A Framework for Generalized Sampled-Data Control Systems and Its Application to Convergence Analysis†

Yasuaki Oishi

METR 97-05 April 1997

†Communicated by Professor M. Fushimi.

A framework for generalized sampled-data control systems and its application to convergence analysis

Yasuaki Oishi

Department of Mathematical Engineering and Information Physics
The University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
oishi@crux.t.u-tokyo.ac.jp

February 26, 1997

Abstract

This paper presents a framework for sampled-data control systems with generalized samplers and holds. This framework is advantageous in the point that it covers a large class of generalized samplers and holds and also in the point that it treats samplers and holds in a symmetrical manner. Using this framework, converging and non-converging properties of sampled-data control are investigated. The best achievable performance of sampled-data control does not always converge to that of time-invariant continuous-time control as a sampling period approaches zero. This paper shows that the non-converging phenomenon occurs when a sampled-data system includes an anti-aliasing filter whose bandwidth is proportional to the Nyquist frequency or when it does not include an anti-aliasing filter. Moreover, this paper proves that one can avoid this non-converging phenomenon for almost all G using a special sampler-hold pair.

1. Introduction

Intuitively, it seems obvious that the best achievable performance of sampled-data control converges to that of time-invariant continuous-time control as a sampling period tends to zero. Recently, this intuition is shown to hold in the $\mathcal{H}^2$-setting [15, 16] and in the $\mathcal{H}^\infty$-setting [6, 12] for some special cases. However, there exists an easy example where this intuition does not hold.

Example 1. Let us control a plant P, whose transfer function is $1/(s-1)$, using sampled-data control and continuous-time control as shown in Figure 1. The purpose of control is to minimize the effect that the sensor noise $d(t)$ gives to the plant output $y(t)$ in the sense of the $\mathcal{L}^2$-induced norm. Here, we let $K_{\rm d}$ be a discrete-time controller having a time-invariant discrete-time state-space representation, while we let K be a continuous-time controller described by a time-invariant continuous-time state-space representation. The symbols $S_\tau^{\rm id}$ and $H_\tau^{\rm zo}$ denote the ideal sampler and the zero-order hold, respectively; both having a sampling period $\tau > 0$. Moreover, F_τ is a low-pass filter whose transfer function is $1/(\tau s+1)$.

The low-pass filter F_τ is introduced to represent two situations. In the first situation, F_τ stands for an anti-aliasing filter inserted to cut off unpreferable aliases. Since these aliases appear mostly at higher frequencies than the Nyquist frequency π/τ, the bandwidth of F_τ is taken proportionally to $1/\tau$. In the second situation, while no anti-aliasing filter exists, a more precise model than the ideal sampler $S_\tau^{\rm id}$ is used for a sampler. Being different from $S_\tau^{\rm id}$, a physically realizable sampler integrates the input signal for a short but finite period of time. If we assume that this integration period is proportional to the sampling period τ, combination of F_τ and $S_\tau^{\rm id}$ reflects properties of an actual sampler better than $S_\tau^{\rm id}$ itself.

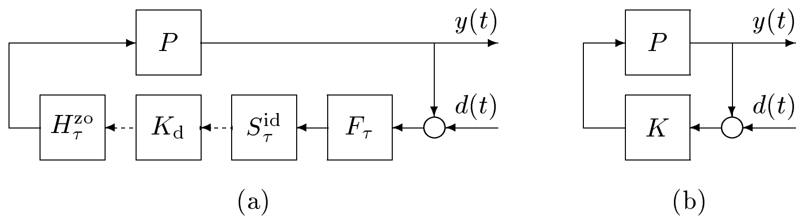

Figure 1. Two methods to control P: (a) sampled-data control; (b) time-invariant continuous-time control.

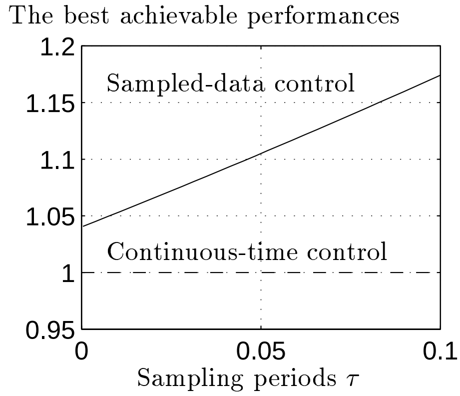

Figure 2. The best achievable performances of the two control methods, which are measured by the $\mathcal{L}^2$-induced norms.

Figure 2 shows simulation results. We consider the operators from $d(t)$ to $y(t)$, minimize their $\mathcal{L}^2$-induced norms by tuning $K_{\rm d}$ and K, respectively, and plot the achieved infima. We can see the performance of sampled-data control does not converge to that of continuous-time control as the sampling period approaches zero. □

This example shows that the above-mentioned intuition is not always correct and implies that we should be careful in applying sampled-data control. Therefore, we want to clarify why this phenomenon occurs and how it can be avoided.

We start by investigating basic properties of sampled-data systems equipped with generalized samplers and holds. Note that combination of F_τ and $S_\tau^{\rm id}$ is a sort of generalized sampler because it converts a continuous-time signal to a discrete-time signal. In fact, using basic properties of this type of systems, we can explain why the observed non-converging phenomenon occurs. Furthermore, we consider how we can prevent the phenomenon.

Another objective of this paper is to present a framework for sampled-data systems with generalized samplers and holds. There is a number of papers on generalized samplers and holds (for example, [1, 9, 10]). Compared with these papers, our framework is useful in the point that it can treat a large class of samplers and holds and in the point that it is symmetrical to samplers and holds.

A part of the results of this paper has been reported in [13, 14].

The notation used in this paper is as follows. The imaginary unit is denoted by i. The symbol ℓ^2 means the set of square-summable sequences. It may be one-sided or two-sided depending on the context. We let $\mathcal{L}^2$ and $\mathcal{L}^2[0,\tau)$ express the sets of Lebesgue square integrable functions defined on $[0,\infty)$ and $[0,\tau)$, respectively. The symbol $\|\cdot\|$ denotes the $\mathcal{L}^2$-induced norm unless specified in other way. Let $\mathcal{H}^\infty$ be a Hardy space and write its norm as $\|\cdot\|_{\mathcal{H}^\infty}$. An element of $\mathcal{H}^\infty$ may be a matrix-valued function. The set $\mathcal{RH}^\infty$ is a subset of $\mathcal{H}^\infty$ and consists of real rational functions only. For a time-invariant continuous-time operator F, its transfer function is defined based on the Laplace transform and is written as $\widehat{F}(s)$; for a time-invariant discrete-time operator F_{d}, its transfer function is written as $\check{F}_{\mathrm{d}}(z)$, which is defined from the z-transform. Here, z corresponds to the unit-time advance operator. Finally, $\mathcal{F}(G, K)$ is the lower linear fractional transform and is defined as $\mathcal{F}(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$.

2. Sampled-Data Control Systems

3. Lifting-Based Transfer Functions and Their Properties

4. Converging and Non-Converging Properties

5. Conclusion

In this paper, we introduced a framework for sampled-data control systems with generalized samplers and holds and investigated basic properties of these systems. Using these properties, we considered when the best achievable performance of sampled-data control systems converges to that of time-invariant continuous-time control systems and when it does not.

Our framework is useful because it covers many practical samplers and holds. Besides, it treats samplers and holds in a symmetrical manner. These advantages are expected to be helpful in considering other advanced problems on sampled-data control such as optimization of samplers and holds.

In the past researches on a sampled-data $\mathcal{H}^\infty$-control theory, insertion of anti-aliasing filters seems to be introduced only for theoretical convenience [1, 2, 3, 4, 18]. However, our results showed that an appropriate choice of the filter has practical importance. Especially, we should remember that the non-converging phenomenon possibly occurs when we determine an anti-aliasing filter according to the sampling period or when we use no anti-aliasing filter. This paper gives a mathematical basis for such an analysis of filters.

Acknowledgment

This research is supported by the Grant-in-Aid for Encouragement of Young Scientists of the Ministry of Education, Science, Sports and Culture of Japan (Grant No. 08750501).

References

[1] M. Araki, Y. Ito, and T. Hagiwara, "Frequency Response of Sampled-Data Systems," *Automatica*, vol. 32, no. 4, pp. 483–497, 1996.

[2] B. A. Bamieh and J. B. Pearson, Jr., "A General Framework for Linear Periodic Systems with Application to $\mathcal{H}^\infty$ Sampled-Data Control," *IEEE Transactions on Automatic Control*, vol. 37, no. 4, pp. 418–435, 1992.

[3] T. Chen and B. A. Francis, "Input-Output Stability of Sampled-Data Systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 1, pp. 50–58, 1991.

[4] T. Chen and B. A. Francis, "$\mathcal{H}^2$-Optimal Sampled-Data Control," *IEEE Transactions on Automatic Control*, vol. 36, no. 4, pp. 387–397, 1991.

[5] B. A. Francis, *A Course in $\mathcal{H}^\infty$ Control Theory*. Berlin: Springer, 1987.

[6] S. Hara, M. Nakajima, and P. T. Kabamba, "Robust Stabilization in Digital Control Systems," in *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing I*, H. Kimura and S. Kodama, Eds. Tokyo: Mita Press, 1991, pp. 481–486.

[7] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*. Providence, RI: American Mathematical Society, 1957.

[8] K. Hoffman, *Banach Spaces of Analytic Functions*. Englewood Cliffs, NJ: Prentice-Hall, 1962.

[9] Y. Ito, M. Araki, and T. Hagiwara, "Use of Generalized Samplers in Sampled-Data Control," in *Proceedings of the 15th SICE Symposium on Dynamical System Theory*, Akita, Japan, December 1992, pp. 165–170.

[10] P. T. Kabamba, "Control of Linear Systems Using Generalized Sampled-Data Hold Functions," *IEEE Transactions on Automatic Control*, vol. 32, no. 9, pp. 772–783, 1987.

[11] P. P. Khargonekar, K. Poolla, and A. Tannenbaum, "Robust Control of Linear Time-Invariant Plants Using Periodic Compensation," *IEEE Transactions on Automatic Control*, vol. 30, no. 11, pp. 1088–1096, 1985.

[12] Y. Oishi, "A Bound of Conservativeness in Sampled-Data Robust Stabilization and Its Dependency on Sampling Periods," Technical Report 95-08, Department of Mathematical Engineering and Information Physics, the University of Tokyo, September 1995; also submitted to *Systems & Control Letters*.

[13] Y. Oishi, "A Performance Bound of Sampled-Data Control with Generalized Samplers and Holds" (in Japanese), in *Proceedings of the 25th SICE Symposium on Control Theory*, Chiba, Japan, May 1996, pp. 115–120.

[14] Y. Oishi, "Converging and Non-Converging Properties of the Best Achievable Performance in Sampled-Data Control," presented at *the International Symposium on the Mathematical Theory of Networks and Systems*, St. Louis, MO, June 1996.

[15] S. L. Osburn and D. S. Bernstein, "An Exact Treatment of the Achievable Closed-Loop $\mathcal{H}^2$ Performance of Sampled-Data Controllers: from Continuous-Time to Open-Loop," *Automatica*, vol. 31, no. 4, pp. 617–620, 1995.

[16] H. L. Trentelman and A. A. Stoorvogel, "Sampled-Data and Discrete-Time $\mathcal{H}^2$ Optimal Control," *SIAM Journal on Control and Optimization*, vol. 33, no. 3, pp. 834–862, 1995.

[17] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: MIT Press, 1985.

[18] Y. Yamamoto, "A Function Space Approach to Sampled Data Control Systems and Tracking Problems," *IEEE Transactions on Automatic Control*, vol. 39, no. 4, pp. 703–713, 1994.

[19] Y. Yamamoto and M. Araki, "Frequency Responses for Sampled-Data Systems—Their Equivalence and Relationships," *Linear Algebra and Its Applications*, vols. 205–206, pp. 1319–1339, 1994.