Berry's phase in the quantum estimation theory, and its intrinsic relation with the complex structure

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Abstract

In this paper, it is pointed out that the Berry's phase is a good index of degree of manifestation of non-commutative nature in the quantum statistical model. Intrinsic relations between the 'complex structure' of the Hilbert space and Berry's phase is also discussed.

Keywords: Berry's phase quantum estimation theory, attainable Cramer-Rao type bound, complex structure, antiunitary operator

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1 Introduction

Berry's phase, discovered by M.V,Berry in 1982, convienced by many experiments, is naturally interpreted as a curvature of natural connection introduced on the line bundle over the space of pure states [1][2][19][20].

Berry's phase is a manifestation of non-commutative nature. Actulally, various quantum mechanical phenomina, such as quantum hall effect, Aharanov-Bohm effect, Yang-Tailor effect, and so on, cannot be predicted by naive analogy of classical mechanics, and are explained in terms of Berry's phase[19] . In the semi classical quantization, we need to add a term propotional to the Berry's phase to the classical action[10]. Hence, it is reasonable to consider that Berry's phase is a good measure of non-commutative nature of the given system.

We shed light on this statement from the estimation theoretical point of view: It is shown that the Berry's phase is a good index of degree of manifestation of non-commutative nature in the quantum statistical model.

It is also pointed out that Berry's phase has intrinsic relation with the 'complex structure' of the Hilbert space.

The paper is organized as follows. Section 2 is review of quantum estimation theory and Berry's phase. In section 3, sections 5-6, relations between Berry's phase and quantum estimation theory are discussed. In section 4 and sections 7-sec:timereversal, relasions between Berry's phase and the 'complex structure' is studied.

2 Preliminaries

2.1 Quantum measurement theory, the unbiased estimator

We denote by $\mathcal{P}_1(\mathcal{H})$ the space of density operators of pure states in a separable Hilbert space $\mathcal{H}$. $\mathcal{P}_1(\mathcal{H})$ is often simply denoted by $\mathcal{P}$, $\mathcal{P}_1$.

Let Ω be a space of all possible outcomes of an experiment, and $\sigma(\Omega)$ be a σ- field in Ω. When the density operator of the system is ρ, the probability that the data $\omega \in \Omega$ lies in $B \in \sigma(\Omega)$ writes

$$\Pr\{\omega \in B|\rho\} = \mathrm{tr}\rho M(B), \tag{1}$$

by use of the map M from $\sigma(\Omega)$ to nonnegative Hermitian operator which satisfies

$$\begin{aligned} &M(\phi) = O, M(\Omega) = I, \\ &M(\bigcup_{i=1}^{\infty} B_i), = \sum_{i=1}^{\infty} M(B_i) \ \ (B_i \cap B_j = \phi, i \neq j), \end{aligned} \tag{2}$$

so that (1) define a probability measure (see Ref.[8], p.53 and Ref.[9], p.50). We call the map M the *measurement*, because there always exist an physical experiment corresponds to the map M which satisfies (2) [21][16].

The purpose of the quantum estimation is to identify the density operator of the given physical system from the data obtained by the appropriately designed experiment. For simplicity, we usually assume that the density operator is a member of a *model*, or a manifold of $\mathcal{M} = \{\rho(\theta)|\theta \in \Theta \subset \mathbf{R}^m\} \subset \mathcal{P}$, and that the parameter θ is to be estimated statistically. For example, $\mathcal{M}$ is the set of spin states with given wave function part and unknown spin part. In this paper, we restrict ourselves to the *pure state model* case, where any member of the model is pure state,

$$\begin{aligned} \rho(\theta) &= \pi(|\phi(\theta)\rangle) \\ &\equiv |\phi(\theta)\rangle\langle\phi(\theta)|. \end{aligned} \tag{3}$$

To estimate the parameter, we performs an experiment to obtain the data ω by which we calculate an estimator $\hat{\theta}$ by *estimator* $\hat{\theta}(\omega)$. A pair $(\hat{\theta}, M, \Omega)$ of a space Ω of data, a measurement M, and an estimator $\hat{\theta}(*)$ is also called an *estimator*. The expectation of $f(\omega)$ with respect to the probability measure (1) is denoted by $E_\theta[f(\omega)|M]$.

The estimator $(\hat{\theta}, M, \Omega)$ is said to be *unbiased* if

$$E_\theta[\hat{\theta}(\omega)|M] = \theta \tag{4}$$

holds for all $\theta \in \Theta$. If (4) and

$$\partial_i E_\theta[\theta^j(\omega)|M] = \delta_i^j \; (i,j = 1,...,m),$$

where ∂_i stands for $\partial/\partial\theta^i$, hold at a particular θ, $(\hat{\theta}, M, \Omega)$ is called *locally unbiased* at θ.

2.2 SLD CR inequality, the attainable CR type bound

Analogically to the estimation theory of probability distributions, in the quantum estimation theory, we have the following *SLD CR inequality*, which is proved for the exact state model by Helstrom [7][8], and is proved for the pure state model by Fujiwara and Nagaoka [5]:

$$V_\theta[\hat{\theta}(\omega) \,|\, M] \geq \left(J^S(\theta)\right)^{-1}, \tag{5}$$

i.e., $V_\theta[\hat{\theta}(\omega) \,|\, M] - (J^S(\theta))^{-1}$ is non-negative definite. Here $V_\theta[\hat{\theta}(\omega) \,|\, M]$ is a covariance matrix of an unbiased measurement M, and $J^S(\theta)$ is called *SLD Fisher information matrix*, and is defined by

$$J^S(\theta) \equiv [\mathrm{Re}\langle l_i(\theta)|l_j(\theta)\rangle]\,, \tag{6}$$

where $|l_i(\theta)\rangle$ $(i = 1,...,m)$ are defined afterward. J^{S-1} is called *SLD CR bound.*

The horizontal lift $|l_X\rangle$ of a tangent vector $X \in \mathcal{T}_{\rho(\theta)}(\mathcal{M})$ to $|\phi(\theta)\rangle$, is an element of $\mathcal{H}$ which satisfies

$$X\rho(\theta) = \frac{1}{2}(|l_X\rangle\langle\phi(\theta)| + |\phi(\theta)\rangle\langle l_X|), \tag{7}$$

and

$$\langle l_X|\phi(\theta)\rangle = 0. \tag{8}$$

Here, X in the left hand side (7) of is to be understood as a differential operator. $|l_i(\theta)\rangle$ is defined to be a horizontal lift of $\partial_i \in \mathcal{T}_{\rho(\theta)}(\mathcal{M})$.

The inequality (5) is of special interest, because J^{S-1} is the one of the best bounds in the sense that for any real hermitian matrix A which is larger than J^{S-1}, there exists such an unbiased estimator M that $V[\hat{\theta}(\omega)\,|\,M]$ is not smaller than A.

However, the bound is not always attainable[12].

Theorem 1 *The SLD CR bound is attainable iff*

$$\mathrm{Im}\langle l_i(\theta)|l_j(\theta)\rangle = 0 \tag{9}$$

for any i, j.

The model is said to be *locally quasi-classical* at θ iff (9) holds true at θ.

If the model is no locally quasi-classical, therefore, we give up to find a matrix which makes attainable lower bound of $V[\hat{\theta}(\omega)\,|\,M]$. and try to determine the *attainable CR type bound* $\mathrm{CR}(G, \theta, \mathcal{M})$, which is defined by

$$\mathrm{CR}(G, \theta, \mathcal{M}) = \min\{\mathrm{Tr}GV_\theta[\hat{\theta}(\omega)\,|\,M]\,|\,M \text{ is locally unbiased at } \theta\} \tag{10}$$

for an arbitrary nonnegative symmetric real matrix G. G is called *weight matrix.* To make the estimational meaning of (10) clear, let us consider the case where G is $diag(g_1, g_2..., g_m)$. Then, $\mathrm{Tr}GV[\hat{\theta}(\omega)\,|\,M]$ is the weighed sum of the covariance of the estimation of θ^i. If one wants to know, for example, θ^1 more precisely than other parameters, then he set g_1 larger than any other g_i, and pick up a measurement which minimize $\mathrm{Tr}GV[\hat{\theta}(\omega)\,|\,M]$.

2.3 Berry's phase

In this section, we review the geometrical theory of Berry's phase.

Let us denote by $\tilde{\mathcal{H}}$ set of all the state vectors, or member of $\mathcal{H}$ with unit length. Because the two state vectors correspond to the same state iff they differ with each other only in their phase factor, it is natural to consider $\tilde{\mathcal{H}}$ as a principal fiber bundle with the base space $\mathcal{P}_1$ and the structure group $U(1)$.

A horizontal lift $\hat{C} = \{|\phi(t)\rangle\,|\,0 \le t \le 1\}$ of the curve $C = \{\rho(t)\,|\,0 \le t \le 1\}$ in $\mathcal{P}_1$ is defined to be a curve in $\tilde{\mathcal{H}}$ which satisfies $\rho(t) = \pi(|\phi(t)\rangle)$ and

$$|l_{d/dt}(t)\rangle = \frac{d}{dt}|\phi(t)\rangle. \tag{11}$$

Then, the Berry's phase $\gamma(C)$ for the curve $C = \{\rho(t) \,|\, 0 \le t \le 1\}$ is defined by

$$|\phi(1)\rangle = e^{i\gamma(C)}|\tilde{\phi}(0)\rangle, \tag{12}$$

where $|\tilde{\phi}(0)\rangle$ satisfies $\pi(|\tilde{\phi}(0)\rangle) = \rho(1)$ and

$$\mathrm{Im}\langle\phi(1)|\tilde{\phi}(0)\rangle = 0. \tag{13}$$

The model is said to be *parallel* iff Berry's phase for any curve in the model vanishes.

The Berry's phase for the infinitesimal loop

$$\begin{array}{ccc} (\theta^1, ..., \theta^i, ..., \theta^j + d\theta^j, ..., \theta^m) & \leftarrow & (\theta^1, ..., \theta^i + d\theta^i, ..., \theta^j + d\theta^j, ..., \theta^m) \\ \downarrow & & \uparrow \\ \theta = (\theta^1, ..., \theta^i, ..., \theta^j, ..., \theta^m) & \rightarrow & (\theta^1, ..., \theta^i + d\theta^i, ..., \theta^j, ..., \theta^m) \end{array} \tag{14}$$

is calculated up to the second order of $d\theta$ as

$$\frac{1}{2}\tilde{J}_{ij} d\theta^i d\theta^j + o(d\theta)^2,$$

where $\tilde{J}_{ij}$ is equal to $\frac{1}{2}\mathrm{Im}\langle l_i|l_j\rangle$.

Geometrically,

$$\sum_{i,j} \tilde{J}_{ij} d\theta^i d\theta^j \tag{15}$$

corresponds to the curvature form.

3 SLD metric

It should be noted that the SLD FIsher information matrix is deeply concerned with this fiber bundle structure. The horizontal lift $\hat{C} = \{|\phi(t)\rangle \,|\, 0 \le t \le 1\}$ of the curve $C = \{\rho(t) \,|\, 0 \le t \le 1\}$ in $\mathcal{P}_1$ is geometrically characterized as

$$\hat{C} = \mathrm{argmin} \left\{ \langle \dot{\phi}(t)|\dot{\phi}(t)\rangle \,\middle|\, |\phi(t)\rangle \in \pi^{-1}(\rho(t)) \right\}. \tag{16}$$

Therefore, SLD Fisher information $J_t^S(t)$ of the parameter t is proportional to 'the shortest distance' between infinitesimally distant two fibers $\pi^{-1}(\rho(t))$ and $\pi^{-1}(\rho(t+dt))$.

Using this fact, SLD CR inequality in the case of one parameter pure state model can be interpreted geometrically. Notice that by virtue of theorem 1, SLD CR bound is always attained in this case. Therefore, if SLD Fisher information is smaller, it is harder to distinguish $\rho(t)$ and $\rho(t+dt)$. Because of geometrical characterization of SLD Fisher information in the end of the last chapter, we can also say that the closer two fibers $\pi^{-1}(\rho(t))$ and $\pi^{-1}(\rho(t+dt))$ are, the harder it is to distinguish $\rho(t)$ from $\rho(t+dt)$.

In the following, we define inner product $\langle *, * \rangle_\theta$ in $\mathcal{T}_\theta(\mathcal{M})$ by

$$\langle \partial_i, \partial_j \rangle_\theta = \left[J^S(\theta) \right]_{ij}, \tag{17}$$

because this metric (*SLD metric*, hereafter) seems to be estimation-theoretically and geometrically natural.

4 D-transform

We define a linear transform $\mathbf{D}_\theta$ in $\mathcal{T}_\theta(\mathcal{M})$ by

$$\langle \partial_i, \mathbf{D}_\theta(\partial_j) \rangle_\theta = \tilde{J}_{ij}(\theta). \tag{18}$$

$\mathbf{D}_\theta$ is called *D-transform*. The non-zero eigenvalues of $\mathbf{D}_\theta$ are denoted by $\pm i\beta_j(\theta)$, where $\beta_j(\theta)$ is positive real number, and j runs from 1 to the half of the rank of D-transform, and $\beta_j(\theta)s$ are sorted so that $\beta_1 \geq \beta_2 \geq ....$

When $\dim \mathcal{M} = 2$,

$$\beta_1(\theta) = \frac{\tilde{J}_{12}(\theta)}{\sqrt{\det J^S(\theta)}}, \tag{19}$$

left hand side of which is Berry's along the curve which encloses unit area, where the unit of area is naturally induced by the SLD metric.

It is worthy of remarking that the $\mathbf{D}_\theta$ is related to the natural 'complex structure' of the Hilbert space $\mathcal{H}$. Actually, D-transform is obtained by the following procedure.

First, we multiply the imaginary unit i to $|l_X\rangle = \mathbf{M}(h(|l_X\rangle))$ and and $\mathbf{M}^{-1}$ and π_* are applied successively to $i|l_X\rangle$. since $\pi_*(\mathbf{M}^{-1}(i|l_X\rangle))$ is not a member of $\mathcal{T}_\rho(\mathcal{M})$ generally, we project $\pi_*(\mathbf{M}^{-1}(i|l_X\rangle)) \in \mathcal{T}_\rho(\mathcal{P}_1)$ to $\mathcal{T}_\rho(\mathcal{M})$ with respect to the metric $\langle *|*\rangle_\rho$, and we obtain $\mathbf{D}X \in \mathcal{T}_\rho(\mathcal{M})$.

Therefore, Berry's phase is deeply related to the 'complex structure ' of the Hilbert space.

$$\begin{array}{ccc} & \text{multiplication of } i & \\ |l_X\rangle \in span_{\mathbf{R}}\mathsf{L} & --\longrightarrow & i|l_X\rangle \in span_{\mathbf{R}}\{\mathsf{L}, i\mathsf{L}\} \\ \uparrow & & \downarrow \mathbf{M}^{-1},\, \pi_* \\ h_{|\phi\rangle},\, \mathbf{M} & & \pi_*(\mathbf{M}^{-1}(i|l_X\rangle)) \in \mathcal{T}_\rho(\mathcal{P}_1) \\ \uparrow & & \text{project} \downarrow \text{w.r.t. } \langle *, *\rangle_\rho \\ X \in \mathcal{T}_\rho(\mathcal{M}) & --\longrightarrow & \mathbf{D}X \in \mathcal{T}_\rho(\mathcal{M}) \\ & \mathbf{D} & \end{array}$$

5 Berry's phase as a measure of local non-commutative nature

In this section, it is pointed out that Berry's phase for infinitesimal loop is good index of non-commutative nature of two parameters. As is already mentioned, if the model has only one parameter, the SLD CR bound is attainable, and the attainable CR type bound is,

$$\mathrm{CR}(G, \theta', \mathcal{M}') = \mathrm{Tr} G J^{S-1}. \tag{20}$$

In the multi-parameter case, however, the SLD CR bound is not necessarily attainable, as an effect of non-commutative nature of quantum mechanics, and

$$\mathrm{CR}(G, \theta', \mathcal{M}') \geq \mathrm{Tr} G J^{S-1}. \tag{21}$$

The author conjectures the difference between both sides of the inequality increases as β_js increase.

As a matter of fact, in the two parameter pure state model, we have the following theorem [12].

Theorem 2 *Let $\mathcal{M}$ and $\mathcal{M}'$ be a two parameter pure state model. Then, if $\beta_1(\theta', \mathcal{M}') \geq \beta_1(\theta, \mathcal{M})$ and $J^S(\theta', \mathcal{M}') = J^S(\theta, \mathcal{M})$, then for any weight matrix G,*

$$\mathrm{CR}(G, \theta', \mathcal{M}') \geq \mathrm{CR}(G, \theta, \mathcal{M}). \tag{22}$$

This theorem means that in the two parameter model, if β_1 is large, it is hard to estimate θ^1 and θ^2 simultaneously.

What about the models with arbitrary number of parameters? By virtue of theorem 1, if Berry's phase for any closed loop vanishes, the gap between both sides of the inequality (21) vanishes.

As for general multi-parameter pure state models, we have [12],

$$\begin{aligned} & \mathrm{CR}\left(\theta, J^S(\theta), \mathcal{M}\right) \\ = & \left\{\mathrm{TrRe}\sqrt{I_m + iJ^{S-1/2}(\theta)\tilde{J}(\theta)J^{S-1/2}(\theta)}\right\}^{-2} \\ = & \sum_j \frac{4}{1 + \sqrt{1 - |\beta_j(\theta)|^2}}. \end{aligned} \tag{23}$$

Though the estimation theoretical significance of $\mathrm{CR}(J^S)$ is hard to verify, the author claims that (23) support the author's conjecture.

6 Berry's phase as a measure of global non-commutative nature

Even if the model is locally quasi-classical at any $\theta \in \Theta$, the best locally unbiased estimator $(\hat{\theta}, M, \Omega)$ is dependent on true value of the parameter θ. Therefore, to attain the bound globally, $\hat{\theta}$, M must be changed adoptively as the number of data increases [15].

A locally quasi-classical model is said to be *globally quasi-classical* when the measurement M in the best triplet $(\hat{\theta}, M, \Omega)$ is independent of true value of the parameter, for, in this case, using the globally best measurement M, and processing data in the way classical estimation theory tells us, the SLD CR bound is attained globally.

In the faithful model case, or the case where the model is consisted of the strictly positive density operator, it is pointed out the model globally quasi-classical iff Uhlmann's RPF(relative phase factor) vanishes for any curve in the modelKeiji:1997a. For Uhlmann's RPF is nothing but a generalization of Berry's phase to the non-pure model, it is of interest to study if the similar fact holds true for the pure state model.

For simplicity, we say that the manifold $\mathcal{N}$ in $\mathcal{P}_1$ is a *horizontal lift* of the model $\mathcal{M}$ if

$$\begin{aligned} \pi(\mathcal{N}) &= \mathcal{M}, \\ \forall|\phi(\theta)\rangle \in \mathcal{N}, \qquad & \frac{1}{2}\frac{\partial}{\partial\theta^i}|\phi(\theta)\rangle \in \mathcal{LS}_{|\phi(\theta)\rangle}. \end{aligned} \tag{24}$$

The horizontal lift $\mathcal{N}$ exists iff $\mathcal{M}$ is locally quasi-classical at any point. The model is said to be *quasi-parallel* iff Berry's for any curve is 0 or π.

Theorem 3 *If the model $\mathcal{M}$ is quasi-parallel, that model is quasi- classical.*

Proof First, apply Schmidt's orthonormalization to the horizontal lift $\mathcal{N}$ of $\mathcal{M}$, to obtain the orthonormal basis $\mathbf{B} = \{|e_i\rangle \,|\, i = 1, 2, ...\}$ such that $\mathcal{N}$ is a subset of the real span of $\mathbf{B}$. We immerse Hilbert space $\mathcal{H}$ into $L^2(\mathbf{R}, \mathbf{C})$ as

$$|\phi(\theta)\rangle = \sum_i a_i|e_i\rangle \mapsto \sum_i a_i(\theta)\psi_i(x),$$

where $\{\ psi_i(x) \,|\, i = 1, 2, ...\}$ is an orthonormal basis in $L^2(\mathbf{R}, \mathbf{C})$. Then, letting $E(dx) = |x\rangle\langle x|dx$, the triplet $(\hat{\theta}(\theta),\, E\,, \mathbf{R})$ is one of the best estimators. This assertion is easily proved by calculating the Fisher information matrix of the family,

$$\left\{ p(x\,,\theta) = \sum_i (a_i(\theta))^2 |\psi_i(x)|^2 \right\} \tag{25}$$

of probability distributions. □

The converse of the latter theorem is, however, not true, because the following counter-examples exist.

Example We consider the *position shifted model* which is defined by

$$\begin{aligned} &\mathcal{M}_x = \pi(\mathcal{N}_x) \\ &\mathcal{N}_x = \left\{ |\phi(\theta)\rangle \,\middle|\, |\phi(\theta)\rangle = cconst. \times (x-\theta)^2 e^{-(x-\theta)^2 + ig(x-\theta)},\ \theta \in \mathbf{R} \right\}, \end{aligned}$$

where g the function such that

$$g(x) = \begin{cases} 0 & (x \geq 0), \\ \alpha & (x < 0). \end{cases}$$

Then, as easily checked, $\mathcal{N}_x$ is a horizontal lift of the model $\mathcal{M}_x$, and $\langle\phi(\theta)|\phi(\theta')\rangle$ is not real unless $\alpha = n\pi$ $(n = 0, 1, ...)$. However, SLD CR bound is uniformly attained by the measurement obtained by the spectral decomposition $E_x(dx) = |x\rangle\langle x|dx$ of the position operator,where $|x_0\rangle = \delta(x - x_0)$. as is checked by comparing SLD Fisher information of the model $\mathcal{M}_x$ and the classical Fisher information of the probability distribution family

$$\left\{ p(x|\theta) \,\middle|\, p(x|\theta) = |\langle\phi(\theta)|x\rangle|^2,\ \theta \in \mathbf{R} \right\}.$$

Note that $|\phi(\theta)\rangle$ is an eigenstate of the Hamiltonian

$$H(\theta) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{m}\left(2(x-\theta)^2 + \frac{1}{(x-\theta)^2} \right),$$

whose potential has two wells with infinite height of wall between them.

Example Let $\mathcal{H}$ be $L^2([0,\ 2\pi],\ \mathbf{C})$, and define a one parameter model $\mathcal{M}$ such that,

$$\begin{aligned} &\mathcal{M} = \pi(\mathcal{N}) \\ &\mathcal{N} = \left\{ |\phi(\theta)\rangle \,\middle|\, |\phi(\theta)\rangle = const. \times (2 - \cos\omega)\, e^{i\alpha(f(\omega-\theta)+\theta)},\ (0 \leq \omega, \theta < 2\pi) \right\}, \end{aligned} \tag{26}$$

where α is a real number and f the function defined by

$$f(\omega - \theta) = \begin{cases} \omega - \theta & (\omega - \theta \geq 0) \\ \omega + 2\pi - \theta & (\omega - \theta < 0) \end{cases}.$$

Physically, (26) is an eigenstate of the Hamiltonian H such that,

$$H(\theta) = -\frac{\hbar^2}{2m}\left(\frac{d}{d\omega} - i\alpha\right)^2 + \frac{A - B\cos(\omega-\theta)}{2-\cos(\omega-\theta)},$$

which characterize the dynamics of an electron confined to the one-dimensional ring which encircles magnetic flax $\Phi = 2\pi\alpha c/e$, where m is the mass of the electron, $-e$ the charge of the electron, c the velocity of light, and A, B the appropriately chosen constant.

It is easily checked that $\mathcal{N}$ is a horizontal lift of the model $\mathcal{M}$, and that the model $\mathcal{M}$ is not parallel unless $\alpha = n\pi\ (n = 0, 1, ...)$. However, consider the projection valued measure E_ω such that

$$E_\omega(d\omega) = |\omega\rangle\langle\omega| d\omega,$$

where $|\omega_0\rangle = \delta(\omega - \omega_0)$. Then, it is easily checked that the classical Fisher information of the probability distribution family

$$\left\{ p(\omega|\theta) \;\middle|\; p(\omega|\theta) = |\langle\phi(\theta)|\omega\rangle|^2,\ 0 \le \omega, \theta < 2\pi \right\}$$

is equal to the SLD Fisher information of $\mathcal{M}$.

7 The antiunitary operator

As is pointed out in the section 4, Berry's phase seems to have some intrinsic relation with the 'complex structure'. In this section, we study this point using the antiunitary operator.

The transformation A

$$|\tilde{a}\rangle = A|a\rangle, \quad |\tilde{b}\rangle = A|b\rangle$$

is said to be *antiunitary* iff

$$\begin{aligned} \langle\tilde{a}|\tilde{b}\rangle &= \overline{\langle a|b\rangle}, \\ A(\alpha|a\rangle + \beta|b\rangle) &= \overline{\alpha}A|a\rangle + \overline{\beta}A|b\rangle, \end{aligned}$$

where $\overline{z}$ means complex conjugate of z (see Ref [17] p.266).

Theorem 4 *The model is quasi-parallel iff the horizontal lift of the model is invariant by some antiunitary operator.*

Proof Suppose that any member of the manifold $\mathcal{N} = \{|\phi\rangle\}$ in $\tilde{\mathcal{H}}$ is invariant by the antiunitary operator A, and let $|\tilde{\phi}\rangle = A|\phi\rangle$, $|\tilde{\phi}'\rangle = A|\phi'\rangle$. Then, we have

$$\langle\phi|\phi'\rangle = \langle\tilde{\phi}'|\tilde{\phi}\rangle = \langle\phi'|\phi\rangle \in \mathbf{R}.$$

Conversely, if $\langle\phi|\phi'\rangle$ is real for any $|\phi\rangle$, $|\phi'\rangle \in \mathcal{N}$, by Schmidt's orthonormalization, we can obtain the orthonormal basis $\mathsf{B} = \{|i\rangle \,|\, i = 1, 2, ..., d\}$ such that $\mathcal{N}$ is subset of the real span of B, which means any member of $\mathcal{N}$ is invariant by the antiunitary operator K_{B}, which is defined by,

$$K_{\mathsf{B}} \sum_i \alpha_i |i\rangle = \sum_i \overline{\alpha}_i |i\rangle.$$

□

8 Time reversal symmetry

As an example of the antiunitary operator, we discuss *time reversal operator* (see Ref.[17], pp. 266-282). The time reversal operator T is an antiunitary operator in $L^2(\mathbf{R}^3, \mathbf{C})$ which transforms the wave function $\psi(x) \in L^2(\mathbf{R}^3, \mathbf{C})$ as:

$$T\psi(x) = \overline{\psi(x)} = K_{\{|\mathbf{x}\rangle\}}\psi(x).$$

The term 'time reversal' came from the fact that if $\psi(\mathbf{x}, t)$ is a solution of the Schödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi,$$

then $\overline{\psi(\mathbf{x}, -t)}$ is also its solution.

The operator T is sometimes called *motion reversal operator*, since it transforms the momentum eigenstate $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$ corresponding to eigenvalue $\mathbf{p}$ to the eigenstate $e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}$ corresponding to eigenvalue $-\mathbf{p}$.

Define the *position shifted model* by

$$\mathcal{M}_{\mathbf{x}} = \{\rho(\theta) \,|\, \rho(\theta) = \pi(\psi(\mathbf{x}-\mathbf{x}_0)\,),\ \mathbf{x}_0 \in \mathbf{R}^3\},$$

and suppose that any member of the horizontal lift $\mathcal{N}_{\mathbf{x}}$ of the model $\mathcal{M}_{\mathbf{x}}$ has time reversal symmetry. Then, since time reversal operator T is antiunitary, the model $\mathcal{M}_{\mathbf{x}}$ is quasi-classical in the wider sense. The spectral decomposition of the position operator gives optimal measurement.

Now, we discuss the generalization of time reversal operator. The antiunitary transform

$$T_\alpha \,:\, e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} \to e^{i\alpha(\mathbf{p})}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}$$

is also called motion reversal operator, or time reversal operator.

If any member $\psi(\mathbf{x}-\mathbf{x}_0)$ of the horizontal lift $\mathcal{N}_{\mathbf{x}}$ of the position shifted model $\mathcal{M}_{\mathbf{x}}$ is invariant by the time reversal operator T_α,

$$\int_{\mathbf{R}^3} \psi(\mathbf{x}-\mathbf{x}_0)\, \overline{\psi(\mathbf{x}-\mathbf{x}_0')}\, d\mathbf{x} \in \mathbf{R} \tag{27}$$

holds true for any $\mathbf{x}_0$, $\mathbf{x}_0'$, which is equivalent to the premise of theorem 3.

Conversely, if (27) holds true, Fourier transform of (27) leads to

$$|\Psi(\mathbf{p})|^2 = |\Psi(-\mathbf{p})|^2,$$

where

$$\Psi(\mathbf{p}) = \frac{1}{\sqrt{2\pi}} \int \psi(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar} d\mathbf{x}.$$

Therefore, any member of $\mathcal{N}_{\mathbf{x}}$ is transformed to itself by the time reversal operator T_α such that

$$T_\alpha \,:\, e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} \to e^{i(\beta(\mathbf{p})+\beta(-\mathbf{p})\,)}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar},$$

where

$$e^{i\beta(\mathbf{p})} = \frac{\Psi(\mathbf{p})}{|\Psi(\mathbf{p})|}.$$

Theorem 5 *A position shifted model is quasi-parallel if and only if there exists the time reversal operator which transforms any member of the horizontal lift* $\mathcal{N}_{\mathbf{x}}$ *to itself.*

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