

0.863-Approximation Algorithm for MAX DICUT

Shiro Matuura¹ and Tomomi Matsui²

¹ Graduate School of Information Science and Technology,
University of Tokyo, Tokyo, Japan
shiro@misojiro.t.u-tokyo.ac.jp

² Graduate School of Information Science and Technology,
University of Tokyo, Tokyo, Japan
tomomi@misojiro.t.u-tokyo.ac.jp

Abstract

In this paper, we propose 0.863-approximation algorithm for MAX DICUT. The approximation ratio is better than the previously known result by Zwick, which is equal to 0.8596434254.

The algorithm solves the SDP relaxation problem proposed by Goemans and Williamson for the first time. We do not use the ‘rotation’ technique proposed by Feige and Goemans. We improve the approximation ratio by using hyperplane separation technique with skewed distribution function on the sphere. We introduce a class of skewed distribution functions defined on the 2-dimensional sphere satisfying that for any function in the class, we can design a skewed distribution functions on any dimensional sphere without decreasing the approximation ratio. We also searched and found a good distribution function defined on the 2-dimensional sphere numerically.

1 Introduction

In this paper we propose an approximation algorithm for the optimization problem called MAX DICUT. Let $D = (V, A)$ be a complete directed graph with vertex set $V = \{1, 2, \dots, n\}$ and arc set $A = \{(i, j) \in V \times V \mid i \neq j\}$. For each arc $(i, j) \in A$, we associate the non-negative arc weight w_{ij} . An arc subset $A' \subseteq A$ is called a *dicut* if and only if there exists a vertex subset $U \subseteq V$ satisfying that $A' = \{(i, j) \in A \mid i \in U \text{ and } j \in V \setminus U\}$. The weight of an arc-subset A' is the sum total of the weights of arcs in A' . The MAX DICUT is the problem for finding a dicut which maximizes its weight. The

MAX DICUT is formulated as follows;

$$(DI) \text{ maximize } \sum_{i \in U, j \in V \setminus U} w_{ij} \text{ subject to } U \subseteq V.$$

This problem is NP-Hard and so there are some algorithms for finding an approximate solution. As known well, Goemans and Williamson [5] proposed a randomized polynomial time algorithm for MAX CUT, MAX 2SAT and MAX DICUT. Their algorithm is based on Semi-Definite Programming (SDP) relaxation and random hyperplane separation technique. The approximation ratio of their algorithm for MAX DICUT is 0.79607. More precisely, their algorithm finds a dicut whose weight is at least 0.79607 times the optimal value.

In the paper [4], Feige and Goemans proposed an approximation algorithm for MAX DICUT which achieves 0.859387 of approximation ratio. Their algorithm based on two ideas. First, they added some constraints introduced by Feige and Lovász in [3] to SDP relaxation problem. Next, they proposed the ‘rotation’ technique which modifies the solution obtained by SDP relaxation. They calculated the approximation ratio of their algorithm numerically.

Recently, Zwick refined the rotation technique and proposed an algorithm whose approximation ratio is 0.8596434254 in [8]. He also showed that the approximation ratio of his algorithm almost completely matches upper bounds that we can obtain by any rotation technique.

As a related work, in the APPROX 2000, Ageev, Hassin and Sviridenko proposed an approximation algorithm for MAX DICUT with given sizes of parts [1].

In this paper, we propose an approximation algorithm without rotation technique whose approximation ratio is 0.863. Our algorithm solves the SDP relaxation problem by Goemans and Williamson with the constraints used in Feige and Goemans’ algorithm. We improve the approximation ratio by using hyperplane separation technique with skewed distribution function on the sphere. Although, the use of hyperplane separation technique with skewed distribution is suggested by Feige and Goemans in the paper [4], there is a non-trivial problem to design a good distribution function. More precisely, the performance of skewed distribution functions depends on the dimension of the corresponding sphere. First, we show a non-trivial relation between the skewed distribution functions on the 2-dimensional sphere and the n -dimensional sphere. We introduce a class of skewed distribution

functions defined on the 2-dimensional sphere satisfying that for any distribution function in the class, we can design a skewed distribution function defined on any dimensional sphere without decreasing the approximation ratio. Second, we searched and found a good distribution function on the 2-dimensional sphere numerically. By using the above results, we can design a good skewed distribution function on any dimensional sphere. It means that the distribution function of our algorithm changes with respect to the dimension of the corresponding sphere.

In Section 2, we review the SDP relaxation and hyperplane separation technique briefly. In Section 3, we describe the outline of our algorithm. In Section 4, we discuss some relations between the skewed distribution functions on the 2-dimensional sphere and the n -dimensional sphere. In Section 5, we describe a numerical method used for finding a good distribution function defined on the 2-dimensional sphere.

2 Semi-Definite Programming Relaxation

Here we describe a SDP relaxation of MAX DICUT and review the hyperplane separation technique. First, we formulate the MAX DICUT problem as an integer programming problem as follows;

$$\begin{aligned}
 \text{(DI')} \quad & \text{maximize} \quad (1/4) \sum_{(i,j) \in A} w_{ij}(1 + v_0 v_i - v_0 v_j - v_i v_j), \\
 & \text{subject to} \quad v_0 = 1, \quad v_i \in \{-1, 1\} \quad (\forall i \in V).
 \end{aligned}$$

The above problem is equivalent to the original problem (DI). In the paper [5], Goemans and Williamson relaxed this problem by replacing each variable $v_i \in \{-1, 1\}$ with a vector on the n -dimensional unit sphere $\mathbf{v}_i \in \mathcal{S}_n$ where $\mathcal{S}_n \stackrel{\text{def.}}{=} \{\mathbf{v} \in \mathbf{R}^{n+1} \mid \|\mathbf{v}\| = 1\}$. This relaxation is proposed by Lovász [6] originally. And We add some valid constraints used in papers [3, 4], and obtain the following relaxation problem;

$$\begin{aligned}
 \text{(\overline{DI})} \quad & \text{maximize} \quad (1/4) \sum_{(i,j) \in A} w_{ij}(1 + \mathbf{v}_0 \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_j - \mathbf{v}_i \cdot \mathbf{v}_j), \\
 & \text{subject to} \quad \mathbf{v}_0 = (1, 0, \dots, 0)^\top, \quad \mathbf{v}_i \in \mathcal{S}_n \quad (\forall i \in V) \\
 & \quad \mathbf{v}_0 \cdot \mathbf{v}_i + \mathbf{v}_0 \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{v}_j \geq -1 \quad (\forall (i, j) \in A), \\
 & \quad -\mathbf{v}_0 \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{v}_j \geq -1 \quad (\forall (i, j) \in A), \\
 & \quad -\mathbf{v}_0 \cdot \mathbf{v}_i + \mathbf{v}_0 \cdot \mathbf{v}_j - \mathbf{v}_i \cdot \mathbf{v}_j \geq -1 \quad (\forall (i, j) \in A).
 \end{aligned}$$

It is well-known that we can transform the above problem to semidefinite programming problem [5] and so we can solve the problem in polynomial time by using an interior point method [2, 7].

Next, we describe the hyperplane separation technique proposed by Goemans and Williamson. Let $(\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \dots, \bar{\mathbf{v}}_n)$ be an optimal solution of $\overline{\text{DI}}$. We generate a vector $\mathbf{r} \in \mathbf{S}_n$ uniformly and construct the vertex-subset $\overline{U} = \{i \in V \mid \text{sign}(\mathbf{r} \cdot \mathbf{v}_0) = \text{sign}(\mathbf{r} \cdot \bar{\mathbf{v}}_i)\}$ and the corresponding dicut $\overline{A} = \{(i, j) \in A \mid i \in U \text{ and } j \notin U\}$. We denote the expected weight of the dicut \overline{A} by $E(\overline{U})$. Then the linearity of the expectation implies that;

$$E(\overline{U}) = \sum_{(i,j) \in A} w_{ij} \frac{\arccos(\bar{\mathbf{v}}_i \cdot \bar{\mathbf{v}}_j) + \arccos(\mathbf{v}_0 \cdot \bar{\mathbf{v}}_j) - \arccos(\mathbf{v}_0 \cdot \bar{\mathbf{v}}_i)}{2\pi}.$$

Then we can estimate the approximation ratio of the algorithm by calculating α defined by;

$$\alpha \stackrel{\text{def.}}{=} \min_{(\mathbf{v}_i, \mathbf{v}_j) \in \Omega} \frac{(1/2\pi)(\arccos(\mathbf{v}_i \cdot \mathbf{v}_j) + \arccos(\mathbf{v}_0 \cdot \mathbf{v}_j) - \arccos(\mathbf{v}_0 \cdot \mathbf{v}_i))}{(1/4)(1 + \mathbf{v}_0 \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_j - \mathbf{v}_i \cdot \mathbf{v}_j)},$$

where

$$\Omega \stackrel{\text{def.}}{=} \left\{ (\mathbf{v}_i, \mathbf{v}_j) \in \mathbf{S}_2 \times \mathbf{S}_2 \left| \begin{array}{l} \mathbf{v}_0 \cdot \mathbf{v}_i + \mathbf{v}_0 \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{v}_j \geq -1, \\ -\mathbf{v}_0 \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{v}_j \geq -1, \\ -\mathbf{v}_0 \cdot \mathbf{v}_i + \mathbf{v}_0 \cdot \mathbf{v}_j - \mathbf{v}_i \cdot \mathbf{v}_j \geq -1 \end{array} \right. \right\},$$

and $\mathbf{v}_0 = (1, 0, 0)^\top$. Clearly, the following inequalities hold;

$$E(\overline{U}) \geq \alpha(\text{optimal value of } (\overline{\text{DI}})) \geq \alpha(\text{optimal value of } (\text{DI})).$$

So, the expected weight $E(\overline{U})$ is greater than or equal to α times the optimal value of (DI). It is known that $\alpha > 0.79607$ [5].

Feige and Goemans' algorithm solves the problem $(\overline{\text{DI}})$ and modifies the obtained optimal solution by using rotation technique. The approximation ratio is 0.859387. Zwick refined the rotation technique and proposed an algorithm whose approximation ratio is equal to 0.8596434254. Our algorithm does not use the rotation technique and so we will not describe the technique here.

3 Hyperplane Separation by Skewed Distribution on Sphere

Goemans and Williamson's algorithm generates a separating hyperplane at random. Our algorithm generates a separating hyperplane with respect to

a distribution function defined on \mathcal{S}_n which is skewed towards \mathbf{v}_0 but is uniform in any direction orthogonal to \mathbf{v}_0 . Given the n -dimensional sphere \mathcal{S}_n , we define the class of skewed distribution function \mathcal{F}_n by;

$$\mathcal{F}_n \stackrel{\text{def.}}{=} \left\{ f : \mathcal{S}_n \rightarrow [0, 1] \mid \begin{array}{l} \int_{\mathcal{S}_n} f(\mathbf{v}) \, ds = 1, \quad f(\mathbf{v}) = f(-\mathbf{v}) \quad (\forall \mathbf{v} \in \mathcal{S}_n), \\ [\mathbf{v}_0 \cdot \mathbf{v} = \mathbf{v}_0 \cdot \mathbf{v}' \rightarrow f(\mathbf{v}) = f(\mathbf{v}')] \quad (\forall \mathbf{v}, \forall \mathbf{v}' \in \mathcal{S}_n) \end{array} \right\}.$$

Let $f \in \mathcal{F}_n$ be a skewed distribution function defined on \mathcal{S}_n . Now consider the probability that arc (i, j) are contained in a dicut obtained by hyperplane separation technique based on f . For any pair $(\mathbf{v}_i, \mathbf{v}_j) \in \mathcal{S}_n$, we define

$$p(\mathbf{v}_i, \mathbf{v}_j \mid f) \stackrel{\text{def.}}{=} \Pr \left[\begin{array}{l} \text{sign}(\mathbf{r} \cdot \mathbf{v}_0) = \text{sign}(\mathbf{r} \cdot \mathbf{v}_i) \text{ and} \\ \text{sign}(\mathbf{r} \cdot \mathbf{v}_0) \neq \text{sign}(\mathbf{r} \cdot \mathbf{v}_j) \end{array} \right].$$

Then the expectation of the weight of the dicut with respect to a feasible solution $(\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \dots, \bar{\mathbf{v}}_n)$ of $\overline{\text{DI}}$ based on the distribution function f is $\sum_{(i,j) \in A} w_{ij} p(\bar{\mathbf{v}}_i, \bar{\mathbf{v}}_j \mid f)$.

When we use a skewed distribution function $f \in \mathcal{F}_n$ defined on \mathcal{S}_n , the approximation ratio can be estimated by the distribution function \hat{f} defined by projection of a vector on \mathcal{S}_n to the linear subspace spanned by $\{\mathbf{v}_0, \mathbf{v}_i, \mathbf{v}_j\}$. We define \hat{f} more precisely. Let H be the 3-dimensional linear subspace including $\{\mathbf{v}_0, \mathbf{v}_i, \mathbf{v}_j\}$. The distribution function $\hat{f} \in \mathcal{F}_2$ is defined as follows;

$$\hat{f}(\mathbf{v}') \stackrel{\text{def.}}{=} \int_{T(\mathbf{v}')} f(\mathbf{v}) \, ds,$$

where

$$T(\mathbf{v}') \stackrel{\text{def.}}{=} \{\mathbf{v} \in \mathcal{S}_n \mid \text{the projection of } \mathbf{v} \text{ to } H \text{ is parallel to } \mathbf{v}'\}.$$

Here we note that the distribution function \hat{f} is invariant with respect to the 3-dimensional subspace H including \mathbf{v}_0 , since f is uniform in any directions orthogonal to \mathbf{v}_0 . For any distribution function $f' \in \mathcal{F}_2$ we define

$$\alpha_{f'} \stackrel{\text{def.}}{=} \min_{(\mathbf{v}_i, \mathbf{v}_j) \in \Omega} \frac{p(\mathbf{v}_i, \mathbf{v}_j \mid f')}{(1/4)(1 + \mathbf{v}_0 \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_j - \mathbf{v}_i \cdot \mathbf{v}_j)},$$

here we note that $p(\mathbf{v}_i, \mathbf{v}_j \mid f')$ is defined on $S_2 = H \cap \mathcal{S}_n$. Then the approximation ratio of the algorithm using skewed distribution function $f \in \mathcal{F}_n$ is bounded by $\alpha_{\hat{f}}$ from below.

For constructing a good skewed distribution function, we need to find a function $f' \in \mathcal{F}_2$ such that the value $\alpha_{f'}$ is large. In Section 5, we describe a numerical method for finding a good skewed distribution function in \mathcal{F}_2 .

Even if we have a good distribution function in \mathcal{F}_2 , a non-trivial problem still remains. For applying hyperplane separation technique, we need a skewed distribution function on the n -dimensional sphere. However, when $n > 2$, not every distribution function $f' \in \mathcal{F}_2$ has a distribution function $f \in \mathcal{F}_n$ satisfying $\hat{f} = f'$. For example, it is easy to show that there does not exist any distribution function $f \in \mathcal{F}_3$ satisfying the conditions that

$$\hat{f}(\mathbf{v}) = \begin{cases} 1/(2\sqrt{2}\pi) & (-0.5 \leq \mathbf{v}_0 \cdot \mathbf{v} \leq 0.5), \\ 0 & (\text{otherwise}). \end{cases}$$

In Section 4, we propose a class of functions in \mathcal{F}_2 such that a corresponding skewed distribution function exists for any sphere \mathcal{S}_n with $n \geq 3$.

4 Main Theorem

For any function $f \in \mathcal{F}_n$, we can characterize f by the function $P_f : [0, \pi/2] \rightarrow [0, 1]$ defined by

$$P_f(\theta) \stackrel{\text{def.}}{=} f(\mathbf{v})|_{\cos \theta = |\mathbf{v}_0 \cdot \mathbf{v}|}.$$

The following theorem gives a class of permitted skewed distribution function in \mathcal{F}_n .

Theorem 1 *Let $f \in \mathcal{F}_n$ be a skewed distribution function with $n \geq 2$ satisfying*

$$P_f(\theta) = \frac{1}{a} \sum_{k=0}^{\infty} a_k \cos^k \theta.$$

Then the function $P_{\hat{f}}(\phi)$ can be described as

$$P_{\hat{f}}(\phi) = \frac{1}{a} \sum_{k=0}^{\infty} \frac{S^{(k+n)}(1)}{S^{(k+2)}(1)} a_k \cos^k \phi,$$

where a is a coefficient used for normalizing the total probability to 1 and $S^{(n)}(r)$ is the area of the n dimensional sphere whose radius is equal to r .

Proof. First, we notate some well-known formulae;

$$\Gamma(0) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(x+1) = x\Gamma(x),$$

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}, \quad S^n(r) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

When we fix ϕ and $d\phi$, we have the following;

$$\begin{aligned} & 2\pi \sin \phi P_{\hat{f}}(\phi) d\phi \\ &= \int_0^1 P_f(\arccos(r \cos \phi))(2\pi r \sin \phi) \left(S^{(n-3)}(\sqrt{1-r^2}) \right) \\ & \quad \left(\frac{\sqrt{1-r^2} \cos^2 \phi}{\sqrt{1-r^2}} r \frac{d\phi}{\cos \phi} \right) \left(\frac{\cos \phi}{\sqrt{1-r^2 \cos^2 \phi}} \right) dr. \end{aligned}$$

Thus we have

$$P_{\hat{f}}(\phi) = \int_0^1 P_f(\arccos(r \cos \phi)) S^{(n-3)}(\sqrt{1-r^2}) r^2 \frac{dr}{\sqrt{1-r^2}}.$$

When we replace r by $\sin \alpha$ and $P_f(\theta)$ by $(1/a) \sum_{k=0}^{\infty} a_k \cos^k \theta$, we can describe $P_{\hat{f}}(\phi)$ as;

$$\begin{aligned} P_{\hat{f}}(\phi) &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{a} \sum_{k=0}^{\infty} a_k \sin^k \alpha \cos^k \phi \right) \frac{2\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)} \cos^{n-3} \alpha \sin^2 \alpha \, d\alpha \\ &= \frac{1}{a} \sum_{k=0}^{\infty} \frac{2\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)} a_k \cos^k \phi \int_0^{\frac{\pi}{2}} \sin^{k+2} \alpha \cos^{n-3} \alpha \, d\alpha \\ &= \frac{1}{a} \sum_{k=0}^{\infty} \frac{2\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)} a_k \cos^k \phi \frac{\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{2\Gamma\left(\frac{n+k+1}{2}\right)} \\ &= \frac{1}{a} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k+3}{2}\right)}{2\pi^{\frac{k+3}{2}}} \frac{2\pi^{\frac{n+k+1}{2}}}{\Gamma\left(\frac{n+k+1}{2}\right)} a_k \cos^k \phi = \frac{1}{a} \sum_{k=0}^{\infty} \frac{S^{(k+n)}(1)}{S^{(k+2)}(1)} a_k \cos^k \phi. \end{aligned}$$

And so we have done. ■

The above theorem directly implies the following.

Corollary 1 *Let $f' \in \mathcal{F}_2$ be a distribution function satisfying*

$$P_{f'} = \frac{1}{b} \sum_{k=0}^{\infty} b_k \cos^k \phi$$

with the condition that $b_k \geq 0$. Then, for any $n \geq 2$, there exists a distribution function $f \in \mathcal{F}_n$ satisfying $\hat{f} = f'$ and

$$P_f = \frac{1}{b} \sum_{k=0}^{\infty} \frac{S^{(k+2)}(1)}{S^{(k+n)}(1)} b_k \cos^k \theta,$$

where b is a coefficient used for normalizing the total probability to 1.

The following theorem extends the class of tractable distribution functions.

Theorem 2 *Let $f \in \mathcal{F}_n$ be a distribution function satisfying that $P_f(\theta) = (1/a) \sum_{p \in X} a_p \cos^p \theta$ and X is a finite set of non-negative real numbers. Then the distribution function \hat{f} satisfies $P_{\hat{f}}(\phi) = (1/a) \sum_{p \in X} c_p a_p \cos^p \phi$,*

where a is a normalization coefficient and $c_p = \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} \int_0^{\frac{\pi}{2}} \sin^{p+2} \alpha \cos^{n-3} \alpha \, d\alpha$ for each $p \in X$.

Proof. We can prove in a similar way with the proof of Theorem 1 and so proof is omitted. ■

This theorem implies the following.

Corollary 2 *Let $f' \in \mathcal{F}_2$ be a distribution function satisfying $P_{f'}(\phi) = (1/b) \sum_{p \in X} b_p \cos^p \phi$ and $b_p \geq 0$ ($\forall p \in X$) where X is a finite set of positive real numbers. Then there exists a distribution function $f \in \mathcal{F}$ satisfying $\hat{f} = f'$ and $P_f(\theta) = (1/b) \sum_{p \in X} d_p b_p \cos^p \theta$ where b is a normalization coefficient and $d_p = \left(\frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} \int_0^{\frac{\pi}{2}} \sin^{p+2} \alpha \cos^{n-3} \alpha \, d\alpha \right)^{-1}$ for each $p \in X$.*

The above corollaries imply that if we have a good distribution function $f' \in \mathcal{F}_2$ satisfying that $P_{f'}(\phi)$ is a finite sum of non-negative power of $\cos \phi$, then we can construct an approximation algorithm for MAX DICUT whose approximation ratio is greater than $\alpha_{f'}$.

5 Numerical Method for Designing Algorithm

First, we designed a distribution function $f' \in \mathcal{F}_2$ satisfying $P_{f'}(\phi) = (1/a)(a_0 + a_1 \cos \phi + a_2 \cos^2 \phi)$, $(a_0, a_1, a_2) \geq \mathbf{0}$ and $a_0 + a_1 + a_2 = 3$. We tried every triplet

$$(a_0, a_1, a_2) \in \{0.01(x_0, x_1, x_2) \mid (x_0, x_1, x_2) \in \mathbf{Z}_+^3, 0.01(x_0 + x_1 + x_2) = 3\}.$$

We choose a triplet (a_0^*, a_1^*, a_2^*) which maximizes the approximation ratio $\alpha_{f'}$. Then we decrease the grid size around (a_0^*, a_1^*, a_2^*) and tried all the triplets in the set

$$\left\{ 0.0001(x_0, x_1, x_2) \mid \begin{array}{l} (x_0, x_1, x_2) \in \mathbf{Z}_+^3, \quad 0.0001(x + y + z) = 3, \\ |0.0001x_i - a_i^*| < 0.01 \quad (\forall i \in \{0, 1, 2\}) \end{array} \right\}.$$

As a result, we found that the following function

$$P_{f'}(\phi) = (1/a)(1.1404 + 0.7754 \cos \phi + 1.0842 \cos^2 \phi)$$

satisfies that the approximation ratio is greater than 0.859.

Next, we designed a distribution function $f' \in \mathcal{F}_2$ satisfying $P_{f'}(\phi) = (1/b) \cos^{1/\beta} \phi$, for each $\beta \in \{1.5, 1.6, \dots, 2.5\}$. As a result, we found that the following function

$$P_{f'}(\phi) = \cos^{(1/1.8)} \phi$$

satisfies that the corresponding approximation ratio is greater than 0.863.

For each function $P_{f'}(\phi)$, we calculate the approximation ratio $\alpha_{f'}$, as follows. We discretize the 2-dimensional sphere \mathbf{S}_2 and choose every pair of points $(\mathbf{v}_i, \mathbf{v}_j)$ from the set

$$\left\{ (x, y, z)^\top \in \mathbf{S}_2 \mid \begin{array}{l} \exists \eta, \exists \xi \in \{-32\pi/64, -31\pi/64, \dots, 32\pi/64\} \\ x = \cos \eta, \quad y = \sin \eta \cos \xi, \quad z = \sin \eta \sin \xi, \end{array} \right\},$$

and calculate the value

$$\frac{p(\mathbf{v}_i, \mathbf{v}_j \mid f')}{(1/4)(1 + \mathbf{v}_0 \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_j - \mathbf{v}_i \cdot \mathbf{v}_j)}.$$

Next, we choose minimum, 2nd minimum and 3rd minimum pairs of points. For each pair $(\mathbf{v}_i^*, \mathbf{v}_j^*)$ of chosen three pairs, we decrease the grid size and checked every pair of points $(\mathbf{v}_i, \mathbf{v}_j)$ satisfying that

$$\mathbf{v}_i \in \left\{ (x, y, z) \in \mathbf{S}_2 \mid \begin{array}{l} \exists \eta, \exists \xi \in \{-64\pi/4096, -63\pi/4096, \dots, 64\pi/4096\} \\ x = \cos(\eta_i^* + \eta), \quad y = \sin(\eta_i^* + \eta) \cos(\xi_i^* + \xi), \\ z = \sin(\eta_i^* + \eta) \sin(\xi_i^* + \xi) \end{array} \right\},$$

and

$$\mathbf{v}_j \in \left\{ (x, y, z) \in \mathbf{S}_2 \mid \begin{array}{l} \exists \eta, \exists \xi \in \{-64\pi/4096, -63\pi/4096, \dots, 64\pi/4096\} \\ x = \cos(\eta_j^* + \eta), \quad y = \sin(\eta_j^* + \eta) \cos(\xi_j^* + \xi), \\ z = \sin(\eta_j^* + \eta) \sin(\xi_j^* + \xi) \end{array} \right\},$$

where $\mathbf{v}_i^* = (\cos \eta_i^*, \sin \eta_i^* \cos \xi_i^*, \sin \eta_i^* \sin \xi_i^*)^\top$ and $\mathbf{v}_j^* = (\cos \eta_j^*, \sin \eta_j^* \cos \xi_j^*, \sin \eta_j^* \sin \xi_j^*)^\top$. For each pair of points $(\mathbf{v}_i, \mathbf{v}_j)$ we calculated the value $p(\mathbf{v}_i, \mathbf{v}_j \mid f)$ by numerical integration.

6 Conclusion

In this paper, we proposed an approximation algorithm for MAX DICUT problem whose approximation ratio is 0.863. Our algorithm solves the SDP relaxation problem proposed by Goemans and Williamson with additional valid constraints introduced in [3, 4]. We generate a dicut by using hyperplane separating technique based on skewed distribution function $f \in \mathcal{F}_n$ satisfying that $P_{\hat{f}}(\theta) = \cos^{(1/1.8)} \phi$.

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