Adjacency on Combinatorial Polyhedra

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Abstract: This paper shows some useful properties of the adjacency structures of a class of combinatorial polyhedra including the equality constrained 0-1 polytopes. The class of polyhedra considered here includes 0-1 polytopes related to some combinatorial optimization problems; e.g., set partitioning polytopes, set packing polytopes, perfect matching polytopes, vertex packing polytopes and all the faces of these polytopes.

First, we establish two fundamental properties of the equality constrained 0-1 polytopes. This paper deals with the polyhedra satisfying these two fundamental properties. We consider a path on the polyhedron satisfying the condition that for each co-ordinate, the vertices in a path form a monotonic sequence. When one of the end vertices of the path is optimal to an optimization problem defined on the polyhedron, the associated objective values form a monotonic sequence and the length of the path is bounded by the dimension of the polytope. In a sense, some of the results in this paper are natural extensions of the properties of the set partitioning polytopes showed by Balas and Padberg. However, different from the studies of Balas and Padberg, our proofs are not based on the pivot operations. Next, we prove the monotone Hirsch conjecture for the combinatorial polyhedra considered here. In the last section, we show that the monotone Hirsch conjecture is true for all the 0-1 polytopes.

1 Introduction

The simplex method is an efficient algorithm for solving a linear programming problem which moves between adjacent vertices of the polyhedron forming the set of feasible solutions. It seems that the success of the simplex method justifies the existence of efficient edge following algorithms for some combinatorial optimization problems. It means that an adjacency criterion for a class of polyhedra could provide a basis for an efficient algorithm which uses some sorts of local search technique. In fact, there exist such algorithms, e.g., Edmonds' blossom algorithm for matching problems [7] and the successful heuristic for traveling salesman problems proposed by Lin and Kernighan [14]. However, it may be very difficult to test adjacency for some combinatorial polyhedra. In [17], Papadimitriou showed that it is NP-complete to decide whether two given vertices of a symmetric traveling salesman polytope are non-adjacent.

In this paper, we discuss the adjacency structures of some combinatorial polyhedra. We begin this paper with an equality constrained 0-1 polytope P, i.e., the convex hull of the set of 0-1 valued vectors $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$, where A is an $m \times n$ matrix and $\boldsymbol{b}$ is an m-dimensional vector. All the entries of the matrix A and the vector $\boldsymbol{b}$ are real numbers. Since all the vertices of P are 0-1 valued, it is clear that $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$ is the set of all vertices of P. In this paper, the convex hull of a set of n-dimensional 0-1 valued vectors is called a *0-1 polytope* (in R^n).

The equality constrained 0-1 polytopes arise very naturally in combinatorial optimization problems. For example, the class of equality constrained 0-1 polytopes contains set partitioning polytopes, set packing polytopes, perfect matching polytopes, vertex packing polytopes and all the faces of these polytopes. In the paper [1], Balas and Padberg discussed the adjacency structures of set partitioning polytopes. The adjacency criterion of perfect matching polytopes was derived by Balinski [5] and Chvátal [6]. Ikura and Nemhauser discussed the adjacency structures of set packing polytopes in [11], and extended some properties of set partitioning polytopes showed by Balas and Padberg in [1, 2, 3]. The adjacency criterion of vertex packing polytopes was discussed by Trubin [20] and Chvátal [6].

Section 2 of this paper considers the equality constrained 0-1 polytopes. In particular, we establish two fundamental properties of the equality constrained 0-1 polytopes, which are useful to discuss the adjacency structures of some combinatorial polyhedra. In Sections 3,4, we consider the polyhedra which satisfy these two properties. Section 3 shows some properties of a path on a polyhedron considered here which satisfies the condition that for each co-ordinate, the vertices in the path form a monotonic sequence. When one of the end vertices of the path is optimal to an optimization problem with a linear objective function defined on the polyhedron, the associated objective values form a monotonic sequence and the length of the path is bounded by the dimension of the polyhedron. Some of the results obtained in Section 3 are related to the properties of set partitioning polytopes showed by Balas and Padberg in [1]. Their proofs are in a sense constructive and based on the pivot operations. On the other hand, our proofs are not based on the pivot operations but on the two properties established in Section 2. Section 4 proves the monotone Hirsch conjecture for the class of polyhedra considered here. In Section 5, we give an idea which leads a local search method for set partitioning problems. At the last of Section 5, we prove the monotone Hirsch conjecture for all the 0-1 polytopes. This proof is obtained by modifying the proof presented in Section 4.

2 Fundamental properties

In this section, we give some definitions and show two properties which we will use throughout this paper.

A sequence $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^K)$ of distinct vertices of a polyhedron $P \subseteq R^n$ is called a *vertex sequence* (of P) from $\boldsymbol{x}^0$ to $\boldsymbol{x}^K$. When a vertex sequence ρ contains $K+1$ vertices, we say the *length* of ρ is K. A vertex sequence $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^K)$ is called a *monotone vertex sequence,* when it satisfies the condition that:

$$\text{for each index } j, \text{ either } x_j^0 \leq x_j^1 \leq x_j^2 \leq \cdots \leq x_j^K \text{ or } x_j^0 \geq x_j^1 \geq x_j^2 \geq \cdots \geq x_j^K.$$

Then it is clear that every vertex sequence consists of two vertices is monotone. If two vertex sequences $(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^K)$ and $(\boldsymbol{x}^{i-1}, \boldsymbol{y}^0, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^L, \boldsymbol{x}^i)$ are monotone $(i \in \{1, 2, \ldots, K\})$, then the vertex sequence $(\boldsymbol{x}^0, \ldots, \boldsymbol{x}^{i-1}, \boldsymbol{y}^0, \ldots, \boldsymbol{y}^L, \boldsymbol{x}^i, \ldots, \boldsymbol{x}^K)$ is also monotone. When a vertex sequence ρ is obtained from a vertex sequence μ by dropping some vertices in μ, we say that ρ is a *subsequence* of μ. Clearly, any subsequence of a monotone vertex sequence is also monotone. If a monotone vertex sequence ρ from $\boldsymbol{x}^0$ to $\boldsymbol{x}^K$ is not a subsequence of any other monotone vertex sequence from $\boldsymbol{x}^0$ to $\boldsymbol{x}^K$, we say ρ is *maximal.*

Now, we show two fundamental properties.

Lemma 2.1 *Let P be a 0-1 polytope. When two distinct vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ of P are not adjacent, there exists a vertex $\boldsymbol{x}'$ of P such that $(\boldsymbol{x}^1, \boldsymbol{x}', \boldsymbol{x}^2)$ is a monotone vertex sequence.*

Proof. Since $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not adjacent, there exist vertices $\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^K$ of P such that $\boldsymbol{x}^1 \neq \boldsymbol{y}^i \neq \boldsymbol{x}^2$ for all $i \in \{1, 2, \ldots, K\}$ and the line segment connecting $\boldsymbol{x}^1$ and

$\boldsymbol{x}^2$ contains a point expressed as a convex combination of the vectors $\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^K$. Put $\boldsymbol{x}' = \boldsymbol{y}^1$. Since the vertices $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{y}^1(=\boldsymbol{x}'), \boldsymbol{y}^2, \ldots, \boldsymbol{y}^K$ of P are 0-1 valued vectors, it is obvious that (1) if $x^1_j = x^2_j$, then $x^1_j = x'_j = x^2_j$; (2) if $x^1_j \neq x^2_j$, then $x^1_j \leq x'_j \leq x^2_j$ or $x^1_j \geq x'_j \geq x^2_j$. Thus, the vertex sequence $(\boldsymbol{x}^1, \boldsymbol{x}', \boldsymbol{x}^2)$ is monotone. □

Lemma 2.2 *Let P be the convex hull of the set $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$, where A is an $m \times n$ matrix and $\boldsymbol{b}$ is an m-dimensional vector. If $(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ is a monotone vertex sequence of P, then $\boldsymbol{x}^1 - \boldsymbol{x}^2 + \boldsymbol{x}^3$ is a vertex of P.*

Proof. Denote the vector $\boldsymbol{x}^1 - \boldsymbol{x}^2 + \boldsymbol{x}^3$ by $\boldsymbol{x}^4$. Clearly, $A\boldsymbol{x}^4 = A(\boldsymbol{x}^1 - \boldsymbol{x}^2 + \boldsymbol{x}^3) = \boldsymbol{b}$. Since $(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ is a monotone vertex sequence of P and $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3$ are 0-1 valued vectors, it is easy to show that $\boldsymbol{x}^4$ is also 0-1 valued. □

Now recall the properties appeared in the above two lemmas.

Property A: If two vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ of P are not adjacent, then there exists a vertex $\boldsymbol{x}'$ of P such that $(\boldsymbol{x}^1, \boldsymbol{x}', \boldsymbol{x}^2)$ is a monotone vertex sequence of P.

Property B: If $(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ is a monotone vertex sequence of P, then the vector $\boldsymbol{x}^1 - \boldsymbol{x}^2 + \boldsymbol{x}^3$ is a vertex of P.

The lemmas above showed that the convex hull of the set $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$ satisfies these two properties. In the following sections, we consider the polyhedra satisfying above two properties in stead of the convex hull of the set $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$.

Before going to the next section, we discuss the relation between a polyhedron satisfying Properties A,B and a *combinatorial polyhedron* defined by Naddef and Pulleyblank in [16]. A polyhedron is called *combinatorial* if all its vertices are 0-1 valued and it satisfies the following property.

Property C: If two vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ of P are not adjacent, then there exists a pair of vertices $\{\boldsymbol{x}^3, \boldsymbol{x}^4\}$ of P such that $\{\boldsymbol{x}^1, \boldsymbol{x}^2\} \neq \{\boldsymbol{x}^3, \boldsymbol{x}^4\}$ and $\boldsymbol{x}^1 + \boldsymbol{x}^2 = \boldsymbol{x}^3 + \boldsymbol{x}^4$.

Hausmann [9] calls Property C the "intersection-union" property. In [10], Ikebe, Matsui and Tamura discussed the adjacency of the best and second best valued solutions of a linear programming problem whose feasible solutions form a combinatorial polytope. It is easy to see the following corollary.

Corollary 2.3

(1) If a polyhedron P satisfies Properties A,B, then it satisfies Property C.

(2) If a 0-1 polytope P satisfies Property B, then it satisfies Property C.

Here we note that the convex hull of the vectors $\{(0,0),(1,0),(1,1)\}$ satisfies Property C but not Property B.

3 Adjacency on combinatorial polyhedra

We will begin this section by considering a polyhedron $P \subseteq R^n$ satisfying Property B. And we shall add Property A later.

Given a vertex sequence $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ of P, the vectors $\boldsymbol{d}^i = \boldsymbol{x}^i - \boldsymbol{x}^{i-1}$ $(i \in \{1, 2, \ldots, K\})$ are called *difference vectors of* ρ. For any permutation $\sigma : \{1, 2, \ldots, K\} \rightarrow \{1, 2, \ldots, K\}$, the sequence $(\boldsymbol{x}^0, \boldsymbol{x}^0 + \boldsymbol{d}^{\sigma(1)}, \boldsymbol{x}^0 + \boldsymbol{d}^{\sigma(1)} + \boldsymbol{d}^{\sigma(2)}, \ldots, \boldsymbol{x}^0 + \sum_{i=1}^K \boldsymbol{d}^{\sigma(i)})$ is denoted by $\rho(\sigma)$.

The following theorem plays an important role in the rest of this paper.

Theorem 3.1 *Let P be a polyhedron satisfying Property B and $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ a monotone vertex sequence of P. For every permutation σ defined on the index set $\{1, 2, \ldots, K\}$, the sequence $\rho(\sigma)$ is a monotone vertex sequence of P.*

Proof. It is sufficient to show the case that

$$(\sigma(1), \sigma(2), \ldots, \sigma(K)) = (1, 2, \ldots, i-1, i+1, i, i+2, \ldots, K),$$

for each $i \in \{0, 1, 2, \ldots, K-1\}$. Put $\rho(\sigma) = (\boldsymbol{y}^0, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^K)$. Then it is clear that $(\boldsymbol{y}^0, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^{i-1}, \boldsymbol{y}^{i+1}, \ldots, \boldsymbol{y}^K)$ is equivalent to $(\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^{i-1}, \boldsymbol{x}^{i+1}, \ldots, \boldsymbol{x}^K)$ and so it is a monotone vertex sequence of P. Thus, we only need to show that $(\boldsymbol{y}^{i-1}, \boldsymbol{y}^i, \boldsymbol{y}^{i+1})$ is a monotone vertex sequence. Since $\boldsymbol{y}^i = \boldsymbol{y}^{i-1} + \boldsymbol{d}^{i+1} = \boldsymbol{x}^{i-1} - \boldsymbol{x}^i + \boldsymbol{x}^{i+1}$, Property B implies that $\boldsymbol{y}^i$ is a vertex of P. The monotonicity of the vertex sequence $(\boldsymbol{y}^{i-1}, \boldsymbol{y}^i, \boldsymbol{y}^{i+1})$ is clear. □

The above theorem says that for every index subset $S \subseteq \{1, 2, \ldots, K\}$, the vector $\boldsymbol{x}^0 + \sum_{i \in S} \boldsymbol{d}^i$ is a vertex of P. Then it seems that the set $\{\boldsymbol{x}^0 + \sum_{i \in S} \boldsymbol{d}^i \mid S \subseteq \{1, 2, \ldots, K\}\}$ contains 2^K "distinct" vertices of P. This question is taken up later.

Next, we consider an optimization problem Q: $\min\{\boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{x} \in P\}$, where $P \subseteq R^n$ and $\boldsymbol{c} \in R^n$. It is well-known that when the polyhedron P has at least one vertex and the problem Q has an optimal solution, there exists at least one optimal vertex solution of Q. The next theorem gives a relation between the monotone vertex sequences of P and the optimization problem Q.

Theorem 3.2 *Let P be a polyhedron satisfying Property B and $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ a monotone vertex sequence of P. If the vertex $\boldsymbol{x}^K$ is an optimal solution of the problem Q: $\min\{\boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{x} \in P\}$, then $\boldsymbol{c}\boldsymbol{x}^0 \geq \boldsymbol{c}\boldsymbol{x}^1 \geq \cdots \geq \boldsymbol{c}\boldsymbol{x}^K$.*

Proof. We have to show that $\boldsymbol{c}\boldsymbol{x}^i - \boldsymbol{c}\boldsymbol{x}^{i-1} = \boldsymbol{c}\boldsymbol{d}^i \leq 0$ for each $i \in \{1, 2, \ldots, K\}$. Let σ_i be the permutation defined on the index set $\{1, 2, \ldots, K\}$ which exchanges two indices

i and K. Denote the vertex sequence $\rho(\sigma_i)$ by $(\boldsymbol{y}^0, \boldsymbol{y}^1, \dots, \boldsymbol{y}^K)$. Since $\boldsymbol{y}^K = \boldsymbol{x}^K$, we have $\boldsymbol{c}\boldsymbol{y}^K = \boldsymbol{c}\boldsymbol{x}^K \le \boldsymbol{c}\boldsymbol{y}^{K-1} = \boldsymbol{c}(\boldsymbol{y}^K - \boldsymbol{d}^i)$ and so $\boldsymbol{c}\boldsymbol{d}^i \le 0$. □

The above theorem implies that if a monotone vertex sequence ρ satisfies the conditions described above, then for every permutation σ defined on the corresponding index set, the associated objective values of the monotone vertex sequence $\rho(\sigma)$ form a non-increasing sequence.

When the given polyhedron P is a 0-1 polytope, the difference vectors of every monotone vertex sequence are mutually orthogonal. This fact directly implies that the length of every monotone vertex sequence is less than or equal to the dimension of P. The following theorem implies that the same property holds for all polyhedra satisfying Property B.

Theorem 3.3 *Let P be a polyhedron satisfying Property B and $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \dots, \boldsymbol{x}^K)$ a monotone vertex sequence of P. Then the set of difference vectors $\{\boldsymbol{d}^i = \boldsymbol{x}^i - \boldsymbol{x}^{i-1} \mid i \in \{1, 2, \dots, K\}\}$ is linearly independent.*

Proof. Suppose that the set of difference vectors is linearly dependent. Then, without loss of generality, we can assume that there exist two index sets S, T and a vector $\boldsymbol{\lambda} \in R^{S \cup T}$ satisfying that $S \cap T = \emptyset$, $S \cup T \subseteq \{1, 2, \dots, K-1\}$, $\lambda_i > 0$ for all $i \in S \cup T$, and

$$\boldsymbol{d}^K = \sum_{i \in S} \lambda_i \boldsymbol{d}^i - \sum_{i \in T} \lambda_i \boldsymbol{d}^i.$$

Put $\boldsymbol{v} = \boldsymbol{x}^0 + \sum_{i \in T} \boldsymbol{d}^i$. From Theorem 3.1, $\boldsymbol{v}$ is a vertex of P. By adding the vector $\boldsymbol{v} + (\sum_{i \in S} \lambda_i + \sum_{i \in T} \lambda_i)(\boldsymbol{d}^K + \boldsymbol{v})$ to the both sides of the above equation, we obtain:

$$\begin{aligned} (1 + \sum_{i \in S} \lambda_i + \sum_{i \in T} \lambda_i)(\boldsymbol{v} + \boldsymbol{d}^K) &= \boldsymbol{v} + \sum_{i \in S} \lambda_i (\boldsymbol{d}^K + \boldsymbol{v} + \boldsymbol{d}^i) + \sum_{i \in T} \lambda_i (\boldsymbol{d}^K + \boldsymbol{v} - \boldsymbol{d}^i), \\ (\boldsymbol{v} + \boldsymbol{d}^K) &= \frac{1}{\delta} \boldsymbol{v} + \sum_{i \in S} \frac{\lambda_i}{\delta} (\boldsymbol{d}^K + \boldsymbol{v} + \boldsymbol{d}^i) + \sum_{i \in T} \frac{\lambda_i}{\delta} (\boldsymbol{d}^K + \boldsymbol{v} - \boldsymbol{d}^i), \end{aligned}$$

where $\delta = 1 + \sum_{i \in S} \lambda_i + \sum_{i \in T} \lambda_i$. From the definition of $\boldsymbol{v}$, it is clear that $\boldsymbol{v} + \boldsymbol{d}^K$ is a vertex of P and $V' = \{\boldsymbol{v}\} \cup \{\boldsymbol{d}^K + \boldsymbol{v} + \boldsymbol{d}^i \mid i \in S\} \cup \{\boldsymbol{d}^K + \boldsymbol{v} - \boldsymbol{d}^i \mid i \in T\}$ is a set of vertices of P. Then the above equation shows that the vertex $\boldsymbol{v} + \boldsymbol{d}^K$ is expressed as a convex combination of the vertices in V'. Since $\boldsymbol{d}^i \neq \boldsymbol{0}$ for all $i \in \{1, 2, \dots, K\}$, it is clear that $\boldsymbol{v} + \boldsymbol{d}^K$ is not contained in V'. Contradiction. □

The above theorem directly implies the following.

Corollary 3.4 *Let P be a polyhedron satisfying Property B. Then the length of every monotone vertex sequence of P is less than or equal to the dimension of P.*

From Theorem 3.3, it is obvious that the set $\{\boldsymbol{x}^0 + \sum_{i \in S} \boldsymbol{d}^i \mid S \subseteq \{1, 2, \ldots, K\}\}$ consists of 2^K distinct vertices of P.

At the last of this section, we consider a relation between the maximal monotone vertex sequences and the edge following paths on a polyhedron satisfying Property A and Property B.

Lemma 3.5 *Let P be a polyhedron satisfying Property B and $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ a monotone vertex sequence of P. If $\boldsymbol{x}^{i-1}$ and $\boldsymbol{x}^i$ are adjacent for all $i \in \{1, 2, \ldots, K\}$, then the sequence ρ is maximal.*

Proof. Suppose that ρ is not maximal. Then there exists a monotone vertex sequence $\mu = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^{i-1}, \boldsymbol{x}', \boldsymbol{x}^i, \ldots, \boldsymbol{x}^K)$ of P for some $i \in \{1, 2, \ldots, K\}$. From Property B, $\boldsymbol{x}'' = \boldsymbol{x}^i - \boldsymbol{x}' + \boldsymbol{x}^{i+1}$ is also a vertex of P. Then $\boldsymbol{x}^i + \boldsymbol{x}^{i+1} = \boldsymbol{x}' + \boldsymbol{x}''$ and so $\boldsymbol{x}^i$ and $\boldsymbol{x}^{i+1}$ are not adjacent. Contradiction. $\square$

When a given polyhedron satisfies Property A, we can show the converse of the above lemma.

Theorem 3.6 *Let P be a polytope satisfying Properties A,B and $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ a monotone vertex sequence of P. Then the monotone vertex sequence ρ is maximal, if and only if $\boldsymbol{x}^{i-1}$ and $\boldsymbol{x}^i$ are adjacent for all $i \in \{1, 2, \ldots, K\}$.*

Proof. What remains to show is the only if part, i.e., the statement that if ρ is maximal, then consecutive two vertices in ρ are adjacent. Suppose $\boldsymbol{x}^{i-1}, \boldsymbol{x}^i$ are not adjacent. Property A implies that there exists a vertex $\boldsymbol{x}'$ such that $(\boldsymbol{x}^i, \boldsymbol{x}', \boldsymbol{x}^{i+1})$ is a monotone vertex sequence. Then ρ is a subsequence of the monotone vertex sequence $(\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \boldsymbol{x}', \boldsymbol{x}^{i+1}, \ldots, \boldsymbol{x}^K)$. Contradiction. $\square$

Summerizing the statements in Theorem 3.1 and Theorem 3.6, we can say that if $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ is a maximal monotone vertex sequence, then for every permutation σ defined on the corresponding index set, the monotone vertex sequence $\rho(\sigma)$ is also maximal and there exist $K!$ distinct paths connecting $\boldsymbol{x}^0$ and $\boldsymbol{x}^K$.

Theorem 3.6 shows that when a polyhedron P satisfies Properties A,B, every maximal monotone vertex sequence of P corresponds to an edge following path on P. Then it seems possible to construct an edge following algorithm for the problem Q which determines any path on the polyhedron P from an initial vertex satisfying the conditions that (1) the sequence of vertices in the path form a monotone vertex sequence, (2) associated objective values form a non-increasing sequence, and (3) its length is less than or equal to the dimension of P.

4 Monotone Hirsch conjecture

In this section, we consider the monotone Hirsch conjecture. Given a polyhedron P, the dimension of P is denoted by $d(P)$ and the number of facets of P is denoted by $f(P)$. The monotone Hirsch conjecture is described as follows.

Monotone Hirsch Conjecture Let P be a polyhedron in R^n. For any vector $\boldsymbol{c} \in R^n$ and for any vertex $\boldsymbol{x}^0$ of P, the following statements hold. If the optimization problem Q: $\min\{\boldsymbol{cx} \mid \boldsymbol{x} \in P\}$ has an optimal vertex solution, then there exists a vertex sequence $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ of P satisfying that:

(1) $\boldsymbol{x}^K$ is an optimal solution of the problem Q: $\min\{\boldsymbol{cx} \mid \boldsymbol{x} \in P\}$,

(2) $\boldsymbol{x}^{i-1}$ and $\boldsymbol{x}^i$ are adjacent for all $i \in \{1, 2, \ldots, K\}$,

(3) $\boldsymbol{cx}^0 \geq \boldsymbol{cx}^1 \geq \cdots \geq \boldsymbol{cx}^K$, and

(4) K (the length of ρ) is less than or equal to $f(P) - d(P)$.

When we drop the condition (3), we obtain the famous Hirsch conjecture. Klee [12] proved the monotone Hirsch conjecture for all 3-dimensional polyhedra. However, Todd [19] showed that the monotone Hirsch conjecture is false in general for polyhedra of dimension 4 or more. There exist some special classes of polyhedra for which the monotone Hirsch conjecture is true [4, 8, 18]. See Klee and Kleinschmidt [13] for survey.

In the rest of this paper, we call a vertex sequence satisfying the conditions (1),(2),(3), a *monotone path*. Here, we prove the monotone Hirsch conjecture for all the polyhedra satisfying Properties A,B. To show this, we need the following lemma.

Lemma 4.1 *If P is a polyhedron satisfying Properties A,B, then every face F of P also satisfies Properties A,B.*

Proof. Let $\boldsymbol{ax} \leq b$ be an inequality such that $F = \{\boldsymbol{x} \in P \mid \boldsymbol{ax} = b\}$ and for all $\boldsymbol{x} \in P$, $\boldsymbol{ax} \leq b$. Property B: Let $(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ be a monotone vertex sequence of F. Then they are also vertices of P and Property B implies that $\boldsymbol{x}^4 = \boldsymbol{x}^1 - \boldsymbol{x}^2 + \boldsymbol{x}^3$ is a vertex of P. Since $\boldsymbol{ax}^4 = b$, $\boldsymbol{x}^4$ is also a vertex of F. Property A: Let $\{\boldsymbol{x}^1, \boldsymbol{x}^2\}$ be a pair of non-adjacent vertices of F. Since both $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are vertices of P, Property A implies that there exists a vertex $\boldsymbol{x}'$ of P such that $(\boldsymbol{x}^1, \boldsymbol{x}', \boldsymbol{x}^2)$ is a monotone vertex sequence of P. From Property B, $\boldsymbol{x}'' = \boldsymbol{x}^1 - \boldsymbol{x}' + \boldsymbol{x}^2$ is a vertex of P and so $\boldsymbol{ax}^3 \leq b$ and $\boldsymbol{ax}^4 \leq b$. The equality $\boldsymbol{x}^1 + \boldsymbol{x}^2 = \boldsymbol{x}' + \boldsymbol{x}''$ implies that $\boldsymbol{ax}' = \boldsymbol{ax}'' = b$. Thus, $\boldsymbol{x}'$ is a vertex of F and $(\boldsymbol{x}^1, \boldsymbol{x}', \boldsymbol{x}^2)$ is a monotone vertex sequence of F. □

Now we show the monotone Hirsch conjecture for the polyhedra satisfying Properties A,B. The following proof similars to that of the Hirsch conjecture for the 0-1 polytopes given by Naddef in [15].

Theorem 4.2 *The monotone Hirsch conjecture is true for all the polyhedra satisfying Properties A,B.*

Proof. The proof is by induction on the dimension of the polyhedron. For one dimensional polyhedron, it is easy to show. Suppose it is true for every polyhedron satisfying Properties A,B whose dimension is $d-1$. Let P be a polyhedron satisfying Properties A,B and $d(P)=d$. Clearly, there exists a maximal monotone vertex sequence ρ from the given initial vertex $\boldsymbol{x}^0$ to an optimal vertex $\boldsymbol{x}^K$ of the problem Q. Theorem 3.2 and Theorem 3.6 show that ρ is a monotone path of P. From Corollary 3.4, the length of ρ is less than or equal to $d(P)$. When $f(P) \geq 2d(P)$, we have done. Suppose $f(P) < 2d(P)$. Since every vertex must belong to at least $d(P)$ facets, there exists at least one facet F containing both $\boldsymbol{x}^0$ and $\boldsymbol{x}^K$. From Lemma 4.1, F satisfies Properties A,B and $d(F) = d(P) - 1 = d - 1$. So, by applying induction to F, we obtain a monotone path on F from $\boldsymbol{x}^0$ to $\boldsymbol{x}^K$ whose length is less than or equal to $f(F) - d(F) \leq (f(P) - 1) - (d(P) - 1) = f(P) - d(P)$. □

5 Discussions

In this paper, we established two properties and discussed the adjacency structures of polyhedra satisfying these two properties. From the historical point of view, some of the results in Section 3 are natural extensions of some properties of set partitioning polytopes showed by Balas and Padberg [1, 2, 3]. However, different from the Balas and Padberg's studies, our proofs are not based on the pivot operations.

Our results indicate that when a polyhedron P satisfies Properties A,B, it is possible to construct an edge following algorithm for the problem $\min\{\boldsymbol{cx} \mid \boldsymbol{x} \in P\}$ which determines any path on P from an initial vertex satisfying the conditions that (1) the sequence of vertices in the path form a monotone vertex sequence, (2) associated objective values form a non-increasing sequence, and (3) its length is less than or equal to $\min\{d(P), f(P) - d(P)\}$.

In Section 3, we showed some properties of the set of difference vectors obtained from a monotone vertex sequence. When a monotone vertex sequence with length K is obtained, we can deal with 2^K distinct vertices implicitly as follows. If we have a monotone vertex sequence $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ of P satisfying Property B, then

$$V' = \{\boldsymbol{x}^0 + \sum_{i \in S} (\boldsymbol{x}^i - \boldsymbol{x}^{i-1}) \mid S \subseteq \{1, 2, \ldots, K\}\}$$

is a set of 2^K distinct vertices of P and we can easily find a vertex in V' which minimizes a linear objective function. More precisely, the vertex $\boldsymbol{x}^0 + \sum_{i \in S'} (\boldsymbol{x}^i - \boldsymbol{x}^{i-1})$ in V' attains the value $\min\{\boldsymbol{cx} \mid \boldsymbol{x} \in V'\}$, where $S' = \{i \in \{1, 2, \ldots, K\} \mid \boldsymbol{c}(\boldsymbol{x}^i - \boldsymbol{x}^{i-1}) < 0\}$.

In the following, we describe a local search method for set partitioning problems. It represents a by-product of our results. Consider the case that P is expressed as the convex hull of the set $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$ where A is a 0-1 valued $m \times n$ matrix and $\boldsymbol{b}$ is the m-dimensional all one vector, i.e., P is a set partitioning polytope. When a pair of distinct vertices $\{\boldsymbol{x}', \boldsymbol{x}''\}$ of P is obtained, we can easily construct a maximal monotone vertex sequence from $\boldsymbol{x}'$ to $\boldsymbol{x}''$ as follows. Let $\overline{A}$ be the matrix consists of columns of A indexed by $I' = \{j \mid x'_j \neq x''_j\}$. Then $\overline{A}$ is the transpose of the incidence matrix of a bipartite graph G' whose vertex set corresponds to I' and edge set corresponds to the rows of $\overline{A}$. It is well-known that two vertices $\boldsymbol{x}'$ and $\boldsymbol{x}''$ of P are adjacent if and only if the bipartite graph G' is connected (see [1, 6, 11, 20] for example). When G' is not connected, we can construct a partition $\{I_1, I_2, \ldots, I_K\}$ of I' such that each index subset I_i represents the set of vertices of a connected component of G'. Let $\boldsymbol{x}^i$ ($i \in \{1, 2, \ldots, K\}$) be the 0-1 valued vector satisfying

$$x^i_j = \begin{cases} 1 - x'_j, & \text{if } j \in I_1 \cup I_2 \cup \cdots \cup I_i, \\ x'_j, & \text{otherwise.} \end{cases}$$

Then it is clear that $\rho = (\boldsymbol{x}', \boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^K = \boldsymbol{x}'')$ is a monotone vertex sequence of P. Since each index subset I_i corresponds to a connected component of G', it is easy to show that each consecutive two vertices in ρ are adjacent, i.e., ρ is maximal.

From the above discussions, we can construct a local search method for set partitioning problems, when we have an algorithm which generates a set of feasible solutions, e.g., a branch and bound method. When a pair of non-adjacent vertices $\{\boldsymbol{x}', \boldsymbol{x}''\}$ are obtained, we construct a maximal vertex sequence $(\boldsymbol{x}', \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K = \boldsymbol{x}'')$ as described above and find a vertex in the set of distinct 2^K vertices $\{\boldsymbol{x}' + \sum_{i \in S} (\boldsymbol{x}^i - \boldsymbol{x}^{i-1}) \mid S \subseteq \{1, 2, \ldots, K\}\}$ which minimizes a given linear objective function. The time complexity required for this local search is bounded by $O(nm)$. When the diameter (the maximum length of a shortest path between any pair of vertices) of the given set partitioning polytope is large, it seems possible to find an improved solution by this local search technique.

In the previous section, we showed that the monotone Hirsch conjecture is true for all the polyhedra satisfying Properties A,B. Recently, Naddef proved the Hirsch conjecture for all 0-1 polytopes [15]. Here, we prove the monotone Hirsch conjecture for all the 0-1 polytopes. To show it, we need the following lemma which similars to Corollary 3.4.

Lemma 5.1 *Let $P \subseteq R^n$ be a 0-1 polytope. For any vector $\boldsymbol{c} \in R^n$ and for any vertex $\boldsymbol{x}^0$ of P there exists a monotone path whose length is less than or equal to the dimension of P.*

Proof. We show that there exists a monotone vertex sequence $\rho = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ of P which is also a monotone path on P. If it exists, since the vertices are 0-1 valued, the difference vectors of ρ are mutually orthogonal and it implies that the length of ρ is less than or equal to $d(P)$.

If the initial vertex is optimal, we have done. Otherwise, let $\boldsymbol{x}^*$ be an optimal vertex solution of the problem $\min\{\boldsymbol{cx} \mid \boldsymbol{x} \in P\}$. Let I' be the index subset $\{j \mid x_j^0 = x_j^*\}$. Then $F = P \cap \{\boldsymbol{x} \in R^n \mid x_j = x_j^* \text{ for all } j \in I'\}$ is a face of P. Since $\boldsymbol{x}^0$ and $\boldsymbol{x}^*$ are vertices of F, there exists a vertex $\boldsymbol{x}^1$ of F which is adjacent with $\boldsymbol{x}^0$ (on F) and satisfies $\boldsymbol{cx}^0 \geq \boldsymbol{cx}^1$. Since F is a face of P, $\{\boldsymbol{x}^0, \boldsymbol{x}^1\}$ is an adjacent pair of vertices of P. From the definition of F, it is obvious that $(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^*)$ is a monotone vertex sequence of P and $|\{j \mid x_j^0 = x_j^*\}| < |\{j \mid x_j^1 = x_j^*\}|$. Thus, by applying this procedure consecutively, we obtain a required monotone vertex sequence of P from $\boldsymbol{x}^0$ to an optimal vertex solution of the problem $\min\{\boldsymbol{cx} \mid \boldsymbol{x} \in P\}$. □

Then we can show the following theorem in the same way as Theorem 4.2 and/or Naddef's proof in [15].

Theorem 5.2 *The monotone Hirsch conjecture is true for the 0-1 polytopes.*

Above proofs show that when P is a 0-1 polytope, it is possible to construct an edge following algorithm for the problem $\min\{\boldsymbol{cx} \mid \boldsymbol{x} \in P\}$ which finds a path on P connecting a given initial vertex and an optimal vertex satisfying the conditions that (1) the sequence of vertices in the path form a monotone vertex sequence, (2) associated objective values form a non-increasing sequence, and (3) its length is less than or equal to $\min\{d(P), f(P) - d(P)\}$.

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