$\mathcal{NP}$-completeness for Calculating Power Indices of Weighted Majority Games

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Abstract: In this paper, we prove that both problems for calculating the Banzhaf power index and the Shapley-Shubik power index for weighted majority games are $\mathcal{NP}$-complete.

Keywords: weighted majority game, Banzhaf power index, Shapley-Shubik power index, $\mathcal{NP}$-complete, knapsack problem

1 Introduction

Weighted voting is frequently used when there is sufficient reason to create or maintain districts which have nontrivial variations in populations. To analyze weighted voting, there is a weighted majority game in the game theory. Banzhaf [1] introduced an index, which is called the Banzhaf power index, for measuring an individual's voting power. Another value concept for measuring voting power was introduced by Shapley and Shubik [8], which is called the Shapley-Shubik power index. The Shapley-Shubik power index is a special application of a more general value concept introduced by Shapley in [7].

In this paper, we prove that both problems for calculating the Banzhaf power index and the Shapley-Shubik power index for weighted majority games are $\mathcal{NP}$-complete.

2 Preliminaries

In this section, we give some definitions and notations. There are n players denoted by $\{1, \ldots, n\}$. The *weighted majority game* is a sequence of nonnegative integers $G = (q; w_1, w_2, \ldots, w_n)$ satisfying the condition that $w_i \geq 0$ and $(1/2)\sum_{i=1}^{n} w_i < q \leq \sum_{i=1}^{n} w_i$, where each w_i denotes the voting weight of player i and the integer q denotes the quota for the game.

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A *coalition* is a subset of players. A coalition S is called a *winning coalition* (respectively a *losing coalition*) when $\sum_{i \in S} w_i \geq q$ (respectively $\sum_{i \in S} w_i < q$).

For any coalition S of players, we say that player i is a *swing* with respect to S if and only if $(S, S \triangle \{i\})$ is a pair of a losing coalition and a winning coalition ($S_1 \triangle S_2$ denotes the symmetric difference of S_1 and S_2). The *raw Banzhaf power index* denotes the vector $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ such that β_i is equal to the number of coalitions for which player i is a swing. The *Banzhaf power index* is the vector $\beta^* = (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ defined by $\beta_i^* = \beta_i / \sum_{i=1}^n \beta_i$.

Given a permutation π defined on $\{1, 2, \ldots, n\}$, we denote $\pi(i)$ by π_i for each $i \in \{1, 2, \ldots, n\}$. For any permutation π on $\{1, 2, \ldots, n\}$, we say that player π_j is the *pivot player* with respect to π if and only if the coalition $S = \{\pi_1, \pi_2, \ldots, \pi_{j-1}\}$ satisfies that S is losing and $S \cup \{\pi_j\}$ is winning. The *raw Shapley-Shubik power index* denotes the vector $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ such that φ_i is equal to the number of permutations defined on the set of players for which player i is the pivot player. The *Shapley-Shubik power index* is the vector $\varphi^* = (\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*)$ defined by $\varphi_i^* = \varphi_i / n!$.

If we calculate the Banzhaf power index conforming to an algorithm by the definition, then the algorithm requires $O(2^n n)$time. Similarly, a naive algorithm for calculating the Shapley-Shubik power index requires $O(n!n)$time. In 1982, Lucas, Maceli, Hillicard and Housman [5] proposed a pseudo polynomial time algorithm which calculates both the Banzhaf power index and the Shapley-Shubik power index simultaneously.

3 Banzhaf index

We discuss the problem for calculating the Banzhaf power index.

BZ1

INSTANCE: A positive integer n and a sequence of nonnegative integers $(q; w_1, \ldots, w_n)$ satisfying $(1/2) \sum_{i=1}^n w_i < q \leq \sum_{i=1}^n w_i$ and $w_1 \geq w_2 \geq \cdots \geq w_n$.

QUESTION: Does the raw Banzhaf power index $(\beta_1, \ldots, \beta_n)$ of the weighted majority game $G = (q; w_1, \ldots, w_n)$ satisfy $\beta_n > 0$?

We prove $\mathcal{NP}$-completeness of **BZ**1 by presenting a polynomial time reduction from the knapsack problem (**KP**), which is a well-known $\mathcal{NP}$-complete problem [3, 4].

KP

INSTANCE: A positive integer k and a sequence of positive integers $(a_1, \ldots, a_k)$ satisfying that $(1/2) \sum_{i=1}^k a_i$ is an integer.

QUESTION: Is there a subset $S \subseteq \{1, 2, \ldots, k\}$ such that $\sum_{i \in S} a_i = (1/2) \sum_{i=1}^k a_i$?

Theorem 1 **BZ**1 *is* $\mathcal{NP}$*-complete.*

Proof. If problem **BZ**1 has *YES* answer, then there exists a coalition for which player n is a swing. The coalition becomes a polynomial size certificate and so problem **BZ**1 is in the class $\mathcal{NP}$.

Given a problem instance of **KP**, we construct a problem instance of **BZ**1 as follows. We put $n = k+1$, $q = (1/2)\sum_{i=1}^{k} a_i + 1$ and

$$w_i = \begin{cases} a_i & (i = 1, 2, \ldots, n-1), \\ 1 & (i = n). \end{cases}$$

The above definitions imply that the quota q is an integer satisfying

$$q = (1/2)\sum_{i=1}^{k} a_i + 1 = (1/2)\sum_{i=1}^{n-1} w_i + 1 = (1/2)\sum_{i=1}^{n} w_i + (1/2) > (1/2)\sum_{i=1}^{n} w_i$$

and so $G = (q; w_1, \ldots, w_n)$ becomes a weighted majority game.

Assume that $\beta_n > 0$. Then there exists a coalition S^* such that player n is a swing with respect to S^*. Without loss of generality, we can assume that S^* does not contain player n. Since n is a swing with respect to S^*,

$$\sum_{i \in S^*} a_i < q \leq \sum_{i \in S^*} a_i + a_n = \sum_{i \in S^*} a_i + 1.$$

The above inequalities and the integrality of weights imply that $\sum_{i \in S^*} a_i = q - 1 = (1/2)\sum_{i=1}^{k} a_i$, and so **KP** has *YES* answer.

Next, we consider the case that there exists a subset $S^* \subseteq \{1, 2, \ldots, n-1\}$ satisfying that $\sum_{i \in S^*} a_i = (1/2)\sum_{i=1}^{k} a_i = q - 1$. Then, it is clear that player n is a swing with respect to S^* and so $\beta_n > 0$. □

The above theorem directly implies the following.

Corollary 1 *Calculating the Banzhaf power index is $\mathcal{NP}$-hard.*

When we are interested in the players with large voting weights, we need to consider the following problem.

<u>BZ</u><u>2</u>

INSTANCE: A positive integer n and a sequence of nonnegative integers $(q; w_1, \ldots, w_n)$ satisfying $(1/2)\sum_{i=1}^{n} w_i < q \leq \sum_{i=1}^{n} w_i$ and $w_1 \geq w_2 \geq \cdots \geq w_n$.

QUESTION: Does the raw Banzhaf power index $(\beta_1, \ldots, \beta_n)$ of the weighted majority game $G = (q; w_1, \ldots, w_n)$ satisfy $\beta_1 > \beta_2$?

Theorem 2 **BZ**2 *is $\mathcal{NP}$-complete.*

Proof. For any coalition S, we define the coalition $\overline{S}$ as follows;

$$\overline{S} = \begin{cases} S & (|\{1,2\} \cap S| = 0), \\ S \triangle \{1,2\} & (|\{1,2\} \cap S| = 1), \\ S & (|\{1,2\} \cap S| = 2). \end{cases}$$

Clearly from the definition, $\overline{(\overline{S})} = S$. We can show easily that if player 2 is a swing with respect to S, then player 1 is a swing with respect to $\overline{S}$. It implies that when $\beta_1 > \beta_2$, there exists a coalition S^* such that player 2 is not a swing with respect to S^* and player 1 is a swing with respect to $\overline{S^*}$. Then the coalition S^* becomes a polynomial size certificate and so **BZ**2 is in the class $\mathcal{NP}$.

To show the $\mathcal{NP}$-completeness, we construct the following weighted majority game G' from a problem instance of **KP**. We assume that $a_1 \geq a_2 \geq \cdots \geq a_k$. Then we put $n = k+2$,

$$w_i = \begin{cases} \sum_{i=1}^{k} a_i + 1 & (i = 1), \\ \sum_{i=1}^{k} a_i & (i = 2), \\ a_{i-2} & (i = 3, 4, \ldots, n), \end{cases}$$

and $q = (3/2)\sum_{i=1}^{k} a_i + 1$. Clearly from the definition, $G' = (q; w_1, \ldots, w_n)$ becomes a weighted majority game. Then it is easy to show that $\beta_1 > \beta_2$ if and only if **KP** has *YES* answer. □

The above theorem implies that it is hard to calculate the Banzhaf power index even if we restrict to the players with large index values. Since $\beta_1 \geq 1/n$, we can decide whether $\beta_1 > \beta_2$ by calculating all the elements of the Banzhaf power index satisfying that corresponding values are greater than or equal to $1/n$. Thus, the problem for calculating all the elements of the Banzhaf power index satisfying that corresponding values are greater than or equal to $1/n$ is $\mathcal{NP}$-hard.

4 Shapley-Shubik index

We consider the following problem.

<u>**SS**1</u>

INSTANCE: A positive integer n and a sequence of nonnegative integers $(q; w_1, \ldots, w_n)$ satisfying $(1/2)\sum_{i=1}^{n} w_i < q \leq \sum_{i=1}^{n} w_i$ and $w_1 \geq w_2 \geq \cdots \geq w_n$.

QUESTION: Does the raw Shapley-Shubik power index $(\varphi_1, \ldots, \varphi_n)$ of the weighted majority game $G = (q; w_1, \ldots, w_n)$ satisfy $\varphi_n > 0$?

We prove $\mathcal{NP}$-hardness of **SS** by presenting a polynomial time reduction from problem **KP** described in the previous section.

Theorem 3 **SS**1 *is* $\mathcal{NP}$*-complete.*

Proof. Assume that problem **SS**1 has *YES* answer. Then there exists a permutation for which player n is the pivot player. The permutation becomes a polynomial size certificate and so problem **SS**1 is in the class $\mathcal{NP}$.

For any problem instance of **KP**, we construct the weighted majority game G with $n = k+1$ players defined in the proof of Theorem 1.

Assume that $\varphi_n > 0$. Then there exists a permutation π^* such that player n is the pivot player with respect to π^*. Let $S = \{\pi^*_1, \pi^*_2, \ldots, \pi^*_{i-1}\}$ where i is the integer satisfying $\pi^*_i = n$. Then the equality $\sum_{i \in S} a_i = a_{\pi^*_1} + a_{\pi^*_2} + \cdots + a_{\pi^*_{i-1}}$ holds. Since n is the pivot player with respect to π^*,

$$\sum_{i \in S} a_i < q \leq \sum_{i \in S} a_i + a_n = \sum_{i \in S} a_i + 1.$$

The above inequalities and the integrality of weights imply that $\sum_{i \in S} a_i = q - 1 = (1/2)\sum_{i=1}^{k} a_i$, and so **KP** has *YES* answer.

Next, we consider the case that there exists a subset $S \subseteq \{1, 2, \ldots, k\}$ satisfying that $\sum_{i \in S} a_i = (1/2)\sum_{i=1}^{k} a_i$. Let π be a permutation satisfying the condition that there exists an integer i such that $\pi_i = n$ and $S = \{\pi_1, \pi_2, \ldots, \pi_{i-1}\}$. Then, it is clear that player $n = \pi_i$ is the pivot player with respect to π and so $\varphi_n > 0$. □

The above theorem directly implies the following.

Corollary 2 *Calculating the Shapley-Shubik power index is $\mathcal{NP}$-hard.*

When we are interested in the players with large voting weights, we need to consider the following problem.

SS2

INSTANCE: A positive integer n and a sequence of nonnegative integers $(q; w_1, \ldots, w_n)$ satisfying $(1/2)\sum_{i=1}^{n} w_i < q \leq \sum_{i=1}^{n} w_i$ and $w_1 \geq w_2 \geq \cdots \geq w_n$.

QUESTION: Does the raw Shapley-Shubik power index $(\varphi_1, \ldots, \varphi_n)$ of the weighted majority game $G = (q; w_1, \ldots, w_n)$ satisfy $\varphi_1 > \varphi_2$?

Theorem 4 **SS**2 *is $\mathcal{NP}$-complete.*

Proof. For any permutation π, $\overline{\pi}$ denotes the permutation obtained from π by exchanging the positions of player 1 and player 2. Clearly from the definition, $\overline{(\overline{\pi})} = \pi$. We can show easily that if player 2 is the pivot player with respect to π, then player 1 is the pivot player with respect to $\overline{\pi}$. It implies that when $\varphi_1 > \varphi_2$, there exists a permutation π^* such that player 2 is not the pivot player with respect to π^* and player 1 is the pivot player with respect to $\overline{\pi^*}$. Then the permutation π^* becomes a polynomial size certificate and so **SS**2 is in the class $\mathcal{NP}$.

To show the $\mathcal{NP}$-completeness, we construct the weighted majority game G' defined in Theorem 2. Then it is easy to show that $\varphi_1 > \varphi_2$ if and only if **KP** has *YES* answer. □

The above corollary implies that it is hard to calculate the Shapley-Shubik power index even if we restrict to the players with large index values. The problem for calculating all the elements of the Shapley-Shubik power index satisfying that corresponding values are greater than or equal to $1/n$ is also $\mathcal{NP}$-hard.

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