A Framework for Generalized Sampled-Data Control Systems with an Analysis of Their Best Achievable Performance†

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Abstract

This paper presents a functional-analytic framework for sampled-data control systems with generalized samplers and holds. Notions of lifting-based transfer functions and their matrix representations are introduced and several properties of the systems are derived. This framework covers a large class of samplers and holds. In the latter half of this paper, the presented framework is applied to an analysis of the best achievable performance of sampled-data control systems. Especially, this paper gives necessary and sufficient conditions in order that the best achievable performance of sampled-data control systems converges to that of continuous-time control systems as a sampling period tends to zero. This research is motivated by the fact that, against our intuition, the best sampled-data performance does not necessarily approach the best continuous-time performance. The conditions for the performance convergence are obtained not only for a general case but also for special cases, which are practically important. The conditions for the special cases have simpler forms and are easy to be tested.

Keywords: Sampled-data control, $\mathcal{H}^\infty$-control, Lifting technique, Sampling period, Generalized hold.

1. Introduction

A functional-analytic technique called lifting, which was introduced by Yamamoto [44, 45], enabled us to analyze and synthesize sampled-data control systems considering not only their behavior at sampling instants but also their intersample behavior. Thanks to intensive study in the subsequent years, our methodology on sampled-data control systems has made large progress. Namely, the $\mathcal{H}^\infty$-synthesis problem was solved in [5, 3, 39, 40, 24, 20], while the $\mathcal{H}^2$-synthesis problem in [8, 26, 4, 17]. A sampled-data $\mathcal{L}^1$-synthesis was considered in [11, 2]. A sampled-data robust stabilization was studied in detail in [12] and the references therein. In References [43, 9, 37], multirate samplers and holds were investigated along this approach.

The objective of this paper is twofold. First, it presents a lifting-based framework for sampled-data control systems with generalized samplers and holds. Second, based on the presented framework, this paper analyzes the best achievable performance of sampled-data control systems. In particular, it gives necessary and sufficient conditions in order that the best achievable performance of sampled-data control systems converges to that of continuous-time control systems as a sampling period approaches zero. This research is motivated by the fact that the former does not necessarily converge to the latter.

There are many papers that treat generalized holds in a lifting-based framework [18, 39, 24, 1, 28]. However, their classes of holds are too small to include even the first-order hold, which is often quoted as an example of a generalized hold. On the other hand, a comparatively small number of papers considered generalized samplers [39, 22, 28]. Again, their frameworks are unsatisfactory because the ideal sampler, which is the most frequently used in practice, cannot be treated there directly.

The framework proposed in this paper enlarges the class of considered samplers and holds by extending the domain of their kernel functions from $[0, \tau]$ to $[0, \infty)$, where τ is the sampling period. Then, this framework can model samplers and holds having memories as well as multirate samplers and holds. Moreover, the framework can treat the ideal sampler by combining it with an anti-aliasing filter. This idea also gives a by-product that we can analyze and synthesize an anti-aliasing filter by regarding it as a part of a generalized sampler. Although an anti-aliasing filter has been modeled as a part of a plant in the lifting-based approach so far, it is appropriate to regard it as a part of a sampler. This is because one can design an anti-aliasing filter to improve a control performance and this feature of anti-aliasing filters resembles that of generalized holds. In this setup, this paper introduces a notion of lifting-based transfer functions with their matrix representations. Based on this notion, some properties of the systems are derived. Especially, this paper shows relationships between kernel functions and lifting-based transfer functions concerning samplers and holds (Propositions **??** and **??**) and presents a strong tool called an approximation lemma (Proposition **??**). There, duality between samplers and holds appears in a clear way.

The second objective of this paper is to investigate convergence of the best sampled-data control performance. Intuitively, it seems obvious that the best achievable performance of sampled-data control systems approaches that of continuous-time control systems as the sampling period goes to zero. Furthermore, because engineers have believed this intuition, they have trusted sampled-data controllers and used them in place of continuous-time controllers. Indeed, this intuition is shown to hold in the $\mathcal{H}^2$-context [35, 41] and in the $\mathcal{H}^\infty$-context [18, 34] but only in special cases. In general, it does not hold as is seen from the following example.

Example 1. Let us control a plant P, whose transfer function is $1/(s-1)$, using a sampled-data control scheme and a continuous-time control scheme as Figure 1. The purpose of control is to minimize the effect that a sensor noise $d(t)$ gives to a plant output $y(t)$ in the sense of the $\mathcal{L}^2$-induced norm. In the sampled-data control system, we carry out this minimization by tuning a discrete-time controller K_{d} that has a time-invariant discrete-time state-space representation. On the other hand, in the continuous-time system, we minimize this effect by tuning a continuous-time controller K, which is described by a time-invariant continuous-time state-space representation. The symbols S_τ^{id} and H_τ^{zo} denote the ideal sampler and the zero-order hold, respectively, both having a sampling period $\tau > 0$. Moreover, R_τ is an anti-aliasing low-pass filter whose transfer function is $1/(\tau s + 1)$. It seems reasonable that the bandwidth of R_τ is taken proportionally to the Nyquist frequency π/τ because undesirable aliases that should be cut off appear mostly at higher frequencies than the Nyquist frequency.

Figure 2 shows simulation results, where the algorithm of [3] is used for evaluation of the best sampled-data control performance. The solid line stands for the best achievable performance of sampled-data control. It is observed that it does not converge to the best performance of continuous-time control even if the sampling period approaches zero. The broken line shows the result when R_τ is replaced by a filter R whose transfer function is $1/(s+1)$. In this case, the sampled-data control performance converges to the continuous-time control performance. □

Such a non-converging phenomenon makes us doubt our confidence in a sampled-data control scheme. Therefore, we have to consider when this non-converging phenomenon occurs and how can be avoided. Actually, there is another experimental result that the performance convergence is related not only to

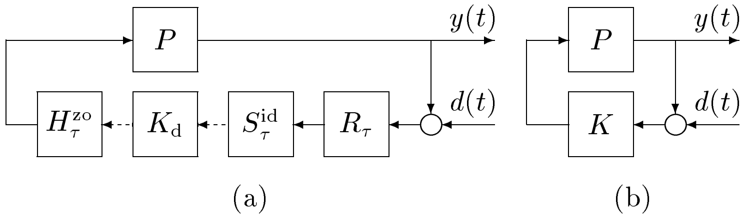

Figure 1. Two systems to control P: (a) a sampled-data control system; (b) a continuous-time control system. All the signs of the adders are positive.

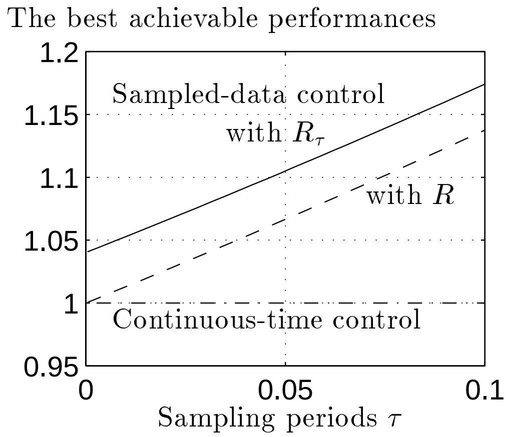

Figure 2. The best achievable performances measured by the $\mathcal{L}^2$-induced norms. The solid line indicates the best sampled-data control performance with R_τ being an anti-aliasing filter and the broken line shows the best sampled-data control performance with R being an anti-aliasing filter. The dot-dash line stands for the best continuous-time control performance.

anti-aliasing filters but also to generalized holds. However, these system components have never been investigated from this viewpoint.

This paper first identifies the set of plants for which we can make the best sampled-data control performance converge to the best continuous-time control performance by choosing an appropriate sequence of samplers and holds. Here, the $\mathcal{L}^2$-induced norm (or the $\mathcal{H}^\infty$-norm) is assumed to measure control performances. Then, this paper gives a necessary and sufficient condition in order that a provided sequence of samplers and holds guarantees convergence of the best sampled-data performance to the best continuous-time performance for any plant in the above set. This condition is split into a condition on holds and a condition on samplers and these two conditions are dual to each other. By noting relationships to the 2-block problem in the $\mathcal{H}^\infty$-control theory [15], we can obtain another necessary and sufficient condition, which is easier to be tested. Moreover, in the case that kernel functions of samplers and holds have special structures, the condition can be simplified more.

The non-converging phenomenon such as observed in Example 1 was first reported and theoretically analyzed by Oishi [30, 31]. The contents of the present paper have partially appeared in [32, 33]. Recently, Hara *et al.* reported several interesting simulation results on sampled-data systems with small sampling periods, which include our non-converging phenomenon [19].

Section 2 is devoted to presenting a framework for sampled-data control systems and deriving their properties. Then, in Section 3, convergence of the best sampled-data performance is investigated. Section 4 concludes this paper.

The notation used in this paper is as follows. The imaginary unit is denoted by i. We write $\mathbb{C} \cup \{\infty\}$ as $\mathbb{C}_{\mathrm{e}}$ and the set $\{ z \mid |z| > \rho \text{ or } z = \infty \}$ as $\mathbb{D}_\rho$ for $\rho > 0$. Especially, $\mathbb{D}_1$ is simply written as $\mathbb{D}$. The complex conjugate of A is denoted by $\overline{A}$. When A is a matrix or a vector, conjugate is taken componentwise. The maximum singular value of a matrix A is expressed by $\overline{\sigma}(A)$. The asterisk (*) stands for a complex-conjugate-transpose matrix or an adjoint operator. The symbol ℓ^2 denotes the set of square-summable one-sided sequences. We let $\mathcal{L}^2$ and $\mathcal{L}^2[0,\tau)$ express the sets of Lebesgue-square-integrable functions defined on $[0,\infty)$ and on $[0,\tau)$, respectively. The elements of ℓ^2, $\mathcal{L}^2$, and $\mathcal{L}^2[0,\tau)$ may be vector-valued functions. When there is a fear of confusion, the dimension of these spaces is explicitly described as $(\ell^2)^n$. In general, the norm of a space X is written as $\|\cdot\|_X$. The symbol $\|\cdot\|$ denotes the $\mathcal{L}^2$-induced norm unless specified in other way. Let $\mathcal{H}^\infty$ be the Hardy space composed of functions analytic and bounded in $\operatorname{Re} s > 0$. (For ease of distinction, the Hardy space that will be defined later regarding $\mathbb{D}$ in place of $\operatorname{Re} s > 0$ is denoted by $\boldsymbol{H}^\infty$ with a bold italic font.) An element of $\mathcal{H}^\infty$ may be a matrix-valued function. The norm associated with $\mathcal{H}^\infty$ is defined as $\|Q\|_{\mathcal{H}^\infty} := \sup_{\operatorname{Re} s > 0} \overline{\sigma}\{Q(s)\}$. The symbol $\mathcal{RH}^\infty$ expresses the subset of $\mathcal{H}^\infty$ that consists of real rational functions only. For a time-invariant continuous-time operator P, its transfer function is defined based on the Laplace transform and is written as $\widehat{P}(s)$; for a time-invariant discrete-time operator P_{d}, its transfer function is written as $\breve{P}_{\mathrm{d}}(z)$, which is defined from the z-transform. Here, z corresponds to the unit-time advance operator. Finally, $\mathcal{F}(G,K)$ is the lower linear fractional transform.

2. A framework for sampled-data control systems

3. Convergence of sampled-data control performance

4. Conclusion

In the former half of this paper, a framework for sampled-data control systems is presented. Our framework is general enough to cover systems with various generalized samplers and holds and can be used for analysis of anti-aliasing filters as well. Furthermore, it has a good property of duality between samplers and holds. Although duality was presented also in the frameworks of [22, 28], our duality holds for a larger class of samplers and holds and appears in more aspects of the theory such as the kernel functions of samplers and holds, their state-space representations, the matrix representations of their transfer functions, and their conditions for convergence.

Based on our framework, established methodologies for analysis and synthesis of sampled-data control systems can be extended to more general system configuration. Moreover, this framework gives a solid basis to consider more advanced problems on sampled-data systems such as analysis and design of sampling periods, anti-aliasing filters, and generalized holds aiming at further improvement of control performance. Especially, notions of lifting-based transfer functions and their matrix representations by use of operators E_m^s and $\widetilde{E}_m^s$ are considered to be useful. An approximation lemma (Proposition **??**) shows a profound relationship between sampled-data control systems and continuous-time control systems. This lemma plays an important role not only in this paper but also in [34], where robust stability of sampled-data systems is considered.

The latter half of this paper is devoted to a convergence analysis of the best sampled-data control performance. First, this paper gives a theoretical bound that the sampled-data control performance can achieve when controllers and sampling environments can be chosen as free parameters. Then, it presents a necessary and sufficient condition in order that a provided sampling environment sequence ensures convergence to this theoretical bound for all plants. If we concentrate only on the plants such that recovery of the best continuous-time control performance is possible, the above condition is necessary and sufficient for convergence to the best continuous-time performance. This condition is made easier to be tested by use of techniques for the 2-block problem in the $\mathcal{H}^\infty$-control theory. For special types of samplers and holds, this condition is further simplified.

It has been believed to be obvious that the best sampled-data control performance converges to the best continuous-time control performance as the sampling period approaches zero. However, the results of this research tell us that this property is more delicate than it appears at first glance. For example, the hold $H_\tau^{\rm tr}$ is similar to the zero-order hold $H_\tau^{\rm zo}$ in the point that their kernel functions change their shapes proportionally to the sampling period. However, the former does not satisfy the condition for convergence whereas the latter does. Without knowing the condition (b) of Theorem **??**, we cannot see what is critically different in these two devices. On the other hand, our results suggest what is important for samplers and holds in order to improve control performance. Namely, it is better that kernel functions of samplers and holds do not have large spectra above the Nyquist frequency. Moreover, we have observed that Hankel norms related to samplers and holds are also important. These knowledges can be a clue to solve more advanced problems on sampled-data control systems mentioned above.

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