An Interpolant Based on Line Segment Voronoi Diagrams

Hisamoto Hiyoshi and Kokichi Sugihara

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Department of Mathematical Engineering and Information Physics,
Graduate School of Engineering, University of Tokyo.
Hisamoto Hiyoshi*and Kokichi Sugihara†

Abstract

This paper considers the interpolation for multi-dimensional data using Voronoi diagrams. Sibson's interpolant is well-known as an interpolation method using Voronoi diagrams for discretely distributed data, and it is extended to continuously distributed data by Gross. On the other hand, the authors studied another interpolation method using Voronoi diagrams recently. This paper outlines the authors' interpolant briefly, and extends the author's interpolant to linearly distributed data based upon the discussion using integrations.

1 Introduction

Let z be a function defined over a set D in the d-dimensional Euclidean space $\mathbb{R}^d$. Suppose that a subset G of D is given, and the values of $z|_G$ are known. Then, the problem to estimate the values of $z|_{D-G}$ from our knowledge about $z|_G$ is called the *interpolation problem*. Each point of G is called a *data site*.

When G is a set consisting of a finite number of points in $\mathbb{R}^1$, the problem is rather simple and a lot of interpolation methods are known, e.g., Lagrange interpolation. However, the problem becomes more difficult when $d > 1$. The finite element method is well-known and very practical to solve the interpolation problem in higher dimensions. On the other hand, there is another approach towards this difficulty, which utilizes Voronoi diagrams.

To see the usability of Voronoi diagrams, recall the piecewise linear interpolant. In order to estimate the value at some point, this scheme uses the values at the left-next data site and the right-next data site of the target point. In higher dimensions, the estimation of the value at some point can be made using the values of the "next" data sites if we can decide which data sites are the "next"

*e-mail: `hiyoshi@simplex.t.u-tokyo.ac.jp`
†e-mail: `sugihara@simplex.t.u-tokyo.ac.jp`

data sites of the target point. The Voronoi diagram can decide the "next" data sites because the Voronoi diagram is the partition of $\mathbb{R}^d$ by the nearest-neighbour rule.

Thiessen proposed the first interpolant using Voronoi diagrams when G consists of a finite number of points in $\mathbb{R}^d$, but his interpolant is not continuous [8]. Sibson made a great progress by proposing an interpolant using Voronoi diagrams, which is globally continuous and continuously differentiable almost everywhere [6]. The property of his method was researched by Farin [1] and Piper [5], and was extended to a more general case where G consists of a finite number of curves such as circles and polygonal curves [2].

Recently, the authors studied another interpolant using Voronoi diagrams when G consists of n points in $\mathbb{R}^2$ [3,7]. In this paper, the authors' interpolant is extended to the case where G consists of a finite number of line segments. The resulting interpolant is based on line segment Voronoi diagrams.

In Sect. 2, we outline the authors' interpolant. In Sect. 3, we extend the authors' interpolant to linearly distributed data. In Sect. 4, we conclude our research.

2 Interpolation for Discretely Distributed Data

First let us outline the authors' interpolant for discretely distributed data. Let $d(\boldsymbol{p}, \boldsymbol{q})$ denote the Euclidean distance between two points $\boldsymbol{p}$ and $\boldsymbol{q}$, $|A|$ denote the length of a curve A, and $\mathrm{CH}(P)$ denote the convex hull of a set P.

The authors' interpolant of this version solves the interpolation problem when $G = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ for $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n \in \mathbb{R}^2$. This means that the values of the function z at the points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are known. Suppose that we want to estimate the value at the target point $\boldsymbol{p}$ which is in $\mathrm{CH}(G)$ but does not belong to G.

In order to estimate the value at $\boldsymbol{p}$, we utilize the Voronoi diagram V for the generator set $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{p}\}$. In the Voronoi diagram V, the Voronoi region of $\boldsymbol{p}$ is a polygon each edge of which is a part of the bisector of the line segment $\overline{\boldsymbol{p}\boldsymbol{p}_i}$ for some $\boldsymbol{p}_i$. Let $e_1, \ldots, e_k$ be the Voronoi edges surrounding the Voronoi region of $\boldsymbol{p}$, and $\boldsymbol{p}_{\iota(1)}, \ldots, \boldsymbol{p}_{\iota(k)}$ denote the generators generating $e_1, \ldots, e_k$, respectively. Now define that

$$\alpha_i = \frac{\sigma_i}{d_i} \ ,$$

where

$$\sigma_i = |e_i| \ ,$$

and

$$d_i = d(\boldsymbol{p}, \boldsymbol{p}_{\iota(i)})$$

for $i = 1, \ldots, k$. Then, the following theorem holds:

Theorem 1 *In the above notations, the following identity holds wherever* $\boldsymbol{p}$ *lies in* $\mathrm{CH}(G) - G$*:*

$$\sum_{i=1}^{k} \alpha_i \boldsymbol{p} = \sum_{i=1}^{k} \alpha_i \boldsymbol{p}_{\iota(i)} \quad . \tag{1}$$

Proof. Refer to Theorem 1 in Sugihara [7]. ■

Equation (1) implies that the point $\boldsymbol{p}$ can be expressed as the convex combination

$$\boldsymbol{p} = \sum_{i=1}^{k} \beta_i \boldsymbol{p}_{\iota(i)} \quad , \tag{2}$$

where

$$\beta_i = \frac{\alpha_i}{\sum_{j=1}^{k} \alpha_j} \quad .$$

The authors' interpolant $\tilde{z}$ is obtained by replacing $\boldsymbol{p}_{\iota(i)}$ by $z(\boldsymbol{p}_{\iota(i)})$ in (2):

$$\tilde{z}(\boldsymbol{p}) = \sum_{i=1}^{k} \beta_i z(\boldsymbol{p}_{\iota(i)}) \quad . \tag{3}$$

Clearly, the function $\tilde{z}$ is continuous and $\tilde{z}(\boldsymbol{p}) \to z(\boldsymbol{p}_i)$ as $\boldsymbol{p} \to \boldsymbol{p}_i$.

Figure 1 shows the obtained function using the author's interpolant when $z = \exp(-(x^2 + y^2)/2)$. Each point in G was selected randomly in $[-1, 1]^2$. The vertical lines denote where the points in G are located. The length of each vertical line denotes the value of z at the lower endpoint of that line.

3 Extension to Linearly Distributed Data

3.1 Strategy

In this section, we extend the authors' interpolant to linearly distributed data. Suppose that the values of z are given over $G = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n^{\mathrm{p}}}\} \cup \lambda_1 \cup \cdots \cup \lambda_{n^{\mathrm{l}}} \subset \mathbb{R}^2$, where $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n^{\mathrm{p}}}$ are points and $\lambda_1, \ldots, \lambda_{n^{\mathrm{l}}}$ are open line segments. Assume that $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n^{\mathrm{p}}}, \lambda_1, \ldots, \lambda_{n^{\mathrm{l}}}$ are disjoint, but the endpoints of each λ_i are two of $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n^{\mathrm{p}}}$. Figure 2 describes an allowable set as the set G.

Now let us describe the strategy for obtaining the extension. At first, each λ_i is replaced by N_i equidistant points $\boldsymbol{q}_{i1}, \ldots, \boldsymbol{q}_{iN_i}$ on λ_i, so that the problem

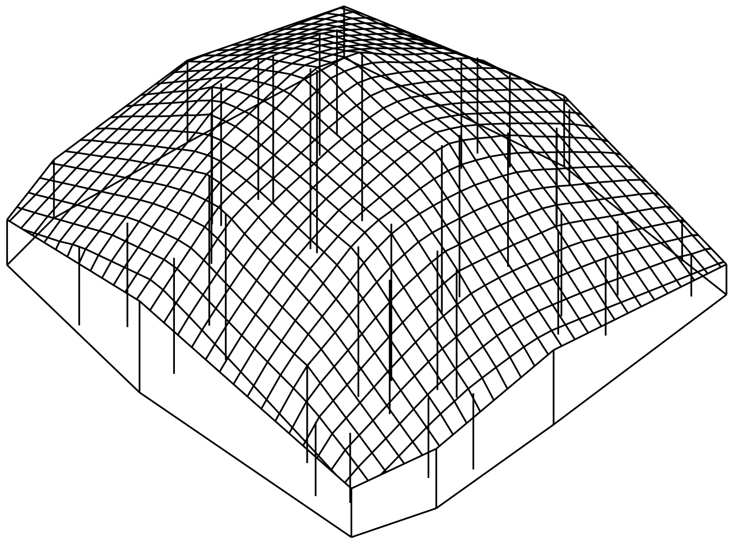

Figure 1. The obtained surface using the authors' interpolant for discretely distributed data

is discretized. Thus, the interpolant given in the previous section is available. Then, the interpolant coincides exactly with z at $\boldsymbol{p}_i$, $i = 1, \ldots, n^{\mathrm{p}}$, and $\boldsymbol{q}_{ij}$, $j = 1, \ldots, N_i$, $i = 1, \ldots, n^{\mathrm{l}}$. Now increase every N_i to the infinity. Then, the obtained interpolant coincides exactly with z at every point on G. The form of the interpolant is expressed using integrations.

3.2 Result

This subsection gives the result of the strategy given in the previous subsection in terms of (decomposed) line segment Voronoi diagrams [4]. Consider the line segment Voronoi diagram for the generator set $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n^{\mathrm{p}}}, \lambda_1, \ldots, \lambda_{n^{\mathrm{l}}}, \boldsymbol{p}\}$. Then, every Voronoi edge surrounding the Voronoi region of $\boldsymbol{p}$ is either

1. a line segment contained in the bisector of the line segment $\overline{\boldsymbol{p}\boldsymbol{p}_i}$ with some $\boldsymbol{p}_i$, or

2. a parabolic arc contained in the parabola whose focus is $\boldsymbol{p}$ and whose directrix is the line containing some λ_i.

Let $e^{\mathrm{p}}_1, \ldots, e^{\mathrm{p}}_{k^{\mathrm{p}}}$ denote the line segments surrounding the Voronoi region of $\boldsymbol{p}$, and $\boldsymbol{p}_{\iota^{\mathrm{p}}(1)}, \ldots, \boldsymbol{p}_{\iota^{\mathrm{p}}(k^{\mathrm{p}})}$ denote the generators generating $e^{\mathrm{p}}_1, \ldots, e^{\mathrm{p}}_{k^{\mathrm{p}}}$, respectively.

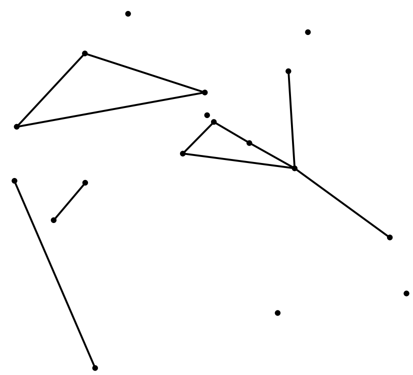

Figure 2. An example of an allowable set as the region where data are given

Moreover, let $e^{\mathrm{l}}_1, \ldots, e^{\mathrm{l}}_{k^{\mathrm{l}}}$ denote the parabolic arcs surrounding the Voronoi region of $\boldsymbol{p}$, and $\lambda_{\iota^{\mathrm{l}}(1)}, \ldots, \lambda_{\iota^{\mathrm{l}}(k^{\mathrm{l}})}$ denote the generators generating $e^{\mathrm{l}}_1, \ldots, e^{\mathrm{l}}_{k^{\mathrm{l}}}$, respectively.

Figure 3 describes the above notations. In this figure, it is assumed that the values of the function z are given over the set $G = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_9\} \cup \lambda_1 \cup \lambda_2 \cup \lambda_3$. Therefore, the line segment Voronoi diagram V for the generator set $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_9, \lambda_1, \lambda_2, \lambda_3, \boldsymbol{p}\}$ is constructed. In V, the Voronoi region of $\boldsymbol{p}$ is surrounded by the line segments $e^{\mathrm{p}}_1, \ldots, e^{\mathrm{p}}_5$ and the parabolic arcs $e^{\mathrm{l}}_1, \ldots, e^{\mathrm{l}}_4$, so $k^{\mathrm{p}} = 5$ and $k^{\mathrm{l}} = 4$. Since the line segment e^{p}_1 is contained in the bisector of $\overline{\boldsymbol{p}\boldsymbol{p}_3}$ and so on,

$$\iota^{\mathrm{p}}(1) = 3, \quad \iota^{\mathrm{p}}(2) = 5, \quad \iota^{\mathrm{p}}(3) = 6, \quad \iota^{\mathrm{p}}(4) = 7, \quad \iota^{\mathrm{p}}(5) = 8,$$
$$\iota^{\mathrm{l}}(1) = 1, \quad , \iota^{\mathrm{l}}(2) = 2, \quad \iota^{\mathrm{l}}(3) = \iota^{\mathrm{l}}(4) = 3 \ .$$

For each line segment e^{p}_i, the discussion in the previous section is available to calculate the contribution of e^{p}_i to the interpolant. Therefore, define that

$$\alpha^{\mathrm{p}}_i = \frac{\sigma^{\mathrm{p}}_i}{d^{\mathrm{p}}_i} \ ,$$

where

$$\sigma^{\mathrm{p}}_i = |e^{\mathrm{p}}_i| \ ,$$

and

$$d^{\mathrm{p}}_i = d(\boldsymbol{p}, \boldsymbol{p}_{\iota^{\mathrm{p}}(i)})$$

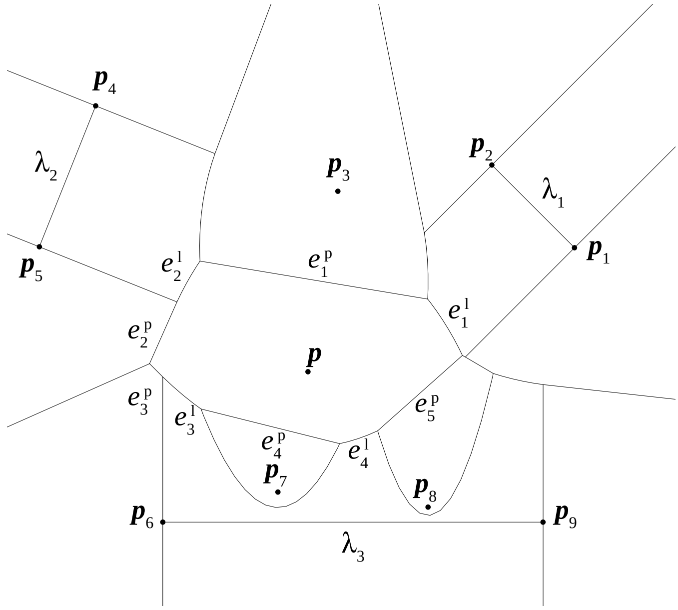

Figure 3. The Voronoi region of the target point

for $i = 1, \ldots, k^{\mathrm{p}}$.

In the following, we will concentrate on calculating the contribution of each parabolic arc e_i^{l} to the interpolant. Let d_i^{l} be the minimum Euclidean distance between $\boldsymbol{p}$ and any point $\boldsymbol{q}$ on the line containing $\lambda_{\iota^{\mathrm{l}}(i)}$, and l_i be the image of e_i^{l} by the projection onto the line containing $\lambda_{\iota^{\mathrm{l}}(i)}$.

For this purpose, we choose the coordinate system such that $\lambda_{\iota^{\mathrm{l}}(i)}$ is on the x-axis, and the coordinates of the target point $\boldsymbol{p}$ are $(0, d_i^{\mathrm{l}})$. Then, the parabola containing the parabolic arc e_i^{l} is written by

$$y = h_i(x) \equiv \frac{1}{2d_i^{\mathrm{l}}} x^2 + \frac{d_i^{\mathrm{l}}}{2} \ .$$

Let a_i and b_i be the x-coordinates of the left and right endpoints of e_i^{l}, respectively. Note that l_i is denoted by $\{(x, 0) \mid a_i < x < b_i\}$ in this coordinate system. Figure 4 describes the coordinate system associated with e_i^{l}.

Suppose that we discretize the line segment $l_i = \{(x, 0) \mid a_i < x < b_i\}$ into N_i equidistant points $\boldsymbol{q}_{i1}, \ldots, \boldsymbol{q}_{iN_i}$, where the coordinates of $\boldsymbol{q}_{ij}$ are $(q_{ij}, 0)$ with $q_{ij} = a_i + (j - 1/2)\Delta_i$ and $\Delta_i = (b_i - a_i)/N_i$. Then, we can interpret the point $\boldsymbol{q}_{ij}$ as the representative point of all the points lying between $(a_i + (j-1)\Delta_i, 0)$ and $(a_i + j\Delta_i, 0)$. The bisector b_{ij} of the line segment $\overline{\boldsymbol{p}\boldsymbol{q}_{ij}}$ is the tangent line of

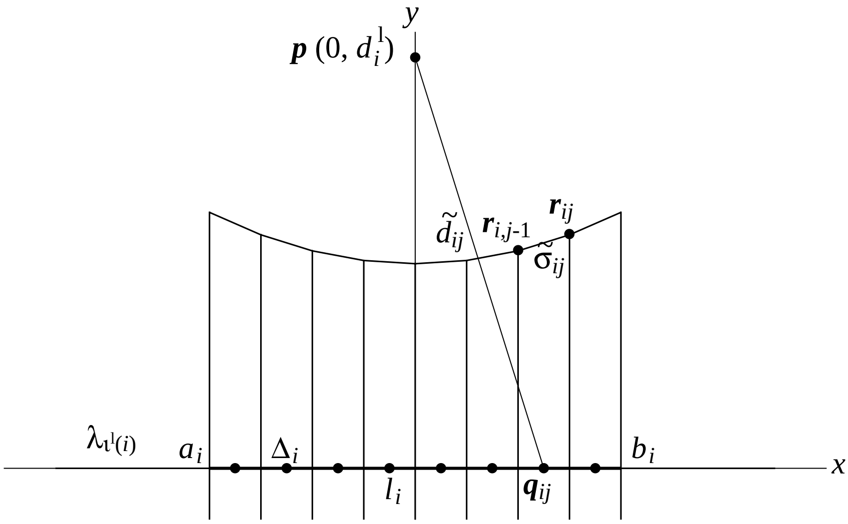

Figure 4. Used coordinate system

$y = h_i(x)$ at the point $(q_{ij}, h_i(q_{ij}))$. Hence, b_{ij} is written by

$$y = \tilde{h}_{ij}(x) \equiv h_i'(q_{ij})(x - q_{ij}) + h_i(q_{ij}) = \frac{q_{ij}}{d_i^1}x - \frac{q_{ij}^2}{2d_i^1} + \frac{d_i^1}{2} \ .$$

Define the piecewise linear function $\tilde{h}_i(x)$ by

$$\tilde{h}_i(x) = h_{ij}(x) \quad \text{for } a_i + (j-1)\Delta_i \le x \le a_i + j\Delta_i,\ 1 \le j \le N_i \ .$$

Note that the piecewise linear curve $y = \tilde{h}_i(x)$ converges to the parabolic arc $y = h_i(x)$, $a_i \le x \le b_i$, as $N_i \to \infty$. Let $\boldsymbol{r}_{ij}$, $0 \le j \le N_i$, be the points whose coordinates are $(a_i + j\Delta_i, \tilde{h}_i(a_i + j\Delta_i))$, respectively. $\boldsymbol{r}_{ij}$ is the Voronoi vertex generated by $\boldsymbol{p}$, $\boldsymbol{q}_{ij}$ and $\boldsymbol{q}_{i,j+1}$.

From the discussion in the previous section, the contribution of the point $\boldsymbol{q}_{ij}$ to the interpolant of discrete version is

$$\tilde{\alpha}_{ij} = \frac{\tilde{\sigma}_{ij}}{\tilde{d}_{ij}} \ ,$$

where

$$\tilde{\sigma}_{ij} = d(\boldsymbol{r}_{i,j-1}, \boldsymbol{r}_{ij}) \ ,$$

and

$$\tilde{d}_{ij} = d(\boldsymbol{q}_{ij}, \boldsymbol{p}_i) \ .$$

From the definition, we obtain that

$$\tilde{\sigma}_{ij} = d(\boldsymbol{r}_{i,j-1}, \boldsymbol{r}_{ij}) = \sqrt{1 + (h_i'(q_{ij}))^2}\,\Delta_i = \frac{\sqrt{q_{ij}^2 + (d_i^{\mathrm{l}})^2}}{d_i^{\mathrm{l}}}\Delta_i \ \ ,$$

and

$$\tilde{d}_{ij} = d(\boldsymbol{q}_{ij}, \boldsymbol{p}_i) = \sqrt{q_{ij}^2 + (d_i^{\mathrm{l}})^2} \ \ ,$$

which yields

$$\tilde{\alpha}_{ij} = \frac{1}{d_i^{\mathrm{l}}}\Delta_i \ \ .$$

Now we increase N_i to the infinity. Then we get the contribution of any point $\boldsymbol{q}$ on l_i to the interpolant as follows:

$$\alpha_i^{\mathrm{l}}(\boldsymbol{q})\mathrm{d}q = \frac{1}{d_i^{\mathrm{l}}}\mathrm{d}q \ \ .$$

Note that $\alpha_i^{\mathrm{l}}(\boldsymbol{q})$ is independent of the location of $\boldsymbol{q}$.

Now let us summarize the above discussion. The resulting identity corresponding to (1) is

$$\sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} \boldsymbol{p} + \sum_{i=1}^{k^{\mathrm{l}}} \int_{l^i} \alpha_i^{\mathrm{l}}(\boldsymbol{q})\boldsymbol{p}|\mathrm{d}\boldsymbol{q}| = \sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} \boldsymbol{p}_{\iota^{\mathrm{p}}(i)} + \sum_{i=1}^{k^{\mathrm{l}}} \int_{l^i} \alpha_i^{\mathrm{l}}(\boldsymbol{q})\boldsymbol{q}|\mathrm{d}\boldsymbol{q}| \ \ ,$$

and the resulting interpolant is

$$\tilde{z}(\boldsymbol{p}) = \frac{\displaystyle\sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} z(\boldsymbol{p}_{\iota^{\mathrm{p}}(i)}) + \sum_{i=1}^{k^{\mathrm{l}}} \int_{l^i} \alpha_i^{\mathrm{l}}(\boldsymbol{q}) z(\boldsymbol{q})|\mathrm{d}\boldsymbol{q}|}{\displaystyle\sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} + \sum_{i=1}^{k^{\mathrm{l}}} \int_{l^i} \alpha_i^{\mathrm{l}}(\boldsymbol{q})|\mathrm{d}\boldsymbol{q}|} \ \ .$$

Here, from the fact that $\alpha_i^{\mathrm{l}}(\boldsymbol{q})$ is independent of the location of $\boldsymbol{q}$, we obtain the identity

$$\sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} \boldsymbol{p} + \sum_{i=1}^{k^{\mathrm{l}}} \frac{|l_i|}{d_i^{\mathrm{l}}} \boldsymbol{p} = \sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} \boldsymbol{p}_{\iota^{\mathrm{p}}(i)} + \sum_{i=1}^{k^{\mathrm{l}}} \frac{1}{d_i^{\mathrm{l}}} \int_{l^i} \boldsymbol{q}|\mathrm{d}\boldsymbol{q}| \ \ , \tag{4}$$

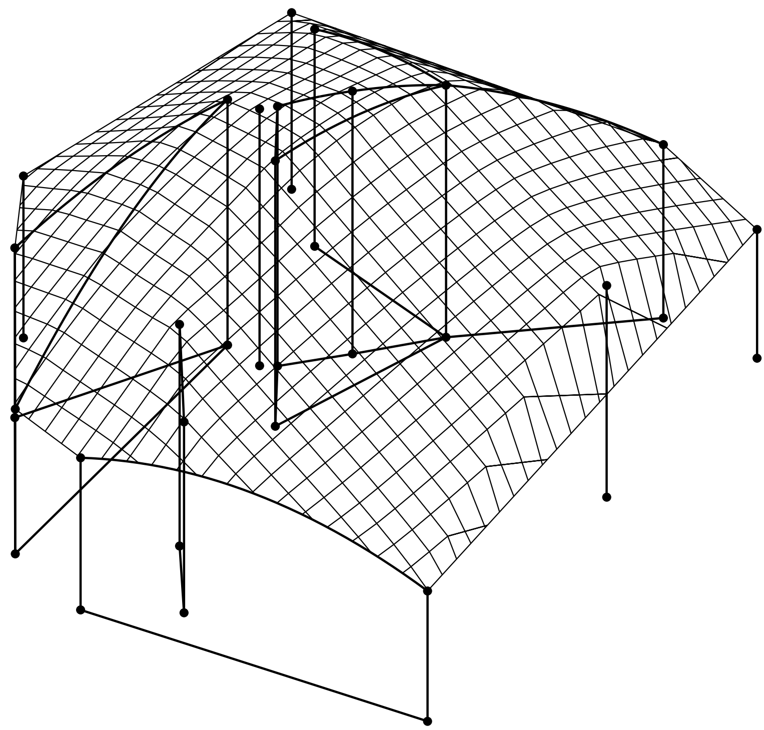

Figure 5. The obtained function using the authors' interpolant of continuous version

and the interpolant

$$\tilde{z}(\boldsymbol{p}) = \frac{\displaystyle\sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} z(\boldsymbol{p}_{\iota^{\mathrm{P}}(i)}) + \sum_{i=1}^{k^{\mathrm{l}}} \frac{1}{d_i^{\mathrm{l}}} \int_{l^i} z(\boldsymbol{q}) |\mathrm{d}\boldsymbol{q}|}{\displaystyle\sum_{i=1}^{k^{\mathrm{p}}} \alpha_i^{\mathrm{p}} + \sum_{i=1}^{k^{\mathrm{l}}} \frac{|l^i|}{d_i^{\mathrm{l}}}} \quad . \tag{5}$$

From the form of (5), we see that the interpolant requires only the integrations of the given data. Therefore, if every primitive function of $z|_{\lambda_i}$ is known, then computing the interpolant does not require numerical integrations.

Figure 5 shows the obtained function using the author's interpolant of continuous version when $z = 1 - (x^2 + y^2)/2$. The set shown in Fig. 2 was used as G.

4 Concluding Remarks

We first outlined the authors' interpolant for discretely distributed data briefly. Next, we extended the authors' interpolant to linearly distributed data according to the discussion using the limitation of the interpolant of discrete version. The obtained interpolant is expressed in terms of line segment Voronoi diagrams. If every given data function has a primitive function, computing the obtained interpolant does not require numerical integrations.

The following are directions of our future research:

1. Comparison of the authors' interpolant with Sibson's interpolant in both of discrete version and of continuous version.

2. Utilizing numerically disturbed Voronoi diagrams constructed by topology-oriented algorithms.

3. Application of the interpolants using Voronoi diagrams. Design of surfaces may be one of the potential applications.

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References

[1] Farin, G.: Surfaces over Dirichlet tessellations. Computer Aided Geometric Design **7** (1990) 281–292.

[2] Gross, L. M.: Transfinite Surface Interpolation over Voronoi Diagrams. PhD thesis, Arizona State University, 1995.

[3] Hiyoshi, H., Sugihara K.: Another interpolant using Voronoi diagrams. IPSJ SIG Notes 98-AL-62 (1998) 33–40. In Japanese.

[4] Okabe, A., Boots, B., Sugihara, K.: Spatial Tessellations: Concepts and Applications of Voronoi Diagrams. John Wiley & Sons, Chichester, 1992.

[5] Piper, B.: Properties of local coordinates based on Dirichlet tessellations. Farin, G., Hagen, H., Noltemeier., H., eds., Geometric Modelling, Computing Suppl. **8**, Springer-Verlag, Wien, 1993, 227–239.

[6] Sibson, R.: A vector identity for the Dirichlet tessellation. Math. Proc. Camb. Phil. Soc. **87** (1980) 151–155.

[7] Sugihara, K.: A new local coordinate system based on the Voronoi diagram. Proceedings of Differential/Topological Techniques in Geometric Modeling and Processing '98 (1998) 27–41.

[8] Thiessen., A. H.: Precipitation averages for large areas. Monthly Weather Report **39** (1911) 1082–1084.