subset such that $X \cap H_i = \emptyset$ for any $i > k$ and $C^k[X]$ is nonsingular. Then, we have*

$$\begin{aligned} p(X \setminus B^*) - p(X \cap B^*) \geq & -\sum \{q(H_i) \mid H_i \in \Lambda_{\rm n},\ |X \cap H_i| \text{ is odd},\ t_i \in X\} \\ & + \sum \{q(H_i) \mid H_i \in \Lambda_{\rm n},\ |X \cap H_i| \text{ is odd},\ t_i \notin X\} \\ & + \sum \{q(H_i) \mid H_i \in \Lambda_{\rm s},\ |X \cap H_i| \text{ is odd}\}. \end{aligned}$$

Proof. Since $C^k[X]$ is nonsingular, there exists a perfect matching $M = \{(u_j, v_j) \mid j = 1, \dots, \mu\}$ between $X \cap B^*$ and $X \setminus B^*$ such that $u_j \in X \cap B^*$, $v_j \in X \setminus B^*$, and $C^k_{u_j v_j} \neq 0$ for $j = 1, \dots, \mu$. Since $X \cap H_i = \emptyset$ for any $i > k$ implies that $i_{u_j v_j} \leq k$, by Lemma 5.3, we have

$$p(v_j) - p(u_j) \geq Q_{u_j v_j}$$

for $j = 1, \dots, \mu$. By combining these inequalities, we obtain

$$p(X \setminus B^*) - p(X \cap B^*) \geq \sum_{j=1}^{\mu} Q_{u_j v_j} = \sum_{j=1}^{\mu} \left(\sum_{i \in I_{u_j v_j} \setminus J_{u_j v_j}} q(H_i) - \sum_{i \in I_{u_j v_j} \cap J_{u_j v_j}} q(H_i) \right). \tag{4}$$

It suffices to show that, for each i, the coefficient of $q(H_i)$ in the right hand side of (4) is

- at least -1 if $H_i \in \Lambda_{\rm n}$, $|X \cap H_i|$ is odd, and $t_i \in X$,
- at least 1 if $H_i \in \Lambda_{\rm n}$, $|X \cap H_i|$ is odd, and $t_i \notin X$,
- at least 1 if $H_i \in \Lambda_{\rm s}$ and $|X \cap H_i|$ is odd, and

- at least 0 if $|X \cap H_i|$ is even.

For each i, since $i \in I_{u_j v_j} \cap J_{u_j v_j}$ implies $t_i \in \{u_j, v_j\}$, there exists at most one index j such that $i \in I_{u_j v_j} \cap J_{u_j v_j}$. This shows that the coefficient of $q(H_i)$ in (4) is at least -1.

Suppose that either (i) $H_i \in \Lambda_{\rm n}$, $|X \cap H_i|$ is odd, and $t_i \notin X$, or (ii) $H_i \in \Lambda_{\rm s}$ and $|X \cap H_i|$ is odd. In both cases, there is no index j with $i \in I_{u_j v_j} \cap J_{u_j v_j}$. Furthermore, since $|X \cap H_i|$ is odd, there exists an index j' such that $i \in I_{u_{j'} v_{j'}}$, which shows that the coefficient of $q(H_i)$ in (4) is at least 1.

If $|X \cap H_i|$ is even, then there exist an even number of indices j such that (u_j, v_j) crosses H_i. Therefore, if there exists an index j such that $i \in I_{u_j v_j} \cap J_{u_j v_j}$, then there exists another index j' such that $i \in I_{u_{j'} v_{j'}} \setminus J_{u_{j'} v_{j'}}$. Thus, the coefficient of $q(H_i)$ in (4) is at least 0 if $|X \cap H_i|$ is even. □

We now consider the tightness of the inequality in Lemma 5.4. For $k = 0, 1, \dots, \lambda$, let $G^k = (V^*, F^k)$ be the graph such that $(u, v) \in F^k$ if and only if $C^k_{uv} \neq 0$ (or $C^k_{vu} \neq 0$). An edge $(u, v) \in F^k$ with $u \in B^*$ and $v \in V^* \setminus B^*$ is said to be *tight* if $p(v) - p(u) = Q_{uv}$. We say that a matching $M \subseteq F^k$ is *consistent with a blossom* $H_i \in \Lambda$ if one of the following three conditions holds:

- $H_i \in \Lambda_{\rm s}$ and $|\{(u, v) \in M \mid i \in I_{uv}\}| \leq 1$,
- $H_i \in \Lambda_{\rm n}$, $t_i \notin \partial M$, and $|\{(u, v) \in M \mid i \in I_{uv}\}| \leq 1$,
- $H_i \in \Lambda_{\rm n}$, $t_i \in \partial M$, and $|\{(u, v) \in M \mid i \in I_{uv} \setminus J_{uv}\}| \leq |\{(u, v) \in M \mid i \in I_{uv} \cap J_{uv}\}|$.

Here, ∂M denotes the set of the end vertices of M. For $k \in \{1, \dots, \lambda\}$, we say that a matching $M \subseteq F^k$ is *tight* if every edge of M is tight and M is consistent with every positive blossom H_i. As the proof of Lemma 5.4 clarifies, if there exists a tight perfect matching M in the subgraph $G^k[X]$ of G^k induced by X, then the inequality of Lemma 5.4 is tight. Furthermore, in such a case, every perfect matching in $G^k[X]$ must be tight, which is stated as follows.

Lemma 5.5. *For $k \in \{0, 1, \dots, \lambda\}$ and a vertex set $X \subseteq V^*$, if $G^k[X]$ has a tight perfect matching, then any perfect matching in $G^k[X]$ is tight.*

We can also see the following lemma by using Lemma 5.4.

Lemma 5.6. *Suppose that p and q are feasible dual variables and $X \subseteq V^*$ is a vertex set such that $C^*[X]$ is nonsingular. Then we have*

$$p(X \setminus B^*) - p(X \cap B^*) \geq -\sum \{q(H_i) \mid H_i \in \Lambda_{\rm n},\ |X \cap H_i| \textit{ is odd}\} + \sum \{q(H_i) \mid H_i \in \Lambda_{\rm s},\ |X \cap H_i| \textit{ is odd}\}.$$

Proof. If none of t_i and $\bar{t}_i$ are contained in X, then $X' := X \cup \{t_i, \bar{t}_i\}$ satisfies that

- $C^*[X]$ is nonsingular if and only if $C^*[X']$ is nonsingular,
- $p(X \setminus B^*) - p(X \cap B^*) = p(X' \setminus B^*) - p(X' \cap B^*)$ by (DF3),
- $|X \cap H_i|$ is odd if and only if $|X' \cap H_i|$ is odd for each i.

Thus it suffices to prove the inequality for X' instead of X. Furthermore, by (BT) and the nonsingularity of $C^*[X]$, $t_i \notin X$ implies that $\bar{t}_i \notin X$. With these observations, it suffices to consider the case when X contains all the tips t_i. Since X contains all the tips t_i, $C^\lambda[X]$ is obtained from $C^*[X]$ by applying elementary operations, and hence $C^\lambda[X]$ is nonsingular. This implies the inequality by Lemma 5.4. □

6 Optimality

In this section, we show that if we obtain a parity base B and feasible dual variables p and q, then B is a minimum-weight parity base.

Theorem 6.1. *If $B := B^* \cap V$ is a parity base and there exist feasible dual variables p and q, then B is a minimum-weight parity base.*

Proof. Since the optimal value of the minimum-weight parity base problem is represented with $\deg_\theta \mathrm{Pf}\, \Phi_A(\theta)$ as shown in Lemma 2.1, we evaluate the value of $\deg_\theta \mathrm{Pf}\, \Phi_A(\theta)$, assuming that we have a parity base B and feasible dual variables p and q.

Recall that A is transformed to $[I\ C]$ by applying row transformations and column permutations, where C is the fundamental circuit matrix with respect to the base B obtained by $C = A[U, B]^{-1} A[U, V \setminus B]$. Note that the identity submatrix gives a one to one correspondence between U and B, and the row set of C can be regarded as U. We now apply the same row transformations and column permutations to $\Phi_A(\theta)$, and then apply also the corresponding column transformations and row permutations to obtain a skew-symmetric polynomial matrix $\Phi'_A(\theta)$, that is,

$$\Phi'_A(\theta) = \left(\begin{array}{c|cc} O & I & C \\ \hline -I & & \\ -C^\top & \multicolumn{2}{c}{D'(\theta)} \end{array}\right) \begin{array}{l} \leftarrow U \\ \leftarrow B \\ \leftarrow V \setminus B \end{array},$$

where $D'(\theta)$ is in a block-diagonal form obtained from $D(\theta)$ by applying row and column permutations simultaneously. Note that $\mathrm{Pf}\, \Phi'_A(\theta) = \pm \mathrm{Pf}\, \Phi_A(\theta) / \det A[U, B]$, where the sign is determined by the ordering of V.

We now define

$$\Phi^*_A(\theta) = \left(\begin{array}{c|c|c|c|c} O & O & O & \multicolumn{2}{c}{\multirow{2}{*}{$C^*[V\cup T]$}} \\ \cline{1-3} O & O & I & \multicolumn{2}{c}{} \\ \hline O & -I & \multicolumn{2}{c|}{\multirow{2}{*}{$D'(\theta)$}} & \multirow{2}{*}{O} \\ \cline{1-2} \multicolumn{2}{c|}{\multirow{2}{*}{$-(C^*[V\cup T])^\top$}} & \multicolumn{2}{c|}{} & \\ \cline{3-5} \multicolumn{2}{c|}{} & \multicolumn{2}{c|}{O} & O \end{array}\right) \begin{array}{l} \leftarrow T\cap B^* \\ \leftarrow U \text{ (identified with } B) \\ \leftarrow B \\ \leftarrow V\setminus B \\ \leftarrow T\setminus B^* \end{array}$$

obtained from $\Phi'_A(\theta)$ by attaching rows and columns corresponding to $T := \{t_i, \bar{t}_i \mid i \in \{1,\ldots,\lambda\}\}$. Note that t_i and $\bar{t}_i$ do exist for each i, as there is no source line and hence $\Lambda = \Lambda_{\mathrm{n}}$. The row and column sets of $\Phi^*_A(\theta)$ are both indexed by $W^* := V \cup U \cup T$. By the definition of $\bar{t}_i$, we have $(\Phi^*_A(\theta))_{\bar{t}_i v} = 0$ for $v \in W^* \setminus \{t_i\}$ and $(\Phi^*_A(\theta))_{\bar{t}_i t_i}$ is a nonzero constant, which shows that $\deg_\theta \operatorname{Pf} \Phi^*_A(\theta) = \deg_\theta \operatorname{Pf} \Phi'_A(\theta)$.

Recall that C^λ is obtained from C^* by adding a row (resp. column) corresponding to t_i to another row (resp. column) repeatedly. By applying the same transformation to $\Phi^*_A(\theta)$, we obtain the following matrix:

$$\Phi^\lambda_A(\theta) = \left(\begin{array}{c|c|c|c|c} O & O & O & \multicolumn{2}{c}{\multirow{2}{*}{$C^\lambda[V\cup T]$}} \\ \cline{1-3} O & O & I & \multicolumn{2}{c}{} \\ \hline O & -I & \multicolumn{2}{c|}{\multirow{2}{*}{$D'(\theta)$}} & \multirow{2}{*}{O} \\ \cline{1-2} \multicolumn{2}{c|}{\multirow{2}{*}{$-(C^\lambda[V\cup T])^\top$}} & \multicolumn{2}{c|}{} & \\ \cline{3-5} \multicolumn{2}{c|}{} & \multicolumn{2}{c|}{O} & O \end{array}\right).$$

Note that $\operatorname{Pf} \Phi^\lambda_A(\theta) = \operatorname{Pf} \Phi^*_A(\theta)$. Thus we have $\deg_\theta \operatorname{Pf} \Phi^\lambda_A(\theta) = \deg_\theta \operatorname{Pf} \Phi_A(\theta)$.

Construct a graph $\Gamma^* = (W^*, E^*)$ with edge set E^* defined by $E^* = \{(u,v) \mid (\Phi^\lambda_A(\theta))_{uv} \neq 0\}$. Each edge $e = (u,v) \in E^*$ has a weight $w(e) := \deg_\theta (\Phi^\lambda_A(\theta))_{uv}$. Then it can be easily seen that the maximum weight of a perfect matching in Γ^* is at least $\deg_\theta \operatorname{Pf} \Phi^\lambda_A(\theta) = \deg_\theta \operatorname{Pf} \Phi_A(\theta)$. Let us recall that the dual linear program of the maximum weight perfect matching problem on Γ^* is formulated as follows.

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{v\in W^*} \pi(v) - \sum_{Z\in\Omega} \xi(Z) & \\ \text{subject to} & \displaystyle\pi(u) + \pi(v) - \sum_{Z\in\Omega_{uv}} \xi(Z) \geq w(e) & (e = (u,v) \in E^*), \\ & \xi(Z) \geq 0 \qquad (Z \in \Omega), & \end{array} \tag{5}$$

where $\Omega = \{Z \mid Z \subseteq W^*, |Z|\text{: odd}, |Z| \geq 3\}$ and $\Omega_{uv} = \{Z \mid Z \in \Omega, |Z \cap \{u,v\}| = 1\}$ (see e.g. [30, Theorem 25.1]). In what follows, we construct a feasible solution (π, ξ) of this linear program. The objective value provides an upper bound on the maximum weight of a perfect matching in Γ^*, and consequently serves as an upper bound on $\deg_\theta \operatorname{Pf} \Phi_A(\theta)$.

Since $\Phi^\lambda_A(\theta)[U,B]$ is the identity matrix, we can naturally define a bijection $\beta : B \to U$ between B and U. For $v \in U \cup (T \cap B^*)$, let v' be the vertex in V^* that corresponds to

v, that is, $v' = \beta^{-1}(v)$ if $v \in U$ and $v' = v$ if $v \in T \cap B^*$. We define $\pi' : W^* \to \mathbb{R}$ by

$$\pi'(v) = \begin{cases} p(v) & \text{if } v \in V \cup (T \setminus B^*), \\ -p(v') & \text{if } v \in U \cup (T \cap B^*), \end{cases}$$

and define $\pi : W^* \to \mathbb{R}$ by

$$\pi(v) = \begin{cases} \pi'(v) + q(H_i) & \text{if } v = t_i \text{ or } v' = t_i \text{ for some } i, \\ \pi'(v) & \text{otherwise.} \end{cases}$$

For $i \in \{1, \ldots, \lambda\}$, let $Z_i = (H_i \cap V) \cup \beta(H_i \cap B) \cup \{\bar{t}_i\}$ and define $\xi(Z_i) = q(H_i)$. See Fig. 2 for an example. For any $i \in \{1, \ldots, \lambda\}$, since $H_i \cap V$ consists of lines and there is no source line in G, we see that both $|H_i \cap V|$ and $|\beta(H_i \cap B)|$ are even, which shows that $|Z_i|$ is odd and $|Z_i| \geq 3$. Define $\xi(Z) = 0$ for any $Z \in \Omega \setminus \{Z_1, \ldots, Z_\lambda\}$. We now show the following claim.

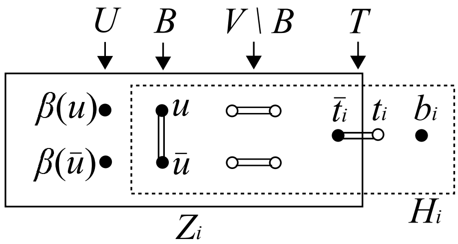

Figure 2: Definition of Z_i. Lines and dummy lines are represented by double bonds.

Claim 6.2. *The dual variables π and ξ defined as above form a feasible solution of the linear program.*

Proof. Suppose that $e = (u, v) \in E^*$. If $u, v \in V$ and $u = \bar{v}$, then (DF1) shows that $\pi(u) + \pi(v) = p(\bar{v}) + p(v) = w_\ell = w(e)$, where $\ell = \{v, \bar{v}\}$. Since $|Z_i \cap \{v, \bar{v}\}|$ is even for any $i \in \{1, \ldots, \lambda\}$, this shows (5). If $u \in U$ and $v \in B$, then $(u, v) \in E^*$ implies that $u = \beta(v)$, and hence $\pi(u) + \pi(v) = 0$, which shows (5) as $|Z_i \cap \{u, v\}|$ is even for any $i \in \{1, \ldots, \lambda\}$.

The remaining case of $(u, v) \in E^*$ is when $u \in U \cup (T \cap B^*)$ and $v \in (V \setminus B) \cup (T \setminus B^*)$. That is, it suffices to show that (u, v) satisfies (5) if $C^\lambda_{uv} \neq 0$. Recall that u' is the vertex in V^* that corresponds to u. By the definition of π, we have

$$\pi(u) + \pi(v) = p(v) - p(u') + \sum_{i \in J_{u'v}} q(H_i). \tag{6}$$

By the definition of Z_i, we have $|Z_i \cap \{u, v\}| = 1$ if and only if $i \in I_{u'v} \triangle J_{u'v}$, which shows that

$$\sum_{i:\, |Z_i \cap \{u,v\}|=1} \xi(Z_i) = \sum_{i \in I_{u'v} \setminus J_{u'v}} q(H_i) + \sum_{i \in J_{u'v} \setminus I_{u'v}} q(H_i). \tag{7}$$

Since $C^\lambda_{uv} \neq 0$, by Lemma 5.3, we have

$$p(v) - p(u') \geq Q_{uv} = \sum_{i \in I_{u'v} \setminus J_{u'v}} q(H_i) - \sum_{i \in I_{u'v} \cap J_{u'v}} q(H_i). \tag{8}$$

By combining (6), (7), and (8), we obtain

$$\pi(u) + \pi(v) - \sum_{i \colon |Z_i \cap \{u,v\}| = 1} \xi(Z_i) \geq 0,$$

which shows that (u, v) satisfies (5). □

The objective value of this feasible solution is

$$\sum_{v \in W^*} \pi(v) - \sum_{i \in \{1,\dots,\lambda\}} \xi(Z_i) = \sum_{v \in W^*} \pi'(v) = \sum_{v \in V \setminus B} p(v) = \sum_{\ell \subseteq V \setminus B} w_\ell, \tag{9}$$

where the first equality follows from the definition of π and ξ, the second one follows from the definition of π' and the fact that $p(\bar{t}_i) = p(t_i)$ for each i, and the third one follows from (DF1). By the weak duality of the maximum weight matching problem, we have

$$\begin{aligned} \sum_{v \in W^*} \pi(v) - \sum_{i \in \{1,\dots,\lambda\}} \xi(Z_i) &\geq (\text{maximum weight of a perfect matching in } \Gamma^*) \\ &\geq \deg_\theta \mathrm{Pf}\, \Phi^\lambda_A(\theta) = \deg_\theta \mathrm{Pf}\, \Phi_A(\theta). \end{aligned} \tag{10}$$

On the other hand, Lemma 2.1 shows that any parity base B' satisfies that

$$\sum_{\ell \subseteq B'} w_\ell \geq \sum_{\ell \in L} w_\ell - \deg_\theta \mathrm{Pf}\, \Phi_A(\theta), \tag{11}$$

Combining (9)–(11), we have $\sum_{\ell \subseteq V \setminus B} w_\ell = \deg_\theta \mathrm{Pf}\, \Phi_A(\theta)$, which means B is a minimum-weight parity base. □

7 Finding an Augmenting Path

In this section, we define an augmenting path and present a procedure for finding one. The validity of our procedure is shown in Section 8.

Suppose we are given V^*, B^*, C^*, Λ, and feasible dual variables p and q. Recall that, for $i = 0, 1, \dots, \lambda$, we denote by $G^i = (V^*, F^i)$ the graph with edge set $F^i := \{(u,v) \mid C^i_{uv} \neq 0\}$. Since $C^0 = C^*$, we use F^* instead of F^0. By Lemma 5.3, we have $p(v) - p(u) \geq Q_{uv}$ if $(u,v) \in F^\lambda$, $u \in B^*$, and $v \in V^* \setminus B^*$. Let $F^\circ \subseteq F^\lambda$ be the set of tight edges in F^λ, that is, $F^\circ = \{(u,v) \in F^\lambda \mid u \in B^*,\ v \in V^* \setminus B^*,\ p(v) - p(u) = Q_{uv}\}$.

Our procedure works primarily on the graph $G^\circ = (V^*, F^\circ)$. For a vertex set $X \subseteq V^*$, $G^\circ[X]$ (resp. $G^i[X]$) denotes the subgraph of G° (resp. G^i) induced by X.

Each normal blossom $H_i \in \Lambda_{\rm n}$ has a specified vertex $g_i \in H_i$, which we call a *generator* of t_i. When we search for an augmenting path, we keep the following properties of g_i and t_i.

(GT1) For each $H_i \in \Lambda_{\rm n}$, there is no edge of F^i between g_i and $V^* \setminus H_i$.

(GT2) For each $H_i \in \Lambda_{\rm n}$, there is no edge of F^* between t_i and $H_i \setminus \{b_i, \bar{t}_i\}$.

By (1-3) and (2-3) of Lemma 4.1, if $i \leq j \leq \lambda$, (GT1) implies that there is no edge of F^j between g_i and $V^* \setminus H_i$. By (GT2), we can see that for each $H_i \in \Lambda_{\rm n}$ and for $j = 0, 1, \ldots, \lambda$, there is no edge in F^j between t_i and $H_i \setminus \{b_i, \bar{t}_i\}$. Furthermore, since (GT2) implies that $C^{i-1}_{uv} = C^i_{uv}$ for each i and $u, v \in H_i$, we have the following observation.

Observation 7.1. *If* (GT2) *holds, then F_Λ coincides with F^λ regardless of the ordering.*

With this observation, it is natural to ask whether one can define the dual feasibility by using F^λ instead of F_Λ. However, (GT2) will be tentatively violated just after the augmentation, which is the reason why we use F_Λ in the definition of the dual feasibility.

Roughly, our procedure finds a part of the augmenting path outside the blossoms. The routing in each blossom H_i is determined by a prescribed vertex set $R_{H_i}(x)$. For $i = 1, \ldots, \lambda$, define $H_i^\circ := (H_i \setminus \{t_i\}) \setminus \{b_j \mid H_j \in \Lambda_{\rm n}\}$, where $\{t_i\} = \emptyset$ if $H_i \in \Lambda_{\rm s}$. For any $i \in \{1, \ldots, \lambda\}$ and for any $x \in H_i^\circ$, the prescribed vertex set $R_{H_i}(x) \subseteq H_i$ is assumed to satisfy the following.

(BR1) $x \in R_{H_i}(x) \subseteq H_i \setminus \{b_j \mid H_j \in \Lambda_{\rm n}\}$.

(BR2) If $H_i \in \Lambda_{\rm n}$, then $R_{H_i}(x)$ consists of lines and dummy lines. If $H_i \in \Lambda_{\rm s}$, then $R_{H_i}(x)$ consists of lines, dummy lines, and a source vertex.

(BR3) For any $j \in \{1, 2, \ldots, i\}$ with $R_{H_i}(x) \cap H_j \neq \emptyset$, it holds that $\{t_j, \bar{t}_j\} \subseteq R_{H_i}(x)$.

We sometimes regard $R_{H_i}(x)$ as a sequence of vertices, and in such a case, the last two vertices are $\bar{x}x$. We also suppose that the first two vertices are $t_i\bar{t}_i$ if $H_i \in \Lambda_{\rm n}$ and the first vertex is the unique source vertex in $R_{H_i}(x)$ if $H_i \in \Lambda_{\rm s}$. Each blossom $H_i \in \Lambda$ is assigned a total order $<_{H_i}$ among all the vertices in H_i°. In the procedure, $R_{H_i}(x)$ keeps additional properties which will be described in Section 8.1.

We say that a vertex set $P \subseteq V^*$ is an augmenting path if it satisfies the following properties.

(AP1) P consists of normal lines, dummy lines, and two vertices from distinct source lines.

(AP2) For each $H_i \in \Lambda$, either $P \cap H_i = \emptyset$ or $P \cap H_i = R_{H_i}(x_i)$ for some $x_i \in H_i^\circ$.

(AP3) $G^\circ[P]$ has a unique tight perfect matching.

In the rest of this section, we describe how to find an augmenting path. Section 7.1 is devoted to the search procedure, which calls two procedures: RBlossom and DBlossom. Here, R and D stand for "regular" and "degenerate," respectively. The details of these procedures are described in Section 7.2.

7.1 Search Procedure

In this subsection, we describe a procedure for searching for an augmenting path. The procedure performs the breadth-first search using a queue to grow paths from source vertices. A vertex $v \in V^*$ is labeled and put into the queue when it is reached by the search. The procedure picks the first labeled element from the queue, and examines its neighbors. A linear order $\prec$ is defined on the labeled vertex set so that $u \prec v$ means u is labeled prior to v.

For each $x \in V^*$, we denote by $K(x)$ the maximal blossom that contains x. If a vertex $x \in V$ is not contained in any blossom, then it is called *single* and we denote $K(x) = \{x, \bar{x}\}$. The procedure also labels some blossoms with $\oplus$ or $\ominus$, which will be used later for modifying dual variables. With each labeled vertex v, the procedure associates a path $P(v)$ and its subpath $J(v)$, where a path is a sequence of vertices. The first vertex of $P(v)$ is a labeled vertex in a source line and the last one is v. The reverse path of $P(v)$ is denoted by $\overline{P(v)}$. For a path $P(v)$ and a vertex r in $P(v)$, we denote by $P(v|r)$ the subsequence of $P(v)$ after r (not including r). We sometimes identify a path with its vertex set. When an unlabeled vertex u is examined in the procedure, we assign a vertex $\rho(u)$ and a path $I(u)$. The procedure is described as follows.

Procedure Search

Step 0: Initialize the objects so that the queue is empty, every vertex is unlabeled, and every blossom is unlabeled.

Step 1: While there exists an unlabeled single vertex x in a source line, label x with $P(x) := J(x) := x$ and put x into the queue. While there exists an unlabeled maximal source blossom $H_i \in \Lambda_{\mathrm{s}}$, label H_i with $\oplus$ and do the following: for each vertex $x \in H_i^\circ$ in the order of $<_{H_i}$, label x with $P(x) := J(x) := R_{H_i}(x)$ and put x into the queue.

Step 2: If the queue is empty, then return $\emptyset$ and terminate the procedure (see Section 9). Otherwise, remove the first element v from the queue.

Step 3: While there exists a labeled vertex u adjacent to v in G° with $K(u) \neq K(v)$, choose such u that is minimum with respect to $\prec$ and do the following steps (3-1) and (3-2).

(3-1) If the first elements in $P(v)$ and in $P(u)$ belong to different source lines, then return $P := P(v)\overline{P(u)}$ as an augmenting path.

(3-2) Otherwise, apply $\mathsf{RBlossom}(v, u)$ to add a new blossom to Λ.

Step 4: While there exists an unlabeled vertex u adjacent to v in G° such that $\rho(u)$ is not assigned, do the following steps (4-1)–(4-5).

(4-1) If u is a single vertex and $(v, \bar{u}) \notin F^\circ$, then label $\bar{u}$ with $P(\bar{u}) := P(v)u\bar{u}$ and $J(\bar{u}) := \{\bar{u}\}$, set $\rho(u) := v$ and $I(u) := \{u\}$, and put $\bar{u}$ into the queue.

(4-2) If u is a single vertex and $(v, \bar{u}) \in F^\circ$, then apply $\mathsf{DBlossom}(v, u)$.

(4-3) If $K(u) = H_i \in \Lambda_{\rm n}$, $(v, t_i) \in F^\circ$, and F^λ contains an edge between v and $H_i \setminus \{t_i\}$, then apply $\mathsf{DBlossom}(v, t_i)$.

(4-4) If $K(u) = H_i \in \Lambda_{\rm n}$, $(v, t_i) \in F^\circ$, and F^λ contains no edge between v and $H_i \setminus \{t_i\}$, then label H_i with $\oplus$, set $\rho(t_i) := v$ and $I(t_i) := \{t_i\}$, and do the following. For each unlabeled vertex $x \in H_i^\circ$ in the order of $<_{H_i}$, label x with $P(x) := P(v)R_{H_i}(x)$ and $J(x) := R_{H_i}(x) \setminus \{t_i\}$, and put x into the queue.

(4-5) If $K(u) = H_i \in \Lambda_{\rm n}$ and $(v, t_i) \notin F^\circ$, then choose $y \in H_i \setminus \{t_i\}$ with $(v, y) \in F^\circ$ that is minimum with respect to $<_{H_i}$, and do the following. Label H_i with $\ominus$, label t_i with $P(t_i) := P(v)\overline{R_{H_i}(y)}$ and $J(t_i) := \{t_i\}$, and put t_i into the queue. For each unlabeled vertex $x \in H_i^\circ$, set $\rho(x) = v$ and $I(x) := \overline{R_{H_i}(x)} \setminus \{t_i\}$.

Step 5: Go back to Step 2.

7.2 Creating a Blossom

In this subsection, we describe two procedures that create a new blossom. The first one is $\mathsf{RBlossom}$ called in Step (3-2) of Search.

Procedure $\mathsf{RBlossom}(v, u)$

Step 1: Let c be the last vertex in $P(v)$ such that $K(c)$ contains a vertex in $P(u)$. Let d be the last vertex in $P(u)$ contained in $K(c)$. Note that $K(c) = K(d)$. If $c = d$, then define $Y := \bigcup\{K(x) \mid x \in P(v|c) \cup P(u|d)\}$ and $r := c$, Otherwise, define $Y := \bigcup\{K(x) \mid x \in P(v|c) \cup P(u|d) \cup \{c\}\}$ and let r be the last vertex in $P(v)$ not contained in Y if exists. See Fig. 3 for an example.

Step 2: If Y contains no source line, then define g to be the vertex subsequent to r in $P(v)$ and introduce new vertices b, t, and $\bar{t}$ (see below for the details).

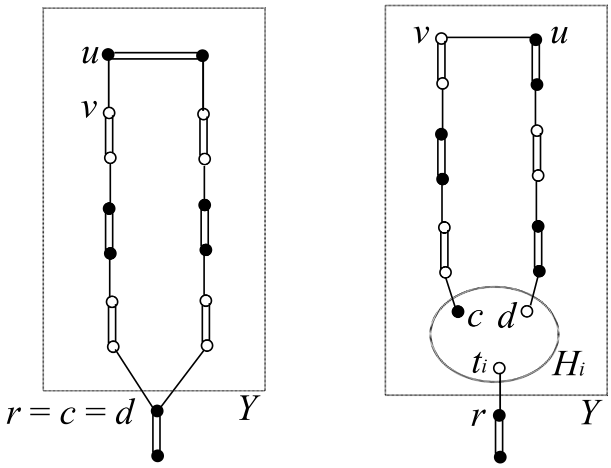

Figure 3: Definition of Y.

Step 3: Define $H := Y \cup \{b, t, \bar{t}\}$ if Y contains no source line, and $H := Y$ otherwise.

Step 4: If H contains no source line, then for each labeled vertex x with $P(x) \cap H \neq \emptyset$, replace $P(x)$ by $P(x) := P(r)t\bar{t}P(x|r)$. Label $\bar{t}$ with $P(\bar{t}) := P(r)t\bar{t}$ and $J(\bar{t}) := \{\bar{t}\}$, and extend the ordering $\prec$ of the labeled vertices so that $\bar{t}$ is just after r, i.e., $r \prec \bar{t}$ and no element is between r and $\bar{t}$. For each vertex $x \in H$ with $\rho(x) = r$, update $\rho(x)$ as $\rho(x) := \bar{t}$. Set $\rho(t) := r$ and $I(t) := \{t\}$.

Step 5: For each unlabeled vertex $x \in H^\circ$, label x with

$$P(x) := \begin{cases} P(v)\overline{P(u|x)}x & \text{if } x \in P(u|d), \\ P(u)\overline{P(v|x)}x & \text{if } x \in P(v|c), \\ P(v)\overline{P(u|t_i)}R_{H_i}(x) & \text{if } K(x) = H_i,\ H_i \text{ is labeled with } \ominus, \text{ and } t_i \in P(u|d), \\ P(u)\overline{P(v|t_i)}R_{H_i}(x) & \text{if } K(x) = H_i,\ H_i \text{ is labeled with } \ominus, \text{ and } t_i \in P(v|c), \end{cases}$$

and $J(x) := P(x|t)$, and put x into the queue. Here, we choose the vertices so that the following conditions hold.

- For two unlabeled vertices $x, y \in H^\circ$, if $\rho(x) \succ \rho(y)$, then we choose x earlier than y.
- For two unlabeled vertices $x, y \in H^\circ$, if $\rho(x) = \rho(y)$, $K(x) = K(y) = H_i$, and $x <_{H_i} y$, then we choose x earlier than y.
- If $r = c = d$, then no element is chosen between g and h, where h is the vertex subsequent to $\bar{t}$ in $P(u)$.

Step 6: Label H with $\oplus$. Define $R_H(x) := P(x|r)$ for each $x \in H^\circ$, where $P(x|r)$ denotes $P(x)$ if r does not exist. Define $<_H$ by the ordering $\prec$ of the labeled vertices in H_i°. Add H to Λ with $q(H) = 0$ regarding b, t, $\bar{t}$, and g, if exist, as the bud of H, the tip of H, the mate of t, and the generator of t, respectively, and update $\Lambda_{\rm n}$, $\Lambda_{\rm s}$, λ, C^λ, G°, and $K(v)$ for $v \in V^*$, accordingly.

We note that, for any $x \in V^*$, if $J(x)$ (resp. $I(x)$) is defined, then it is equal to either $\{x\}$ or $R_{H_i}(x) \setminus \{t_i\}$ (resp. either $\{x\}$ or $\overline{R_{H_i}(x)} \setminus \{t_i\}$) for some $H_i \in \Lambda$. In particular, the last element of $J(x)$ and the first element of $I(x)$ are x. We also note that $J(x)$ and $I(x)$ are not used in the procedure explicitly, but we introduce them to show the validity of the procedure. We now describe details in Step 2.

Definition of b, t, and $\bar{t}$ (Step 2). Let $\widehat{V}^*, \widehat{B}^*, \widehat{C}^*$, and $\widehat{p}$ denote the objects obtained from V^*, B^*, C^*, and p by adding b, t, and $\bar{t}$. We consider the following two cases separately.

If $r \in B^*$ and $g \in V^* \setminus B^*$, then define $\widehat{V}^*, \widehat{B}^*, \widehat{C}^*$, and $\widehat{p}$ as follows.

- $\widehat{V}^* := V^* \cup \{b, t, \bar{t}\}$, $\widehat{B}^* := B^* \cup \{b, \bar{t}\}$, and let $\widehat{p} : \widehat{V}^* \to \mathbb{R}$ be an extension of p such that $\widehat{p}(b) = \widehat{p}(t) = \widehat{p}(\bar{t}) = p(r) + Q_{rb}$.
- $\widehat{C}^\lambda_{by} = C^\lambda_{ry}$ for any $y \in Y \setminus B^*$ and $\widehat{C}^\lambda_{by} = 0$ for any $y \in (V^* \setminus B^*) \setminus Y$.
- $\widehat{C}^\lambda_{xt} = C^\lambda_{xg}$ for any $x \in (B^* \setminus Y) \cup \{b\}$ and $\widehat{C}^\lambda_{xt} = 0$ for any $x \in B^* \cap Y$.
- $\widehat{C}^\lambda_{\bar{t}t} = 1$ and $\widehat{C}^\lambda_{\bar{t}y} = 0$ for any $y \in (\widehat{V}^* \setminus \widehat{B}^*) \setminus \{t\}$.
- $\widehat{C}^\lambda$ naturally defines $\widehat{C}^*$.

If $r \in V^* \setminus B^*$ and $g \in B^*$, then define $\widehat{V}^*, \widehat{B}^*, \widehat{C}^*$, and $\widehat{p}$ as follows.

- $\widehat{V}^* := V^* \cup \{b, t, \bar{t}\}$, $\widehat{B}^* := B^* \cup \{t\}$, and let $\widehat{p} : \widehat{V}^* \to \mathbb{R}$ be an extension of p such that $\widehat{p}(b) = \widehat{p}(t) = \widehat{p}(\bar{t}) = p(r) - Q_{rb}$.
- $\widehat{C}^\lambda_{xb} = C^\lambda_{xr}$ for any $x \in B^* \cap Y$ and $\widehat{C}^\lambda_{xb} = 0$ for any $x \in B^* \setminus Y$.
- $\widehat{C}^\lambda_{ty} = C^\lambda_{gy}$ for any $y \in ((V^* \setminus B^*) \setminus Y) \cup \{b\}$ and $\widehat{C}^\lambda_{ty} = 0$ for any $y \in Y \setminus B^*$.
- $\widehat{C}^\lambda_{t\bar{t}} = 1$ and $\widehat{C}^\lambda_{x\bar{t}} = 0$ for any $x \in \widehat{B}^* \setminus \{t\}$.
- $\widehat{C}^\lambda$ naturally defines $\widehat{C}^*$.

Then, we rename $\widehat{V}^*$, $\widehat{B}^*$, $\widehat{C}^*$ and $\widehat{p}$ to V^*, B^*, C^*, and p, respectively.

The next one is DBlossom, called in Steps (4-2) and (4-3) of Search.

Procedure $\mathsf{DBlossom}(v,u)$

Step 1: Set $Y := K(u)$, $r := v$, and $g := u$. Introduce new vertices b, t, and $\bar{t}$ in the same say as Step 2 of $\mathsf{RBlossom}(v,u)$, and define $H := Y \cup \{b, t, \bar{t}\}$. Label $\bar{t}$ with $P(\bar{t}) := P(v)t\bar{t}$ and $J(\bar{t}) := \{\bar{t}\}$, and extend the ordering $\prec$ of the labeled vertices so that $\bar{t}$ is just after v, i.e., $v \prec \bar{t}$ and no element is between v and $\bar{t}$. Set $\rho(t) := v$ and $I(t) := \{t\}$.

Step 2: If Y is a line, then for each vertex $x \in Y$, label x with $P(x) := P(v)t\bar{t}\bar{x}x$ and $J(x) := \bar{t}\bar{x}x$, and put x into the queue.

If $Y = H_i$ for some positive blossom $H_i \in \Lambda_{\mathrm{n}}$, then do the following. For each vertex $x \in H_i^\circ$ in the order of $<_{H_i}$, label x with $P(x) := P(v)t\bar{t}R_{H_i}(x)$ and $J(x) := \bar{t}R_{H_i}(x)$, and put x into the queue.

Step 3: Label H with $\oplus$. Define $R_H(x) := P(x|v)$ for each $x \in H^\circ$. Define $<_H$ by the ordering $\prec$ of the labeled vertices in H°. Add H to Λ with $q(H) = 0$ regarding b, t, $\bar{t}$, and g as the bud of H, the tip of H, the mate of t, and the generator of t, respectively, and update Λ_{n}, λ, C^λ, G°, and $K(v)$ for $v \in V^*$, accordingly.

Step 4: If $Y = H_i$ for some positive blossom $H_i \in \Lambda_{\mathrm{n}}$, then set $\epsilon := q(H_i)$ and modify the dual variables as follows: $q(H_i) := q(H_i) - \epsilon$, $q(H) := q(H) + \epsilon$,

$$p(t) := \begin{cases} p(t) - \epsilon & \text{if } t \in V^* \setminus B^*, \\ p(t) + \epsilon & \text{if } t \in B^*, \end{cases}$$

$$p(\bar{t}) := \begin{cases} p(\bar{t}) - \epsilon & \text{if } \bar{t} \in B^*, \\ p(\bar{t}) + \epsilon & \text{if } \bar{t} \in V^* \setminus B^*. \end{cases}$$

Since $q(H_i)$ becomes zero, we delete H_i from Λ (see Lemma 5.2). We also remove b_i, t_i, and $\bar{t}_i$ from V^* and update Λ_{n}, λ, C^λ, G°, and $K(v)$ for $v \in V^*$, accordingly.

We note that Step 4 of $\mathsf{DBlossom}(v,u)$ is executed to keep the condition $H_i \cap V \neq H_j \cap V$ for distinct $H_i, H_j \in \Lambda$.

8 Validity

This section is devoted to the validity proof of the procedures described in Section 7. In Section 8.1, we introduce properties (BR4) and (BR5) of the routing in blossoms. The procedures are designed so that they keep the conditions (GT1), (GT2), (BT), (DF1)–(DF3), and (BR1)–(BR5). Assuming these conditions, we show in Section 8.2 that a nonempty output of Search is indeed an augmenting path. In Sections 8.3 and 8.4, we show that these conditions hold when a new blossom is created.

8.1 Properties of Routings in Blossoms

In this subsection, we introduce properties (BR4) and (BR5) of $R_{H_i}(x)$ kept in the procedure. For $H_i \in \Lambda$, we denote $H_i^- := H_i^\circ \setminus \{\bar{t}_j, g_j \mid H_j \in \Lambda_{\mathrm{n}},\ H_j \subseteq H_i\}$. We note that there is no edge of F^λ connecting $H_i \setminus H_i^-$ and $V^* \setminus H_i$ by Lemma 4.1 and (GT1). This shows that we can ignore $H_i \setminus H_i^-$ when we consider edges in F^λ (or F°) connecting H_i and $V^* \setminus H_i$.

Recall that if $H_i \in \Lambda_{\mathrm{s}}$, then $\{b_i\}$, $\{t_i\}$, and $\{\bar{t}_i\}$ denote $\emptyset$. In particular, for $H_i \in \Lambda_{\mathrm{s}}$, $H_i \setminus \{t_i\}$ and $R_{H_i}(x) \setminus \{t_i\}$ denote H_i and $R_{H_i}(x)$, respectively. In addition to (BR1)–(BR3), we assume that $R_{H_i}(x)$ satisfies the following (BR4) and (BR5) for any $H_i \in \Lambda$ and $x \in H_i^\circ$.

(BR4) $G^\circ[R_{H_i}(x) \setminus \{x, t_i\}]$ has a unique tight perfect matching.

(BR5) If $x \in H_i^-$, then we have the following. Suppose that $Z \subseteq (R_{H_i}(x) \setminus \{t_i\}) \cap H_i^-$ satisfies that $z \geq_{H_i} x$ for any $z \in Z$, $Z \neq \{x\}$, and $|(H_j \setminus \{t_j\}) \cap Z| \leq 1$ for any positive blossom $H_j \in \Lambda$. Then, $G^\circ[(R_{H_i}(x) \setminus \{t_i\}) \setminus Z]$ has no tight perfect matching.

Here, we suppose that $G^\circ[\emptyset]$ has a unique tight perfect matching $M = \emptyset$ to simplify the description. Lemma 5.5 implies the following lemma, which guarantees that it suffices to focus on tight perfect matchings in $G^\circ[X]$ when we consider the nonsingularity of $C^\lambda[X]$.

Lemma 8.1. *If $G^\circ[X]$ has a unique tight perfect matching, then $C^\lambda[X]$ is nonsingular.*

Suppose that P is an augmenting path. Then Lemma 8.1 together with (AP3) implies that $C^\lambda[P]$ is nonsingular. It follows from (AP2) and (BR3) that $t_i \in P$ holds for any $H_i \in \Lambda_{\mathrm{n}}$ with $P \cap H_i \neq \emptyset$. Therefore, $C^*[P]$ is nonsingular. By the same argument, one can derive from (BR4) that $C^\lambda[R_{H_i}(x) \setminus \{x, t_i\}]$ and $C^*[R_{H_i}(x) \setminus \{x, t_i\}]$ are nonsingular.

8.2 Finding an Augmenting Path

This subsection is devoted to the validity of Step (3-1) of Search. We first show the following lemma.

Lemma 8.2. *In each step of* Search*, for any labeled vertex x, $P(x)$ is decomposed as*

$$P(x) = J(x_k)I(y_k) \cdots J(x_1)I(y_1)J(x_0)$$

with $x_k \prec \cdots \prec x_1 \prec x_0 = x$ such that, for $i = 1, \ldots, k$,

(PD1) *x_i is adjacent to y_i in G°,*

(PD2) *the first element of $J(x_{i-1})$ is the mate of the last element of $I(y_i)$,*

(PD3) *any labeled vertex z with $z \prec x_i$ is not adjacent to $I(y_i) \cup J(x_{i-1})$ in G°, and*

(PD4) *x_i is not adjacent to $J(x_{i-1})$ in G°. Furthermore, if $I(y_i) = R_{H_j}(y_i) \setminus \{t_j\}$, then x_i is not adjacent to $\{z \in I(y_i) \mid z \notin H_j^\circ$ or $z <_{H_j} y_i\}$ in G°.*

Proof. The procedure Search naturally defines the decomposition

$$P(x) = J(x_k)I(y_k) \cdots J(x_1)I(y_1)J(x_0).$$

Since we can easily see that Steps (4-1), (4-4), and (4-5) of Search do not violate the conditions (PD1)–(PD4), it suffices to show that RBlossom(v, u) and DBlossom(v, u) do not violate these conditions.

We first consider the case when RBlossom(v, u) is applied to obtain a new blossom H. In RBlossom(v, u), $P(x)$ is updated or defined as $P(x) := P(x)$, $P(x) := P(r)t\bar{t}P(x|r)$, or $P(x) := P(r)R_H(x)$. Let F° (resp. $\widehat{F}^\circ$) be the tight edge sets before (resp. after) adding H to Λ in Step 6 of RBlossom(v, u). We show the following claim.

Claim 8.3. *If $(x, y) \in F^\circ \triangle \widehat{F}^\circ$, then either* (i) *$\{x, y\} \cap \{b, \bar{t}\} \neq \emptyset$, or* (ii) *exactly one of $\{x, y\}$, say x, is contained in H, and $(x, b), (t, y) \in F^\circ$.*

Proof. Suppose that $\{x, y\} \cap \{b, \bar{t}\} = \emptyset$. By the definition of C^λ, we have $(x, y) \in F^\circ \triangle \widehat{F}^\circ$ only when $(x, b), (t, y) \in F^\lambda$ or $(y, b), (t, x) \in F^\lambda$ holds before H is added to Λ. By (BT) and (GT2), this shows that exactly one of $\{x, y\}$, say x, is contained in H. Suppose that $x \in B^*$. In this case, if $(x, b), (t, y) \in F^\lambda$ holds before H is added to Λ and $(x, y) \in F^\circ \triangle \widehat{F}^\circ$, then we have

$$\begin{aligned} p(y) - p(x) &= Q_{xy}, \\ p(b) - p(t) &= Q_{tb}, \\ p(b) - p(x) &\geq Q_{xb}, \\ p(y) - p(t) &\geq Q_{ty}. \end{aligned}$$

Since the last two inequalities above must be tight, we have $(x, b), (t, y) \in F^\circ$. The same argument can be applied to the case when $x \in V^* \setminus B^*$. □

Suppose that $P(x)$ is defined by $P(x) := P(r)I(t)J(x)$, where $I(t) = \{t\}$ and $J(x) = R_H(x) \setminus \{t\}$. In this case, (PD1) and (PD2) are trivial. We now consider (PD3). Since $P(r)$ satisfies (PD3), in order to show that any labeled vertex z with $z \prec x_i$ is not adjacent to $I(y_i) \cup J(x_{i-1})$ in $\widehat{G}^\circ$, it suffices to consider the case when $x_i = r$, $y_i = t$, and $x_{i-1} = x$. Assume to the contrary that $z \prec r$ is adjacent to $I(t) \cup J(x)$ in $\widehat{G}^\circ$. Since z is not adjacent to $I(t) \cup J(x)$ in G° by the procedure, Claim 8.3 shows that $(z, t) \in F^\circ$, which implies that $(z, g) \in F^\circ$. This contradicts that $z \prec x_i$ and the definition of H.

To show (PD4), it suffices to consider the case when $x_i = r$. In this case, since r is not adjacent to $H \setminus \{t\}$ in $\widehat{G}^\circ$, $P(x)$ satisfies (PD4).

Suppose that $P(x)$ is updated as $P(x) := P(x)$ or $P(x) := P(r)I(t)J(\bar{t})P(x|r)$, where $I(t) = \{t\}$ and $J(\bar{t}) = \{\bar{t}\}$. In this case, (PD1) and (PD2) are trivial. We now consider (PD3). Since (PD3) holds before creating H, in order to show that any labeled vertex z with $z \prec x_i$ is not adjacent to $w \in I(y_i) \cup J(x_{i-1})$ in $\widehat{G}^\circ$, it suffices to consider the case when (i) $z = \bar{t}$, or (ii) $w \in I(t) \cup J(\bar{t})$, or (iii) $(z,t) \in F^\circ$ and $(w,b) \in F^\circ$, or (iv) $(w,t) \in F^\circ$ and $(z,b) \in F^\circ$ by Claim 8.3. In the first case, if $(\bar{t}, w) \in \widehat{F}^\circ$, then $(r,w) \in F^\circ$, which contradicts that (PD3) holds before creating H. In the second case, if $w = t$, then $(z,w) \in \widehat{F}^\circ$ implies that $(z,g) \in F^\circ$, which contradicts that $z \prec x_i = r$ and the definition of H. If $w = \bar{t}$, then $(w,z) \in \widehat{F}^\circ$ implies that $(r,z) \in F^\circ$, which contradicts that r and z are labeled. In the third case, $(w,b) \in F^\circ$ implies $(w,r) \in F^\circ$, and hence $x_i \preceq r$ as (PD3) holds before creating H. Furthermore, $(z,t) \in F^\circ$ implies $(z,g) \in F^\circ$, which contradicts that $z \prec x_i \preceq r$ and the definition of H. In the fourth case, $(z,b) \in F^\circ$ implies $(z,r) \in F^\circ$, which contradicts that r and z are labeled. By these four cases, we obtain (PD3).

We next consider (PD4). Since (PD4) holds before creating H, in order to show that x_i is not adjacent to $w \in J(x_{i-1})$ or $w \in \{z \in I(y_i) \mid z \notin H_j^\circ \text{ or } z <_{H_j} y_i\}$ in $\widehat{F}^\circ$ it suffices to consider the case when (i) $x_i = r$, or (ii) $x_i = \bar{t}$, or (iii) (x_i, w) crosses H. In the first case, the claim is obvious. In the second case, if $(\bar{t}, w) \in \widehat{F}^\circ$, then $(r,w) \in F^\circ$, which contradicts that (PD4) holds before creating H. In the third case, since $x_i \in H$ and $w \notin H$, it suffices to consider the case when $(w,t) \in F^\circ$ and $(x_i, b) \in F^\circ$ by Claim 8.3. This shows that $(x_i, r) \in F^\circ$, which contradicts that x_i and r are labeled. By these three cases, we obtain (PD4).

We can show that $\mathsf{DBlossom}(v,u)$ does not violate (PD1)–(PD4) in a similar manner by observing that $P(x)$ is updated or defined as $P(x) := P(x)$ or $P(x) := P(v)R_H(x)$ in $\mathsf{DBlossom}(v,u)$. ◻

We are now ready to show the validity of Step (3-1) of Search.

Lemma 8.4. *If* Search *returns* $P := P(v)\overline{P(u)}$ *in Step* (3-1)*, then* P *is an augmenting path.*

Proof. It suffices to show that $G^\circ[P]$ has a unique tight perfect matching. By Lemma 8.2, $P(v)$ and $P(u)$ are decomposed as $P(v) = J(v_k)I(s_k)\cdots J(v_1)I(s_1)J(v_0)$ and $P(u) = J(u_l)I(r_l)\cdots J(u_1)I(r_1)J(u_0)$. For each pair of $i \le k$ and $j \le l$, let X_{ij} denote the set of vertices in the subsequence

$$J(v_i)I(s_i)\cdots J(v_1)I(s_1)J(v_0)\overline{J(u_0)}\,\overline{I(r_1)}\,\overline{J(u_1)}\cdots\overline{I(r_j)}\,\overline{J(u_j)}$$

of P. We intend to show inductively that $G^\circ[X_{ij}]$ has a unique tight perfect matching.

We first show that $G^\circ[X_{00}] = G^\circ[J(u) \cup J(v)]$ has a unique tight perfect matching. Let M be an arbitrary tight perfect matching in $G^\circ[J(u) \cup J(v)]$, and let Z be the set of vertices in $J(v)$ adjacent to $J(u)$ in M. If $J(v) = \{v\}$, then it is obvious that $Z = \{v\}$. Otherwise, $J(v) = R_{H_i}(v) \setminus \{t_i\}$ for some $H_i \in \Lambda$. For any positive blossom $H_j \in \Lambda$, since M is consistent with H_j, we have that $|(H_j \setminus \{t_j\}) \cap Z| \leq 1$. Since there are no edges of G° between $J(u)$ and $\{y \in J(v) \mid y \prec v\}$, we have that $z \geq_{H_i} v$ for any $z \in Z$. Furthermore, since there is an edge in M connecting each $z \in Z$ and $J(u)$, we have $Z \subseteq J(v) \cap H_i^-$. Then it follows from (BR5) that $G^\circ[J(v) \setminus Z]$ has no tight perfect matching unless $Z = \{v\}$. This means v is the only vertex in $J(v)$ adjacent to $J(u)$ in M. Note that $G^\circ[J(v) \setminus \{v\}]$ has a unique tight perfect matching by (BR4), which must form a part of M. Let z be the vertex adjacent to v in M. Since the vertices in $\{y \in J(u) \mid y \prec u\}$ are not adjacent to v in G°, we have $z \geq_{H_j} u$ if $J(u) = R_{H_j}(u) \setminus \{t_j\}$ for some $H_j \in \Lambda$. By (BR5) again, $G^\circ[J(u) \setminus \{z\}]$ has no tight perfect matching unless $z = u$. This means M must contain the edge (u, v). Note that $G^\circ[J(u) \setminus \{u\}]$ has a unique tight perfect matching by (BR4), which must form a part of M. Thus M must be the unique tight perfect matching in $G^\circ[J(u) \cup J(v)]$.

We now show the statement for general i and j assuming that the same statement holds if either i or j is smaller. Suppose that $v_i \prec u_j$. Then there are no edges of G° between $X_{ij} \setminus J(v_i)$ and $\{y \in J(v_i) \mid y \prec v_i\}$ by (PD3) of Lemma 8.2. Let M be an arbitrary tight perfect matching in $G^\circ[X_{ij}]$, and let Z be the set of vertices in $J(v_i)$ adjacent to $X_{ij} \setminus J(v_i)$ in M. Then, by the same argument as above, $G^\circ[J(v_i) \setminus Z]$ has no tight perfect matching unless $Z = \{v_i\}$. Thus v_i is the only vertex in $J(v_i)$ matched to $X_{ij} \setminus J(v_i)$ in M. Since v_i is not adjacent to $X_{i-1,j}$ in G° by (PD3) and (PD4) of Lemma 8.2, an edge connecting v_i and $I(s_i)$ must belong to M. We note that it is the only edge in M between $I(s_i)$ and $X_{ij} \setminus I(s_i)$ since M is tight. Let z be the vertex adjacent to v_i in M. By (BR5), $G^\circ[I(s_i) \setminus \{z\}]$ has no tight perfect matching unless $z = s_i$. This means that M contains the edge (v_i, s_i). Note that each of $G^\circ[J(v_i) \setminus \{v_i\}]$ and $G^\circ[I(s_i) \setminus \{s_i\}]$ has a unique tight perfect matching by (BR4), and so does $G^\circ[X_{i-1,j}]$ by induction hypothesis. Therefore, M is the unique tight perfect matching in $G^\circ[X_{ij}]$. The case of $v_i \succ u_j$ can be dealt with similarly. Thus, we have seen that $G^\circ[X_{kl}] = G^\circ[P]$ has a unique tight perfect matching. $\square$

This proof implies the following as a corollary.

Corollary 8.5. *For any labeled vertex $v \in V^*$, $G^\circ[P(v) \setminus \{v\}]$ has a unique tight perfect matching.*

8.3 Routing in Blossoms

When we create a new blossom H in $\mathsf{DBlossom}(v, u)$, for each $x \in H^\circ$, $R_H(x)$ clearly satisfies (BR1)–(BR5). Suppose that a new blossom H is created in $\mathsf{RBlossom}(v, u)$. For each $x \in H^\circ$, $R_H(x)$ defined in $\mathsf{RBlossom}(v,u)$ also satisfies (BR1), (BR2), and (BR3). We will show (BR4) and (BR5) in this subsection.

Lemma 8.6. *Suppose that* $\mathsf{RBlossom}(v,u)$ *creates a new blossom* H*. Then, for each* $x \in H^\circ$*,* $R_H(x)$ *satisfies* (BR4) *and* (BR5)*.*

Proof. We only consider the case when H° contains no source line, since the case with a source line can be dealt with in a similar way. We note that a vertex $v \in H^\circ$ is adjacent to r in G° before creating H° if and only if v is adjacent to $\bar{t}$ in G° after adding H° to Λ. If $x = \bar{t}$, the claim is obvious. We consider the other cases separately.

Case (i): Suppose that $x \in H$ was not labeled before H is created.

We consider the case, in which either $x \in P(u|d)$ or $K(x) = H_i$, H_i is labeled with $\ominus$, and $t_i \in P(u|d)$. The case, in which either $x \in P(v|c)$ or $K(x) = H_i$, H_i is labeled with $\ominus$, and $t_i \in P(v|c)$, can be dealt with in a similar manner.

By Lemma 8.2, $P(v)$ can be decomposed as

$$P(v) = P(r)t\bar{t}I(s_k)J(v_{k-1})I(s_{k-1})\cdots J(v_1)I(s_1)J(v_0)$$

with $v = v_0$. In addition, if $x \in P(v|c)$, then $P(u|x)$ can be decomposed as $P(u|x) = J(u_l)I(r_l)\cdots J(u_1)I(r_1)J(u_0)$ with $u_0 = u$, where the first element of $J(u_l)$ is the mate of x. Thus, if $x \in P(v|c)$, then we have

$$R_H(x) = tJ(v_k)I(s_k)J(v_{k-1})\cdots I(s_1)J(v_0)\overline{J(u_0)}\,\overline{I(r_1)}\cdots\overline{I(r_l)}\,\overline{J(u_l)}x$$

with $v_k = \bar{t}$. Similarly, if $K(x) = H_i$, H_i is labeled with $\ominus$, and $t_i \in P(v|c)$, then

$$R_H(x) = tJ(v_k)I(s_k)J(v_{k-1})\cdots I(s_1)J(v_0)\overline{J(u_0)}\,\overline{I(r_1)}\cdots\overline{I(r_l)}R_{H_i}(x).$$

In both cases, we have

$$R_H(x) = tJ(v_k)I(s_k)J(v_{k-1})\cdots I(s_1)J(v_0)\overline{J(u_0)}\,\overline{I(r_1)}\cdots\overline{I(r_l)}\,\overline{J(u_l)}\,\overline{I(r_{l+1})}$$

with $r_{l+1} = x$ (see Fig. 4 for an example).

We now intend to show that $R_H(x)$ satisfies (BR5), that is, $G^\circ[(R_H(x) \setminus \{t\}) \setminus Z]$ has no tight perfect matching if $Z \subseteq (R_H(x) \setminus \{t\}) \cap H^-$ satisfies that $z \geq_H x$ for any $z \in Z$, $Z \neq \{x\}$, and $|(H_j \setminus \{t_j\}) \cap Z| \leq 1$ for any positive blossom $H_j \in \Lambda$. Suppose to the contrary that $G^\circ[(R_H(x) \setminus \{t\}) \setminus Z]$ has a tight perfect matching M. Note that $Z \subseteq I(r_{l+1}) \cup \bigcup_i I(s_i)$, because $z \geq_H x$ for any $z \in Z$. For each i, since either $I(s_i) = \{s_i\}$ or $I(s_i) = R_{H_j}(s_i) \setminus \{t_j\}$ for some positive blossom $H_j \in \Lambda$, we have $|I(s_i) \cap Z| \leq 1$. Similarly, $|I(r_{l+1}) \cap Z| \leq 1$. Furthermore, since M is a tight perfect matching, $|I(s_i) \cap Z| = 1$ (resp. $|I(r_{l+1}) \cap Z| = 1$) implies that there is no edge in M between $I(s_i)$ (resp. $I(r_{l+1})$) and its outside. If $Z \subseteq I(r_{l+1})$, then $|I(r_{l+1}) \cap Z| = 1$

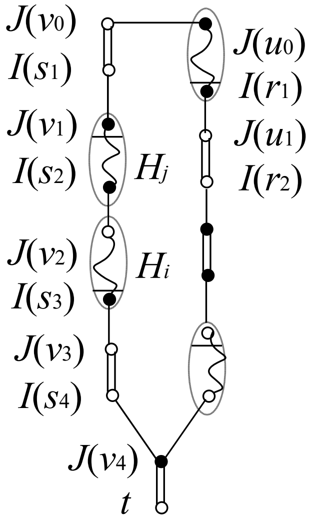

Figure 4: A decomposition of $R_H(x)$. In this example, $J(v_1) = \{t_j\}$, $I(s_2) = R_{H_j}(s_2)$, $J(v_2) = R_{H_i}(v_2)$, $I(s_3) = \{t_i\}$, and $J(v_4) = \bar{t}$.

and M contains no edge between $I(r_{l+1})$ and the outside of $I(r_{l+1})$, which contradicts that $G^\circ[I(r_{l+1}) \setminus Z]$ has no tight perfect matching by (BR5). Thus, we may assume that $Z \cap \bigcup_i I(s_i) \neq \emptyset$. In this case, we can take the largest number j such that $(v_j, s_j) \notin M$. We consider the following two cases separately.

Case (i)-a: Suppose that $j = k$. In this case, since $J(v_k) = \{\bar{t}\}$, there exists an edge in M between $\bar{t}$ and $I(r_{l+1}) \cup (I(s_k) \setminus \{s_k\})$. If this edge is incident to $z \in I(s_k) \setminus \{s_k\}$, then $z \prec s_k$ by the procedure, and hence $G^\circ[I(s_k) \setminus \{z\}]$ has no tight perfect matching by (BR5), which is a contradiction. Otherwise, since $v_k = \bar{t}$ is incident to some $y \in I(r_{l+1})$, we have $Z \subseteq I(r_{l+1}) \cup I(s_k)$ by $z \geq_H x$ for any $z \in Z$. Then, since M is a tight perfect matching, we have $I(r_{l+1}) \cap Z = \emptyset$, $Z = \{z\}$ for some $z \in I(s_k)$, each of $G^\circ[I(r_{l+1}) \setminus \{y\}]$ and $G^\circ[I(s_k) \setminus \{z\}]$ has a tight perfect matching, and $s_k \in \{g, h\}$. By (BR5), we have that $z \leq_H s_k$ and $y \leq_H r_{l+1}$, which shows that $y \leq_H r_{l+1} = x \leq_H z \leq_H s_k$. By Step 5 of $\mathsf{RBlossom}(v, u)$, this means that $y = r_{l+1} = x$, $z = s_k$, and $\{x, z\} = \{g, h\}$, which contradicts that $x, z \in H^-$ and $g \notin H^-$.

Case (i)-b: Suppose that $j \leq k-1$. In this case, since M is a tight perfect matching, for $i = j+1, \ldots, k$, we have $I(s_i) \cap Z = \emptyset$ and (v_i, s_i) is the only edge in M between $I(s_i)$ and the outside of $I(s_i)$. We can also see that $Z \cap J(v_j) = \emptyset$, since $z \geq_H x$ for any $z \in Z$. We denote by Z_j the set of vertices in $J(v_j)$ matched by M to the outside of $J(v_j)$. Since $z \geq_H x$ for any $z \in Z$ and $Z \cap I(s_i) \neq \emptyset$ for some $i \leq j-1$, we have $v_j \prec u_{l+1}$, where u_{l+1} is the vertex naturally defined by the decomposition of $P(u)$. Note that the assumption $j \leq k-1$ is used here. Hence, if a vertex $z \in (R_H(x) \setminus \{t\}) \setminus J(v_j)$

is adjacent to $\{y \in J(v_j) \mid y <_H v_j\}$ in G°, then $z \in I(s_i)$ with $i > j$ by (PD3) of Lemma 8.2. Since (v_i, s_i) is the only edge in M between $I(s_i)$ and its outside for $i > j$, this shows that $Z_j = (Z \cap J(v_j)) \cup Z_j \subseteq \{y \in J(v_j) \mid y \geq_H v_j\}$. Therefore, by (BR5), if $G^\circ[J(v_j) \setminus (Z \cup Z_j)]$ has a tight perfect matching, then $Z_j = \{v_j\}$. The vertex v_j is not adjacent to the vertices in $R_H(x) \setminus (J(v_j) \cup I(s_j) \cup \cdots \cup I(s_k) \cup \{t\})$ by (PD3) and (PD4) of Lemma 8.2. Since $|J(v_j)|$ is odd and (v_i, s_i) is the only edge in M between $I(s_i)$ and its outside for $i > j$, v_j has to be adjacent to $I(s_j)$. Furthermore, by $(v_j, s_j) \notin M$ and by (PD4) of Lemma 8.2, we have that v_j is incident to a vertex $z \in I(s_j)$ with $z >_{H_i} s_j$, where $I(s_j) = R_{H_i}(s_j) \setminus \{t_i\}$ for some positive blossom $H_i \in \Lambda$. Since $G^\circ[I(s_j) \setminus \{z\}]$ has no tight perfect matching by (BR5), we obtain a contradiction.

We next show that $R_H(x)$ satisfies (BR4), that is, $G^\circ[R_H(x) \setminus \{x, t\}]$ has a unique tight perfect matching. Let M be an arbitrary tight perfect matching in $G^\circ[R_H(x) \setminus \{x, t\}]$. Recall that $r_{l+1} = x$ and either $I(r_{l+1}) = \{r_{l+1}\}$ or $I(r_{l+1}) = R_{H_j}(r_{l+1}) \setminus \{t_j\}$ for some positive blossom $H_j \in \Lambda$. Since M is a tight perfect matching and $|I(r_{l+1}) \setminus \{x\}|$ is even, there is no edge in M between $I(r_{l+1})$ and its outside. By (BR4), $G^\circ[I(r_{l+1}) \setminus \{x\}]$ has a unique tight perfect matching, which must form a part of M. On the other hand,

$$G^\circ[J(v_k)I(s_k)J(v_{k-1})I(s_{k-1}) \cdots J(v_1)I(s_1)J(v_0)\overline{J(u_0)}\,\overline{I(r_1)}\,\overline{J(u_1)} \cdots \overline{I(r_l)}\,\overline{J(u_l)}]$$

has a unique tight perfect matching by the same argument as Lemma 8.4. By combining them, we have that $G^\circ[R_H(x) \setminus \{x, t\}]$ has a unique tight perfect matching.

Case (ii): Suppose that $x \in H$ was labeled before H is created.

We consider the case of $x \in K(y)$ with $y \in P(v|c)$. The case of $x \in K(y)$ with $y \in P(u|d)$ can be dealt with in a similar manner. By Lemma 8.2, $R_H(x)$ can be decomposed as

$$R_H(x) = tJ(v_k)I(s_k)J(v_{k-1})I(s_{k-1}) \cdots J(v_{l+1})I(s_{l+1})J(v_l)$$

with $x = v_l$.

We first show that $R_H(x)$ satisfies (BR5), that is, $G^\circ[(R_H(x) \setminus \{t\}) \setminus Z]$ has no tight perfect matching if $Z \subseteq (R_H(x) \setminus \{t\}) \cap H^-$ satisfies that $z \geq_H x$ for any $z \in Z$, $Z \neq \{x\}$, and $|(H_j \setminus \{t_j\}) \cap Z| \leq 1$ for any positive blossom $H_j \in \Lambda$. Since $z \geq_H x$ for any $z \in Z$, we have that $Z \subseteq J(v_l) \cup \bigcup_i I(s_i)$, which shows that we can apply the same argument as Case (i) to obtain (BR5).

We next show that $R_H(x)$ satisfies (BR4), that is, $G^\circ[R_H(x) \setminus \{x, t\}]$ has a unique tight perfect matching. Let M be an arbitrary tight perfect matching in $G^\circ[R_H(x) \setminus \{x, t\}]$. By the same argument as Lemma 8.4,

$$G^\circ[J(v_k)I(s_k)J(v_{k-1})I(s_{k-1}) \cdots J(v_1)I(s_1)J(v_0)\overline{J(u_0)}]$$

has a unique tight perfect matching M, and a part of M forms a tight perfect matching in $G^\circ[R_H(x)\setminus\{x,t\}]$. Thus, this matching is a unique tight perfect matching in $G^\circ[R_H(x)\setminus\{x,t\}]$. □

8.4 Creating a Blossom

When we create a new blossom H in $\mathsf{RBlossom}(v,u)$ or $\mathsf{DBlossom}(v,u)$, (GT2) holds by the definition. In this subsection, we show that (GT1), (BT), (DF1), (DF2), and (DF3) hold when a new blossom is created.

Lemma 8.7. *Suppose that* $\mathsf{RBlossom}(v,u)$ *creates a new blossom* H *containing no source line. Then, there is no edge in* F^λ *between* g *and* $V^*\setminus H$*, that is,* g *satisfies (GT1), after* H *is added to* Λ*.*

Proof. If $g\in B^*$, then we have that $\widehat{C}^\lambda_{xt}=C^\lambda_{xg}$ for any $x\in(B^*\setminus H)\cup\{b\}$ before H is added to Λ, by the definition of $\widehat{C}^\lambda$. This shows that, after H is added to Λ, $C^\lambda_{xg}=0$ for any $x\in B^*\setminus H$, that is, there is no edge of F^λ between $x=g$ and $V^*\setminus H$. We can deal with the case of $g\in V^*\setminus B^*$ in the same way. □

Lemma 8.8. *Suppose that* $\mathsf{RBlossom}(v,u)$ *creates a new blossom* H *containing no source line. Then,* b*,* t*, and* $\bar{t}$ *satisfy the conditions in (BT).*

Proof. As in Step 2 of $\mathsf{RBlossom}(v,u)$, we use the notation $\widehat{V}^*,\widehat{B}^*$, and $\widehat{C}^*$ to represent the objects after adding b, t, and $\bar{t}$. We only consider the case when $b,\bar{t}\in\widehat{B}^*$ and $t\in\widehat{V}^*\setminus\widehat{B}^*$, since the case when $b,\bar{t}\in\widehat{V}^*\setminus\widehat{B}^*$ and $t\in\widehat{B}^*$ can be dealt with in a similar way.

In Step 2 of $\mathsf{RBlossom}(v,u)$, we have $\widehat{C}^*_{bt}=\widehat{C}^\lambda_{bt}=\widehat{C}^\lambda_{rg}\neq 0$ and $\widehat{C}^*_{\bar{t}t}=\widehat{C}^\lambda_{\bar{t}t}=1\neq 0$. Since $\widehat{C}^\lambda_{by}=0$ for any $y\in(V^*\setminus B^*)\setminus Y$, $\widehat{C}^*_{by}=0$ for any $y\in(\widehat{V}^*\setminus\widehat{B}^*)\setminus H$. Similarly, since $\widehat{C}^\lambda_{\bar{t}y}=0$ for any $y\in(\widehat{V}^*\setminus\widehat{B}^*)\setminus\{t\}$, $\widehat{C}^*_{\bar{t}y}=0$ for any $y\in(\widehat{V}^*\setminus\widehat{B}^*)\setminus\{t\}$. These conditions show that b,t, and $\bar{t}$ satisfy the conditions in (BT). □

Lemma 8.9. *Suppose that* $\mathsf{RBlossom}(v,u)$ *creates a new blossom* H *and the dual variables are feasible before executing* $\mathsf{RBlossom}(v,u)$*. Then, the dual variables are feasible after executing* $\mathsf{RBlossom}(v,u)$*.*

Proof. We use the notation $\widehat{V}^*,\widehat{B}^*,\widehat{C}^*,\widehat{p}$, and $\widehat{\Lambda}$ to represent the objects after H is added to Λ, and use the notation V^*,B^*,C^*,p, and Λ to represent the objects before H is added to Λ. We only consider the case when $b,\bar{t}\in\widehat{B}^*$ and $t\in\widehat{V}^*\setminus\widehat{B}^*$, since the case when $b,\bar{t}\in\widehat{V}^*\setminus\widehat{B}^*$ and $t\in\widehat{B}^*$ can be dealt with in a similar way.

Since there is an edge in F° between r and g, we have $p(g)-p(r)=Q_{rg}$, and hence

$$\widehat{p}(t)=p(r)+Q_{rb}=p(g)+Q_{rb}-Q_{rg}=p(g)-Q_{gt}. \tag{12}$$

By the definition of $\widehat{C}^*$, we have the following.

- If $(x,t) \in F_\Lambda$ for $x \in B^*$, then $x \in V^* \setminus H$ and $(x,g) \in F_\Lambda$. Thus,

$$\widehat{p}(t) - \widehat{p}(x) = p(g) - p(x) - Q_{gt} \geq Q_{xg} - Q_{gt} = Q_{xt}$$

  by (12), $q(H) = 0$, and the dual feasibility before executing $\mathsf{RBlossom}(v,u)$.

- If $(b,y) \in F_\Lambda$ for $y \in V^* \setminus B^*$, then $y \in H$ and $(r,y) \in F_\Lambda$. Thus,

$$\widehat{p}(y) - \widehat{p}(b) = \widehat{p}(y) - p(r) - Q_{rb} \geq Q_{ry} - Q_{rb} = Q_{by}$$

  by the dual feasibility before executing $\mathsf{RBlossom}(v,u)$.

- $\widehat{p}(b) = \widehat{p}(t) = \widehat{p}(\bar{t})$, and $\bar{t}$ is incident only to t in F_Λ.

These facts show that $\widehat{p}$ and $\widehat{q}$ are feasible with respect to Λ. By Lemma 5.2, they are also feasible with respect to $\widehat{\Lambda}$. □

By the same argument as Lemmas 8.7, 8.8, and 8.9, (GT1), (BT), (DF1), (DF2), and (DF3) hold when we create a new blossom in Step 3 of $\mathsf{DBlossom}(v,u)$. Furthermore, we can see that Step 4 of $\mathsf{DBlossom}(v,u)$ keeps the dual feasibility, since there is no edge in F^λ between $g = t_i$ and $V^* \setminus H_i$ by (GT1). Therefore, $\mathsf{RBlossom}(v,u)$ and $\mathsf{DBlossom}(v,u)$ keep the conditions (GT1), (GT2), (BT), (DF1), (DF2), and (DF3).

9 Dual Update

In this section, we describe how to modify the dual variables when Search returns $\emptyset$ in Step 2. In Section 9.1, we show that the procedure keeps the dual variables finite as long as the instance has a parity base. In Section 9.2, we bound the number of dual updates per augmentation.

Let $R \subseteq V^*$ be the set of vertices that are reached or examined by the search procedure and not contained in any blossoms, i.e., $R = R^+ \cup R^-$, where R^+ is the set of labeled vertices that are not contained in any blossom, and R^- is the set of unlabeled vertices whose mates are in R^+. Let Z denote the set of vertices in V^* contained in labeled blossoms. The set Z is partitioned into Z^+ and Z^-, where

$$\begin{aligned}
Z^+ = {} & \{t_i \mid H_i \text{ is a maximal blossom labeled with } \ominus\} \\
& \cup \bigcup \{H_i \setminus \{t_i\} \mid H_i \text{ is a maximal blossom labeled with } \oplus\}, \\
Z^- = {} & \{t_i \mid H_i \text{ is a maximal blossom labeled with } \oplus\} \\
& \cup \bigcup \{H_i \setminus \{t_i\} \mid H_i \text{ is a maximal blossom labeled with } \ominus\}.
\end{aligned}$$

We denote by Y the set of vertices that do not belong to these subsets, i.e., $Y = V^* \setminus (R \cup Z)$.

For each vertex $v \in R$, we update $p(v)$ as

$$p(v) := \begin{cases} p(v) + \epsilon & (v \in R^+ \cap B^*) \\ p(v) - \epsilon & (v \in R^+ \setminus B^*) \\ p(v) - \epsilon & (v \in R^- \cap B^*) \\ p(v) + \epsilon & (v \in R^- \setminus B^*). \end{cases}$$

We also modify $q(H)$ for each maximal blossom H by

$$q(H) := \begin{cases} q(H) + \epsilon & (H : \text{labeled with } \oplus) \\ q(H) - \epsilon & (H : \text{labeled with } \ominus) \\ q(H) & (\text{otherwise}). \end{cases}$$

To keep the feasibility of the dual variables, ϵ is determined by $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, where

$$\begin{aligned} \epsilon_1 &= \frac{1}{2} \min\{p(v) - p(u) - Q_{uv} \mid (u,v) \in F_\Lambda,\ u,v \in R^+ \cup Z^+,\ K(u) \neq K(v)\}, \\ \epsilon_2 &= \min\{p(v) - p(u) - Q_{uv} \mid (u,v) \in F_\Lambda,\ u \in R^+ \cup Z^+,\ v \in Y\}, \\ \epsilon_3 &= \min\{p(v) - p(u) - Q_{uv} \mid (u,v) \in F_\Lambda,\ u \in Y,\ v \in R^+ \cup Z^+\}, \\ \epsilon_4 &= \min\{q(H) \mid H\text{: a maximal blossom labeled with } \ominus\}. \end{aligned}$$

We note that F_Λ coincides with F^λ as seen in Observation 7.1. If $\epsilon = +\infty$, then we terminate Search and conclude that there exists no parity base. If there are any blossoms whose values of q become zero, then the algorithm deletes those blossoms from Λ, which is possible by Lemma 5.2. Then, apply the procedure Search again.

9.1 Detecting Infeasibility

By the definition of ϵ, we can easily see that the updated dual variables are feasible if ϵ is a finite value. We now show that we can conclude that the instance has no parity base if $\epsilon = +\infty$.

A skew-symmetric matrix is called an *alternating matrix* if all the diagonal entries are zero. Note that any skew-symmetric matrix is alternating unless the underlying field is of characteristic two. By a congruence transformation, an alternating matrix can be brought into a block-diagonal form in which each nonzero block is a 2×2 alternating matrix. This shows that the rank of an alternating matrix is even, which plays an important role in the proof of the following lemma.

Lemma 9.1. *Suppose that there is a source line, and suppose also that $\epsilon = +\infty$ when we update the dual variables. Then, the instance has no parity base.*

Proof. Recall that Y is the set of vertices v such that $K(v)$ contains no labeled vertices. In the proof, we use the following properties of F^{λ}:

(A) there exists no edge in F^{λ} between two labeled vertices $u, v \in V^*$ with $K(u) \neq K(v)$, and

(B) there exists no edge in F^{λ} between a labeled vertex in $V^* \setminus Y$ and a vertex in Y,

but do not use the dual feasibility. Therefore, we may assume that Y contains no blossom, because removing such blossoms from Λ does not create a new edge in F^{λ} between a labeled vertex in $V^* \setminus Y$ and a vertex in Y. Note that this operation might violate the dual feasibility. Let $\Lambda_{\max} \subseteq \Lambda$ be the set of maximal blossoms. Since $\epsilon_4 = +\infty$, any blossom $H_i \in \Lambda_{\max}$ is labeled with $\oplus$. Let L_{s} be the set of source lines that are not contained in any blossom. Let L_{n} be the set of normal lines ℓ such that ℓ is not contained in any blossom and ℓ contains a labeled vertex. We can see that for each line $\ell \in L_{\mathrm{n}}$, exactly one vertex v_ℓ in ℓ is unlabeled and the other vertex $\bar{v}_\ell$ is labeled.

In order to show that there is no parity base, by Lemma 2.1, it suffices to show that $\operatorname{Pf} \Phi_A(\theta) = 0$. We construct the matrix

$$
\Phi_A^{\lambda}(\theta) = \left(\begin{array}{cc|c|cc}
O & O & O & \multicolumn{2}{c}{\multirow{2}{*}{$C^{\lambda}[V \cup T]$}} \\ \cline{1-3}
O & O & I & & \\ \hline
O & -I & \multicolumn{2}{c|}{\multirow{2}{*}{$D'(\theta)$}} & \multirow{2}{*}{O} \\ \cline{1-2}
\multicolumn{2}{c|}{\multirow{2}{*}{$-(C^{\lambda}[V \cup T])^{\top}$}} & & & \\ \cline{3-5}
& & \multicolumn{2}{c|}{O} & O
\end{array}\right)
\begin{array}{l}
\leftarrow T \cap B^* \\
\leftarrow U \text{ (identified with } B) \\
\leftarrow B \\
\leftarrow V \setminus B \\
\leftarrow T \setminus B^*
\end{array}
$$

in the same way as Section 6, where $T := \{t_i, \bar{t}_i \mid H_i \in \Lambda_{\mathrm{n}}\}$. Since $\operatorname{Pf} \Phi_A(\theta) = 0$ is equivalent to $\operatorname{Pf} \Phi_A^{\lambda}(\theta) = 0$, it suffices to show that $\Phi_A^{\lambda}(\theta)$ is singular, i.e., $\operatorname{rank} \Phi_A^{\lambda}(\theta) < |U| + |V| + |T|$.

In order to evaluate $\operatorname{rank} \Phi_A^{\lambda}(\theta)$, we consider a skew-symmetric matrix $\tilde{\Phi}_A(\theta)$ obtained from $\Phi_A^{\lambda}(\theta)$ by attaching rows and column and by applying row and column transformations as follows.

- For each line $\ell \in L_{\mathrm{n}}$, regard v_ℓ as a vertex in U if $\ell \subseteq B$ and regard v_ℓ as a vertex in $V \setminus B$ if $\ell \subseteq V \setminus B$. Add a row and a column indexed by a new element z_ℓ such that $\tilde{\Phi}_A(\theta)_{v z_\ell} = -\tilde{\Phi}_A(\theta)_{z_\ell v} = 0$ if $v \neq v_\ell$ and $\tilde{\Phi}_A(\theta)_{v_\ell z_\ell} = -\tilde{\Phi}_A(\theta)_{z_\ell v_\ell} = 1$. Then, sweep out nonzero entries $\tilde{\Phi}_A(\theta)_{v_\ell x}$ and $\tilde{\Phi}_A(\theta)_{x v_\ell}$ with $x \neq z_\ell$ using the row and the column indexed by z_ℓ.

- For each blossom $H_i \in \Lambda_{\max} \cap \Lambda_{\mathrm{n}}$, add a row and a column indexed by a new element z_i such that $\tilde{\Phi}_A(\theta)_{v z_i} = -\tilde{\Phi}_A(\theta)_{z_i v} = 0$ if $v \neq t_i$ and $\tilde{\Phi}_A(\theta)_{t_i z_i} = -\tilde{\Phi}_A(\theta)_{z_i t_i} = 1$. Then, sweep out nonzero entries $\tilde{\Phi}_A(\theta)_{t_i x}$ and $\tilde{\Phi}_A(\theta)_{x t_i}$ with $x \neq z_i$ using the row and the column indexed by z_i.

Note that we apply the above operations for each $\ell \in L_{\rm n}$ and for each $H_i \in \Lambda_{\max} \cap \Lambda_{\rm n}$ in an arbitrary order. Since attaching rows and columns does not decrease the rank of a matrix, we have $\operatorname{rank} \tilde{\Phi}_A(\theta) \geq \operatorname{rank} \Phi_A^\lambda(\theta)$. The above operations sweep out all nonzero entries C_{xy}^λ if either x or y is unlabeled. This together with (A) and (B) shows that $\tilde{\Phi}_A(\theta)$ is a block-diagonal skew-symmetric matrix, where the index set of each block corresponds to one of the following vertex sets: (i) $\ell \cup \{z_\ell\}$ for $\ell \in L_{\rm n}$, (ii) $\ell \in L_{\rm s}$, (iii) $(H_i \cap (V \cup T)) \cup \{z_i\}$ for $H_i \in \Lambda_{\max} \cap \Lambda_{\rm n}$, (iv) $H_i \cap (V \cup T)$ for $H_i \in \Lambda_{\max} \cap \Lambda_{\rm s}$, or (v) Y. Note that a vertex in B corresponds to two indices (i.e., two rows and two columns) of $\tilde{\Phi}_A(\theta)$, where one is in B and the other is in U, and a vertex in $(V \setminus B) \cup T$ corresponds to one index of $\tilde{\Phi}_A(\theta)$. We denote this partition of the index set by $V_1, \ldots, V_k$. Then, we have

$$\operatorname{rank} \tilde{\Phi}_A(\theta) = \sum_{j=1}^{k} \operatorname{rank} \tilde{\Phi}_A(\theta)[V_j],$$

where $\tilde{\Phi}_A(\theta)[V_j]$ is the principal submatrix of $\tilde{\Phi}_A(\theta)$ whose rows and columns are both indexed by V_j. In what follows, we evaluate $\operatorname{rank} \tilde{\Phi}_A(\theta)[V_j]$ for each j.

If V_j corresponds to (i) $\ell \cup \{z_\ell\}$ for $\ell \in L_{\rm n}$, (ii) $\ell \in L_{\rm s}$, (iii) $(H_i \cap (V \cup T)) \cup \{z_i\}$ for $H_i \in \Lambda_{\max} \cap \Lambda_{\rm n}$, or (iv) $H_i \cap (V \cup T)$ for $H_i \in \Lambda_{\max} \cap \Lambda_{\rm s}$, then we have that $|V_j|$ is odd. Since $\tilde{\Phi}_A(\theta)[V_j]$ is an alternating matrix, this implies that $\operatorname{rank} \tilde{\Phi}_A(\theta)[V_j] \leq |V_j| - 1$. If V_j corresponds to Y, then $\operatorname{rank} \tilde{\Phi}_A(\theta)[V_j] \leq |V_j|$. Hence, we have that

$$\begin{aligned}
\operatorname{rank} \Phi_A^\lambda(\theta) \leq \operatorname{rank} \tilde{\Phi}_A(\theta) = \sum_{j=1}^{k} \operatorname{rank} \tilde{\Phi}_A(\theta)[V_j] &\leq \sum_{j=1}^{k} |V_j| - (k-1) \\
&\leq 2|B| + |V \setminus B| + |T| + |L_{\rm n}| + |\Lambda_{\max} \cap \Lambda_{\rm n}| - (k-1) \\
&= |U| + |V| + |T| - |L_{\rm s}| - |\Lambda_{\max} \cap \Lambda_{\rm s}|.
\end{aligned}$$

We note that $|L_{\rm s}| + |\Lambda_{\max} \cap \Lambda_{\rm s}|$ is equal to the number of source lines. Therefore, since there exists at least one source line, we have that $\operatorname{rank} \Phi_A^\lambda(\theta) < |U| + |V| + |T|$. Thus, $\operatorname{Pf} \Phi_A(\theta) = \operatorname{Pf} \Phi_A^\lambda(\theta) = 0$ and there is no parity base by Lemma 2.1. □

9.2 Bounding Iterations

We next show that the dual variables are updated $O(n)$ times per augmentation. To see this, roughly, we show that this operation increases the number of labeled vertices. Although Search contains flexibility on the ordering of vertices, it does not affect the set of the labeled vertices. This is guaranteed by the following lemma.

Lemma 9.2. *A vertex $x \in V^* \setminus \{b_i \mid H_i \in \Lambda_{\rm n}\}$ is labeled in* Search *if and only if there exists a vertex set $X \subseteq V^*$ such that*

- *$X \cup \{x\}$ consists of normal lines, dummy lines, and a source vertex s,*

- $C^*[X]$ *is nonsingular, and*
- *the following equality holds:*

$$\begin{aligned} p(X \setminus B^*) - p(X \cap B^*) = &-\sum\{q(H_i) \mid H_i \in \Lambda_{\mathrm{n}},\ |X \cap H_i| \textit{ is odd},\ t_i \in X\} \\ &+\sum\{q(H_i) \mid H_i \in \Lambda_{\mathrm{n}},\ |X \cap H_i| \textit{ is odd},\ t_i \notin X\} \\ &+\sum\{q(H_i) \mid H_i \in \Lambda_{\mathrm{s}},\ |X \cap H_i| \textit{ is odd}\}. \end{aligned} \tag{13}$$

Proof. If x is labeled in Search, then we obtain $P(x)$ and $X := P(x) \setminus \{x\}$ satisfies the conditions by Corollary 8.5.

Suppose that X satisfies the above conditions, and assume to the contrary that x is not labeled. If x is not labeled, then we can update the dual variables keeping the dual feasibility. We now see how the dual update affects (13).

- If s is not contained in any blossom, then the left hand side of (13) decreases by ϵ by updating $p(s)$. Otherwise, the right hand side of (13) increases by ϵ by updating $q(K(s))$.
- If x is not contained in any blossom, then the left hand side of (13) decreases by ϵ or does not change by updating $p(\bar{x})$. Otherwise, the right hand side of (13) increases by ϵ or does not change by updating $q(K(x))$.
- Updating the other dual variables does not affect the equality (13), since $s, x \notin H_i$ implies that $|X \cap H_i|$ is even.

By combining these facts, after updating the dual variables, we have that the left hand side of (13) is strictly less than its right hand side, which contradicts Lemma 5.4. □

By using this lemma, we bound the number of dual updates as follows.

Lemma 9.3. *If there exists a parity base, then the dual variables are updated at most $O(n)$ times before* Search *finds an augmenting path.*

Proof. Suppose that we update the dual variables more than once, and we consider how the value of

$$\kappa(V^*, \Lambda) := |\{v \in V^* \mid v \text{ is labeled}\}| + |\Lambda_{\mathrm{s}}| - 2|\{H_i \in \Lambda_{\mathrm{n}} \mid H_i^\circ \text{ contains no labeled vertex}\}|$$

will change between two consecutive dual updates. By Lemma 9.2, if $v \in V^*$ is labeled at the time of the first dual update, then it is labeled again at the time of the second dual update unless v is removed in the procedure. Note that if a labeled vertex $v \in V^*$ is removed, then $v = t_i$, $H_i \in \Lambda_{\mathrm{n}}$ is a maximal blossom labeled with $\ominus$, and $q(H_i) = \epsilon$, which shows that $|\{H_i \in \Lambda_{\mathrm{n}} \mid H_i^\circ \text{ contains no labeled vertex}\}|$ will decrease by one. We also observe that if a new blossom H_i is created in the procedure, then either it is in Λ_{s} or $\bar{t}_i$ is a new labeled vertex, which shows that $\kappa(V^*, \Lambda)$ will increase. Thus, the value of $\kappa(V^*, \Lambda)$ increases by at least one between two consecutive dual updates. Since the range of $\kappa(V^*, \Lambda)$ is at most $O(n)$, the dual variables are updated at most $O(n)$ times. $\square$

10 Augmentation

The objective of this section is to describe how to update the primal solution using an augmenting path P. In Sections 10.1 and 10.2, we present an augmentation procedure that primarily replaces B^* with $B^* \triangle P$. In addition, it updates the bud, the tip, and its mate carefully. In Section 10.3, we show that the augmentation procedure keeps the dual feasibility. After the augmentation, the algorithm applies Search in each blossom H to obtain a new routing and ordering in H, which will be described in Section 10.4.

10.1 Definition of the augmentation

Suppose we are given V^*, B^*, C^*, Λ, and feasible dual variables p and q. Let P be an augmenting path. In the augmentation along P, we update V^*, B^*, C^*, Λ, b_i, t_i, $\bar{t}_i$, p, and q. The new objects are denoted by $\widehat{V}^*$, $\widehat{B}^*$, $\widehat{C}^*$, $\widehat{\Lambda}$, $\widehat{b}_i$, $\widehat{t}_i$, $\widehat{\bar{t}}_i$, $\widehat{p} : \widehat{V}^* \to \mathbb{R}$, and $\widehat{q} : \widehat{\Lambda} \to \mathbb{R}_+$, respectively. The procedure for augmentation is described as follows (see Fig. 5).

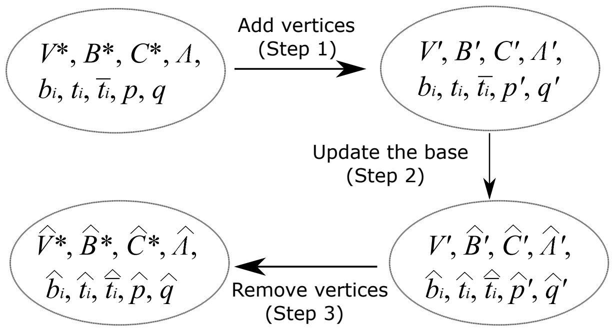

Figure 5: Each step of the augmentation

Procedure Augment

Step 0: Remove each blossom $H_i \in \Lambda$ with $q(H_i) = 0$ from Λ. Note that we also remove b_i, t_i, and $\bar{t}_i$ if $H_i \in \Lambda_{\mathrm{n}}$, and update V^*, B^*, C^*, Λ, p, and q, accordingly.

Step 1: Let $I_P := \{i \in \{1, \dots, \lambda\} \mid P \cap H_i \neq \emptyset\}$, and introduce three new vertices $\widehat{b}_i$, $\widehat{t}_i$, and $\widehat{\bar{t}}_i$ for each $i \in I_P$, which will be a bud, a tip, and the mate of $\widehat{t}_i$ after the augmentation. By the definition of augmenting paths, for each $i \in I_P$, there exists a vertex $x_i \in H_i^\circ$ such that $P \cap H_i = R_{H_i}(x_i)$.

Define

$$\begin{aligned}
V' &:= V^* \cup \{\widehat{b}_i, \widehat{t}_i, \widehat{\bar{t}}_i \mid i \in I_P\},\\
B' &:= B^* \cup \{\widehat{b}_i, \widehat{\bar{t}}_i \mid i \in I_P,\ x_i \in B^*\} \cup \{\widehat{t}_i \mid i \in I_P,\ x_i \in V^* \setminus B^*\},\\
H_i' &:= H_i \cup \{\widehat{t}_j, \widehat{\bar{t}}_j \mid j \in I_P,\ H_j \subseteq H_i\} \cup \{\widehat{b}_j \mid j \in I_P,\ H_j \subsetneq H_i\} \quad (i = 1, \dots, \lambda),\\
\Lambda' &:= \{H_1', \dots, H_\lambda'\},
\end{aligned}$$

and define $q' : \Lambda' \to \mathbb{R}_+$ as $q'(H_i') = q(H_i)$ for $i = 1, \dots, \lambda$. Let Λ'_{n} (resp. Λ'_{s}) be the set of blossoms in Λ' without (resp. with) a source line. For $H_i' \in \Lambda'_{\mathrm{n}}$, the bud of H_i', the tip of H_i', and the mate of t_i are b_i, t_i, and $\bar{t}_i$, respectively. For $u, v \in V'$, define Q'_{uv} by using q' in the same way as Q_{uv}. Let $p' : V' \to \mathbb{R}$ be the extension of p defined as follows.

- If $x_i \in B^*$, then $p'(\widehat{t}_i) = p'(\widehat{\bar{t}}_i) := p(x_i) + Q'_{x_i \widehat{t}_i}$ and $p'(\widehat{b}_i) := p'(\widehat{t}_i) + q'(H_i')$.
- If $x_i \in V^* \setminus B^*$, then $p'(\widehat{t}_i) = p'(\widehat{\bar{t}}_i) := p(x_i) - Q'_{x_i \widehat{t}_i}$ and $p'(\widehat{b}_i) := p'(\widehat{t}_i) - q'(H_i')$.

We also define a matrix C' with row set B' and column set $V' \setminus B'$ such that $C'[V^*] = C^*$ and

$$p' \text{ and } q' \text{ are feasible with respect to } C' \text{ and } \Lambda'. \tag{14}$$

The construction of C' satisfying this condition is given in Section 10.2.

Step 2: We consider a matrix $A' := [I\ C']$, where I is the identity matrix. In order to distinguish rows and columns, the row set of A' is denoted by U'. Let $\widehat{B}' := B' \triangle P$. By applying row transformations to A' and by changing the ordering of columns, we obtain a matrix $\widehat{A}'$ such that $\widehat{A}'[U', \widehat{B}']$ is the identity matrix. Note that this transformation is possible, because $C'[P] = C^*[P]$ is nonsingular. Let $\widehat{C}' := \widehat{A}'[U', V' \setminus \widehat{B}']$ and identify the row set of $\widehat{C}'$ with $\widehat{B}'$. Let

$$\begin{aligned}
\widehat{H}_i' &:= \begin{cases} H_i' \cup \{\widehat{b}_i\} & \text{if } i \in I_P,\\ H_i' & \text{if } i \in \{1, \dots, \lambda\} \setminus I_P, \end{cases}\\
\widehat{\Lambda}' &:= \{\widehat{H}_1', \dots, \widehat{H}_\lambda'\},
\end{aligned}$$

and define $\widehat{q}' : \widehat{\Lambda}' \to \mathbb{R}_+$ as $\widehat{q}'(\widehat{H}_i') = q'(H_i')$ for $i = 1, \dots, \lambda$. We also define $\widehat{p}' : V' \to \mathbb{R}_+$ as

$$\widehat{p}'(v) = \begin{cases} p'(v) - \widehat{q}'(\widehat{H}_i') & \text{if } v = \widehat{b}_i \text{ for some } i \in I_P \text{ and } v \in \widehat{B}',\\ p'(v) + \widehat{q}'(\widehat{H}_i') & \text{if } v = \widehat{b}_i \text{ for some } i \in I_P \text{ and } v \in V' \setminus \widehat{B}',\\ p'(v) & \text{otherwise.} \end{cases}$$

Let $\widehat{\Lambda}'_{\rm n}$ (resp. $\widehat{\Lambda}'_{\rm s}$) be the set of blossoms in $\widehat{\Lambda}'$ without (resp. with) a source line. Define $\widehat{b}_i := b_i$, $\widehat{t}_i := t_i$, and $\widehat{\bar{t}}_i := \bar{t}_i$ for $i \in \{1, \ldots, \lambda\} \setminus I_P$ with $\widehat{H}'_i \in \widehat{\Lambda}'_{\rm n}$. For $\widehat{H}'_i \in \widehat{\Lambda}'_{\rm n}$, regard $\widehat{b}_i$, $\widehat{t}_i$, and $\widehat{\bar{t}}_i$ as the bud of $\widehat{H}'_i$, the tip of $\widehat{H}'_i$, and the mate of $\widehat{t}_i$, respectively.

Step 3: Let $I'_P := \{i \in \{1, \ldots, \lambda\} \mid P \cap H_i \neq \emptyset,\ H_i \in \Lambda_{\rm n}\}$ and remove $\{b_i, t_i, \bar{t}_i \mid i \in I'_P\}$ from each object. More precisely, let $\widehat{V}^* := V' \setminus \{b_i, t_i, \bar{t}_i \mid i \in I'_P\}$ and $\widehat{B}^* := \widehat{B}' \setminus \{b_i, t_i, \bar{t}_i \mid i \in I'_P\}$. Let $\widehat{H}_j := \widehat{H}'_j \setminus \{b_i, t_i, \bar{t}_i \mid i \in I'_P\}$ for each j and let $\widehat{\Lambda} := \{\widehat{H}_1, \ldots, \widehat{H}_\lambda\}$. Let $\widehat{\Lambda}_{\rm n}$ (resp. $\widehat{\Lambda}_{\rm s}$) be the set of blossoms in $\widehat{\Lambda}$ without (resp. with) a source line. Define $\widehat{C}^* := \widehat{C}'[\widehat{V}^*]$. Let $\widehat{p}$ be the restriction of $\widehat{p}'$ to $\widehat{V}^*$ and define $\widehat{q} : \widehat{\Lambda} \to \mathbb{R}_+$ by $\widehat{q}(\widehat{H}_i) = \widehat{q}'(\widehat{H}'_i)$ for each i. For $\widehat{H}_i \in \widehat{\Lambda}_{\rm n}$, the bud of $\widehat{H}_i$, the tip of $\widehat{H}_i$, and the mate of $\widehat{t}_i$ are $\widehat{b}_i$, $\widehat{t}_i$, and $\widehat{\bar{t}}_i$, respectively.

10.2 Construction of C'

In this subsection, we give the definition of C' over $\mathbf{K}$ with row set B' and column set $V' \setminus B'$ such that (14) and the following property hold:

$$\widehat{C}',\ \widehat{b}_i,\ \widehat{t}_i, \text{ and } \widehat{\bar{t}}_i \text{ satisfy (BT).} \tag{15}$$

Recall that C^i is the matrix obtained from C by elementary column and row transformations (1) and (2) using nonzero entries at $\{(b_j, t_j) \mid H_j \in \Lambda_{\rm n}, j \in \{1, \ldots, i\}\}$ for $i = 1, \ldots, \lambda$. Since C^i is a submatrix of the matrix obtained from C' by elementary column and row transformations using nonzero entries at $\{(b_j, t_j) \mid H'_j \in \Lambda'_{\rm n}, j \in \{1, \ldots, i\}\}$, this larger matrix is also denoted by C^i if no confusion may arise. Since C' has to satisfy that $C'[V^*] = C^*$, it suffices to define rows and columns corresponding to $\{\widehat{b}_i, \widehat{t}_i, \widehat{\bar{t}}_i \mid i \in I_P\}$. We add the rows and the columns in two steps.

In the first step, we add rows and columns corresponding to $\{\widehat{t}_i, \widehat{\bar{t}}_i \mid i \in I_P\}$. By abuse of notation, let C', V', and B' be the objects obtained from C, V, and B by adding $\{\widehat{t}_i, \widehat{\bar{t}}_i \mid i \in I_P\}$. Recall that, for $i \in I_P$, $P \cap H_i = R_{H_i}(x_i)$ for some $x_i \in H_i^\circ$. If $\widehat{t}_i \in B'$ and $\widehat{\bar{t}}_i \in V' \setminus B'$, then we add a row corresponding to $\widehat{t}_i$ and a column corresponding to $\widehat{\bar{t}}_i$ so that $C^{i-1}_{\widehat{t}_i v} = 0$ for $v \in (V' \setminus B') \setminus \{x_i, \widehat{\bar{t}}_i\}$, $C^{i-1}_{u \widehat{\bar{t}}_i} = 0$ for $u \in B' \setminus \{\widehat{t}_i\}$, and $C^{i-1}_{\widehat{t}_i \widehat{\bar{t}}_i} = C^{i-1}_{\widehat{t}_i x_i} = 1$. If $\widehat{t}_i \in V' \setminus B'$ and $\widehat{\bar{t}}_i \in B'$, then we add a column corresponding to $\widehat{t}_i$ and a row corresponding to $\widehat{\bar{t}}_i$ so that $C^{i-1}_{u \widehat{t}_i} = 0$ for $u \in B' \setminus \{x_i, \widehat{\bar{t}}_i\}$, $C^{i-1}_{\widehat{\bar{t}}_i v} = 0$ for $v \in (V' \setminus B') \setminus \{\widehat{t}_i\}$, and $C^{i-1}_{\widehat{\bar{t}}_i \widehat{t}_i} = C^{i-1}_{x_i \widehat{t}_i} = 1$. The matrix C^{i-1} naturally defines C'. This definition means that $(\widehat{\bar{t}}_i, \widehat{t}_i) \in F_{\Lambda'}$, $(\widehat{t}_i, x_i) \in F_{\Lambda'}$ (or $(\widehat{t}_i, \widehat{\bar{t}}_i) \in F_{\Lambda'}$, $(x_i, \widehat{t}_i) \in F_{\Lambda'}$), and the other edges in $F_{\Lambda'}$ are incident to neither $\widehat{\bar{t}}_i$ nor $\widehat{t}_i$.

By the definition, we can easily see that the dual feasibility holds after adding $\{\widehat{t}_i, \widehat{\bar{t}}_i \mid i \in I_P\}$. Furthermore, by the construction of $\widehat{C}'$, we have that $\widehat{C}'_{u \widehat{\bar{t}}_i} = 0$ for any $u \in$

$\widehat{B}' \setminus \{\widehat{t}_i\}$ and $\widehat{C}'_{\widehat{t}_i\widehat{\bar{t}}_i} \neq 0$ if $\widehat{t}_i \in B'$ and $\widehat{\bar{t}}_i \in V' \setminus B'$, and $\widehat{C}'_{\widehat{\bar{t}}_i v} = 0$ for $v \in (V' \setminus B') \setminus \{\widehat{t}_i\}$ and $\widehat{C}'_{\widehat{\bar{t}}_i\widehat{t}_i} \neq 0$ if $\widehat{t}_i \in V' \setminus B'$ and $\widehat{\bar{t}}_i \in B'$. This shows that $\widehat{C}'$ satisfies a part of the condition (BT) concerning $\widehat{t}_i$ and $\widehat{\bar{t}}_i$.

In the second step, for $i = \lambda, \lambda - 1, \ldots, 1$ in this order, we add a row or a column corresponding to $\widehat{b}_i$ if $i \in I_P$ one by one. Suppose that we have already added rows and columns corresponding to $\{\widehat{t}_i, \widehat{\bar{t}}_i \mid i \in I_P\} \cup \{\widehat{b}_i \mid i \in \{\lambda, \lambda-1, \ldots, k+1\} \cap I_P\}$, and consider $\widehat{b}_k$. In what follows in this subsection, we define a row or a column corresponding to $\widehat{b}_k$ and show the validity of the definition, i.e., (14) and (15). By abuse of notation, let C^* be the matrix after adding rows and columns corresponding to $\{\widehat{t}_i, \widehat{\bar{t}}_i \mid i \in I_P\} \cup \{\widehat{b}_i \mid i \in \{\lambda, \lambda - 1, \ldots, k + 1\} \cap I_P\}$, and let C' be the matrix obtained from this matrix by adding a row or a column corresponding to $\widehat{b}_k$ if $k \in I_P$. Note that $\widehat{b}_k = b_k$ and $C' = C^*$ if $k \notin I_P$. We abuse $V^*, B^*, H_i, p, q, Q, V', B', H'_i, p', q'$, and Q' in a similar way.

We first consider the case when $k \notin I_P$. Since (14) is obvious, it suffices to show (15). For any $u, v \in V'$, it holds that

$$\begin{aligned} \widehat{C}'_{uv} \neq 0 &\iff \widehat{A}'[U', \{u, v\} \triangle \widehat{B}'] \text{ is nonsingular} \\ &\iff A'[U', \{u, v\} \triangle B' \triangle P] \text{ is nonsingular} \\ &\iff C'[\{u, v\} \triangle P] \text{ is nonsingular.} \end{aligned} \tag{16}$$

Since $P \cap H'_k = \emptyset$, either $C'_{b_k y} = 0$ or $C'_{y b_k} = 0$ holds for any $y \in P$. Thus, for $x \in \widehat{V}^*$, $C'[\{b_k, x\} \triangle P]$ is nonsingular if and only if either $C'_{b_k x} \neq 0$ or $C'_{x b_k} \neq 0$, because $C'[P]$ is nonsingular. By combining this with (16), we have

- either $\widehat{C}'_{\widehat{b}_k\widehat{t}_k} \neq 0$ or $\widehat{C}'_{\widehat{t}_k\widehat{b}_k} \neq 0$, and
- either $\widehat{C}'_{\widehat{b}_k x} \neq 0$ or $\widehat{C}'_{x\widehat{b}_k} \neq 0$ for any $x \in \widehat{V}^* \setminus \widehat{H}_k$.

By applying the same argument to $\widehat{\bar{t}}_k = \bar{t}_k$, we have

- either $\widehat{C}'_{\widehat{\bar{t}}_k\widehat{t}_k} \neq 0$ or $\widehat{C}'_{\widehat{t}_k\widehat{\bar{t}}_k} \neq 0$, and
- either $\widehat{C}'_{\widehat{\bar{t}}_k x} \neq 0$ or $\widehat{C}'_{x\widehat{\bar{t}}_k} \neq 0$ for any $x \in \widehat{V}^* \setminus \widehat{t}_k$.

This shows (15).

We next consider the case when $k \in I_P$. Let $X := R_{H_k}(x_k) \setminus \{x_k, t_k\}$. By (BR4), we have that $G^\circ[X]$ has a unique tight perfect matching, and hence $C^\lambda[X]$ is nonsingular by Lemma 8.1. Note that one can easily see that adding $\widehat{t}_k$, $\widehat{\bar{t}}_k$, and $\widehat{b}_k$ does not violate (GT2), which shows that $C^\lambda[X] = C^k[X]$. Let s be the unique source vertex in X if $H_k \in \Lambda_{\rm s}$, and let $s = b_k$ if $H_k \in \Lambda_{\rm n}$. Since $|X \cap H_i|$ is odd if and only if $|H_i \cap \{x_k, s\}| = 1$, the existence of a tight perfect matching shows that

$$\begin{aligned} p(X \setminus B^*) - p(X \cap B^*) = &- \sum \{q(H_i) \mid H_i \in \Lambda_{\rm n},\ x_k \in H_i \subseteq H_k \setminus \{s\},\ x_k \neq t_i\} \\ &+ \sum \{q(H_i) \mid H_i \in \Lambda_{\rm n},\ x_k \in H_i \subseteq H_k \setminus \{s\},\ x_k = t_i\} \\ &+ \sum \{q(H_i) \mid H_i \in \Lambda_{\rm s},\ s \in H_i \subseteq H_k \setminus \{x_k\}\}. \end{aligned} \tag{17}$$

We consider the following two cases separately. We note that if a matrix contains exactly one column (resp. row), it can be regarded as a column vector (resp. row vector).

10.2.1 Case (i): $x_k \in V^* \setminus B^*$

Suppose that $x_k \in V^* \setminus B^*$. In this case, $\widehat{b}_k, \widehat{d}_k \in V' \setminus B'$, and $\widehat{t}_k \in B'$. The nonsingularity of $C^k[X] = C^\lambda[X]$ shows that a column vector $C^k[X \cap B^*, \{x_k\}]$ can be represented as a linear combination of $C^k[X \cap B^*, \{v\}]$, where $v \in X \setminus B^*$. That is, there exists a constant $\alpha_v \in \mathbf{K}$ for each $v \in X \setminus B^*$ such that

$$C^k[X \cap B^*, \{x_k\}] = \sum_{v \in X \setminus B^*} \alpha_v \cdot C^k[X \cap B^*, \{v\}]. \tag{18}$$

Since $G^\lambda[X]$ has a unique perfect matching, $C^k[X] = C^\lambda[X]$ is an upper triangular matrix by changing the ordering of the rows and the columns, and hence α_v's can be computed by $O(n^2)$ operations. Define a column vector η by

$$\eta := C^k[B^*, \{x_k\}] - \sum_{v \in X \setminus B^*} \alpha_v \cdot C^k[B^*, \{v\}],$$

and define the column vector of C^k corresponding to $\widehat{b}_k$ by

$$C^k_{u\widehat{b}_k} = \begin{cases} 0 & \text{if } u \in (H_k \setminus (X \cup \{t_k\})) \cap B^*, \\ \eta_u & \text{otherwise} \end{cases}$$

for $u \in B^*$, where C^k is the matrix obtained from C' by elementary column and row transformations using nonzero entries at $\{(b_i, t_i) \mid H'_i \in \Lambda'_{\rm n},\ i \in \{1, \dots, k\}\}$. This naturally defines the column vector of C' corresponding to $\widehat{b}_k$. With this definition, we will show that (14) and (15) hold.

Claim 10.1. *For any $u \in B' = B^*$, if $(u, \widehat{b}_k) \in F_{\Lambda'}$, then $p'(\widehat{b}_k) - p'(u) \geq Q'_{u\widehat{b}_k}$.*

Proof. By the definition, $C^\lambda_{u\widehat{b}_k} = C^k_{u\widehat{b}_k} = 0$ for any $u \in B' \cap (H'_k \setminus \{t_k\})$, which shows that $(u, \widehat{b}_k) \notin F_{\Lambda'}$. Fix $u \in B' \setminus (H'_k \setminus \{t_k\})$ with $(u, \widehat{b}_k) \in F_{\Lambda'}$, which implies that $C^\lambda_{u\widehat{b}_k} \neq 0$ by Observation 7.1. Since

$$C^\lambda_{u\widehat{b}_k} = C^\lambda_{ux_k} - \sum_{v \in X \setminus B^*} \alpha_v \cdot C^\lambda_{uv},$$

we have either $C^{\lambda}_{ux_k} \neq 0$, or $C^{\lambda}_{uv} \neq 0$ and $\alpha_v \neq 0$ for some $v \in X \setminus B^*$.

If $C^{\lambda}_{ux_k} \neq 0$, then $p(x_k) - p(u) \geq Q_{ux_k}$ by Lemma 5.3, and hence $p'(x_k) - p'(u) \geq Q'_{ux_k}$. Combining this with $p'(\widehat{b}_k) = p'(x_k) - Q'_{x_k \widehat{t}_k} - q'(H'_k)$, we have

$$p'(\widehat{b}_k) - p'(u) \geq Q'_{ux_k} - Q'_{x_k \widehat{t}_k} - q'(H'_k) = Q'_{u\widehat{b}_k},$$

which shows the claim.

Otherwise, since $C^{\lambda}_{uv} \neq 0$ and $\alpha_v \neq 0$ for some $v \in X \setminus B^*$, in a similar way as the above argument, we obtain $p'(v) - p'(u) \geq Q'_{uv}$ and

$$p'(\widehat{b}_k) - p'(u) + p'(v) - p'(x_k) \geq Q'_{uv} - Q'_{x_k \widehat{t}_k} - q'(H'_k). \tag{19}$$

On the other hand, since $C^k[X]$ is nonsingular, (18) and $\alpha_v \neq 0$ show that $C^k[(X \setminus \{v\}) \cup \{x_k\}]$ is nonsingular. By Lemma 5.4, we obtain

$$\begin{aligned} &p(((X \setminus B^*) \setminus \{v\}) \cup \{x_k\}) - p(X \cap B^*) \\ &\quad \geq -\sum \{q(H_i) \mid H_i \in \Lambda_{\mathrm{n}},\ v \in H_i \subseteq H_k \setminus \{s\},\ v \neq t_i\} \\ &\quad\quad + \sum \{q(H_i) \mid H_i \in \Lambda_{\mathrm{n}},\ v \in H_i \subseteq H_k \setminus \{s\},\ v = t_i\} \\ &\quad\quad + \sum \{q(H_i) \mid H_i \in \Lambda_{\mathrm{s}},\ s \in H_i \subseteq H_k \setminus \{v\}\}, \end{aligned} \tag{20}$$

because $|((X \setminus \{v\}) \cup \{x_k\}) \cap H_i|$ is odd if and only if $|H_i \cap \{v, s\}| = 1$. Recall that s is the unique source vertex in X if $H_k \in \Lambda_{\mathrm{s}}$ and $s = b_k$ if $H_k \in \Lambda_{\mathrm{n}}$. By (17) and (20), we have that

$$\begin{aligned} p'(x_k) - p'(v) &= p(x_k) - p(v) \\ &= -Q'_{vs} + 2\sum \{q(H_i) \mid H_i \in \Lambda_{\mathrm{s}},\ s \in H_i \subseteq H_k \setminus \{v\}\} \\ &\quad + Q'_{sx_k} - 2\sum \{q(H_i) \mid H_i \in \Lambda_{\mathrm{s}},\ s \in H_i \subseteq H_k \setminus \{x_i\}\}. \end{aligned} \tag{21}$$

By (19) and (21), we obtain

$$\begin{aligned} p'(\widehat{b}_k) - p'(u) &\geq Q'_{uv} - Q'_{vs} + 2\sum \{q(H_i) \mid H_i \in \Lambda_{\mathrm{s}},\ s \in H_i \subseteq H_k \setminus \{v\}\} \\ &\quad + Q'_{sx_k} - 2\sum \{q(H_i) \mid H_i \in \Lambda_{\mathrm{s}},\ s \in H_i \subseteq H_k \setminus \{x_i\}\} - Q'_{x_k \widehat{t}_k} - q'(H'_k) \\ &= Q'_{us} - Q'_{s\widehat{t}_k} - q'(H'_k) = Q'_{u\widehat{b}_k}, \end{aligned}$$

which shows the claim. □

Claim 10.2. *$\widehat{C}'_{u\widehat{b}_k} = 0$ for any $u \in \widehat{B}' \setminus \widehat{H}'_k$ and $\widehat{C}'_{\widehat{t}_k \widehat{b}_k} \neq 0$.*

Proof. Let $Y := B^* \setminus (H_k \setminus (X \cup \{t_k\}))$. By the definition, $C^k[Y, \{\widehat{b}_k\}]$ is a linear combination of $C^k[Y, \{v\}]$, where $v \in (X \setminus B^*) \cup \{x_k\}$. However, since $\eta_{\widehat{t}_k} = C^k_{\widehat{t}_k x_k} \neq 0 = C^k_{\widehat{t}_k \widehat{b}_k}$, this relation does not hold if we add a row corresponding to $\widehat{t}_k$, that is, $C^k[Y \cup \{\widehat{t}_k\}, \{\widehat{b}_k\}]$ is not a linear combination of $C^k[Y \cup \{\widehat{t}_k\}, \{v\}]$, where $v \in (X \setminus B^*) \cup \{x_k\}$. Since $t_i \in X \cup \{x_k\}$ for any $i < k$ with $H_i \in \Lambda_{\rm n}$ and $H_i \cap (X \cup \{x_k\}) \neq \emptyset$, we have that $C'[Y, \{\widehat{b}_k\}]$ is a linear combination of $C'[Y, \{v\}]$, where $v \in ((X \cup \{t_k\}) \setminus B^*) \cup \{x_k\}$. Furthermore, this relation does not hold if we add a row corresponding to $\widehat{t}_k$.

Recall that $A' = [I\ C']$, the row set of A' is denoted by U', and $B' = B^*$. Then, $A'[U', \{\widehat{b}_k\}]$ can be represented as a linear combination of $A'[U', \{v\}]$, where $v \in ((X \cup \{t_k\}) \setminus B') \cup \{x_k\} \cup ((H_k \setminus (X \cup \{t_k\})) \cap B')$. Furthermore, the coefficient of $A'[U', \{\widehat{t}_k\}]$ in this linear combination is not zero, because $C'[Y \cup \{\widehat{t}_k\}, \{\widehat{b}_k\}]$ is not a linear combination of $C'[Y \cup \{\widehat{t}_k\}, \{v\}]$, where $v \in ((X \cup \{t_k\}) \setminus B') \cup \{x_k\}$.

Since $\widehat{B}' = B' \triangle P$ and $P \cap H_k = X \cup \{x_k, t_k\}$, we have that $((X \cup \{t_k\}) \setminus B') \cup \{x_k\} \cup ((H_k \setminus (X \cup \{t_k\})) \cap B') = \widehat{B}' \cap \widehat{H}'_k$. Recall that $\widehat{A}'$ is obtained from A' by applying row transformations and by changing the ordering of columns. Thus, $\widehat{A}'[U', \{\widehat{b}_k\}]$ is a linear combination of $\widehat{A}'[U', \{v\}]$, where $v \in \widehat{B}' \cap \widehat{H}'_k$, and the coefficient of $\widehat{A}'[U', \{\widehat{t}_k\}]$ is not zero. This shows that $\widehat{C}'_{u\widehat{b}_k} = 0$ for $u \in \widehat{B}' \setminus \widehat{H}'_k$ and $\widehat{C}'_{\widehat{t}_k \widehat{b}_k} \neq 0$, where we recall that $\widehat{C}' = \widehat{A}'[U', V' \setminus \widehat{B}']$ and the row set of $\widehat{C}'$ is identified with $\widehat{B}'$. $\square$

Note that we add $\{\widehat{b}_i \mid i \in \{1, \dots, k-1\},\ i \in I_P\}$ after this procedure, but adding these vertices does not affect $\widehat{C}'_{u\widehat{b}_k}$ for any u. By Claims 10.1 and 10.2, we have (14) and (15).

10.2.2 Case (ii): $x_k \in B^*$

Suppose that $x_k \in B^*$. In this case, $\widehat{b}_k, \widehat{d}_k \in B'$, and $\widehat{t}_k \in V' \setminus B'$. The nonsingularity of $C^k[X] = C^\lambda[X]$ shows that a row vector $C^k[\{x_k\}, X \setminus B^*]$ can be represented as a linear combination of $C^k[\{u\}, X \setminus B^*]$, where $u \in X \cap B^*$. That is, there exists a constant $\alpha_u \in \mathbf{K}$ for each $u \in X \cap B^*$ such that

$$C^k[\{x_k\}, X \setminus B^*] = \sum_{u \in X \cap B^*} \alpha_u \cdot C^k[\{u\}, X \setminus B^*]. \tag{22}$$

Note that α_u's can be computed by $O(n^2)$ operations. Define a row vector η by

$$\eta := C^k[\{x_k\}, V^* \setminus B^*] - \sum_{u \in X \cap B^*} \alpha_u \cdot C^k[\{u\}, V^* \setminus B^*],$$

and define the row vector of C^k corresponding to $\widehat{b}_k$ by

$$C^k_{\widehat{b}_k v} = \begin{cases} 0 & \text{if } v \in (H_k \setminus (X \cup \{t_k\})) \setminus B^*, \\ \eta_v & \text{otherwise} \end{cases}$$

for $v \in V^* \setminus B^*$, where C^k is the matrix obtained from C' by elementary column and row transformations using nonzero entries at $\{(b_i, t_i) \mid H'_i \in \Lambda'_{\rm n},\ i \in \{1, \dots, k\}\}$. This naturally defines the row vector of C' corresponding to $\widehat{b}_k$. By a similar argument as Claim 10.1, in which we change the role of rows and columns of C', we obtain the following claim, which shows (14).

Claim 10.3. *For any $v \in V' \setminus B' = V^* \setminus B^*$, if $(\widehat{b}_k, v) \in F_{\Lambda'}$, then $p'(v) - p'(\widehat{b}_k) \geq Q'_{\widehat{b}_k v}$.*

We next show that (15) holds.

Claim 10.4. *$\widehat{C}'_{\widehat{b}_k v} = 0$ for any $v \in (V' \setminus \widehat{B}') \setminus \widehat{H}'_k$ and $\widehat{C}'_{\widehat{b}_k \widehat{t}_k} \neq 0$.*

Proof. Let $Y := (V^* \setminus B^*) \setminus (H_k \setminus (X \cup \{t_k\}))$. By the definition, $C^k[\{\widehat{b}_k\}, Y]$ is a linear combination of $C^k[\{u\}, Y]$, where $u \in (X \cap B^*) \cup \{x_k\}$. However, since $\eta_{\widehat{t}_k} = C^k_{x_k \widehat{t}_k} \neq 0 = C^k_{\widehat{b}_k \widehat{t}_k}$, this relation does not hold if we add a column corresponding to $\widehat{t}_k$, that is, $C^k[\{\widehat{b}_k\}, Y \cup \{\widehat{t}_k\}]$ is not a linear combination of $C^k[\{u\}, Y \cup \{\widehat{t}_k\}]$, where $u \in (X \cap B^*) \cup \{x_k\}$.

Since $t_i \in X \cup \{x_k\}$ for any $i < k$ with $H_i \in \Lambda_{\rm n}$ and $H_i \cap (X \cup \{x_k\}) \neq \emptyset$, we have that $C'[\{\widehat{b}_k\}, Y]$ is a linear combination of $C'[\{u\}, Y]$, where $u \in ((X \cup \{t_k\}) \cap B^*) \cup \{x_k\}$. That is, there exists a constant α'_u for each $u \in ((X \cup \{t_k\}) \cap B^*) \cup \{x_k\}$ such that

$$C'[\{\widehat{b}_k\}, Y] = \sum_{u \in ((X \cup \{t_k\}) \cap B^*) \cup \{x_k\}} \alpha'_u \cdot C'[\{u\}, Y].$$

Furthermore, this relation does not hold if we add a column corresponding to $\widehat{t}_k$, that is,

$$C'_{\widehat{b}_k \widehat{t}_k} \neq \sum_{u \in ((X \cup \{t_k\}) \cap B^*) \cup \{x_k\}} \alpha'_u \cdot C'_{u \widehat{t}_k}. \tag{23}$$

Recall that $A' = [I\ C']$, the row set of A' is denoted by U', and $B' = B^* \cup \{\widehat{b}_k\}$.

We first replace the row $A'[\{\widehat{b}_k\}, V']$ with

$$A''[\{\widehat{b}_k\}, V'] := A'[\{\widehat{b}_k\}, V'] - \sum_{u \in ((X \cup \{t_k\}) \cap B^*) \cup \{x_k\}} \alpha'_u \cdot A'[\{u\}, V'] \tag{24}$$

to obtain a new matrix A'', which is obtained from A' by a row transformation. By the definition of A'', we obtain $A''_{\widehat{b}_k \widehat{b}_k} = 1$, $A''_{\widehat{b}_k u} = -\alpha'_u$ for $u \in ((X \cup \{t_k\}) \cap B^*) \cup \{x_k\}$, and $A''_{\widehat{b}_k u} = 0$ for $u \in B^* \setminus (X \cup \{t_k, x_k\})$. Furthermore, by the definitions of α'_u, it holds that $A''_{\widehat{b}_k v} = 0$ for $v \in Y$. Therefore, we have that $A''_{\widehat{b}_k v} \neq 0$ only if $v \in ((X \cup \{t_k\}) \cap B^*) \cup \{x_k, \widehat{b}_k\} \cup ((H_k \setminus (X \cup \{t_k\})) \setminus B^*) = (\widehat{H}'_k \setminus \widehat{B}') \cup \{\widehat{b}_k\}$, where we note that $\widehat{B}' = B' \triangle P$ and $P \cap H_k = X \cup \{t_k, x_k\}$. Furthermore, by (23) and (24), we obtain $A''_{\widehat{b}_k \widehat{t}_k} \neq 0$.

Recall that $\widehat{A}'$ is obtained from A' by applying row transformation and by changing the ordering of columns so that $\widehat{A}'[U', \widehat{B}']$ is the identity matrix. Since $A''_{\widehat{b}_k \widehat{b}_k} = 1$ and $A''_{\widehat{b}_k v} = 0$ for $v \in \widehat{B}' \setminus \{\widehat{b}_k\}$, it holds that $\widehat{A}'[\{\widehat{b}_k\}, V'] = A''[\{\widehat{b}_k\}, V']$. Therefore, $\widehat{C}'_{\widehat{b}_k v} = \widehat{A}'_{\widehat{b}_k v} = A''_{\widehat{b}_k v} = 0$ for any $v \in (V' \setminus \widehat{B}') \setminus \widehat{H}'_k$ and $\widehat{C}'_{\widehat{b}_k \widehat{t}_k} = \widehat{A}'_{\widehat{b}_k \widehat{t}_k} = A''_{\widehat{b}_k \widehat{t}_k} \neq 0$, where we recall that $\widehat{C}' = \widehat{A}'[U', V' \setminus \widehat{B}']$ and the row set of $\widehat{C}'$ is identified with $\widehat{B}'$. $\square$

Note that we add $\{\widehat{b}_i \mid i \in \{1, \dots, k-1\},\ i \in I_P\}$ after this procedure, but adding these vertices does not affect $\widehat{C}'_{\widehat{b}_k v}$ for any v. By Claims 10.3 and 10.4, we have (14) and (15).

10.3 Feasibility

In this subsection, we show that the dual feasibility holds after the augmentation.

Lemma 10.5. *Assume that $\widehat{V}^*$, $\widehat{B}^*$, $\widehat{C}^*$, $\widehat{\Lambda}$, $\widehat{p}$, and $\widehat{q}$ are defined as in Sections 10.1 and 10.2. Then, dual variables $\widehat{p}$ and $\widehat{q}$ are feasible with respect to $\widehat{C}^*$ and $\widehat{\Lambda}$.*

Proof. It suffices to prove that dual variables $\widehat{p}'$ and $\widehat{q}'$ are feasible with respect to $\widehat{C}'$ and $\widehat{\Lambda}'$, because removing $\{b_i, t_i, \bar{t}_i \mid i \in I'_P\}$ does not affect the dual feasibility.

Since p and q are feasible, it is obvious that $\widehat{p}'$ and $\widehat{q}'$ satisfy (DF1). Since $\widehat{q}'(\widehat{H}'_i) = q'(H'_i)$ for each i, we can also see that (DF3) holds by the definition of $\widehat{p}'$. To show (DF2), take a pair $(u, v) \in F_{\widehat{\Lambda}'}$. Then, by Lemma 5.1, $\widehat{C}'[X]$ is nonsingular, where $X := \{u, v\} \cup \bigcup \{\{\widehat{b}_i, \widehat{t}_i\} \mid i \in \widehat{I}'_{uv} \setminus \widehat{J}'_{uv},\ \widehat{H}'_i \in \widehat{\Lambda}'_{\rm n}\}$. Note that $\widehat{I}'_{uv}$ and $\widehat{J}'_{uv}$ are defined with respect to $\widehat{\Lambda}'$ in the same way as I_{uv} and J_{uv}. Since $\widehat{A}'[U', \widehat{B}']$ is the identity matrix and $\widehat{C}' = \widehat{A}'[U', V' \setminus \widehat{B}']$, the nonsingularity of $\widehat{C}'[X]$ shows that $\widehat{A}'[U', (X \setminus \widehat{B}') \cup (\widehat{B}' \setminus X)]$ is nonsingular. Let $Y := (X \setminus \widehat{B}') \cup (\widehat{B}' \setminus X) = X \triangle \widehat{B}'$. Since $\widehat{A}'$ is obtained from A' by row transformations, it holds that $A'[U', Y]$ is nonsingular, which shows that $C'[B' \setminus Y, Y \setminus B'] = C'[Y \triangle B']$ is nonsingular. Since p' and q' are dual feasible with respect to C' and Λ', by Lemma 5.6, we have that

$$
\begin{aligned}
p'(Y \setminus B') - p'(B' \setminus Y) \geq & -\sum \{q'(H'_i) \mid H'_i \in \Lambda'_{\rm n},\ |(Y \triangle B') \cap H'_i| \text{ is odd}\} \\
& + \sum \{q'(H'_i) \mid H'_i \in \Lambda'_{\rm s},\ |(Y \triangle B') \cap H'_i| \text{ is odd}\}.
\end{aligned}
$$

On the other hand, by the definition of P, we have

$$
p'(P \setminus B') - p'(P \cap B') = Q_{rs} = \sum \{q'(H'_i) \mid i \in I'_{rs}\},
$$

where r and s are the source vertices in P, and I'_{rs} is defined with respect to Λ'. By these inequalities, we have

$$
\begin{aligned}
&p'(Y \setminus B') + p'(P \cap B') - p'(B' \setminus Y) - p'(P \setminus B') \\
&\quad \geq -\sum\{q'(H'_i) \mid H'_i \in \Lambda'_{\mathrm{n}},\ |(Y \triangle B') \cap H'_i| \text{ is odd}\} \\
&\qquad + \sum\{q'(H'_i) \mid H'_i \in \Lambda'_{\mathrm{s}},\ |(Y \triangle B') \cap H'_i| \text{ is odd}\} - \sum\{q'(H'_i) \mid i \in I'_{rs}\}. \qquad (25)
\end{aligned}
$$

Recall that $\widehat{B}' = B' \triangle P$. This shows that

$$Y \setminus B' = (\widehat{B}' \triangle X) \setminus (\widehat{B}' \triangle P) = (X \setminus (\widehat{B}' \cup P)) \cup ((\widehat{B}' \cap P) \setminus X), \qquad (26)$$

$$B' \setminus Y = (\widehat{B}' \triangle P) \setminus (\widehat{B}' \triangle X) = (P \setminus (\widehat{B}' \cup X)) \cup ((\widehat{B}' \cap X) \setminus P), \qquad (27)$$

$$Y \triangle B' = X \triangle P, \qquad (28)$$

$$P \setminus B' = P \setminus (\widehat{B}' \triangle P) = P \cap \widehat{B}', \qquad (29)$$

$$P \cap B' = P \cap (\widehat{B}' \triangle P) = P \setminus \widehat{B}'. \qquad (30)$$

By combining (25)–(30), we obtain

$$
\begin{aligned}
&p'(X \setminus \widehat{B}') - p'(X \cap \widehat{B}') \\
&\quad = p'(Y \setminus B') + p'(P \cap B') - p'(B' \setminus Y) - p'(P \setminus B') \\
&\quad \geq -\sum\{q'(H'_i) \mid H'_i \in \Lambda'_{\mathrm{n}},\ |(X \triangle P) \cap H'_i| \text{ is odd}\} \\
&\qquad + \sum\{q'(H'_i) \mid H'_i \in \Lambda'_{\mathrm{s}},\ |(X \triangle P) \cap H'_i| \text{ is odd}\} - \sum\{q'(H'_i) \mid i \in I'_{rs}\} \\
&\quad = -\sum\{q'(H'_i) \mid H'_i \in \Lambda'_{\mathrm{n}},\ |X \cap H'_i| \text{ is odd}\} + \sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\mathrm{s}},\ |X \cap H'_i| \text{ is odd}\} \\
&\qquad + \sum\{q'(H'_i) \mid i \in I'_{rs},\ |X \cap H'_i| \text{ is even}\} - \sum\{q'(H'_i) \mid i \in I'_{rs}\} \\
&\quad = -\sum\{q'(H'_i) \mid H'_i \in \Lambda'_{\mathrm{n}},\ |X \cap H'_i| \text{ is odd}\} + \sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\mathrm{s}},\ |X \cap H'_i| \text{ is odd}\} \\
&\qquad - \sum\{q'(H'_i) \mid i \in I'_{rs},\ |X \cap H'_i| \text{ is odd}\} \\
&\quad = -\sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\mathrm{n}},\ |X \cap H'_i| \text{ is odd}\} + \sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\mathrm{s}},\ |X \cap H'_i| \text{ is odd}\} \\
&\quad = -\sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\mathrm{n}},\ i \in I'_{uv} \setminus (\widehat{I}'_{uv} \setminus \widehat{J}'_{uv})\} + \sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\mathrm{s}},\ i \in I'_{uv}\}. \\
&\hfill (31)
\end{aligned}
$$

Note that, in the second equality above, we use the facts that

- $\widehat{H}'_i \in \widehat{\Lambda}'_{\mathrm{s}}$ if and only if both $H'_i \in \Lambda'_{\mathrm{s}}$ and $i \notin I'_{rs}$ hold, and
- $|P \cap H'_i|$ is odd if and only if $i \in I'_{rs}$.

On the other hand, we have

$$
\begin{aligned}
p'(\{\widehat{b}_i, \widehat{t}_i\} \setminus \widehat{B}') - p'(\{\widehat{b}_i, \widehat{t}_i\} \cap \widehat{B}') &= p'(\{\widehat{b}_i, \widehat{t}_i\} \setminus B') - p'(\{\widehat{b}_i, \widehat{t}_i\} \cap B') \\
&= -q'(H'_i) = -\widehat{q}'(\widehat{H}'_i) \qquad (32)
\end{aligned}
$$

for $i \in \widehat{I}'_{uv} \setminus \widehat{J}'_{uv}$ with $\widehat{H}'_i \in \widehat{\Lambda}'_{\rm n}$.

Since $(u, v) \in F_{\widehat{\Lambda}'}$, by (15) and Lemma 4.1, we have either

(i) $\widehat{b}_i \notin \{u, v\}$ for any $i \in \{1, \ldots, \Lambda\}$, or

(ii) $\widehat{b}_k \in \{u, v\}$ for some $k \in \{1, \ldots, \Lambda\}$, $\widehat{b}_i \notin \{u, v\}$ for any $i \in \{1, \ldots, \Lambda\} \setminus \{k\}$, and $\{u, v\} \subseteq \widehat{H}_k$.

In the case (i), we have $I'_{uv} = \widehat{I}'_{uv}$, and hence the right hand side of (31) is equal to

$$-\sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\rm n},\ i \in \widehat{I}'_{uv} \cap \widehat{J}'_{uv}\} + \sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\rm s},\ i \in \widehat{I}'_{uv}\}.$$

By combining this with (32), we obtain

$$\widehat{p}'(v) - \widehat{p}'(u) = p'(v) - p'(u) \geq \sum_{i \in \widehat{I}'_{uv} \setminus \widehat{J}'_{uv}} \widehat{q}'(\widehat{H}'_i) - \sum_{i \in \widehat{I}'_{uv} \cap \widehat{J}'_{uv}} \widehat{q}'(\widehat{H}'_i),$$

which shows that (DF2) holds.

In the case (ii), we have $I'_{uv} = \widehat{I}'_{uv} \cup \{k\}$, and hence the right hand side of (31) is equal to

$$-\sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\rm n},\ i \in (\widehat{I}'_{uv} \cap \widehat{J}'_{uv}) \cup \{k\}\} + \sum\{\widehat{q}'(\widehat{H}'_i) \mid \widehat{H}'_i \in \widehat{\Lambda}'_{\rm s},\ i \in \widehat{I}'_{uv}\}.$$

By combining this with (32), we obtain

$$\widehat{p}'(v) - \widehat{p}'(u) = p'(v) - p'(u) + \widehat{q}'(\widehat{H}'_k) \geq \sum_{i \in \widehat{I}'_{uv} \setminus \widehat{J}'_{uv}} \widehat{q}'(\widehat{H}'_i) - \sum_{i \in \widehat{I}'_{uv} \cap \widehat{J}'_{uv}} \widehat{q}'(\widehat{H}'_i),$$

which shows that (DF2) holds. □

10.4 Search in Each Blossom

We have already given in Section 10.1 the definition of $\widehat{V}^*$, $\widehat{B}^*$, $\widehat{C}^*$, $\widehat{\Lambda}$, $\widehat{p}$, and $\widehat{q}$, which are the objects after the augmentation. However, we do not have a new generator $\widehat{g}_i$ of $\widehat{t}_i$, a new routing $R_{\widehat{H}_i}(x)$ for each $x \in \widehat{H}_i^\circ$, and a new ordering $<_{\widehat{H}_i}$ in $\widehat{H}_i^\circ$ after the augmentation. To overcome these problems, we apply Search in each blossom.

Let $\widehat{\lambda} := |\widehat{\Lambda}|$. For $i = 1, 2, \ldots, \widehat{\lambda}$ in this order, we apply Search in $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$, where

- the blossom set is $\{\widehat{H}_j \in \widehat{\Lambda} \mid \widehat{H}_j \subsetneq \widehat{H}_i\}$,
- $\widehat{C}^*$, $\widehat{p}$, and $\widehat{q}$ are restricted to $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$, and
- $\widehat{b}_i$ is regarded as the unique source vertex if $\widehat{H}_i$ has no source line.

Note that if $\widehat{H}_i$ is a source blossom, then it contains a unique source line, and $\{\widehat{b}_i, \widehat{t}_i, \widehat{\bar{t}}_i\}$ does not exist. Note also that we have already computed $\widehat{g}_j$ (if exists), $R_{\widehat{H}_j}(x)$, and $<_{\widehat{H}_j}$ for each $\widehat{H}_j \in \widehat{\Lambda}$ with $\widehat{H}_j \subsetneq \widehat{H}_i$. Since there may be vertices outside $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$, we slightly modify the procedure Search as follows.

> Although Search is basically working on $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$, we consider all the vertices in $\widehat{V}^*$ when we introduce new vertices in Step 2 of RBlossom(v, u) or Step 1 of DBlossom(v, u). That is, in these steps, we suppose that $V^* := \widehat{V}^*$ and $B^* := \widehat{B}^*$.

If Search returns $\emptyset$, then we update the dual variables (see below) and repeat the procedure as long as $\widehat{q}(\widehat{H}_i)$ is positive. Since there exists no augmenting path in $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$, this procedure terminates only when $\widehat{q}(\widehat{H}_i)$ becomes zero. We note that this procedure may create new blossoms in $\widehat{H}_i$, and such a blossom $\widehat{H}_j$ is accompanied by $\widehat{g}_j$ (if exists), $R_{\widehat{H}_j}(x)$, and $<_{\widehat{H}_j}$ satisfying (GT1), (GT2), and (BR1)–(BR5) by the argument in Sections 7 and 8. We can also see that $\widehat{p}$ and $\widehat{q}$ are feasible after creating a new blossom by the same argument as Lemma 8.9. Then, since $\widehat{q}(\widehat{H}_i) = 0$ after the procedure, we remove $\widehat{H}_i$ from $\widehat{\Lambda}$.

Updating Dual Variables. In what follows in this subsection, we describe how we update the dual variables. We first show the following lemma.

Lemma 10.6. *When we apply* Search *in $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$ as above, every vertex in $\widehat{H}_i \cap V$ is labeled without updating the dual variables.*

Proof. By Lemma 9.2, it suffices to show that for every vertex $x \in \widehat{H}_i \cap V$, there exists a vertex set $X \subseteq \widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$ with the conditions in Lemma 9.2. We fix $x \in \widehat{H}_i \cap V$ and consider the following two cases separately.

Case (i): Suppose that $P \cap H_i = \emptyset$. In this case, $\widehat{b}_i = b_i$. Since b_i is incident only to t_i in G°, for any $x \in H_i^\circ$, we have that $G^\circ[(R_{H_i}(x) \setminus \{x\}) \cup \{b_i\}]$ has a unique tight perfect matching. Thus, $G^\circ[P \cup (R_{H_i}(x) \setminus \{x\}) \cup \{b_i\}]$ also has a unique tight perfect matching. Since this shows that $C'[P \cup (R_{H_i}(x) \setminus \{x\}) \cup \{b_i\}]$ is nonsingular, we have that $C'[P \cup (R_{H_i}(x) \setminus \{x, t_i, \bar{t}_i\}) \cup \{b_i\}]$ is nonsingular, and hence $\widehat{C}'[X]$ is nonsingular, where $X := (R_{H_i}(x) \setminus \{x, t_i, \bar{t}_i\}) \cup \{b_i\} \subseteq \widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$. Furthermore, the tightness of the perfect matching shows that X satisfies (13) after the augmentation. Therefore, X satisfies the conditions in Lemma 9.2.

Case (ii): Suppose that $P \cap H_i \neq \emptyset$. In this case, $P \cap H_i = R_{H_i}(x_i)$ for some $x_i \in H_i$ and $\widehat{b}_i$ is defined as in Section 10.2. Let $y \in P$ be the vertex matched with x_i in the unique tight perfect matching in $G^\circ[P]$. Then, each of $G^\circ[P \setminus ((R_{H_i}(x_i) \setminus \{t_i\}) \cup \{y\})]$ and $G^\circ[(R_{H_i}(x_i) \setminus \{t_i\}) \cup \{y\}]$ also has a unique tight perfect matching. By the definition of

$\widehat{b}_i$, $C'[(R_{H_i}(x_i) \setminus \{t_i, x_i\}) \cup \{\widehat{b}_i, y\}]$ is nonsingular, and hence either $C'_{\widehat{b}_i y} \neq 0$ or $C'_{y \widehat{b}_i} \neq 0$. We can also see that for any $v \in V' \setminus H'_i$, there is a tight edge in G° between $\widehat{b}_i$ and v only if there is a tight edge in G° between $\widehat{b}_i$ and z for some $z \in R_{H_i}(x_i) \setminus \{t_i\}$ (see the proof of Claim 10.1). With these facts, we have that $G^\circ[(P \setminus (R_{H_i}(x_i) \setminus \{t_i\})) \cup \{\widehat{b}_i\}]$ has a unique tight perfect matching by using the same argument as Lemma 8.4. Let $Y := (P \setminus (R_{H_i}(x_i) \setminus \{t_i\})) \cup \{\widehat{b}_i\}$. For the given vertex $x \in \widehat{H}_i \cap V$, define $Z \subseteq H'_i$ as follows:

$$Z := \begin{cases} R_{H_i}(x) \setminus \{x, t_i\} & \text{if } x \notin P, \\ R_{H_i}(\bar{x}) \setminus \{\bar{x}, t_i\} & \text{if } x \in P. \end{cases}$$

Then, since $G^\circ[Z]$ has a unique tight perfect matching, $G^\circ[Y \cup Z]$ also has a unique tight perfect matching. Since this shows that $C'[Y \cup Z]$ is nonsingular, we have that $\widehat{C}'[X]$ is nonsingular, where $X := (Y \cup Z) \triangle P$. If X contains $\{t_j, \bar{t}_j\} \not\subseteq \widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$, then we remove them from X. Note that this does not affect the nonsingularity of $\widehat{C}'[X]$. Then, $X \subseteq \widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$ and X consists of lines, dummy lines, and a source vertex $\widehat{b}_i$. Furthermore, the tightness of the perfect matching in $G^\circ[Y \cup Z]$ shows that X satisfies (13) after the augmentation. Therefore, X satisfies the conditions in Lemma 9.2.

By these two cases, x is labeled without updating the dual variables by Lemma 9.2, which completes the proof. $\square$

Suppose that Search returns $\emptyset$ when it is applied in $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$. Define R^+, R^-, Z^+, Z^-, Y, and $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ as in Section 9. By Lemma 10.6, we have that $R^+ = \{\widehat{b}_i\}$, $R^- = Y = \emptyset$, and $\epsilon_2 = \epsilon_3 = \epsilon_4 = +\infty$. We now modify the dual variables in $\widehat{V}^*$ (not only in $\widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$) as follows. Let $\epsilon' := \min\{\epsilon, \widehat{q}(\widehat{H}_i)\}$, which is a finite positive value. If $\widehat{H}_i$ contains no source line, then update $\widehat{p}(\widehat{b}_i)$ as

$$\widehat{p}(\widehat{b}_i) := \begin{cases} \widehat{p}(\widehat{b}_i) + \epsilon' & (\widehat{b}_i \in \widehat{B}^*), \\ \widehat{p}(\widehat{b}_i) - \epsilon' & (\widehat{b}_i \in \widehat{V}^* \setminus \widehat{B}^*), \end{cases}$$

and update $\widehat{p}(\widehat{t}_i)$ and $\widehat{p}(\widehat{\bar{t}}_i)$ as $\widehat{p}(\widehat{t}_i) := \widehat{p}(\widehat{\bar{t}}_i) := \widehat{p}(\widehat{b}_i)$. For each maximal blossom H in $\{\widehat{H}_j \in \widehat{\Lambda} \mid \widehat{H}_j \subsetneq \widehat{H}_i\}$, which must be labeled with $\oplus$, update $\widehat{q}(H)$ as $\widehat{q}(H) := \widehat{q}(H) + \epsilon'$. We also update $\widehat{q}(\widehat{H}_i)$ by $\widehat{q}(\widehat{H}_i) := \widehat{q}(\widehat{H}_i) - \epsilon'$. We now prove the following claim, which shows the validity of this procedure.

Claim 10.7. *The obtained dual variables $\widehat{p}$ and $\widehat{q}$ are feasible in $\widehat{V}^*$ with respect to $\widehat{\Lambda}$.*

Proof. If suffices to show (DF2). Suppose that $u \in \widehat{B}^*$, $v \in \widehat{V}^* \setminus \widehat{B}^*$, and $(u, v) \in F_{\widehat{\Lambda}}$. If $u, v \in \widehat{H}_i \setminus \{\widehat{t}_i, \widehat{\bar{t}}_i\}$, then the inequality in (DF2) holds by the argument in Section 10.3. In the other cases, we will show that updating the dual variables does not decrease the value of $\widehat{p}(v) - \widehat{p}(u) - \widehat{Q}_{uv}$.

- If $u = \widehat{t_i}$ and $v \in \widehat{H}_i$, then $\widehat{p}(u)$ decreases by ϵ and $\widehat{p}(v) - \widehat{Q}_{uv}$ decreases by at most ϵ, which shows that $\widehat{p}(v) - \widehat{p}(u) - \widehat{Q}_{uv}$ does not decrease.
- If $v = \widehat{t_i}$ and $u \in \widehat{H}_i$, then $\widehat{p}(v)$ increases by ϵ and $\widehat{p}(u) + \widehat{Q}_{uv}$ increases by at most ϵ, which shows that $\widehat{p}(v) - \widehat{p}(u) - \widehat{Q}_{uv}$ does not decrease.
- If $u \in \widehat{H}_i$ and $v \notin \widehat{H}_i$, then $\widehat{Q}_{uv}$ decreases by ϵ by updating $\widehat{q}(\widehat{H}_i)$ and $\widehat{p}(v)$ does not change. Since $\widehat{p}(u) + \widehat{Q}_{uv}$ increases by at most ϵ by updating $\widehat{p}(u)$ or $\widehat{q}(H)$ where H is the maximal blossom containing u, we have that $\widehat{p}(v) - \widehat{p}(u) - \widehat{Q}_{uv}$ does not decrease.
- If $u \notin \widehat{H}_i$ and $v \in \widehat{H}_i$, then $\widehat{Q}_{uv}$ decreases by ϵ by updating $\widehat{q}(\widehat{H}_i)$ and $\widehat{p}(u)$ does not change. Since $\widehat{p}(v) - \widehat{Q}_{uv}$ decreases by at most ϵ by updating $\widehat{p}(v)$ or $\widehat{q}(H)$ where H is the maximal blossom containing v, we have that $\widehat{p}(v) - \widehat{p}(u) - \widehat{Q}_{uv}$ does not decrease.
- If $u, v \notin \widehat{H}_i$, then it is obvious that $\widehat{p}(v) - \widehat{p}(u) - \widehat{Q}_{uv}$ does not change.

Note that $\widehat{\widehat{t_i}}$ is incident only to $\widehat{t_i}$ in $F_{\widehat{\Lambda}}$. This shows that the inequality in (DF2) holds for any $(u, v) \in F_{\widehat{\Lambda}}$ after updating the dual variables. □

11 Algorithm Description and Complexity

Our algorithm for the minimum-weight parity base problem is described as follows.

Algorithm Minimum-Weight Parity Base

Step 1: Split the weight w_ℓ into $p(v)$ and $p(\bar{v})$ for each line $\ell = \{v, \bar{v}\} \in L$, i.e., $p(v) + p(\bar{v}) = w_\ell$. Execute the greedy algorithm for finding a base $B \in \mathcal{B}$ with minimum value of $p(B) = \sum_{u \in B} p(u)$. Set $\Lambda = \emptyset$.

Step 2: If there is no source line, then return $B := B^* \cap V$ as an optimal solution. Otherwise, apply Search. If Search returns $\emptyset$, then go to Step 3. If Search finds an augmenting path, then go to Step 4.

Step 3: Update the dual variables as in Section 9. If $\epsilon = +\infty$, then conclude that there exists no parity base and terminate the algorithm. Otherwise, delete all blossoms H_i with $q(H_i) = 0$ from Λ and go to Step 2.

Step 4: Apply Augment to obtain a new base $\widehat{B}^*$, a family $\widehat{\Lambda}$ of blossoms, and feasible dual variables $\widehat{p}$ and $\widehat{q}$. For each $\widehat{H}_i$ in increasing order of i, do the following.

While $\widehat{q}(\widehat{H}_i) > 0$, apply Search in $\widehat{H}_i \setminus \{\widehat{t_i}, \widehat{\widehat{t_i}}\}$ and update the dual variables as in Section 10.4. Then, remove $\widehat{H}_i$ from $\widehat{\Lambda}$.

Replace B^*, Λ, p, and q with $\widehat{B}^*$, $\widehat{\Lambda}$, $\widehat{p}$, and $\widehat{q}$, respectively, and go back to Step 2.

We have already seen the correctness of this algorithm. We now analyze the complexity. Since $|V^*| = O(n)$, an execution of the procedure Search as well as the dual update requires $O(n^2)$ arithmetic operations. By Lemma 9.3, Step 3 is executed at most $O(n)$ times per augmentation. In Step 4, we create a new blossom or remove $\widehat{H}_i$ from $\widehat{\Lambda}$ when we update the dual variables, which shows that the number of dual updates as well as executions of Search in Step 4 is also bounded by $O(n)$. Thus, Search and dual update are executed $O(n)$ times per augmentation, which requires $O(n^3)$ operations. We note that it also requires $O(n^3)$ operations to update C^i and G^i after augmentation. Since each augmentation reduces the number of source lines by two, the number of augmentations during the algorithm is $O(m)$, where $m = \operatorname{rank} A$, and hence the total number of arithmetic operations is $O(n^3 m)$.

Theorem 11.1. *Algorithm* Minimum-Weight Parity Base *finds a parity base of minimum weight or detects infeasibility with $O(n^3 m)$ arithmetic operations over $\mathbf{K}$.*

If $\mathbf{K}$ is a finite field of fixed order, each arithmetic operation can be executed in $O(1)$ time. Hence Theorem 11.1 implies the following.

Corollary 11.2. *The minimum-weight parity base problem over an arbitrary fixed finite field $\mathbf{K}$ can be solved in strongly polynomial time.*

When $\mathbf{K} = \mathbb{Q}$, it is not obvious that a direct application of our algorithm runs in polynomial time. This is because we do not know how to bound the number of bits required to represent the entries of C^i. However, the minimum-weight parity base problem over $\mathbb{Q}$ can be solved in polynomial time by applying our algorithm over a sequence of finite fields.

Theorem 11.3. *The minimum-weight parity base problem over $\mathbb{Q}$ can be solved in time polynomial in the binary encoding length $\langle A \rangle$ of the matrix representation A.*

Proof. By multiplying each entry of A by the product of the denominators of all entries, we may assume that each entry of A is an integer. Let γ be the maximum absolute value of the entries of A, and put $N := \lceil m \log(m\gamma) \rceil$. Note that N is bounded by a polynomial in $\langle A \rangle$. We compute the N smallest prime numbers $p_1, \dots, p_N$. Since it is known that $p_N = O(N \log N)$ by the prime number theorem, they can be computed in polynomial time by the sieve of Eratosthenes.

For $i = 1, \dots, N$, we consider the minimum-weight parity base problem over $\mathrm{GF}(p_i)$ where each entry of A is regarded as an element of $\mathrm{GF}(p_i)$. In other words, we consider the problem in which each operation is executed modulo p_i. Since each arithmetic operation over $\mathrm{GF}(p_i)$ can be executed in polynomial time, we can solve the minimum-weight parity base problem over GF(p_i) in polynomial time by Theorem 11.1. Among all optimal solutions of these problems, the algorithm returns the best one B. That is, B is the minimum weight parity set subject to $|B| = m$ and $\det A[U, B] \not\equiv 0 \pmod{p_i}$ for some $i \in \{1, \dots, N\}$.

To see the correctness of this algorithm, we evaluate the absolute value of the subdeterminant of A. For any subset $X \subseteq V$ with $|X| = m$, we have

$$|\det A[U, X]| \le m!\gamma^m \le (m\gamma)^m \le 2^N < \prod_{i=1}^{N} p_i.$$

This shows that $\det A[U, X] = 0$ if and only if $\det A[U, X] \equiv 0 \pmod{\prod_{i=1}^{N} p_i}$. Therefore, $\det A[U, X] \neq 0$ if and only if $\det A[U, X] \not\equiv 0 \pmod{p_i}$ for some $i \in \{1, \dots, N\}$, which shows that the output B is an optimal solution. □

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A Omitted Proofs

This appendix is devoted to proofs of Lemmas 5.2 and 5.3.

A.1 Proof of Lemma 5.2

It is easy to see that p and q satisfy (DF1) and (DF3) with respect to Λ if and only if p and q' satisfy (DF1) and (DF3) with respect to Λ'. Thus, it suffices to consider (DF2).

Necessity (only if part). Assume that p and q are feasible with respect to Λ, and suppose that $(u,v) \in F_{\Lambda'}$. If $(u,v) \in F_\Lambda$, then

$$\begin{aligned} p(v) - p(u) &\geq \sum_{j \in I_{uv} \setminus J_{uv}} q(H_j) - \sum_{j \in I_{uv} \cap J_{uv}} q(H_j) \\ &= \sum_{j \in (I_{uv} \setminus J_{uv}) \setminus \{i\}} q'(H_j) - \sum_{j \in (I_{uv} \cap J_{uv}) \setminus \{i\}} q'(H_j), \end{aligned}$$

which shows that p and q' satisfy the inequality in (DF2) for (u,v).

Thus, it suffices to consider the case when $(u,v) \in F_{\Lambda'}$ and $(u,v) \notin F_\Lambda$. Note that $(u,v) \in F_{\Lambda'}$ implies that $u \in B^*$ and $v \in V^* \setminus B^*$. By Lemma 5.1, $(u,v) \in F_{\Lambda'}$ shows that $C^*[X]$ is nonsingular, where $X := \{u,v\} \cup \bigcup\{\{b_j,t_j\} \mid j \in (I_{uv} \setminus J_{uv}) \setminus \{i\},\ H_j \in \Lambda_\mathrm{n}\}$. Furthermore, by Lemma 5.1, $(u,v) \notin F_\Lambda$ shows that $i \in I_{uv} \setminus J_{uv}$, $H_i \in \Lambda_\mathrm{n}$, and $C^*[X \cup \{b_i,t_i\}]$ is singular. By applying row and column transformations, we have that $C^{i-1}[X \cup \{b_i,t_i\}]$ is singular and $C^{i-1}[X]$ is nonsingular. Let

$$X' := X \setminus \bigcup\{\{b_j,t_j\} \mid j \in (I_{uv} \setminus J_{uv}) \setminus \{i\},\ H_j \in \Lambda_\mathrm{n},\ H_j \subseteq H_i\}.$$

By Lemma 4.1, if $b_j \in (X \setminus X') \cap B^*$, then $C^{i-1}_{b_j v'} = 0$ for any $v' \in (V^* \setminus B^*) \setminus \{t_j\}$ and $C^{i-1}_{b_j t_j} \neq 0$. Similarly, if $b_j \in (X \setminus X') \setminus B^*$, then $C^{i-1}_{u' b_j} = 0$ for any $u' \in B^* \setminus \{t_j\}$ and $C^{i-1}_{t_j b_j} \neq 0$. By these properties, the singularity of $C^{i-1}[X \cup \{b_i,t_i\}]$ is equivalent to that of $C^{i-1}[X' \cup \{b_i,t_i\}]$, and the nonsingularity of $C^{i-1}[X]$ is equivalent to that of $C^{i-1}[X']$. We consider the following two cases separately.

Case (i): Suppose that $v \in H_i$ and $u \in V^* \setminus H_i$. If $b_i \in V^* \setminus B^*$ and $t_i \in B^*$, then

$$\det\left(C^{i-1}[X' \cup \{b_i,t_i\}]\right) = \pm C^{i-1}_{t_i b_i} \cdot \det\left(C^{i-1}[X']\right),$$

because $C^{i-1}_{u' b_i} = 0$ for any $u' \in X' \cap B^*$ by Lemma 5.1. This contradicts that $C^{i-1}[X' \cup \{b_i,t_i\}]$ is singular, $C^{i-1}[X']$ is nonsingular, and $C^{i-1}_{t_i b_i} \neq 0$. Otherwise, $b_i \in B^*$ and $t_i \in V^* \setminus B^*$. In this case,

$$\det\left(C^{i-1}[X' \cup \{b_i,t_i\}]\right) = \pm C^{i-1}_{b_i t_i} \cdot \det\left(C^{i-1}[X']\right) \pm C^{i-1}_{b_i v} \cdot \det\left(C^{i-1}[(X' \setminus \{v\}) \cup \{t_i\}]\right),$$

because $C^{i-1}_{b_i v'} = 0$ for any $v' \in (X' \setminus B^*) \setminus \{v\}$ by Lemma 5.1. Since $C^{i-1}[X' \cup \{b_i,t_i\}]$ is singular, $C^{i-1}[X']$ is nonsingular, and $C^{i-1}_{b_i t_i} \neq 0$, we obtain $C^{i-1}_{b_i v} \neq 0$ and $C^{i-1}[(X' \setminus$

$\{v\}) \cup \{t_i\}]$ is nonsingular. By the dual feasibility of p and q, $C^{i-1}_{b_i v} \neq 0$ implies that $(b_i, v) \in F_\Lambda$ and

$$p(v) - p(b_i) \geq Q_{b_i v}. \tag{33}$$

Since $C^{i-1}[(X' \setminus \{v\}) \cup \{t_i\}] = C^*[(X' \setminus \{v\}) \cup \{t_i\}]$, its nonsingularity shows that $(u, t_i) \in F_\Lambda$ and

$$p(t_i) - p(u) \geq Q_{u t_i} \tag{34}$$

by Lemma 5.1. By combining (33), (34), $p(b_i) = p(t_i)$, and $q(H_i) = 0$, we obtain $p(v) - p(u) \geq Q_{uv}$, which shows that p and q' satisfy the inequality in (DF2) for (u, v).

Case (ii): Suppose that $u \in H_i$ and $v \in V^* \setminus H_i$. If $b_i \in B^*$ and $t_i \in V^* \setminus B^*$, then

$$\det\left(C^{i-1}[X' \cup \{b_i, t_i\}]\right) = \pm C^{i-1}_{b_i t_i} \cdot \det\left(C^{i-1}[X']\right),$$

because $C^{i-1}_{b_i v'} = 0$ for any $v' \in X' \setminus B^*$ by Lemma 5.1. This contradicts that $C^{i-1}[X' \cup \{b_i, t_i\}]$ is singular, $C^{i-1}[X']$ is nonsingular, and $C^{i-1}_{b_i t_i} \neq 0$. Otherwise, $b_i \in V^* \setminus B^*$ and $t_i \in B^*$. In this case,

$$\det\left(C^{i-1}[X' \cup \{b_i, t_i\}]\right) = \pm C^{i-1}_{t_i b_i} \cdot \det\left(C^{i-1}[X']\right) \pm C^{i-1}_{u b_i} \cdot \det\left(C^{i-1}[(X' \setminus \{u\}) \cup \{t_i\}]\right),$$

because $C^{i-1}_{u' b_i} = 0$ for any $u' \in (X' \cap B^*) \setminus \{u\}$ by Lemma 5.1. Since $C^{i-1}[X' \cup \{b_i, t_i\}]$ is singular, $C^{i-1}[X']$ is nonsingular, and $C^{i-1}_{t_i b_i} \neq 0$, we obtain $C^{i-1}_{u b_i} \neq 0$ and $C^{i-1}[(X' \setminus \{u\}) \cup \{t_i\}]$ is nonsingular. By the dual feasibility of p and q, $C^{i-1}_{u b_i} \neq 0$ implies that $(u, b_i) \in F_\Lambda$ and

$$p(b_i) - p(u) \geq Q_{u b_i}. \tag{35}$$

Since $C^{i-1}[(X' \setminus \{u\}) \cup \{t_i\}] = C^*[(X' \setminus \{u\}) \cup \{t_i\}]$, its nonsingularity shows that $(t_i, v) \in F_\Lambda$ and

$$p(v) - p(t_i) \geq Q_{t_i v} \tag{36}$$

by Lemma 5.1. By combining (35), (36), $p(b_i) = p(t_i)$, and $q(H_i) = 0$, we obtain $p(v) - p(u) \geq Q_{uv}$, which shows that p and q' satisfy the inequality in (DF2) for (u, v).

Sufficiency (if part). Assume that p and q' are feasible with respect to Λ', and suppose that $(u, v) \in F_\Lambda$. If $(u, v) \in F_{\Lambda'}$, then

$$\begin{aligned} p(v) - p(u) &\geq \sum_{j \in (I_{uv} \setminus J_{uv}) \setminus \{i\}} q'(H_j) - \sum_{j \in (I_{uv} \cap J_{uv}) \setminus \{i\}} q'(H_j) \\ &= \sum_{j \in I_{uv} \setminus J_{uv}} q(H_j) - \sum_{j \in I_{uv} \cap J_{uv}} q(H_j), \end{aligned}$$

which shows that p and q satisfy the inequality in (DF2) for (u, v).

Thus, it suffices to consider the case when $(u,v) \notin F_{\Lambda'}$ and $(u,v) \in F_\Lambda$. We define X and X' in the same way as the necessity part. Then, we have that $C^{i-1}[X' \cup \{b_i, t_i\}]$ is nonsingular and $C^{i-1}[X']$ is singular. We consider the following two cases separately.

Case (i): Suppose that $v \in H_i$ and $u \in V^* \setminus H_i$. If $b_i \in V^* \setminus B^*$ and $t_i \in B^*$, then

$$\det\left(C^{i-1}[X' \cup \{b_i, t_i\}]\right) = \pm C^{i-1}_{t_i b_i} \cdot \det\left(C^{i-1}[X']\right).$$

This contradicts that $C^{i-1}[X' \cup \{b_i, t_i\}]$ is nonsingular and $C^{i-1}[X']$ is singular. Otherwise, $b_i \in B^*$ and $t_i \in V^* \setminus B^*$. In this case,

$$\det\left(C^{i-1}[X' \cup \{b_i, t_i\}]\right) = \pm C^{i-1}_{b_i t_i} \cdot \det\left(C^{i-1}[X']\right) \pm C^{i-1}_{b_i v} \cdot \det\left(C^{i-1}[(X' \setminus \{v\}) \cup \{t_i\}]\right).$$

Since $C^{i-1}[X' \cup \{b_i, t_i\}]$ is nonsingular and $C^{i-1}[X']$ is singular, we obtain $C^{i-1}_{b_i v} \neq 0$ and $C^{i-1}[(X' \setminus \{v\}) \cup \{t_i\}]$ is nonsingular. By the dual feasibility of p and q', $C^{i-1}_{b_i v} \neq 0$ implies that $(b_i, v) \in F_{\Lambda'}$ and

$$p(v) - p(b_i) \geq Q_{b_i v}. \tag{37}$$

Since $C^{i-1}[(X' \setminus \{v\}) \cup \{t_i\}] = C^*[(X' \setminus \{v\}) \cup \{t_i\}]$, its nonsingularity shows that $(u, t_i) \in F_{\Lambda'}$ and

$$p(t_i) - p(u) \geq \sum_{j \in (I_{ut_i} \setminus J_{ut_i}) \setminus \{i\}} q'(H_j) = Q_{ut_i} \tag{38}$$

by Lemma 5.1. By combining (37), (38), and $p(b_i) = p(t_i)$, we obtain $p(v) - p(u) \geq Q_{uv}$, which shows that p and q satisfy the inequality in (DF2) for (u,v).

Case (ii): Suppose that $u \in H_i$ and $v \in V^* \setminus H_i$. In this case, in the same way as Case (ii) of the necessity part, p and q satisfy the inequality in (DF2) for (u,v). □

A.2 Proof of Lemma 5.3

We prove the lemma by induction on k. If $k = 0$, then the statement is obvious since $I_{uv} = \emptyset$. Let $u, v \in V^*$ be vertices such that $i_{uv} \leq k$ and $C^k_{uv} \neq 0$.

Suppose that $i_{uv} = k$. In this case, since $C^k_{uv} \neq 0$, (3) is immediately obtained from the dual feasibility.

In what follows, we consider the case when $i_{uv} \leq k-1$. If $C^{k-1}_{uv} \neq 0$, then (3) holds by the induction hypothesis. Otherwise, since $C^k_{uv} \neq 0$ and $C^{k-1}_{uv} = 0$, we have $\{u, v\} \subseteq H_k$ and one of the following.

(A) $b_k \in B^*$, $t_k \in V^* \setminus B^*$, $C^{k-1}_{b_k v} \neq 0$, and $C^{k-1}_{ut_k} \neq 0$, or

(B) $t_k \in B^*$, $b_k \in V^* \setminus B^*$, $C^{k-1}_{t_k v} \neq 0$, and $C^{k-1}_{ub_k} \neq 0$.

In the case (A), by the induction hypothesis and (DF3), we obtain

$$\begin{aligned}
p(v) - p(b_k) &\geq Q_{b_k v} = \sum_{i \in I_{b_k v} \setminus J_{b_k v}} q(H_i) - \sum_{i \in I_{b_k v} \cap J_{b_k v}} q(H_i), \\
p(t_k) - p(u) &\geq Q_{u t_k} = \sum_{i \in I_{u t_k} \setminus J_{u t_k}} q(H_i) - \sum_{i \in I_{u t_k} \cap J_{u t_k}} q(H_i), \\
p(t_k) - p(b_k) &= 0.
\end{aligned}$$

By combining these inequalities, we obtain

$$\begin{aligned}
p(v) - p(u) \geq & \sum_{i \in I_{b_k v} \setminus J_{b_k v}} q(H_i) - \sum_{i \in I_{b_k v} \cap J_{b_k v}} q(H_i) \\
& + \sum_{i \in I_{u t_k} \setminus J_{u t_k}} q(H_i) - \sum_{i \in I_{u t_k} \cap J_{u t_k}} q(H_i). \qquad (39)
\end{aligned}$$

We will show that, for each i, the coefficient of $q(H_i)$ in the right hand side of (39) is greater than or equal to that in the right hand side of (3).

- Suppose that $t_i \notin \{u, v, t_k\}$. In this case i is not contained in $I_{uv} \cap J_{uv}$, $I_{b_k v} \cap J_{b_k v}$, and $I_{u t_k} \cap J_{u t_k}$. If i is contained in $I_{uv} \setminus J_{uv}$, then exactly one of $I_{b_k v} \setminus J_{b_k v}$ and $I_{u t_k} \setminus J_{u t_k}$ contains i, which shows that the coefficient of $q(H_i)$ in (39) is greater than or equal to that in (3).
- Suppose that $t_i = t_k$, i.e., $i = k$. In this case, i is not contained in I_{uv}, $I_{b_k v}$, and $I_{u t_k}$, which shows that the coefficient of $q(H_k)$ is equal to zero in both (3) and (39).
- Suppose that $i \neq k$ and $t_i = u$. If $v \in H_i$, then i is contained in $I_{b_k v} \setminus J_{b_k v}$ and $I_{u t_k} \cap J_{u t_k}$, and it is not contained in the other sets. Otherwise, i is contained in $I_{uv} \cap J_{uv}$ and $I_{b_k v} \cap J_{b_k v}$ and it is not contained in the other sets. In both cases, the coefficient of $q(H_i)$ in (39) is greater than or equal to that in (3). The same argument can be applied to the case with $t_i = v$.

Thus, we obtain (3) from (39).

The same argument can be applied to the case (B), which completes the proof. □