0.935-Approximation Randomized Algorithm for MAX 2SAT and its Derandomization

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Abstract. In this paper, we propose 0.935-approximation algorithm for MAX 2SAT. The approximation ratio is better than the previously known result by Zwick, which is equal to 0.93109.
The algorithm solves the SDP relaxation problem proposed by Goemans and Williamson for the first time. We do not use the 'rotation' technique proposed by Feige and Goemans. We improve the approximation ratio by using hyperplane separation technique with skewed distribution function on the sphere. We introduce a class of skewed distribution functions defined on the 2-dimensional sphere satisfying that for any function in the class, we can design a skewed distribution functions on any dimensional sphere without decreasing the approximation ratio. We also searched and found a good distribution function defined on the 2-dimensional sphere numerically. And we propose the derandomized algorithm for the introduced distribution functions.

1 Introduction

In this paper we propose an approximation algorithm for the optimization problem called MAX 2SAT. There are n boolean variables x_i ($i \in \{1, \dots, n\}$) and m clauses $C_s(s \in \{1, \dots, m\})$. Each clause consists of two literals, where each literal is either boolean variable x_i or its negation $\neg x_i$. We associate a non-negative weight w_s for each clause C_s. The MAX 2SAT is the problem for finding an assignment of x_t's value which maximizes the total weight of satisfied clauses. The MAX 2SAT is formulated as follows;

$$\text{(P) maximize} \sum_{s: C_s \text{ is satisfied}} w_s \quad \text{subject to} \quad x_i \in \{\text{True,False}\}.$$

This problem is NP-hard [5] and so there are some algorithms for finding an approximate solution. As known well, Goemans and Williamson [6] proposed a randomized polynomial time algorithm for MAX CUT, MAX 2SAT and MAX DICUT. Their algorithm is based on Semi-Definite Programming (SDP) relaxation and random hyperplane separation technique. The approximation ratio of their algorithm for MAX 2SAT is 0.87856. More precisely, their algorithm finds an assignment whose total weight is at least 0.87856 times the optimal value.

In the paper [4], Feige and Goemans proposed an approximation algorithm for MAX 2SAT which achieves 0.93109 of approximation ratio. Their algorithm based on two ideas. First, they added some constraints introduced by Feige and Lovász in [3] to SDP relaxation problem. Next, they proposed the 'rotation' technique which modifies the solution obtained by SDP relaxation. They calculated the approximation ratio of their algorithm numerically. Recently, Zwick refined the rotation technique and proposed an algorithm whose approximation ratio is 0.9310900680 in [12]. He also showed that the approximation ratio of his algorithm almost completely matches upper bounds that we can obtain by any rotation technique.

As a related work, in SODA 2001, Halperin and Zwick proposed a combinatorial approximation algorithm for MAX DICUT [7]. In APPROX+RANDOM 2001, authors proposed a randomized algorithm for MAX DICUT [10].

In this paper, we propose an approximation algorithm without rotation technique whose approximation ratio is 0.935. Our algorithm solves the SDP relaxation problem proposed by Goemans and Williamson with the constraints used in Feige and Goemans' algorithm. We improve the approximation ratio by using hyperplane separation technique with skewed distribution function on the sphere. Although, the use of hyperplane separation technique with skewed distribution is suggested by Feige and Goemans in the paper [4], there is a non-trivial problem to design a good distribution function. More precisely, the performance of skewed distribution functions depends on the dimension of the corresponding sphere. First, we show a non-trivial relation between the skewed distribution functions on the 2-dimensional sphere and the n-dimensional sphere. We introduce a class of skewed distribution functions defined on the 2-dimensional sphere satisfying that for any distribution function in the class, we can design a skewed distribution function defined on any dimensional sphere without decreasing the approximation ratio. Second, we searched and found a good distribution function on the 2-dimensional sphere numerically. By using the above results, we can design a good skewed distribution function on any dimensional sphere. It means that the distribution function of our algorithm changes with respect to the dimension of the corresponding sphere. And finally, we propose a derandomized algorithm for proposed distribution functions. The derandomized algorithm is based on the technique proposed by Mahajan and Ramesh in the paper [9].

In Section 2, we review the SDP relaxation and hyperplane separation technique briefly. In Section 3, we describe the outline of our algorithm. In Section 4, we discuss some relations between the skewed distribution functions on the 2-dimensional sphere and the n-dimensional sphere. In Section 5, we describe a numerical method used for finding a good distribution function defined on the 2-dimensional sphere. In Section 6, we describe how to derandomize our algorithm.

2 Semi-Definite Programming Relaxation

Here we describe an SDP relaxation of MAX 2SAT and review the hyperplane separation technique. First, we formulate the MAX 2SAT problem as an integer programming problem. Let v_i be a $\{-1,1\}$-variable associate with x_i and v_{i+n} be a $\{-1,1\}$-variable associate with $\neg x_i$. Let C be the set of index pairs of clauses, i.e., $C = \{(i,j) | \exists s\,; C_s = (x_i \vee x_j)\} \cup \{(i,j) | \exists s\,; C_s = (x_i \vee \neg x_{j-n})\} \cup \{(i,j) | \exists s\,; C_s = (\neg x_{i-n} \vee x_j)\} \cup \{(i,j) | \exists s\,; C_s = (\neg x_{i-n} \vee \neg x_{j-n})\}$, and w_{ij} be the weight associate with corresponding clause.

The next problem is equivalent to the original problem (P).

$$
\begin{aligned}
\text{(P')}\ \ \text{maximize}\ \ & (1/4) \sum_{(i,j)\in C} w_{ij}(3 + v_0 v_i + v_0 v_j - v_i v_j), \\
\text{subject to}\ \ & v_0 = 1,\ \ v_i + v_{i+n} = 0\ \ (\forall i \in \{1,\ldots,n\}), \\
& v_i \in \{-1,1\}\ \ \ (\forall i \in \{1,\ldots,n,n+1,\ldots,2n\}).
\end{aligned}
$$

In the paper [6], Goemans and Williamson relaxed the above problem by replacing each variable $v_i \in \{-1,1\}$ with a vector on the n-dimensional unit sphere $\boldsymbol{v}_i \in \boldsymbol{S}_n$ where $\boldsymbol{S}_n \stackrel{\text{def.}}{=} \{\boldsymbol{v} \in \mathrm{R}^{n+1} \mid ||\boldsymbol{v}|| = 1\}$. This relaxation is proposed by Lovász [8] originally. By introducing some valid constraints used in papers [3, 4], we obtain the following relaxation problem;

$$
\begin{aligned}
(\overline{\text{P}})\ \ \text{maximize}\ \ & (1/4) \sum_{(i,j)\in C} w_{ij}(3 + \boldsymbol{v}_0 \cdot \boldsymbol{v}_i + \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j), \\
\text{subject to}\ & \boldsymbol{v}_0 = (1,0,\ldots,0)^\top,\ \ \ \boldsymbol{v}_i + \boldsymbol{v}_{i+n} = 0\ \ (\forall i \in \{1,\ldots,n\}), \\
& \boldsymbol{v}_i \in \boldsymbol{S}_n\ \ \ (\forall i \in \{1,\ldots,n,n+1,\ldots,2n\}), \\
& \boldsymbol{v}_0 \cdot \boldsymbol{v}_i + \boldsymbol{v}_0 \cdot \boldsymbol{v}_j + \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1\ \ (\forall (i,j)), \\
& -\boldsymbol{v}_0 \cdot \boldsymbol{v}_i - \boldsymbol{v}_0 \cdot \boldsymbol{v}_j + \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1\ \ (\forall (i,j)), \\
& -\boldsymbol{v}_0 \cdot \boldsymbol{v}_i + \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1\ \ (\forall (i,j)), \\
& \boldsymbol{v}_0 \cdot \boldsymbol{v}_i - \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1\ \ (\forall (i,j)).
\end{aligned}
$$

It is well-known that we can transform the above problem to a semidefinite programming problem [6] and so we can solve the problem in polynomial time by using an interior point method [2, 11].

Next, we describe the hyperplane separation technique proposed by Goemans and Williamson. Let $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \ldots, \overline{\boldsymbol{v}}_{2n})$ be an optimal solution of $\overline{\text{P}}$. We generate a vector $\boldsymbol{r} \in \boldsymbol{S}_n$ uniformly and construct the index-set $\overline{U} = \{i \in \{1,\ldots,n\} \mid \text{sign}(\boldsymbol{r} \cdot \boldsymbol{v}_0) = \text{sign}(\boldsymbol{r} \cdot \overline{\boldsymbol{v}}_i)\}$ and the corresponding assignment $\overline{A} : x_i \mapsto \{\text{True if } i \in \overline{U},\ \text{False if } i \notin \overline{U}\}$. We denote the expected weight of the assignment $\overline{A}$ by $\text{E}(\overline{U})$. Then the linearity of the expectation implies that;

$$
\text{E}(\overline{U}) = \sum_{(i,j)\in A} w_{ij} \frac{2\pi - \arccos(\boldsymbol{v}_0 \cdot \overline{\boldsymbol{v}}_i) - \arccos(\boldsymbol{v}_0 \cdot \overline{\boldsymbol{v}}_j) + \arccos(\overline{\boldsymbol{v}}_i \cdot \overline{\boldsymbol{v}}_j)}{2\pi}.
$$

Then we can estimate the approximation ratio of the algorithm by calculating α defined by;

$$\alpha \stackrel{\text{def.}}{=} \min_{(\boldsymbol{v}_i, \boldsymbol{v}_j) \in \Omega} \frac{(1/2\pi)(2\pi - \arccos(\boldsymbol{v}_0 \cdot \boldsymbol{v}_i) - \arccos(\boldsymbol{v}_0 \cdot \boldsymbol{v}_j) + \arccos(\boldsymbol{v}_i \cdot \boldsymbol{v}_j))}{(1/4)(3 + \boldsymbol{v}_0 \cdot \boldsymbol{v}_i + \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j)},$$

where

$$\Omega \stackrel{\text{def.}}{=} \left\{ (\boldsymbol{v}_i, \boldsymbol{v}_j) \in \boldsymbol{S}_2 \times \boldsymbol{S}_2 \;\middle|\; \begin{array}{r} \boldsymbol{v}_0 \cdot \boldsymbol{v}_i + \boldsymbol{v}_0 \cdot \boldsymbol{v}_j + \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1, \\ -\boldsymbol{v}_0 \cdot \boldsymbol{v}_i - \boldsymbol{v}_0 \cdot \boldsymbol{v}_j + \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1, \\ -\boldsymbol{v}_0 \cdot \boldsymbol{v}_i + \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1, \\ \boldsymbol{v}_0 \cdot \boldsymbol{v}_i - \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j \geq -1 \end{array} \right\},$$

and $\boldsymbol{v}_0 = (1, 0, 0)^\top$. Clearly, the following inequalities hold;

$$\mathrm{E}(\overline{U}) \geq \alpha(\text{optimal value of } (\overline{\mathrm{P}})) \geq \alpha(\text{optimal value of } (\mathrm{P})).$$

So, the expected weight $\mathrm{E}(\overline{U})$ is greater than or equal to α times the optimal value of (P). It is known that $\alpha > 0.87856$ [6].

Feige and Goemans' algorithm solves the problem $(\overline{\mathrm{P}})$ and modifies the obtained optimal solution by using rotation technique. The approximation ratio is 0.93109. Zwick refined the rotation technique and proposed an algorithm whose approximation ratio is equal to 0.9310900680. Our algorithm does not use the rotation technique and so we will not describe the technique here.

3 Hyperplane Separation by Skewed Distribution on Sphere

Goemans and Williamson's algorithm generates a separating hyperplane at random. Our algorithm generates a separating hyperplane with respect to a distribution function defined on $\boldsymbol{S}_n$ which is skewed towards $\boldsymbol{v}_0$ but is uniform in any direction orthogonal to $\boldsymbol{v}_0$. Given the n-dimensional sphere $\boldsymbol{S}_n$, we define the class of skewed distribution function $\mathcal{F}_n$ by;

$$\mathcal{F}_n \stackrel{\text{def.}}{=} \left\{ f : \boldsymbol{S}_n \to \mathrm{R}_+ \;\middle|\; \begin{array}{l} \int_{\boldsymbol{S}_n} f(\boldsymbol{v})\, \mathrm{d}s = 1, \;\; f(\boldsymbol{v}) = f(-\boldsymbol{v}) \; (\forall \boldsymbol{v} \in \boldsymbol{S}_n), \\ [\boldsymbol{v}_0 \cdot \boldsymbol{v} = \boldsymbol{v}_0 \cdot \boldsymbol{v}' \to f(\boldsymbol{v}) = f(\boldsymbol{v}')] \; (\forall \boldsymbol{v}, \forall \boldsymbol{v}' \in \boldsymbol{S}_n) \end{array} \right\}.$$

Let $f \in \mathcal{F}_n$ be a skewed distribution function defined on $\boldsymbol{S}_n$. Now consider the probability that $w_{i,j}$ is counted in a assignment obtained by hyperplane separation technique based on f. For any pair $(\boldsymbol{v}_i, \boldsymbol{v}_j) \in \boldsymbol{S}_n$, we define

$$p(\boldsymbol{v}_i, \boldsymbol{v}_j \mid f) \stackrel{\text{def.}}{=} \Pr \begin{bmatrix} \mathrm{sign}(\boldsymbol{r} \cdot \boldsymbol{v}_0) = \mathrm{sign}(\boldsymbol{r} \cdot \boldsymbol{v}_i) \text{ or} \\ \mathrm{sign}(\boldsymbol{r} \cdot \boldsymbol{v}_0) = \mathrm{sign}(\boldsymbol{r} \cdot \boldsymbol{v}_j) \end{bmatrix}.$$

Then the expectation of the weight of the assignment with respect to a feasible solution $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \ldots, \overline{\boldsymbol{v}}_n)$ of $\overline{\mathrm{P}}$ based on the distribution function f is $\sum_{(i,j) \in A} w_{ij}\; p(\overline{\boldsymbol{v}_i}, \overline{\boldsymbol{v}_j} \mid f)$.

When we use a skewed distribution function $f \in \mathcal{F}_n$ defined on $\boldsymbol{S}_n$, the approximation ratio can be estimated by the distribution function $\widehat{f}$ defined by projection of a vector on $\boldsymbol{S}_n$ to the linear subspace spanned by $\{\boldsymbol{v}_0, \boldsymbol{v}_i, \boldsymbol{v}_j\}$. We define $\widehat{f}$ more precisely. Let H be the 3-dimensional linear subspace including $\{\boldsymbol{v}_0, \boldsymbol{v}_i, \boldsymbol{v}_j\}$. The distribution function $\widehat{f} \in \mathcal{F}_2$ is defined as follows;

$$\widehat{f}(\boldsymbol{v}') \stackrel{\text{def.}}{=} \int_{T(\boldsymbol{v}')} f(\boldsymbol{v})\, \mathrm{d}s,$$

where

$$T(\boldsymbol{v}') \stackrel{\text{def.}}{=} \{\boldsymbol{v} \in \boldsymbol{S}_n \mid \text{ the projection of } \boldsymbol{v} \text{ to } H \text{ is parallel to } \boldsymbol{v}'\}.$$

Here we note that the distribution function $\widehat{f}$ is invariant with respect to the 3-dimensional subspace H including $\boldsymbol{v}_0$, since f is uniform in any directions orthogonal to $\boldsymbol{v}_0$. For any distribution function $f' \in \mathcal{F}_2$ we define

$$\alpha_{f'} \stackrel{\text{def.}}{=} \min_{(\boldsymbol{v}_i, \boldsymbol{v}_j) \in \Omega} \frac{p(\boldsymbol{v}_i, \boldsymbol{v}_j \mid f')}{(1/4)(3 + \boldsymbol{v}_0 \cdot \boldsymbol{v}_i + \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j)},$$

here we note that $p(\boldsymbol{v}_i, \boldsymbol{v}_j \mid f')$ is defined on $S_2 = H \cap S_n$. Then the approximation ratio of the algorithm using skewed distribution function $f \in \mathcal{F}_n$ is bounded by $\alpha_{\widehat{f}}$ from below.

For constructing a good skewed distribution function, we need to find a function $f' \in \mathcal{F}_2$ such that the value $\alpha_{f'}$ is large. In Section 5, we describe a numerical method for finding a good skewed distribution function in $\mathcal{F}_2$.

Even if we have a good distribution function in $\mathcal{F}_2$, a non-trivial problem still remains. For applying hyperplane separation technique, we need a skewed distribution function on the n-dimensional sphere. However, when $n > 2$, not every distribution function $f' \in \mathcal{F}_2$ has a distribution function $f \in \mathcal{F}_n$ satisfying $\widehat{f} = f'$. For example, it is easy to show that there does not exists any distribution function $f \in \mathcal{F}_3$ satisfying the conditions that

$$\widehat{f}(\boldsymbol{v}) = \begin{cases} 1/(2\sqrt{2}\pi) & (-0.5 \le \boldsymbol{v}_0 \cdot \boldsymbol{v} \le 0.5), \\ 0 & (\text{otherwise}). \end{cases}$$

In Section 4, we propose a class of functions in $\mathcal{F}_2$ such that a corresponding skewed distribution function exists for any sphere $\boldsymbol{S}_n$ with $n \ge 3$.

4 Main Theorem

For any function $f \in \mathcal{F}_n$, we can characterize f by the function $P_f : [0, \pi/2] \to \mathrm{R}_+$ defined by

$$P_f(\theta) \stackrel{\text{def.}}{=} f(\boldsymbol{v})|_{\cos\theta = |\boldsymbol{v}_0 \cdot \boldsymbol{v}|}.$$

The following theorem gives a class of permitted skewed distribution function in $\mathcal{F}_n$.

Theorem 1 *Let $f \in \mathcal{F}_n$ be a skewed distribution function with $n \geq 2$ satisfying*

$$P_f(\theta) = \frac{1}{a}\sum_{k=0}^{\infty} a_k \cos^k \theta.$$

Then the function $P_{\widehat{f}}(\phi)$ can be described as

$$P_{\widehat{f}}(\phi) = \frac{1}{a}\sum_{k=0}^{\infty} \frac{S^{(k+n)}(1)}{S^{(k+2)}(1)} a_k \cos^k \phi,$$

where a is the coefficient used for normalizing the total probability to 1 and $S^{(n)}(r)$ is the area of the n dimensional sphere whose radius is equal to r.

Proof. First, we notate some well-known formulae;

$$\Gamma(0) = 1, \;\; \Gamma(\frac{1}{2}) = \sqrt{\pi}, \;\; \Gamma(x+1) = x\Gamma(x),$$
$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \,\mathrm{d}x = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}, \;\; S^n(r) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}.$$

When we fix ϕ and $\mathrm{d}\phi$, we have the following;

$$\begin{aligned} & 2\pi \sin\phi P_{\widehat{f}}(\phi)\mathrm{d}\phi \\ & = \int_0^1 P_f(\arccos(r\cos\phi))(2\pi r \sin\phi)\left(S^{(n-3)}\left(\sqrt{1-r^2}\right)\right) \\ & \quad \left(\frac{\sqrt{1-r^2\cos^2\phi}}{\sqrt{1-r^2}} r \frac{\mathrm{d}\phi}{\cos\phi}\right)\left(\frac{\cos\phi}{\sqrt{1-r^2\cos^2\phi}}\right)\mathrm{d}r. \end{aligned}$$

Thus we have

$$P_{\widehat{f}}(\phi) = \int_0^1 P_f(\arccos(r\cos\phi)) S^{(n-3)}\left(\sqrt{1-r^2}\right) r^2 \frac{\mathrm{d}r}{\sqrt{1-r^2}}.$$

When we replace r by $\sin\alpha$ and $P_f(\theta)$ by $(1/a)\sum_{k=0}^{\infty} a_k \cos^k\theta$, we can describe $P_{\widehat{f}}(\phi)$ as;

$$\begin{aligned} P_{\widehat{f}}(\phi) &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{a}\sum_{k=0}^{\infty} a_k \sin^k\alpha\cos^k\phi\right) \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} \cos^{n-3}\alpha \; \sin^2\alpha \,\mathrm{d}\alpha \\ &= \frac{1}{a}\sum_{k=0}^{\infty} \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} a_k \cos^k\phi \int_0^{\frac{\pi}{2}} \sin^{k+2}\alpha \; \cos^{n-3}\alpha \,\mathrm{d}\alpha \\ &= \frac{1}{a}\sum_{k=0}^{\infty} \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} a_k \cos^k\phi \frac{\Gamma(\frac{k+3}{2})\Gamma(\frac{n-2}{2})}{2\Gamma(\frac{n+k+1}{2})} \\ &= \frac{1}{a}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{k+3}{2})}{2\pi^{\frac{k+3}{2}}} \frac{2\pi^{\frac{n+k+1}{2}}}{\Gamma(\frac{n+k+1}{2})} a_k \cos^k\phi = \frac{1}{a}\sum_{k=0}^{\infty} \frac{S^{(k+n)}(1)}{S^{(k+2)}(1)} a_k \cos^k\phi. \end{aligned}$$

And so we have done. □

The above theorem directly implies the following.

Corollary 1 *Let $f' \in \mathcal{F}_2$ be a distribution function satisfying*

$$P_{f'} = \frac{1}{b} \sum_{k=0}^{\infty} b_k \cos^k \phi$$

with the condition that $b_k \geq 0$. Then, for any $n \geq 2$, there exists a distribution function $f \in \mathcal{F}_n$ satisfying $\widehat{f} = f'$ and

$$P_f = \frac{1}{b} \sum_{k=0}^{\infty} \frac{S^{(k+2)}(1)}{S^{(k+n)}(1)} b_k \cos^k \theta,$$

where b is the coefficient used for normalizing the total probability to 1.

The following theorem extends the class of tractable distribution functions.

Theorem 2 *Let $f \in \mathcal{F}_n$ be a distribution function satisfying that*

$$P_f(\theta) = (1/a) \sum_{t \in T} a_t \cos^t \theta$$

and T is a finite set of non-negative real numbers. Then the distribution function $\widehat{f}$ satisfies

$$P_{\widehat{f}}(\phi) = (1/a) \sum_{t \in T} c_t a_t \cos^t \phi,$$

where a is the normalization coefficient and

$$c_t = \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} \int_0^{\frac{\pi}{2}} \sin^{t+2} \alpha \cos^{n-3} \alpha \, \mathrm{d}\alpha$$

for each $t \in T$.

Proof. We can prove in a similar way with the proof of Theorem 1 and so proof is omitted. □

This theorem implies the following.

Corollary 2 *Let $f' \in \mathcal{F}_2$ be a distribution function satisfying*

$$P_{f'}(\phi) = (1/b) \sum_{t \in T} b_t \cos^t \phi \text{ and } b_t \geq 0 \ (\forall t \in T)$$

where T is a finite set of positive real numbers. Then there exists a distribution function $f \in \mathcal{F}$ satisfying

$$\widehat{f} = f' \text{ and } P_f(\theta) = (1/b) \sum_{t \in T} d_t b_t \cos^t \theta$$

where b is the normalization coefficient and

$$d_t = \left(\frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} \int_0^{\frac{\pi}{2}} \sin^{t+2} \alpha \cos^{n-3} \alpha \, \mathrm{d}\alpha \right)^{-1}$$

for each $t \in T$.

The above corollaries imply that if we have a good distribution function $f' \in \mathcal{F}_2$ satisfying that $P_{f'}(\phi)$ is a finite sum of non-negative power of $\cos\phi$, then we can construct an approximation algorithm for MAX 2SAT whose approximation ratio is greater than or equal to $\alpha_{f'}$.

5 Numerical Method for Designing Algorithm

We designed a distribution function $f' \in \mathcal{F}_2$ satisfying $P_{f'}(\phi) = (1/b)\cos^{1/\beta}\phi$, where $\beta \in \{1.0, 1.1, \ldots, 2.0\}$. As a result, we found that the following function

$$P_{f'}(\phi) = \cos^{(1/1.3)}\phi$$

satisfies that the corresponding approximation ratio is greater than 0.935.

For each function $P_{f'}(\phi)$, we calculate the approximation ratio $\alpha_{f'}$ as follows. We discretize the 2-dimensional sphere $\boldsymbol{S}_2$ and choose every pair of points $(\boldsymbol{v}_i, \boldsymbol{v}_j)$ from the set

$$\left\{ (x,y,z)^\top \in \boldsymbol{S}_2 \,\middle|\, \begin{array}{l} \exists\eta, \exists\xi \in \{-32\pi/64, -31\pi/64, \ldots, 32\pi/64\} \\ x = \cos\eta,\ y = \sin\eta\cos\xi,\ z = \sin\eta\sin\xi, \end{array} \right\},$$

and calculate the value

$$\frac{p(\boldsymbol{v}_i, \boldsymbol{v}_j \mid f')}{(1/4)(1 + \boldsymbol{v}_0 \cdot \boldsymbol{v}_i - \boldsymbol{v}_0 \cdot \boldsymbol{v}_j - \boldsymbol{v}_i \cdot \boldsymbol{v}_j)}.$$

Next, we choose minimum, 2nd minimum and 3rd minimum pairs of points. For each pair $(\boldsymbol{v}_i^*, \boldsymbol{v}_j^*)$ of chosen three pairs, we decrease the grid size and checked every pair of points $(\boldsymbol{v}_i, \boldsymbol{v}_j)$ satisfying that

$$\boldsymbol{v}_i \in \left\{ (x,y,z) \in \boldsymbol{S}_2 \,\middle|\, \begin{array}{l} \exists\eta, \exists\xi \in \{-64\pi/4096, -63\pi/4096, \ldots, 64\pi/4096\} \\ x = \cos(\eta_i^* + \eta),\ y = \sin(\eta_i^* + \eta)\cos(\xi_i^* + \xi), \\ z = \sin(\eta_i^* + \eta)\sin(\xi_i^* + \xi) \end{array} \right\},$$

and

$$\boldsymbol{v}_j \in \left\{ (x,y,z) \in \boldsymbol{S}_2 \,\middle|\, \begin{array}{l} \exists\eta, \exists\xi \in \{-64\pi/4096, -63\pi/4096, \ldots, 64\pi/4096\} \\ x = \cos(\eta_j^* + \eta),\ y = \sin(\eta_j^* + \eta)\cos(\xi_j^* + \xi), \\ z = \sin(\eta_j^* + \eta)\sin(\xi_j^* + \xi) \end{array} \right\},$$

where

$$\boldsymbol{v}_i^* = (\cos\eta_i^*, \sin\eta_i^*\cos\xi_i^*, \sin\eta_i^*\sin\xi_i^*)^\top$$

and

$$\boldsymbol{v}_j^* = (\cos\eta_j^*, \sin\eta_j^*\cos\xi_j^*, \sin\eta_j^*\sin\xi_j^*)^\top.$$

For each pair of points $(\boldsymbol{v}_i, \boldsymbol{v}_j)$ we calculated the value $p(\boldsymbol{v}_i, \boldsymbol{v}_j | f)$ by numerical integration.

6 Derandomization

We generate the random vector $\boldsymbol{r} = (Z, X_0, \ldots, X_n)/\sqrt{Z^2 + X_0^2 + \cdots + X_n^2}$ as follows. Let $Z, X_0, \ldots, X_n$ be independent random variables, and $X_0, \ldots, X_n$ have an identical distribution function. We define a density function of $X_0, \ldots, X_n$ by $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and of Z by $\frac{1}{a} \sum_{t \in T} a_t |z|^t e^{-z^2/2}$.

Then the distribution function $f : \boldsymbol{S}_n \to \mathrm{R}$ of $\boldsymbol{r}$ satisfies the conditions of $\mathcal{F}_n$ described in Section 3 clearly. From the above definitions,

$$\int 2\pi \sin\phi P_{\widehat{f}}(\phi) \mathrm{d}\phi = \frac{1}{2\pi a} \sum_{t \in T} a_t \int\int\int |z|^t e^{-z^2/2} e^{-(x_i^2 + x_j^2)/2} \mathrm{d}z \mathrm{d}x_i \mathrm{d}x_j.$$

By replacing (x_i, x_j) with $(r\cos\psi, r\sin\psi)$, we have

$$= \frac{1}{a} \sum_{t \in T} a_t \int\int |z|^t e^{-z^2/2} r e^{-r^2/2} \mathrm{d}z \mathrm{d}r$$

and by replacing r with $z\cos\phi$,

$$= \frac{1}{a} \sum_{t \in T} a_t \int\int |z|^t z^2 \tan\phi (1 + \tan^2\phi) e^{-z^2 \tan^2\phi/2} \mathrm{d}z \mathrm{d}\phi$$

$$= \frac{1}{a} \int \left\{ \sum_{t \in T} a_t \frac{\sin\phi}{\cos^3\phi} \mathrm{d}\phi \int_{-\infty}^{\infty} |z|^{t+2} e^{-z^2/(2\cos^2\phi)} \mathrm{d}z \right\}$$

$$= \frac{1}{a} \int \left\{ \sum_{t \in T} a_t \frac{\sin\phi}{\cos^3\phi} \cos^{t+3} \sqrt{2}^{t+3} \mathrm{d}\phi \; 2 \int_0^{\infty} \left(\frac{z}{\sqrt{2}\cos\phi}\right)^{t+2} e^{-(z/\sqrt{2}\cos\phi)^2} \frac{\mathrm{d}z}{\sqrt{2}\cos\phi} \right\}$$

$$= \frac{1}{a} \int \sin\phi \sum_{t \in T} a_t b_t \cos^t\phi \; \mathrm{d}\phi,$$

where

$$b_t = 2^{(t+5)/2} \int_0^{\infty} z^{t+2} e^{-z^2} \mathrm{d}z,$$

is a constant. From these equations,

$$P_{\widehat{f}}(\phi) = \frac{1}{2\pi a} \sum_{t \in T} a_t b_t \cos^t\phi.$$

So, by choosing appropriate distribution for Z, we can obtain any distribution functions for applying Corollary 2.

Then we can derandomize our algorithm in a similar way with the method by Mahajan and Ramesh [9]. We fix each variables step by step by calculating expectation with conditional probabilities. The only thing to pay attention is that we need to fix Z first.

This result leads that we do not need to select or search a good distribution in derandomized algorithm. A good distribution is needed only for evaluating the approximation ratio.

7 Conclusion

In this paper, we proposed an approximation algorithm for MAX 2SAT problem whose approximation ratio is 0.935. Our algorithm solves the SDP relaxation problem proposed by Goemans and Williamson with additional valid constraints introduced in [3, 4]. We generate an assignment by using hyperplane separation technique based on skewed distribution function $f \in \mathcal{F}_n$ satisfying that $P_{\widehat{f}}(\theta) = \cos^{(1/1.3)} \phi$. Lastly, we derandomized our algorithm, and showed that the above distribution function is not needed to solve the problem in the derandomized algorithm.

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