NP-Completeness of Non-Adjacency Relations on Some 0-1 Polytopes

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Abstract: In this paper, we discuss the adjacency structures of some classes of 0-1 polytopes including knapsack polytopes, set covering polytopes and 0-1 polytopes represented by complete sets of implicants. We show that for each class of 0-1 polytope, non-adjacency test problems are NP-complete. For equality constrained knapsack polytopes, we can solve adjacency test problems in pseudo polynomial time.

1 Introduction

It seems that an adjacency criterion for a class of polyhedra could provide a basis of an efficient algorithm which uses some sorts of local search technique. For this purpose, it is necessary to have an efficient algorithm for checking adjacency. In [16], Papadimitriou showed that the problem of checking non-adjacency on the travelling salesman polytope is NP-complete. So, one cannot expect an efficient edge-following type algorithm for the travelling salesman problem. However, there exist some classes of combinatorial polytopes, including matching polytopes [4, 6], vertex packing polytopes [17, 6], set partitioning polytopes [1, 2, 3] and set packing polytopes [12], such that we can decide the adjacency of two given vertices in polynomial time.

In this paper, we show that for some well-known classes of combinatorial polytopes, the non-adjacency test problems are NP-complete. We deal with the following classes of polytopes (equality constrained) knapsack polytopes, set covering polytopes and 0-1 polytopes given by complete sets of implicants, In the last section, we show that the adjacency test problems defined on equality constrained knapsack polytopes are solvable in pseudo polynomial time.

2 Preliminaries

In this section, we describe some fundamental properties without proofs.

First, we describe a necessary and sufficient condition of non-adjacency.

Lemma 2.1 *Let* $\Omega \subseteq \{0,1\}^n$ *be a set of 0-1 vectors and* $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ *two vectors in* Ω. *The vertices* $\boldsymbol{x}^1$ *and* $\boldsymbol{x}^2$ *are non-adjacent on the convex hull of* Ω *if and only if there exists a set of vectors* $\{\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^k\} \subseteq \Omega \setminus \{\boldsymbol{x}^1, \boldsymbol{x}^2\}$ *such that* $k \leq n$ *and the line segment connecting* $\boldsymbol{x}^1$ *and* $\boldsymbol{x}^2$ *intersects the convex hull of* $\{\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^k\}$.

Since $k \leq n$, the above lemma implies that the property of being non-adjacent is in NP.

Next, we give a necessary condition of non-adjacency.

Lemma 2.2 *Let* $\Omega \subseteq \{0,1\}^n$ *be a set of 0-1 vectors and* $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ *two vectors in* Ω. *If* $\boldsymbol{x}^1$ *and* $\boldsymbol{x}^2$ *are non-adjacent on the convex hull of* Ω, *then there exists a vector* $\boldsymbol{y} \in \Omega$ *such that* $\boldsymbol{x}^1 \neq \boldsymbol{y} \neq \boldsymbol{x}^2$ *and*

$$\text{for each index } j, \text{ either } x_j^1 \leq y_j \leq x_j^2 \text{ or } x_j^1 \geq y_j \geq x_j^2. \tag{2.1}$$

When a sequence $\rho = (\boldsymbol{x}^1, \boldsymbol{y}, \boldsymbol{x}^2)$ of distinct 0-1 vectors in $\{0,1\}^n$ satisfies (2.1), we say ρ is *monotone.*

At last of this section, we consider the equality constrained 0-1 polytope.

Lemma 2.3 *Let* $\Omega = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$ *where* A *is an* $m \times n$ *matrix and* $\boldsymbol{b}$ *is an* m*-dimensional vector. For any pair of 0-1 vectors* $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \Omega$, *the following three statements are equivalent.*
(1) Two vertices $\boldsymbol{x}^1$ *and* $\boldsymbol{x}^2$ *are non-adjacent on the convex hull of* Ω.
(2) There exists a vector $\boldsymbol{y} \in \Omega$ *such that* $\boldsymbol{x}^1 \neq \boldsymbol{y} \neq \boldsymbol{x}^2$ *and* $(\boldsymbol{x}^1, \boldsymbol{y}, \boldsymbol{x}^2)$ *is monotone.*
(3) There exists a pair of vectors $\boldsymbol{y}^1, \boldsymbol{y}^2 \in \Omega \setminus \{\boldsymbol{x}^1, \boldsymbol{x}^2\}$ *satisfying* $\boldsymbol{x}^1 + \boldsymbol{x}^2 = \boldsymbol{y}^1 + \boldsymbol{y}^2$.

The equality constrained 0-1 polytopes are discussed in [15, 14].

3 Knapsack Polytopes

In this section, we show that the following two problems are NP-complete.

EQUALITY KNAPSACK NON-ADJACENCY (EKN)

INSTANCE : An n-dimensional positive integer vector $\boldsymbol{a}$, a positive integer b and a pair of 0-1 vectors $\boldsymbol{x}^1, \boldsymbol{x}^2$ in $\Omega^{\text{EKN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{a}^T\boldsymbol{x} = b\}$.

QUESTION : Are the vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ non-adjacent on the convex hull of Ω^{EKN} ?

KNAPSACK NON-ADJACENCY (KN)

INSTANCE : An n-dimensional positive integer vector $\boldsymbol{a}$, a positive integer b and a pair of 0-1 vectors $\boldsymbol{x}^1, \boldsymbol{x}^2$ in $\Omega^{\text{KN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{a}^T\boldsymbol{x} \leq b\}$.

QUESTION : Are the vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ non-adjacent on the convex hull of Ω^{KN} ?

First, we discuss **EKN**.

Theorem 3.1 **EKN** *is NP-complete.*

Proof. Lemma 2.1 implies that the problem is in NP. We shall now transform the following NP-complete problem to **EKN**.

PARTITION [8, 13]

INSTANCE : A k-dimensional positive integer vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_k)$.

QUESTION : Let L be the sum of the elements of $\boldsymbol{c}$. Does there exist a 0-1 vector $\boldsymbol{z} \in \{0,1\}^k$ satisfying $\boldsymbol{c}^T\boldsymbol{z} = L/2$?

Put $\boldsymbol{a} = (c_1, \ldots, c_k, L/2, L/2)$ and $\Omega^{\mathrm{EKN}} = \{\boldsymbol{x} \in \{0,1\}^{k+2} \mid \boldsymbol{a}^T\boldsymbol{x} = L\}$. Let $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \{0,1\}^{k+2}$ be the pair of vectors

$$\boldsymbol{x}^1 = (1,1,\ldots,1,0,0)^T \text{ and } \boldsymbol{x}^2 = (0,0,\ldots,0,1,1)^T.$$

Clearly, $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \Omega^{\mathrm{EKN}}$. We shall show that $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are non-adjacent on the convex hull of Ω^{EKN} if and only if there exists a 0-1 vector $\boldsymbol{z} \in \{0,1\}^k$ satisfying $\boldsymbol{c}^T\boldsymbol{z} = L/2$. Lemma 2.2 shows that when the pair is non-adjacent, there exists a 0-1 vector $\boldsymbol{y} \in \Omega^{\mathrm{EKN}}$ such that $\boldsymbol{x}^1 \neq \boldsymbol{y} \neq \boldsymbol{x}^2$. Since $\boldsymbol{c}$ is a positive vector, it is clear that exactly one of the last two elements of $\boldsymbol{y}$ is 1. Thus the 0-1 vector $\boldsymbol{z} \in \{0,1\}^k$ corresponding to the first k elements of $\boldsymbol{y}$ satisfies the equality $\boldsymbol{c}^T\boldsymbol{z} = L/2$. The converse implication is easy. □

From the above theorem, it is easy to show that **KN** is also NP-complete.

Corollary 3.2 **KN** *is NP-complete.*

Proof. Clearly, **KN** is in the class NP. Let $\Omega^{\mathrm{EKN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{a}^T\boldsymbol{x} = b\}$. Then the convex hull of Ω^{EKN} is a face of the convex hull of Ω^{KN}. It implies that for any pair of vectors $\boldsymbol{x}^1, \boldsymbol{x}^2$ in Ω^{EKN}, $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are adjacent on the convex hull of Ω^{EKN} if and only if $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are adjacent on the convex hull of Ω^{KN}. Thus **EKN** is polynomially reducible to **KN** and so, **KN** is NP-complete. □

4 Set Covering Polytopes

In this section, we show that the following three problems are NP-complete. In the rest of this paper, the d-dimensional all one vector is denoted by $\mathbf{1}_d$.

EQUALITY INTEGER PROGRAMMING NON-ADJACENCY (EIPN)

INSTANCE : An $m \times n$ 0-1 matrix A, an m-dimensional positive integer vector $\boldsymbol{b}$ and a pair of 0-1 vectors $\boldsymbol{x}^1, \boldsymbol{x}^2$ in $\Omega^{\mathrm{EIPN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$.

QUESTION : Are the vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ non-adjacent on the convex hull of Ω^{EIPN} ?

INTEGER PROGRAMMING NON-ADJACENCY (IPN)

INSTANCE : An $m \times n$ 0-1 matrix A, an m-dimensional positive integer vector $\boldsymbol{b}$ and a pair of 0-1 vectors $\boldsymbol{x}^1, \boldsymbol{x}^2$ in $\Omega^{\mathrm{IPN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\}$

QUESTION : Are the vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ non-adjacent on the convex hull of Ω^{IPN} ?

SET COVERING NON-ADJACENCY (SCN)

INSTANCE : An $m \times n$ 0-1 matrix A, and a pair of 0-1 vectors $\boldsymbol{x}^1, \boldsymbol{x}^2$ in the set $\Omega^{\mathrm{SCN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} \geq \mathbf{1}_m\}$.

QUESTION : Are the vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ non-adjacent on the convex hull of Ω^{SCN} ?

First, we show the following theorem.

Theorem 4.1 **EIPN** *is NP-complete. The problem remains NP-complete, even if each row of the matrix A contains exactly four 1's and the right-hand-side vector is the all two vector. If each row of the matrix A contains at most three 1's, we can decide whether given two vertices are adjacent or not in polynomial time.*

Proof. We know that **EIPN** is in NP. Hence it will suffice to show that the following problem polynomially transforms to **EIPN**.

SET PARTITIONING [8, 13]

INSTANCE : A $p \times q$ 0-1 matrix M.

QUESTION : Does there exist a 0-1 vector in $\{\boldsymbol{z} \in \{0,1\}^q \mid M\boldsymbol{z} = \mathbf{1}_p\}$?

This problem remains NP-complete, even if the matrix M satisfies the condition that each row of M contains exactly three 1's [8, 13]. (Here we note that the problem is different from the well-known **EXACT THREE COVER** problem.) Given such a 0-1 matrix M, we construct a $(p+5q) \times (6q+3)$ 0-1 matrix A as follows.

First, we prepare three artificial variables x_0^0, x_0^1 and x_0^2. For each variable z_j of the **SET PARTITIONING** instance, we prepare six variables $x_j^1, x_j^2, \ldots, x_j^6$ and construct following five constraints;

$$\begin{array}{lllllll} x_0^1 + x_0^2 & & +x_j^3 & +x_j^4 & & & = 2, \\ x_0^1 + x_0^2 & & & +x_j^4 & +x_j^5 & & = 2, \\ x_0^1 + x_0^2 & & & & +x_j^5 & +x_j^6 & = 2, \\ x_0^1 + x_0^2 & & +x_j^3 & & & +x_j^6 & = 2, \\ & x_j^1 + x_j^2 & +x_j^3 & & +x_j^5 & & = 2. \end{array}$$

For each constraint of the **SET PARTITIONING** instance, we construct one constraint as follows. If the **SET PARTITIONING** instance contains the constraint $z_i + z_j + z_k = 1$, where the indices satisfy $i < j < k$, we construct the constraint $x_0^0 + x_i^1 + x_j^4 + x_k^4 = 2$.

Here, we give an example. Assume that we have the following **SET PARTITIONING-ING** instance,

$$\{\boldsymbol{z} \in \{0,1\}^4 \mid z_1 + z_2 + z_3 = 1, z_2 + z_3 + z_4 = 1\}.$$

Then, by the above procedure, we obtain the matrix A illustrated in Fig.1.

Let $\Omega^{\mathrm{EIPN}} = \{\boldsymbol{x} \in \{0,1\}^{(6q+3)} \mid A\boldsymbol{x} = \mathbf{2}_{(p+5q)}\}$ where $\mathbf{2}_{(p+5q)}$ denotes the $(p+5q)$-dimensional all two vector.

If $\overline{\boldsymbol{x}} \in \Omega^{\mathrm{EIPN}}$ and $\overline{x_0^1} = \overline{x_0^2} = 1$, then the vector $\overline{\boldsymbol{x}}$ is uniquely determined as

$$\overline{x_i^j} = \begin{cases} 1 & \text{if } i = 0, \\ 1 & \text{if } i \neq 0, j = 1,2, \\ 0 & \text{if } i \neq 0, j = 3,4,5,6. \end{cases} \tag{4.1}$$

In the case that $\overline{\boldsymbol{x}} \in \Omega^{\mathrm{EIPN}}$ and $\overline{x_0^1} = \overline{x_0^2} = 0$, the vector $\overline{\boldsymbol{x}}$ is uniquely determined as

$$\overline{x_i^j} = \begin{cases} 0 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, j = 1,2, \\ 1 & \text{if } i \neq 0, j = 3,4,5,6. \end{cases} \tag{4.2}$$

Let $\boldsymbol{x}' \in \Omega^{\mathrm{EIPN}}$ be the vector satisfying the condition (4.1) and $\boldsymbol{x}'' \in \Omega^{\mathrm{EIPN}}$ the vector satisfying the condition (4.2). Now we show that $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are non-adjacent on the convex hull of Ω^{EIPN} if and only if the set $\{\boldsymbol{z} \in \{0,1\}^q \mid M\boldsymbol{z} = \mathbf{1}_p\}$ is non-empty.

Assume that $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are non-adjacent. Then Lemma 2.3 implies that there exists a vector $\widetilde{\boldsymbol{x}} \in \Omega^{\mathrm{EIPN}}$ satisfying $\widetilde{x_0^1} \neq \widetilde{x_0^2}$ and $\widetilde{x_0^0} = 1$. Then it is clear that $\widetilde{\boldsymbol{x}}$ satisfies the condition that for each index $i \ (\neq 0)$, $\widetilde{x_i^1} = \widetilde{x_i^2} = \widetilde{x_i^4} = \widetilde{x_i^6} \neq \widetilde{x_i^3} = \widetilde{x_i^5}$. Let $\widetilde{\boldsymbol{z}} \in \{0,1\}^p$ be the vector satisfying $\widetilde{z}_i = \widetilde{x_i^1}$. Since $\widetilde{x_0^0} = 1$, $\widetilde{\boldsymbol{z}}$ is contained in $\{\boldsymbol{z} \in \{0,1\}^q \mid M\boldsymbol{z} = \mathbf{1}_p\}$.

Now consider the case that $\exists \widetilde{\boldsymbol{z}} \in \{\boldsymbol{z} \in \{0,1\}^q \mid M\boldsymbol{z} = \mathbf{1}_p\}$. Let $\widetilde{\boldsymbol{x}} \in \{0,1\}^{(3+6q)}$ be the vector satisfying

$$\widetilde{x_i^j} = \begin{cases} 1 & \text{if } i = 0, j = 0 \\ 1 & \text{if } i = 0, j = 1 \\ 0 & \text{if } i = 0, j = 2 \\ \widetilde{z}_i & \text{if } i \neq 0, j = 1,2,4,6 \\ 1 - \widetilde{z}_i & \text{if } i \neq 0, j = 3,5. \end{cases}$$

x_0^0	x_0^1	x_0^2	x_1^1	x_1^2	x_1^3	x_1^4	x_1^5	x_1^6	x_2^1	x_2^2	x_2^3	x_2^4	x_2^5	x_2^6	x_3^1	x_3^2	x_3^3	x_3^4	x_3^5	x_3^6	x_4^1	x_4^2	x_4^3	x_4^4	x_4^5	x_4^6
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Figure 1: The matrix A.

Then it is obvious that $\boldsymbol{x}' \neq \tilde{\boldsymbol{x}} \neq \boldsymbol{x}''$, $\tilde{\boldsymbol{x}} \in \Omega^{\mathrm{EIPN}}$ and $(\boldsymbol{x}', \tilde{\boldsymbol{x}}, \boldsymbol{x}'')$ is a monotone sequence. Thus, Lemma 2.3 implies that $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are non-adjacent.

From the above discussions, it is clear that **EIPN** remains NP-complete, even if each row of the matrix A contains exactly four 1's and the right-hand-side vector is the all two vector.

Lastly, we show that if each row of the matrix A contains at most three 1's, we can decide whether given two vertices $\boldsymbol{x}^1, \boldsymbol{x}^2$ are adjacent or not in polynomial time. Since each row of A contains at most three 1's, we can assume that $\boldsymbol{b} \in \{1,2\}^m$. Let I^1 be the set of indices satisfying $x_j^1 = x_j^2 = 1$, I^0 the set of indices with $x_j^1 = x_j^2 = 0$ and $I = \{1, 2, \ldots, n\} \setminus (I^1 \cup I^0)$. Set $\Omega^{\mathrm{EIPN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$. and $\Omega' = \{\boldsymbol{x} \in \Omega^{\mathrm{EIPN}} \mid \forall j \in I^1, x_j = 1 \text{ and } \forall j \in I^0, x_j = 0\}$. It is clear that $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are adjacent on $\mathrm{conv}(\Omega^{\mathrm{EIPN}})$ if and only if $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are adjacent on $\mathrm{conv}(\Omega')$.

Let A' be the submatrix of A consists of the column vectors of A indexed by I and $\boldsymbol{b}' = \boldsymbol{b} - \boldsymbol{b}''$ where $\boldsymbol{b}''$ is the sum of the column vectors of A indexed by I^1. Then it is clear that the adjacency structure of $\mathrm{conv}(\Omega')$ is equivalent to that of $\mathrm{conv}(\Omega'')$ where $\Omega'' = \{\boldsymbol{y} \in \{0,1\}^I \mid A'\boldsymbol{y} = \boldsymbol{b}'\}$. The definition of $\boldsymbol{b}'$ directly implies that $\boldsymbol{b}' \in \{0,1,2\}^m$.

From the definition of A' and $\boldsymbol{b}'$, it is clear that for each row vector $\boldsymbol{a}^T$ of A', either $\boldsymbol{a}^T$ contains exactly two 1's or $\boldsymbol{a}^T$ is the zero-vector. Then the convex hull of Ω'' is essentially equivalent to a face of a stable set polytope. Thus the adjacency test problem is polynomially solvable (see [6] for example). $\square$

We can show the following in a similar way with Corollary 3.2.

Corollary 4.2 IPN *is NP-complete. It remains NP-complete, even if each row of the given matrix A contains exactly three 1's and the right-hand-side vector is the all two vector. If* *each row of matrix A contains at most two 1's, we can decide whether given two vertices are adjacent or not in polynomial time.*

Proof. Clearly, **IPN** is in the class NP. Let $\Omega^{\text{EIPN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$. Then the convex hull of Ω^{EIPN} is a face of the convex hull of Ω^{IPN}. It implies that **IPN** is NP-complete.

Now we show the case that each row of A contains exactly three 1's. Theorem 4.1 implies that both **EIPN** and **IPN** are NP-complete even if each row of the matrix A contains exactly four 1's and the right-hand-side vector is the all two vector. For each inequality constraint

$$x_h + x_i + x_j + x_k \leq 2 \tag{4.3}$$

of the **IPN** instance, we construct four constraints

$$x_h + x_i + x_j \leq 2, \;\; x_i + x_j + x_k \leq 2, \;\; x_j + x_k + x_h \leq 2, \;\; x_k + x_h + x_i \leq 2. \tag{4.4}$$

Then, it is clear that a 0-1 vector satisfies the constraint (4.3) if and only if it satisfies (4.4). Thus, **IPN** is NP-complete even if each row of the given matrix A contains exactly three 1's and the right-hand-side vector is the all two vector.

When each row of A contains at most two 1's, the system of constraints is essentially equivalent to that of a stable set problem. So, we can decide the adjacency in polynomial time (see [6] for example). □

At the last of this section, we discuss the set covering polytope. In [7], Etcheberry proposed a sufficient (but not necessary) conditions for the adjacency on set covering polytopes. His conditions are discussed in [10] (see Chapter 1, Theorem 1.4.11 and Theorem 1.4.12). However, the following theorem indicates that we cannot expect an appropriate necessary and sufficient condition for adjacency on set covering polytopes.

Theorem 4.3 SCN *is NP-complete even if each row of the given matrix A contains exactly three 1's. If each row of matrix A contains at most two 1's, we can decide whether given two vertices are adjacent or not in polynomial time.*

Proof. Clearly, **SCN** is in the class NP.

Now we discuss the relation between **SCN** and **IPN**. Let A be a 0-1 matrix satisfying each row of the given matrix A contains exactly three 1's. Put

$$\Omega^{\text{IPN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} \leq \boldsymbol{2}_m\} \;\text{ and }\; \Omega^{\text{SCN}} = \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} \geq \boldsymbol{1}_m\}$$

where $\boldsymbol{2}_m$ be the m-dimensional all two vector. Then it is clear that for any 0-1 vector $\boldsymbol{x} \in \{0,1\}^n$, $\boldsymbol{x} \in \Omega^{\text{SCN}}$ if and only if $\boldsymbol{1}_n - \boldsymbol{x} \in \Omega^{\text{IPN}}$. So, given two vertices $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \Omega^{\text{SCN}}$ are adjacent on $\text{conv}(\Omega^{\text{SCN}})$ if and only if $\boldsymbol{1}_n - \boldsymbol{x}^1$ and $\boldsymbol{1}_n - \boldsymbol{x}^2$ in Ω^{IPN} are adjacent on $\text{conv}(\Omega^{\text{IPN}})$. Corollary 4.2 implies that **SCN** is NP-complete even if each row of A contains exactly three 1's.

When each row of A contains at most two 1's, Corollary 4.2 also implies that the adjacency test problem is polynomially solvable, □

5 Implicant Systems

In [9, 10], Hausmann and Korte introduced a implicant system for representing a system of subsets. Let E be a finite set and $\mathcal{S} \subseteq 2^E$ a system of subsets of E. An *implicant* on E is an ordered pair $(I^+; I^-)$ of disjoint subsets I^+, I^- of E. We say $(I^+; I^-)$ is an implicant of $\mathcal{S}$ when for any $F \subseteq E$,

$$\text{if } I^+ \subseteq F,\; I^- \cap F = \emptyset, \text{ then } F \notin \mathcal{S}.$$

For any implicant $(I^+; I^-)$, we say that the *size* of the implicant is $|I^+| + |I^-|$. A set of implicants of $\mathcal{S}$ is called a *complete set of implicants* of $\mathcal{S}$ when it satisfies the condition that for all subset $F \notin \mathcal{S}$, there exists an implicant $(I^+; I^-)$ satisfying $I^+ \subseteq F$, $I^- \cap F = \emptyset$. Clearly, any system of subsets $\mathcal{S}$ has a complete set of implicants $\{(F; E \setminus F) \mid F \notin \mathcal{S}\}$.

Let $\Omega \subseteq \{0,1\}^E$ be the set of characteristic vectors corresponding to the subsets in $\mathcal{S} \subseteq 2^E$. Then it is clear that, a pair $(I^+; I^-)$ is a implicant of $\mathcal{S}$ if and only if the inequality

$$\sum_{i \in I^+} x_i - \sum_{j \in I^-} x_j \leq |I^+| - 1$$

is a valid inequality of Ω. In this paper, we say that an inequality $\boldsymbol{a}^T \boldsymbol{x} \leq b$ is an *implicant type inequality* when it satisfies the conditions that $\boldsymbol{a}$ is a $\{0,1,-1\}$-vector and b is equivalent to the number of 1's in $\boldsymbol{a}$ minus 1. When $\mathcal{I}$ is a complete set of implicants of a system of subsets $\mathcal{S} \subseteq 2^E$, we denote the set of characteristic vectors corresponding to the subsets in $\mathcal{S}$ by $\Omega(\mathcal{I})$.

For many well-known combinatorial polytope, e.g., stable set polytopes, partial ordering polytopes, complete sets of implicants for the feasible solutions can easily be obtained (see [9, 10, 11]). In this section, we discuss the adjacency relation of a 0-1 polytope represented by a complete set of implicants.

IMPLICANT NON-ADJACENCY (IN)

INSTANCE : A finite set E, a set $\mathcal{I}$ of implicants on E and 0-1 vectors $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \Omega(\mathcal{I})$.

QUESTION : Are the vertices $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ non-adjacent on the convex hull of $\Omega(\mathcal{I})$?

In [9, 10], Hausmann and Korte showed that when the size of each implicant is equal to two, we can decide the adjacency of two given vectors in $\Omega(\mathcal{I})$ in polynomial time. In [11], Ikebe and Tamura discussed a system of subset represented by a set of implicants with size 2.

Theorem 5.1 **IN** *is NP-complete even if the size of each implicant is equal to three.*

Proof. We know that **IN** is in NP. Here we show that **IPN** polynomially transforms to **IN**.

Corollary 4.1 showed that **IPN** is NP-complete even if each row of given matrix contains exactly three 1's and the right-hand-side vector is the all two vector. In such a case, each constraint is an implicant type inequality. So, **IPN** is polynomially reducible to **IN** such that the size of each implicant is equal to three. □

6 Equality Constrained 0-1 Polytopes

Lastly, we discuss the relation between the adjacency test problems and the optimization problems defined on the equality constrained 0-1 polytopes.

Theorem 6.1 *Let A be an $m \times n$ matrix and $\boldsymbol{b}$ an m-dimensional vector. The set of 0-1 vectors $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$ is denoted by Ω. If the optimization problem $\max\{\boldsymbol{c}^T \boldsymbol{x} \mid \boldsymbol{x} \in \Omega\}$ is polynomially solvable for any n-dimensional vector $\boldsymbol{c}$, then we can decide whether two given vectors $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \Omega$ are adjacent or not in polynomial time.*

Proof. Let $\boldsymbol{x}^1, \boldsymbol{x}^2$ be a pair of distinct vectors in $\{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$. Without loss of generality, we can assume that there exists an index j such that $x^1_j = 1$ and $x^2_j = 0$.

Let $\boldsymbol{c}$ be the cost vector satisfying

$$c_i = \begin{cases} (n+1)^4 & \text{if } x_i^1 = x_i^2 = 1, \\ -(n+1)^3 & \text{if } x_i^1 = x_i^2 = 0, \\ (n+1)^2 & \text{if } i = j, \\ -1 & \text{if } x_i^1 = 1, x_i^2 = 0, i \neq j, \\ 1 & \text{if } x_i^1 = 0, x_i^2 = 1. \end{cases}$$

We shall show that the optimal value of the problem $\max\{\boldsymbol{c}^T\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n\}$ is greater than $\boldsymbol{c}^T\boldsymbol{x}^1$ if and only if $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are non-adjacent. Clearly, $\boldsymbol{c}^T\boldsymbol{x}^1 > \boldsymbol{c}^T\boldsymbol{x}^2$. Lemma 2.3 implies that when $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not adjacent, there exists a vector $\boldsymbol{y} \in \{\boldsymbol{x} \in \{0,1\}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$ such that $\boldsymbol{x}^1 \neq \boldsymbol{y} \neq \boldsymbol{x}^2$, $(\boldsymbol{x}^1, \boldsymbol{y}, \boldsymbol{x}^2)$ is a monotone vertex sequence and $\boldsymbol{y}_j = 1$. Then it is clear that $\boldsymbol{c}^T\boldsymbol{y} > \boldsymbol{c}^T\boldsymbol{x}^1$. Thus the optimal value of the problem $\max\{\boldsymbol{c}^T\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n\}$ is greater than $\boldsymbol{c}^T\boldsymbol{x}^1$. The inverse implication is now clear.

Thus, by solving the problem $\max\{\boldsymbol{c}^T\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n\}$, we can test the adjacency of given two vertices. □

The above proof says that we can test the adjacency of two vertices on equality constrained 0-1 polytope by solving an optimization problem defined on the given polytope. Since there exists a pseudo-polynomial time algorithm for the equality constrained knapsack problem [5], we can check the adjacency of two vertices on an equality constrained knapsack polytope in pseudo-polynomial time.

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