M-convex and L-convex Fuctions over the Real Space
— Two Conjugate Classes of Combinatorial Convex Functions —[1]

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Abstract

By extracting combinatorial structures in well-solved nonlinear combinatorial optimization problems, Murota (1996,1998) introduced the concepts of M-convexity and L-convexity to functions defined over the integer lattice. Recently, Murota–Shioura (2000, 2001) extended these concepts to polyhedral convex functions and quadratic functions defined over the real space. In this paper, we consider a further extension to more general convex functions defined over the real space. The main aim of this paper is to provide rigorous proofs for the fundamental results of general M-convex and L-convex functions over the real space. In particular, we prove that the conjugacy relationship holds for general M-convex and L-convex functions over the real space.

Keywords: combinatorial optimization, matroid, base polyhedron, convex function, convex analysis.

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Contents

1 Introduction

Combinatorial optimization problems with nonlinear objective functions have been dealt with more often than before due to theoretical interest and needs of practical applications. Extensive studies have been done for revealing the essence of the well-solvability in nonlinear combinatorial optimization problems [3, 5, 13, 16, 17, 29]. By extracting combinatorial structures in well-solved nonlinear combinatorial optimization problems, Murota [18, 19] introduced the concepts of M-convexity and L-convexity to functions defined over the integer lattice; subsequently, their variants called M$^\natural$-convexity and L$^\natural$-convexity were introduced by Murota–Shioura [23] and by Fujishige–Murota [10], respectively. Applications of M-/L-convexity can be found in mathematical economics with indivisible commodities [4, 26, 27], system analysis by mixed polynomial matrices [20], etc. Recently, Murota–Shioura [24, 25] extended these concepts to polyhedral convex functions and quadratic functions defined over the real space. In this paper, we consider a further extension to more general convex functions defined over the real space. The main aim of this paper is to provide rigorous proofs for the fundamental results of general M-convex and L-convex functions over the real space.

The concepts of M-convexity and L-convexity are defined for polyhedral convex functions and quadratic functions as follows. A polyhedral convex function (or quadratic function) $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]),$$

where $\mathrm{supp}^+(x - y) = \{i \mid x(i) > y(i)\}$ and $\mathrm{supp}^-(x - y) = \{i \mid x(i) < y(i)\}$, and $\chi_i \in \{0, 1\}^n$ is the i-th unit vector for $i = 1, 2, \cdots, n$. A polyhedral convex function (or quadratic function) $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M$^\natural$-convex if the function $\widehat{f} : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) & ((x_0, x) \in \mathbf{R} \times \mathbf{R}^n,\ x_0 = -\sum_{i=1}^n x(i)), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex. On the other hand, a polyhedral convex function (or quadratic function) $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,

(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g,\ \lambda \in \mathbf{R})$,

and L$^\natural$-convex if the function $\widehat{g} : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p_0, p) = g(p - p_0 \mathbf{1}) \qquad ((p_0, p) \in \mathbf{R} \times \mathbf{R}^n)$$

is L-convex.

To fully cover the well-solved nonlinear combinatorial optimization problems, it is desirable to further extend these concepts to more general convex functions defined over the real space on the basis of (M-EXC), and (LF1) and (LF2), respectively. It can be easily imagined that the previous results of M-/L-convexity for polyhedral convex functions and quadratic functions naturally extend to more general M-/L-convex functions. However, the proofs cannot be extended so directly to general M-/L-convex functions, but some technical difficulties such as topological issues arise. By taking such technical difficulties into consideration, we define M-convex and L-convex functions over the real space as closed proper convex functions satisfying (M-EXC), and (LF1) and (LF2), respectively. The contribution of this paper is to provide rigorous proofs of the fundamental results of general M-convex and L-convex functions over the real space. In particular, we prove that the conjugacy relationship holds for general M-convex and L-convex functions over the real space, as in the cases of functions over the integer lattice [19], polyhedral convex functions [24], and quadratic functions [25].

Theorem 1.1. *For* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, *define its conjugate function* $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *by*

$$f^\bullet(p) = \sup\{\sum_{i=1}^{n} p(i)x(i) - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n).$$

(i) *If* f *is M-convex, then* $f^\bullet$ *is L-convex and* $(f^\bullet)^\bullet = f$.
(ii) *If* g *is L-convex, then* $g^\bullet$ *is M-convex and* $(g^\bullet)^\bullet = g$.
(iii) *The mappings* $f \mapsto f^\bullet$ (f : *M-convex*) *and* $g \mapsto g^\bullet$ (g : *L-convex*) *provide a one-to-one correspondence between the classes of M-convex and L-convex functions, and are the inverse of each other.*

The organization of this paper is as follows. Section 2 provides a number of natural classes of M-/M$^\natural$-convex and L-/L$^\natural$-convex functions. To develop the theory of M-/L-convexity, we also consider the set version of M-/L-convexity in addition to M-/L-convex functions. In Sections 3.2 and 4.2. we give definitions of M-convex and L-convex sets, and show that these sets are in fact polyhedra. We also investigate positively homogeneous M-/L-convex functions, which are shown to be polyhedral convex functions. Fundamental properties of M-/L-convex functions are shown in Sections 3.3 and 4.3. We present equivalent axioms for M-/L-convex functions, and give some properties on local combinatorial structure of M-/L-convex functions such as directional derivatives, subdifferentials, minimizers, etc. Finally, the conjugacy relationship between M-/L-convexity is shown in Section 5.

2 Preliminaries

2.1 Notation and Definitions

Throughout this paper, we assume that n is a positive integer, and put $N = \{1, 2, \ldots, n\}$. The cardinality of a finite set X is denoted by $|X|$. The characteristic vector of a subset $X \subseteq N$ is denoted by χ_X ($\in \{0,1\}^n$), i.e., $\chi_X(i) = 1$ for $i \in X$ and $\chi_X(i) = 0$ for $i \in N \setminus X$. We denote $\mathbf{0} = \chi_\emptyset$, $\mathbf{1} = \chi_N$.

The set of reals and the set of nonnegative reals are denoted by $\mathbf{R}$ and by $\mathbf{R}_+$, respectively. For $x = (x(i) \mid i = 1, 2, \ldots, n) \in \mathbf{R}^n$, we define

$$\mathrm{supp}^+(x) = \{i \in N \mid x(i) > 0\}, \quad \mathrm{supp}^-(x) = \{i \in N \mid x(i) < 0\},$$
$$\|x\|_1 = \sum_{i=1}^{n} |x(i)|, \qquad \langle p, x\rangle = \sum_{i=1}^{n} p(i)x(i) \quad (p \in \mathbf{R}^n), \qquad x(X) = \sum_{i \in X} x(i) \quad (X \subseteq N).$$

For $p, q \in \mathbf{R}^n$, we define $p \wedge q, p \vee q \in \mathbf{R}^n$ by

$$(p \wedge q)(i) = \min\{p(i), q(i)\}, \quad (p \vee q)(i) = \max\{p(i), q(i)\} \qquad (i \in N).$$

For $\alpha \in \mathbf{R} \cup \{-\infty\}$ and $\beta \in \mathbf{R} \cup \{+\infty\}$ with $\alpha \leq \beta$, we define the intervals $[\alpha, \beta] = \{\gamma \in \mathbf{R} \mid \alpha \leq \gamma \leq \beta\}$, $(\alpha, \beta) = \{\gamma \in \mathbf{R} \mid \alpha < \gamma < \beta\}$, and $[\alpha, \beta) = \{\gamma \in \mathbf{R} \mid \alpha \leq \gamma < \beta\}$.

A set $S \subseteq \mathbf{R}^n$ is said to be *convex* if it satisfies

$$(1 - \alpha)x + \alpha y \in S \qquad (\forall x, y \in S,\ \forall \alpha \in [0, 1]),$$

and *polyhedral* if there exist some $\{p_i\}_{i=1}^k$ ($\subseteq \mathbf{R}^n$) and $\{\alpha_i\}_{i=1}^k$ ($\subseteq \mathbf{R}$) ($k \geq 0$) such that

$$S = \{x \in \mathbf{R}^n \mid \langle p_i, x\rangle \leq \alpha_i \ (i = 1, 2, \ldots, k)\}.$$

A polyhedral set is also called a *polyhedron*.

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ be a function. The *effective domain* and the *epigraph* of f, denoted by $\mathrm{dom}\, f$ and $\mathrm{epi} f$, respectively, are given by

$$\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid -\infty < f(x) < +\infty\}, \qquad \mathrm{epi} f = \{(x, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq f(x)\}.$$

We denote by $\arg\min f$ the set of minimizers of f, i.e., $\arg\min f = \{x \in \mathbf{R}^n \mid f(x) \leq f(y)\ (\forall y \in \mathbf{R}^n)\}$, which can be an empty set.

A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *convex* if $\mathrm{epi} f$ is a convex set. If $f > -\infty$, then f is convex if and only if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y) \quad (\forall x, y \in \mathbf{R}^n,\ \forall \alpha \in [0, 1]). \tag{2.1}$$

In contrast to (2.1), we consider the following weaker condition:

$$f((x + y)/2) \leq \{f(x) + f(y)\}/2 \quad (\forall x, y \in \mathbf{R}^n), \tag{2.2}$$

called *mid-point convexity*. For a continuous function, mid-point convexity (2.2) is equivalent to convexity (2.1).

Remark 2.1. Mid-point convexity does not imply convexity in general. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a function satisfying so-called *Jensen's equation*:

$$\varphi(\alpha) + \varphi(\beta) = 2\varphi((\alpha + \beta)/2) \qquad (\forall \alpha, \beta \in \mathbf{R}). \tag{2.3}$$

It is known that there are discontinuous and nonconvex functions satisfying Jensen's equation (see, e.g., [1, pp. 43–48], [30, p. 217]). Such a function φ is mid-point convex, but not convex. □

For any set $S \subseteq \mathbf{R}^n$, its *indicator function* $\delta_S : \mathbf{R}^n \to \{0, +\infty\}$ is defined by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases}$$

Note that S is a convex set if and only if δ_S is a convex function.

A convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *proper* if $\mathrm{dom}\, f \neq \emptyset$, and *closed* if its epigraph $\mathrm{epi} f$ is a closed set. For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, any level set $\{x \in \mathbf{R}^n \mid f(x) \leq \eta\}$ $(\eta \in \mathbf{R})$ is a closed set, and $\arg\min f \neq \emptyset$ if $\mathrm{dom}\, f$ is bounded. A convex function is said to be *polyhedral* if its epigraph is polyhedral. Any polyhedral convex function is closed convex.

A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ is said to be *positively homogeneous* if

$$f(\alpha x) = \alpha f(x) \qquad (\forall x \in \mathbf{R}^n,\ \forall \alpha > 0).$$

For a positively homogeneous convex function f with $\mathrm{dom}\, f \neq \emptyset$, we have $f(\mathbf{0}) = 0$ and

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \qquad (\forall x, y \in \mathbf{R}^n,\ \forall \alpha, \beta \in \mathbf{R}_+).$$

For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$, $x \in \mathrm{dom}\, f$, and $y \in \mathbf{R}^n$, we define the *directional derivative* of f at x with respect to y by

$$f'(x; y) = \lim_{\alpha \downarrow 0} \{f(x + \alpha y) - f(x)\}/\alpha$$

if the limit exists. If f is a convex function, then $f'(x; y)$ is well-defined and a positively homogeneous convex function in y with $f'(x; \mathbf{0}) = 0$. For a convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$, the *subdifferential* of f at x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) = \{p \in \mathbf{R}^n \mid f(y) \geq f(x) + \langle p, y - x \rangle \ (\forall y \in \mathbf{R}^n)\}.$$

2.2 Examples

M-/M$^\natural$-convex and L-/L$^\natural$-convex functions have rich examples.

Example 2.2 (affine functions). For $p_0 \in \mathbf{R}^n$ and $\beta \in \mathbf{R}$, the function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by

$$f(x) = \langle p_0, x \rangle + \beta \quad (x \in \mathrm{dom}\, f)$$

is M-convex or M$^\natural$-convex according as $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid \sum_{i=1}^n x(i) = 0\}$ or $\mathrm{dom}\, f = \mathbf{R}^n$.

For $x_0 \in \mathbf{R}^n$ and $\nu \in \mathbf{R}$, the function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by

$$g(p) = \langle p, x_0 \rangle + \nu \quad (p \in \mathbf{R}^n)$$

is L-convex as well as L$^\natural$-convex. □

We denote by $\mathcal{C}^1$ the class of univariate closed proper convex functions, i.e.,

$$\mathcal{C}^1 = \{\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\} \mid \varphi : \text{ closed proper convex}\}.$$

Example 2.3. For $\varphi, \psi \in \mathcal{C}^1$, the functions $f, g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ given by

$$\mathrm{dom}\, f = \{(x_1, x_2) \in \mathbf{R}^n \mid x_1 + x_2 = 0\}, \qquad f(x_1, x_2) = \varphi(x_1) \quad ((x_1, x_2) \in \mathrm{dom}\, f), \tag{2.4}$$

$$g(p_1, p_2) = \psi(p_1 - p_2) \quad ((p_1, p_2) \in \mathbf{R}^2) \tag{2.5}$$

are M-convex and L-convex, respectively. Moreover, if φ and ψ are conjugate to each other, then f and g are conjugate to each other. □

Example 2.4 (separable-convex functions). Let $f_i \in \mathcal{C}^1$ $(i = 1, 2, \ldots, n)$ be a family of univariate convex functions. The function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \sum_{i=1}^{n} f_i(x(i)) \qquad (x \in \mathbf{R}^n)$$

is M$^\natural$-convex as well as L$^\natural$-convex. The restriction of f over the hyperplane $\{x \in \mathbf{R}^n \mid \sum_{i=1}^n x(i) = 0\}$ is M-convex if its effective domain is nonempty.

For functions $g_{ij} \in \mathcal{C}^1$ indexed by $i, j \in \{1, 2, \ldots, n\}$, the function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(p(j) - p(i)) \qquad (p \in \mathbf{R}^n)$$

is L-convex with $r = 0$ in (LF2) if $\mathrm{dom}\, g \neq \emptyset$. □

Example 2.5 (quadratic functions). Let $A = (a(i,j))_{i,j=1}^n \in \mathbf{R}^{n \times n}$ be a symmetric matrix. Define a quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x) = (1/2) x^{\mathrm{T}} A x$ $(x \in \mathbf{R}^n)$. Then, f is M$^\natural$-convex if and only if

$$x^{\mathrm{T}} a_i \geq \min\{x^{\mathrm{T}} a_j \mid j \in \mathrm{supp}^-(x)\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)),$$

where a_i denotes the i-th column of A for $i = 1, 2, \ldots, n$. The function f is L$^\natural$-convex if and only if

$$a(i,j) \leq 0 \quad (\forall i, j,\ i \neq j), \qquad \sum_{i=1}^{n} a(i,j) \geq 0 \quad (\forall j = 1, 2, \ldots, n).$$

See [25] for proofs. □

Example 2.6 (minimum cost flow/tension problems). M-convex and L-convex functions arise from the minimum cost flow/tension problems with nonlinear cost functions.

Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Suppose that we are given a family of convex functions $f_a \in \mathcal{C}^1$ $(a \in A)$, each of which represents the cost of flow on the arc a. A vector $\xi \in \mathbf{R}^A$ is called a flow, and the boundary $\partial\xi \in \mathbf{R}^V$ of a flow ξ is given by

$$\partial\xi(v) = \sum\{\xi(a) \mid \text{ arc } a \text{ leaves } v\} - \sum\{\xi(a) \mid \text{ arc } a \text{ enters } v\} \qquad (v \in V).$$

Then, the minimum cost of a flow that realizes a supply/demand vector $x \in \mathbf{R}^T$ is represented by a function $f : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$f(x) = \inf_{\xi}\{\sum_{a\in A} f_a(\xi(a)) \mid (\partial\xi)(v) = -x(v)\ (v \in T),\ (\partial\xi)(v) = 0\ (v \in V \setminus T)\} \quad (x \in \mathbf{R}^T).$$

On the other hand, suppose that we are given another family of convex functions $g_a \in \mathcal{C}^1$ $(a \in A)$, each of which represents the cost of tension on the arc a. Any vector $\widetilde{p} \in \mathbf{R}^V$ is called a potential, and the coboundary $\delta\widetilde{p} \in \mathbf{R}^A$ of a potential $\widetilde{p}$ is defined by $\delta\widetilde{p}(a) = \widetilde{p}(u) - \widetilde{p}(v)$ for $a = (u, v) \in A$. Then, the minimum cost of a tension that realizes a potential vector $p \in \mathbf{R}^T$ is represented by a function $g : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$g(p) = \inf_{\eta,\widetilde{p}}\{\sum_{a\in A} g_a(\eta(a)) \mid \eta(a) = -\delta\widetilde{p}(a)\ (a \in A),\ \widetilde{p}(v) = p(v)\ (v \in T)\} \quad (p \in \mathbf{R}^T).$$

It can be shown that both f and g are closed proper convex if $f(x_0)$ and $g(p_0)$ are finite for some $x_0 \in \mathbf{R}^T$ and $p_0 \in \mathbf{R}^T$, which is a direct extension of the results in Iri [13] and Rockafellar [29] for the case of $|T| = 2$. These functions, however, are equipped with different combinatorial structures; f is M-convex and g is L-convex, as follows. j

Theorem 2.7. *Suppose that f_a and g_a are conjugate to each other for all $a \in A$. Then, f and g are M-convex and L-convex, respectively, and conjugate to each other if one of the following conditions holds:*

$$-\infty < f(x_0) < +\infty \text{ for some } x_0 \in \mathbf{R}^T, \tag{2.6}$$

$$-\infty < g(p_0) < +\infty \text{ for some } p_0 \in \mathbf{R}^T, \tag{2.7}$$

$$f(x_0) < +\infty, g(p_0) < +\infty \text{ for some } x_0 \in \mathbf{R}^T, p_0 \in \mathbf{R}^T. \tag{2.8}$$

We first prove the closedness of f and g and the conjugacy relationship. For this, we use the following duality theorem for the minimum cost flow and tension problems:

Theorem 2.8 (cf. [29, Sec. 8H]). *Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Also, let $f_a, g_a \in \mathcal{C}^1$ $(a \in A)$ and $f_v, g_v \in \mathcal{C}^1$ $(v \in T)$ be conjugate pairs of closed convex functions. Then, we have*

$$\inf_{\xi\in\mathbf{R}^A,\ x\in\mathbf{R}^T} \left\{ \sum_{a\in A} f_a(\xi(a)) + \sum_{v\in T} f_v(x(v)) \;\middle|\; (\partial\xi)(v) = -x(v)\ (v \in T),\ (\partial\xi)(v) = 0\ (v \in V \setminus T) \right\}$$
$$= \sup_{\eta\in\mathbf{R}^A,\ \widetilde{p}\in\mathbf{R}^V} \left\{ -\sum_{a\in A} g_a(\eta(a)) - \sum_{v\in T} g_v(-\widetilde{p}(v)) \;\middle|\; \eta(a) = -\delta\widetilde{p}(a)\ (a \in A) \right\}$$

unless $\inf = +\infty$ *and* $\sup = -\infty$.

For $p, x \in \mathbf{R}^T$, we denote $\langle p, x\rangle_T = \sum_{v\in T} p(v)x(v)$.

Lemma 2.9. *Let $x \in \mathbf{R}^n$ and $p \in \mathbf{R}^n$.*
(i) $f(x) = \sup\{\langle p, x\rangle_T - g(p) \mid p \in \mathbf{R}^T\}$ *if $f(x) < +\infty$ or $g(p_0) < +\infty$ for some $p_0 \in \mathbf{R}^T$.*
(ii) $g(p) = \sup\{\langle p, x\rangle_T - f(x) \mid x \in \mathbf{R}^T\}$ *if $g(p) < +\infty$ or $f(x_0) < +\infty$ for some $x_0 \in \mathbf{R}^T$.*

We see from Lemma 2.9 that three conditions (2.6), (2.7), and (2.8) are equivalent to each other. Hence, if one of these conditions holds, then Lemma 2.9 implies that $f = g^\bullet$, $g = f^\bullet$, and both f and g are closed convex functions with $f, g > -\infty$.
[Proof of Lemma 2.9] To prove (i), consider functions $f_v, g_v \in \mathcal{C}^1$ $(v \in T)$ given as

$$f_v(\alpha) = \begin{cases} 0 & (\alpha = x(v)), \\ +\infty & (\alpha \neq x(v)), \end{cases} \qquad g_v(\beta) = x(v)\beta \quad (\beta \in \mathbf{R})$$

for the given $x \in \mathbf{R}^T$. The functions f_v and g_v are conjugate to each other for each $v \in T$. If $f(x) < +\infty$ or $g(p_0) < +\infty$ for some $p_0 \in \mathbf{R}^T$, then Theorem 2.8 implies that

$$\begin{aligned} f(x) &= \inf_{\xi\in\mathbf{R}^A} \{\sum_{a\in A} f_a(\xi(a)) \mid (\partial\xi)(v) = -x(v)\ (v \in T),\ (\partial\xi)(v) = 0\ (v \in V \setminus T)\} \\ &= \sup_{\eta\in\mathbf{R}^A,\ \widetilde{p}\in\mathbf{R}^V} \left\{ \sum_{v\in T} \widetilde{p}(v)x(v) - \sum_{a\in A} g_a(\eta(a)) \;\middle|\; \eta(a) = -\delta\widetilde{p}(a)\ (a \in A) \right\} \\ &= \sup\{\langle p, x\rangle_T - g(p) \mid p \in \mathbf{R}^T\}. \end{aligned}$$

To prove (ii), consider functions $f_v, g_v \in \mathcal{C}^1$ $(v \in T)$ given as

$$f_v(\alpha) = -p(v)\alpha\ (\alpha \in \mathbf{R}), \qquad g_v(\beta) = \begin{cases} 0 & (\beta = -p(v)), \\ +\infty & (\beta \neq -p(v)). \end{cases}$$

If $g(p) < +\infty$ or $f(x_0) < +\infty$ for some $x_0 \in \mathbf{R}^T$, then Theorem 2.8 implies that

$$\begin{aligned} g(p) &= \inf_{\eta\in\mathbf{R}^A} \left\{ \sum_{a\in A} g_a(\eta(a)) \;\middle|\; \eta(a) = -\delta\widetilde{p}(a)\ (a \in A),\ \widetilde{p}(v) = p(v)\ (v \in T) \right\} \\ &= \sup_{\xi\in\mathbf{R}^A,\ x\in\mathbf{R}^T} \left\{ \langle p, x\rangle_T - \sum_{a\in A} f_a(\xi(a)) \;\middle|\; (\partial\xi)(v) = -x(v)\ (v \in T),\ (\partial\xi)(v) = 0\ (v \in V \setminus T) \right\} \\ &= \sup\{\langle p, x\rangle_T - f(x) \mid x \in \mathbf{R}^T\}. \end{aligned}$$

We then prove the M-convexity of f and the L-convexity of g.
[(M-EXC) for f] Let $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$. For any $\varepsilon > 0$ there exist $\xi_x, \xi_y \in \mathbf{R}^A$ with

$$\begin{aligned} &\sum_{a\in A} f_a(\xi_x(a)) \le f(x) + \varepsilon, \qquad (\partial\xi_x)(v) = -x(v)\ (v \in T), \qquad (\partial\xi_x)(v) = 0\ (v \in V \setminus T), \\ &\sum_{a\in A} f_a(\xi_y(a)) \le f(y) + \varepsilon, \qquad (\partial\xi_y)(v) = -y(v)\ (v \in T), \qquad (\partial\xi_y)(v) = 0\ (v \in V \setminus T). \end{aligned}$$

By a standard augmenting path argument, there exist $\pi : A \to \{0, \pm 1\}$ and $v \in \mathrm{supp}^-(x - y)$ $(\subseteq T)$ such that

$$\mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi_y - \xi_x),\ \mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi_y - \xi_x),\ \partial\pi = \chi_u - \chi_v,$$

where we can assume the following inequality with $m = |A|$, $n = |V|$:

$$\min\{|\xi_x(a) - \xi_y(a)| \mid a \in A,\ \pi(a) = \pm 1\} \geq \{x(u) - y(u)\}/m^n.$$

Putting $\alpha_0 = \{x(u) - y(u)\}/m^n$, we have

$$f_a(\xi_x(a) + \alpha\pi(a)) + f_a(\xi_y(a) - \alpha\pi(a)) \leq f_a(\xi_x(a)) + f_a(\xi_y(a)) \quad (\alpha \in [0, \alpha_0])$$

for all $a \in A$. Hence follows that

$$\begin{aligned} & f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \\ \leq\ & \sum_{a \in A} \{f_a(\xi_x(a) + \alpha\pi(a)) + f_a(\xi_y(a) - \alpha\pi(a))\} \\ \leq\ & \sum_{a \in A} \{f_a(\xi_x(a)) + f_a(\xi_y(a))\} \ \leq\ f(x) + f(y) + 2\varepsilon \qquad (\alpha \in [0, \alpha_0]). \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily and T is a finite set, there exists some $v = v_*$ satisfying

$$f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \leq f(x) + f(y) \qquad (\alpha \in [0, \alpha_0]),$$

implying (M-EXC) for f.
[L-convexity for g] Let $p, q \in \mathrm{dom}\, g$. For any $\varepsilon > 0$ there exist $\widetilde{p}, \widetilde{q} \in \mathbf{R}^V$ with

$$\begin{aligned} & \sum_{a \in A} g_a(-\delta\widetilde{p}(a)) \leq g(p) + \varepsilon, \qquad \widetilde{p}(v) = p(v)\ (v \in T), \\ & \sum_{a \in A} g_a(-\delta\widetilde{q}(a)) \leq g(q) + \varepsilon, \qquad \widetilde{q}(v) = q(v)\ (v \in T). \end{aligned}$$

It holds that $(\widetilde{p} \wedge \widetilde{q})(v) = (p \wedge q)(v)$, $(\widetilde{p} \vee \widetilde{q})(v) = (p \vee q)(v)$ for all $v \in T$, and

$$g_a(-\delta(\widetilde{p} \wedge \widetilde{q})(a)) + g_a(-\delta(\widetilde{p} \vee \widetilde{q})(a)) \leq g_a(-\delta\widetilde{p}(a)) + g_a(-\delta\widetilde{q}(a)) \qquad (a \in A)$$

by convexity of g_a. Hence follows that

$$g(p \wedge q) + g(p \vee q) \leq \sum_{a \in A} g_a(-\delta(\widetilde{p} \wedge \widetilde{q})(a)) + \sum_{a \in A} g_a(-\delta(\widetilde{p} \vee \widetilde{q})(a)) \leq g(p) + g(q) + 2\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily, we have $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q)$.

The property (LF2) for g is immediate from the equation $\delta(\widetilde{p} + \lambda\mathbf{1}) = \delta\widetilde{p}$ for $\widetilde{p} \in \mathbf{R}^V$ and $\lambda \in \mathbf{R}$. $\square$

3 M-convex Functions over the Real Space

3.1 Definitions of M-convex and M$^\natural$-convex Functions

We call a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *M-convex* if it is closed proper convex and satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]). \tag{3.1}$$

As shown later in Proposition 3.2, the effective domain $\mathrm{dom}\, f$ of an M-convex function f is contained in a hyperplane $\{x \in \mathbf{R}^n \mid x(N) = r\}$ for some $r \in \mathbf{R}$. Hence, no information is lost when an M-convex function is projected onto an $(n-1)$-dimensional space. We call a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *M$^\natural$-convex* if the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) & ((x_0, x) \in \mathbf{R}^{\widehat{N}},\ x_0 = -x(N)), \\ +\infty & (\text{otherwise}) \end{cases} \tag{3.2}$$

is M-convex, where $\widehat{N} = \{0\} \cup N$. M$^\natural$-convexity of f is characterized by the following exchange property (cf. [23, 24, 25]):

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]),$$

where $\chi_0 = \mathbf{0}$ by convention.

Theorem 3.1. *A closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is M$^\natural$-convex if and only if it satisfies* (M$^\natural$-EXC)*.*

Proof. The "only if" part is obvious from the definition. The "if" part is proven later in Section 5. □

We denote by $\mathcal{M}_n$ (resp. $\mathcal{M}_n^\natural$) the class of M-convex (resp. M$^\natural$-convex) functions in n variables:

$$\mathcal{M}_n = \{f \mid f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{M-convex}\}, \qquad \mathcal{M}_n^\natural = \{f \mid f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{M}^\natural\text{-convex}\}.$$

As is obvious from the definitions, M$^\natural$-convex function is essentially equivalent to M-convex function, whereas the class of M$^\natural$-convex functions contains that of M-convex functions as a proper subclass. This relationship between M-convexity and M$^\natural$-convexity can be summarized as

$$\mathcal{M}_n \subset \mathcal{M}_n^\natural \simeq \mathcal{M}_{n+1}.$$

From the definition of M$^\natural$-convex functions, every property of M-convex functions can be restated in terms of M$^\natural$-convex functions, and vice versa. In this paper, we primarily work with M-convex functions, making explicit statements for M$^\natural$-convex functions when appropriate.

We also define the set version of M-convexity and M$^\natural$-convexity. We call a set $B \subseteq \mathbf{R}^n$ *M-convex* (resp. *M$^\natural$-convex*) if its indicator function $\delta_B : \mathbf{R}^n \to \{0, +\infty\}$ is an M-convex (resp. M$^\natural$-convex) function. Equivalently, an M-convex set is defined as a nonempty closed set satisfying the exchange property (B-EXC):

(B-EXC) $\forall x, y \in B$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$x - \alpha(\chi_i - \chi_j) \in B, \quad y + \alpha(\chi_i - \chi_j) \in B \qquad (\forall \alpha \in [0, \alpha_0]).$$

In fact, M-convex and M$^\natural$-convex sets are polyhedral as shown later in Section 3.2. Hence, these concepts coincide with those of M-convex and M$^\natural$-convex polyhedra introduced in [24]. In particular, M-convex and M$^\natural$-convex sets are nothing but the base polyhedra of submodular systems [9] and generalized polymatroids [7, 8], respectively.

We now prove that the effective domain of an M-convex function is contained in a hyperplane of the form $\{x \in \mathbf{R}^n \mid x(N) = r\}$ for some $r \in \mathbf{R}$.

Proposition 3.2.
(i) *For an M-convex set* $B \subseteq \mathbf{R}^n$*, we have* $x(N) = y(N)$ $(\forall x, y \in B)$.
(ii) *For an M-convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$*, we have* $x(N) = y(N)$ $(\forall x, y \in \mathrm{dom}\, f)$*. In particular,* $\mathrm{dom}\, f$ *satisfies* (B-EXC)*, and is an M-convex set if it is a closed set.*

Proof. Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function, and $x, y \in \mathrm{dom}\, f$. Assume, to the contrary, that $x(N) > y(N)$ holds for some $x, y \in \mathrm{dom}\, f$. Put

$$S = \{z \in \mathbf{R}^n \mid x \wedge y \leq z \leq x \vee y,\ f(z) \leq \max\{f(x), f(y)\}\},$$

which is a bounded closed set. Let $x_*, y_* \in S$ minimize the value $||x_* - y_*||_1$ among all pairs of vectors in S with $x_*(N) = x(N)$ and $y_*(N) = y(N)$. The property (M-EXC) for x_* and y_* implies

$$f(x_*) + f(y_*) \geq f(x_* - \alpha(\chi_i - \chi_j)) + f(y_* + \alpha(\chi_i - \chi_j))$$

for some $i \in \mathrm{supp}^+(x_* - y_*)$, $j \in \mathrm{supp}^-(x_* - y_*)$, and a sufficiently small $\alpha > 0$. Putting $\widehat{x} = x_* - \alpha(\chi_i - \chi_j)$ and $\widehat{y} = y_* + \alpha(\chi_i - \chi_j)$, we have $\widehat{x} \in S$ or $\widehat{y} \in S$, a contradiction to the choice of x_* and y_* since $||\widehat{x} - y_*||_1 < ||x_* - y_*||_1$ and $||x_* - \widehat{y}||_1 < ||x_* - y_*||_1$. The property (B-EXC) for $\mathrm{dom}\, f$ is immediate from (M-EXC) for f, and (i) is a special case of (ii). □

Remark 3.3. The property (B-EXC) alone, without closedness, is independent of good properties such as convexity and connectivity. An example is the set

$$\{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 + x_2 = 0, x_1 < 0\} \cup \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 + x_2 = 1, x_1 > 1\},$$

which satisfies (B-EXC); however, it is neither convex nor connected, and not contained in a hyperplane $\{x \in \mathbf{R}^n \mid x(N) = r\}$ for some $r \in \mathbf{R}$ (cf. Proposition 3.2 (i)). Even if convexity assumption is added, (B-EXC) is still independent of nice combinatorial properties. A typical example is $\{x \in \mathbf{R}^n \mid \sum_{i=1}^n x(i)^2 < \gamma,\ x(N) = 0\}$ $(\gamma > 0)$, which is an open ball in a hyperplane. □

Remark 3.4. The property (M-EXC) alone is independent of good properties such as convexity and continuity. Consider a function $\varphi : \mathbf{R} \to \mathbf{R}$ satisfying Jensen's equation (2.3) such that φ is neither continuous nor convex, and define a function $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ as in (2.4). Then, (M-EXC) for f follows immediately from Jensen's equation for φ; however, f is neither convex nor continuous. □

Remark 3.5. The effective domain of an M-convex function is not a closed set in general. An example is the function $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ given by

$$\mathrm{dom}\, f = \{(x_1, x_2) \mid x_1 + x_2 = 0,\ x_1 > 0\}, \qquad f(x_1, x_2) = 1/x_1 \quad ((x_1, x_2) \in \mathrm{dom}\, f).$$

□

3.2 M-convex Sets and Positively Homogeneous M-convex Functions

M-convex sets and positively homogeneous M-convex functions constitute important subclasses of M-convex functions, which appear as local structure of general M-convex functions such as minimizers and directional derivative functions (see Theorem 3.16). We show that M-convex sets and positively homogeneous M-convex functions are polyhedral sets and polyhedral convex functions, respectively.

Theorem 3.6.
(i) *Any M-convex set is a polyhedron.*
(ii) *Any positively homogeneous M-convex function is a polyhedral convex function.*

In the following, we prove Theorem 3.6. For a nonempty set $B \subseteq \mathbf{R}^n$, we define a set function $\rho_B : 2^N \to \mathbf{R} \cup \{+\infty\}$ by $\rho_B(X) = \sup\{x(X) \mid x \in B\}$ $(X \subseteq N)$.

Lemma 3.7. *Let $B \subseteq \mathbf{R}^n$ be a nonempty bounded closed set with* (B-EXC)*, and $\{X_k\}_{k=1}^h$ be subsets of N with $X_1 \subset X_2 \subset \cdots \subset X_h$. Then, there exists $x_* \in B$ with $x_*(X_k) = \rho_B(X_k)$ $(\forall k = 1, 2, \ldots, h)$.*

Proof. The claim is shown by induction on the value h. Let $x_h \in B$ be a vector with $x_h(X_h) = \rho_B(X_h)$. By the induction hypothesis, there exists a vector $x \in B$ satisfying $x(X_k) = \rho_B(X_k)$ $(k = 1, 2, \cdots, h-1)$, and assume that x minimizes the value $\|x - x_h\|_1$ of all such vectors. Suppose that $x(X_h) < \rho_B(X_h)$. By (B-EXC), there exist $i \in \mathrm{supp}^+(x - x_h) \setminus X_h$ and $j \in \mathrm{supp}^-(x - x_h)$ such that $x' = x - \alpha(\chi_i - \chi_j) \in B$ for a sufficiently small $\alpha > 0$. Here $j \in N \setminus X_{h-1}$ holds since $i \in N \setminus X_{h-1}$ and $x'(X_{h-1}) \leq \rho_B(X_{h-1}) = x(X_{h-1})$. Therefore, x' satisfies $x'(X_k) = \rho_B(X_k)$ $(\forall k = 1, 2, \cdots, h-1)$ and $\|x' - x_h\|_1 = \|x - x_h\|_1 - 2\alpha$, a contradiction. Therefore, $x(X_h) = \rho_B(X_h)$. □

Let $\rho : 2^N \to \mathbf{R} \cup \{+\infty\}$ be a set function. We call ρ *submodular* if it satisfies

$$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y) \quad (\forall X, Y \in \mathrm{dom}\, \rho), \tag{3.3}$$

where $\mathrm{dom}\, \rho = \{X \subseteq N \mid \rho(N) < +\infty\}$. We define a polyhedron $\mathrm{B}(\rho) \subseteq \mathbf{R}^n$ by

$$\mathrm{B}(\rho) = \{x \in \mathbf{R}^n \mid x(X) \leq \rho(X)\ (X \subseteq N),\ x(N) = \rho(N)\}. \tag{3.4}$$

For any convex set $S \subseteq \mathbf{R}^n$, a point $x \in S$ is called an *extreme point* of S if there are no $y_1, y_2 \in S \setminus \{x\}$ and $\alpha \in (0, 1)$ such that $x = \alpha y_1 + (1 - \alpha) y_2$.

Theorem 3.8 ([9, Theorem 3.22]). *Let $\rho : 2^N \to \mathbf{R} \cup \{+\infty\}$ be a submodular function with $\rho(\emptyset) = 0$ and $\rho(N) < +\infty$. Then, $x \in \mathbf{R}^n$ is an extreme point of $\mathrm{B}(\rho)$ if and only if there exists $\{X_k\}_{k=0}^n \subseteq \mathrm{dom}\, \rho$ such that $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = N$ and $x(X_k) = \rho(X_k)$ for all $k = 0, 1, \ldots, n$.*

Proof of Theorem 3.6 (i). Let $B \subseteq \mathbf{R}^n$ be a nonempty closed set with (B-EXC). It suffices to show $B = \mathrm{B}(\rho_B)$ since $\mathrm{B}(\rho_B)$ is a polyhedron. The inclusion $B \subseteq \mathrm{B}(\rho_B)$ is easy to see; therefore we prove the reverse inclusion.

We first assume that B is bounded. Let $X, Y \in \mathrm{dom}\,\rho_B$ $(= 2^N)$. From Lemma 3.7, there exists $x_* \in B$ with $x_*(X \cap Y) = \rho_B(X \cap Y)$ and $x_*(X \cup Y) = \rho_B(X \cup Y)$, which implies the inequality

$$\rho_B(X) + \rho_B(Y) \geq x_*(X) + x_*(Y) = x_*(X \cap Y) + x_*(X \cup Y) = \rho_B(X \cap Y) + \rho_B(X \cup Y).$$

Therefore, ρ_B is a submodular function. By Lemma 3.7 and Theorem 3.8, any extreme point of $\mathrm{B}(\rho_B)$ is contained in B. Hence we have $\mathrm{B}(\rho_B) \subseteq B$.

Next, assume that B is unbounded. For a fixed $x_0 \in B$, define $B_k = \{x \in B \mid x(i) - x_0(i) \leq k \ (i \in N)\}$ and put $\rho_k = \rho_{B_k}$ for $k = 0, 1, 2, \ldots$. Since each B_k is a bounded M-convex set, we have $B_k = \mathrm{B}(\rho_k)$. To prove $\mathrm{B}(\rho_B) \subseteq B$, let y be any vector in $\mathrm{B}(\rho_B)$. Then, there exists a sequence of vectors $\{x_k\}_{k=0,}^{\infty}$ such that $x_k \in \mathrm{B}(\rho_k) \subseteq B$ $(k = 0, 1, 2, \cdots)$ and $\lim_{k\to\infty} x_k = y$. Since B is a close set, we have $y \in B$, i.e., $\mathrm{B}(\rho_B) \subseteq B$. □

We then prove Theorem 3.6 (ii). For a convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, $x \in \mathrm{dom}\, f$, and $i, j \in N$, we define

$$f'(x; j, i) = f'(x; \chi_j - \chi_i).$$

For a function $\gamma : N \times N \to \mathbf{R} \cup \{\pm\infty\}$, we define $f_\gamma : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_\gamma(x) = \inf\{ \sum_{(i,j)\in A} \lambda_{ij}\gamma(i,j) \mid \sum_{(i,j)\in A} \lambda_{ij}(\chi_j - \chi_i) = x,\ \lambda_{ij} \geq 0\ ((i,j) \in A)\} \quad (x \in \mathbf{R}^n), \tag{3.5}$$

where $A = \{(i,j) \mid i, j \in N,\ \gamma(i,j) < +\infty\}$.

Proof of Theorem 3.6 (ii). Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a positively homogeneous M-convex function, and define $\gamma : N \times N \to \mathbf{R} \cup \{+\infty\}$ by $\gamma(i,j) = f(\chi_j - \chi_i)$ $(i, j \in N)$. Then, it suffices to show that $f(x) = f_\gamma(x)$ holds for any $x \in \mathbf{R}^n$.

We first claim that there exists $\{\lambda_{ij}\}_{i,j\in N}$ $(\subseteq \mathbf{R}_+)$ satisfying $\sum_{i,j\in N} \lambda_{ij}(\chi_j - \chi_i) = x$ and $f(x) \geq \sum_{i,j\in N} \lambda_{ij} f'(x; j, i)$, which implies $f(x) \geq f_\gamma(x)$. We prove this claim by induction on the cardinality of the set $\{i \in N \mid x(i) \neq 0\}$, where we may assume $x \neq \mathbf{0}$. By (M-EXC) for f, there exist some $h \in \mathrm{supp}^+(x)$ and $k \in \mathrm{supp}^-(x)$ such that

$$f(x) = f(x) + f(\mathbf{0}) \geq f(x - \alpha(\chi_h - \chi_k)) + f(\alpha(\chi_h - \chi_k)) = f(x - \alpha(\chi_h - \chi_k)) + \alpha\gamma(k, h)$$

for any $\alpha \in [0, \min\{x(h), -x(k)\}]$, where we use the fact that f is positively homogeneous. Put $x' = x - \alpha_0(\chi_h - \chi_k)$ with $\alpha_0 = \min\{x(h), -x(k)\}$. By the induction hypothesis, there exists $\{\lambda'_{ij}\}_{i,j\in N}$ $(\subseteq \mathbf{R}_+)$ satisfying $\sum_{i,j\in N} \lambda'_{ij}(\chi_j - \chi_i) = x'$ and $f(x') \geq \sum_{i,j\in N} \lambda'_{ij} f'(x; j, i)$. Putting $\lambda_{hk} = \lambda'_{hk} + \alpha_0$ and $\lambda_{ij} = \lambda'_{ij}$ for $(i,j) \neq (h,k)$, we have the claim.

On the other hand, we have $f(x) \leq \sum_{i,j\in N} \lambda_{ij} f(\chi_j - \chi_i)$ for any $\{\lambda_{ij}\}_{i,j\in N}$ $(\subseteq \mathbf{R}_+)$ with $\sum_{i,j\in N} \lambda_{ij}(\chi_j - \chi_i) = x$ since f is positively homogeneous convex. Hence $f(x) \leq f_\gamma(x)$ follows. □

3.3 Fundamental Properties of M-convex Functions

We firstly show that an M$^\natural$-convex function has supermodularity over $\mathbf{R}^n$.

Proposition 3.9 ([28, Cor. 7.5.1]). *For any closed proper convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ $x \in \mathbf{R}^n$*, and* $y \in \operatorname{dom} f$*, we have* $f(x) = \lim_{\lambda \uparrow 1} f(\lambda x + (1-\lambda)y)$.

Lemma 3.10. *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be an M-convex function, and* $x, y \in \operatorname{dom} f$ *satisfy* $\operatorname{supp}^-(x - y) = \{k\}$ *for some* $k \in N$*. Then, for any* $i \in \operatorname{supp}^+(x-y)$ *we have* $x - \{x(i) - y(i)\}(\chi_i - \chi_k) \in \operatorname{dom} f$*,* $y + \{x(i) - y(i)\}(\chi_i - \chi_k) \in \operatorname{dom} f$*, and*

$$f(x) + f(y) \geq f(x - \{x(i) - y(i)\}(\chi_i - \chi_k)) + f(y + \{x(i) - y(i)\}(\chi_i - \chi_k)). \qquad (3.6)$$

Proof. Put $\overline{\alpha} = x(i) - y(i)$, and define functions $\varphi_x, \varphi_y : [0, \overline{\alpha}] \to \mathbf{R} \cup \{+\infty\}$ by

$$\varphi_x(\alpha) = f(x - \alpha(\chi_i - \chi_k)), \quad \varphi_y(\alpha) = f(y + \{\overline{\alpha} - \alpha\}(\chi_i - \chi_k)) \quad (\alpha \in [0, \overline{\alpha}]).$$

Claim 1. Let $\alpha \in [0, \overline{\alpha}]$.
(i) If $\varphi_x(\alpha) < +\infty$, then $\varphi_x((\alpha + \overline{\alpha})/2) < +\infty$. (ii) If $\varphi_y(\alpha) < +\infty$, then $\varphi_y(\alpha/2) < +\infty$.

[Proof of Claim 1] We prove (i) only. It may be assumed that $\alpha < \overline{\alpha}$. Put $\widehat{x} = x - \alpha(\chi_i - \chi_k)$ and

$$\alpha_* = \sup\{\beta \mid f(\widehat{x} - \beta(\chi_i - \chi_k)) + f(y + \beta(\chi_i - \chi_k)) \leq f(\widehat{x}) + f(y),\ \beta \leq \{\widehat{x}(i) - y(i)\}/2\}.$$

It follows from Proposition 3.9 that $f(\widehat{x} - \alpha_*(\chi_i - \chi_k)) + f(y + \alpha_*(\chi_i - \chi_k)) \leq f(\widehat{x}) + f(y)$. Put $\widetilde{x} = \widehat{x} - \alpha_*(\chi_i - \chi_k)$, $\widetilde{y} = y + \alpha_*(\chi_i - \chi_k)$. Assume, to the contrary, that $\alpha_* < \{\widehat{x}(i) - y(i)\}/2$. Then, we have $i \in \operatorname{supp}^+(\widetilde{x} - \widetilde{y})$. By Proposition 3.2 for $\widehat{x}, y \in \operatorname{dom} f$, we also have $y(k) - \widehat{x}(k) \geq \widehat{x}(i) - y(i)$, from which $\operatorname{supp}^-(\widetilde{x} - \widetilde{y}) = \{k\}$ follows. Hence, (M-EXC) for $\widetilde{x}$ and $\widetilde{y}$ implies that there exists a sufficiently small $\beta > 0$ satisfying

$$f(\widehat{x}) + f(y) \geq f(\widetilde{x}) + f(\widetilde{y}) \geq f(\widetilde{x} - \beta(\chi_i - \chi_k)) + f(\widetilde{y} + \beta(\chi_i - \chi_k)),$$

a contradiction to the choice of α_*. Hence, $\alpha_* = \{\widehat{x}(i) - y(i)\}/2$, and we have $\varphi_x((\alpha + \overline{\alpha})/2) = f(\widehat{x} - \alpha_*(\chi_i - \chi_k)) < +\infty$. [End of Claim 1]

We also define a function $\varphi : [0, \overline{\alpha}] \to \mathbf{R} \cup \{\pm\infty\}$ by $\varphi(\alpha) = \varphi_x(\alpha) - \varphi_y(\alpha)$ $(\alpha \in [0, \overline{\alpha}])$. Since φ_x and φ_y are closed convex functions with $\varphi_x(0) < +\infty$, $\varphi_y(\overline{\alpha}) < +\infty$, we have $\varphi_x(\alpha) < +\infty$ $(\forall \alpha \in [0, \overline{\alpha}))$ and $\varphi_y(\alpha) < +\infty$ $(\forall \alpha \in (0, \overline{\alpha}])$ by Claim 1. Hence, Proposition 3.9 for φ_x and φ_y implies that φ is continuous on $(0, \overline{\alpha})$, and $\varphi(0) = \lim_{\alpha \downarrow 0} \varphi(\alpha)$, $\varphi(\overline{\alpha}) = \lim_{\alpha \uparrow \overline{\alpha}} \varphi(\alpha)$. To prove (3.6), it suffices to show that $\varphi(\alpha)$ is nonincreasing on $[0, \overline{\alpha}]$, which follows from Claim 2 below:

Claim 2. $\varphi'(\alpha; 1) \leq 0$ and $\varphi'(\alpha; -1) \geq 0$ for all $\alpha \in (0, \overline{\alpha})$.

We now prove Claim 2. It is noted that the values $\varphi'(\alpha; \pm 1) = \varphi'_x(\alpha; \pm 1) - \varphi'_y(\alpha; \pm 1)$ are well-defined for all $\alpha \in (0, \overline{\alpha})$. We here prove $\varphi'(\alpha; 1) \leq 0$ only since $\varphi'(\alpha; -1) \geq 0$ can be proven similarly. Put $x' = x - \alpha(\chi_i - \chi_k)$ and $y'_\delta = y + \{\overline{\alpha} - \alpha - \delta\}(\chi_i - \chi_k)$ for $\delta > 0$. Then, we have $i \in \operatorname{supp}^+(x' - y'_\delta)$ and $\operatorname{supp}^-(x' - y'_\delta) = \{k\}$. By (M-EXC), there exists some $\beta_0 > 0$ such that

$$f(x') + f(y'_\delta) \geq f(x' - \beta(\chi_i - \chi_k)) + f(y'_\delta + \beta(\chi_i - \chi_k)) \qquad (\forall \beta \in [0, \beta_0]),$$

implying $\varphi'_x(\alpha; 1) \leq -\varphi'_y(\alpha + \delta; -1)$. Hence follows $\varphi'_x(\alpha; 1) \leq -\lim_{\delta \downarrow 0} \varphi'_y(\alpha + \delta; -1) = \varphi'_y(\alpha; 1)$. □

Theorem 3.11. *An* $\mathrm{M}^\natural$*-convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *satisfies the supermodular inequality:*

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y) \qquad (x, y \in \mathbf{R}^n). \tag{3.7}$$

Proof. We show (3.7) by induction on the cardinality of the sets $\mathrm{supp}^+(x - y)$ and $\mathrm{supp}^-(x - y)$. If $|\mathrm{supp}^+(x - y)| \leq 1$ or $|\mathrm{supp}^-(x - y)| \leq 1$, then (3.7) follows from Lemma 3.10 and the definition of $\mathrm{M}^\natural$-convex functions. Hence, we consider the case when $|\mathrm{supp}^+(x - y)| > 1$ and $|\mathrm{supp}^-(x - y)| > 1$. We may assume $x \wedge y, x \vee y \in \mathrm{dom}\, f$, since otherwise (3.7) holds immediately. Let $j \in \mathrm{supp}^-(x - y)$. Then, we have $(x \wedge y) + \{y(j) - x(j)\}\chi_j \in \mathrm{dom}\, f$ by Lemma 3.10, and the induction assumption implies

$$f(x) - f(x \wedge y) \leq f(x + \{y(j) - x(j)\}\chi_j) - f((x \wedge y) + \{y(j) - x(j)\}\chi_j) \leq f(x \vee y) - f(y).$$

□

We then present two equivalent definitions of M-convex functions. The property (M-EXC) is equivalent to the following stronger property:

(M-EXC$_\mathbf{s}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 \leq \forall \alpha \leq \{x(i) - y(i)\}/2t),$$

where $t = |\mathrm{supp}^-(x - y)|$.

Theorem 3.12. *For a closed proper convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, (M-EXC) $\iff$ (M-EXC$_\mathrm{s}$).

Proof. We assume (M-EXC) and show (M-EXC$_\mathrm{s}$) for f. Let $x_0, y_0 \in \mathrm{dom}\, f$, and $i \in \mathrm{supp}^+(x_0 - y_0)$. Put $\mathrm{supp}^-(x_0 - y_0) = \{j_1, j_2, \cdots, j_t\}$. For $h = 1, 2, \cdots, t$, we iteratively define a function $\varphi_h : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$, a real number $\alpha_h \in \mathbf{R}$, and vectors $x_h, y_h \in \mathbf{R}^n$ by

$$\begin{aligned}
&\varphi_h(\alpha) = f(x_{h-1} - \alpha(\chi_i - \chi_{j_h})) + f(y_{h-1} + \alpha(\chi_i - \chi_{j_h})) \qquad (\alpha \in \mathbf{R}),\\
&\alpha_h \;=\; \sup\{\alpha \mid \varphi_h(\alpha) \leq \varphi_h(0),\ \alpha \leq \min[x_{h-1}(i) - y_{h-1}(i), y_{h-1}(j_h) - x_{h-1}(j_h)]/2\},\\
&x_h = x_{h-1} - \alpha_h(\chi_i - \chi_{j_h}),\ y_h = y_{h-1} + \alpha_h(\chi_i - \chi_{j_h}).
\end{aligned}$$

Since each φ_h is closed proper convex, Proposition 3.9 implies

$$x_h, y_h \in \mathrm{dom}\, f,\ \ f(x_h) + f(y_h) \leq f(x_{h-1}) + f(y_{h-1}) \quad (h = 1, 2, \ldots, t). \tag{3.8}$$

Assume, to the contrary, that $\sum_{h=1}^{t} \alpha_h < \{x_0(i) - y_0(i)\}/2$. Since $i \in \mathrm{supp}^+(x_t - y_t)$, there exist some $j_h \in \mathrm{supp}^-(x_t - y_t) \subseteq \mathrm{supp}^-(x_0 - y_0)$ and a sufficiently small $\alpha > 0$ such that

$$f(x_t) + f(y_t) \geq f(x_t - \alpha(\chi_i - \chi_{j_h})) + f(y_t + \alpha(\chi_i - \chi_{j_h})). \tag{3.9}$$

Putting $\widetilde{x}_h = x_h - \alpha(\chi_i - \chi_{j_h})$ and $\widetilde{x}_t = x_t - \alpha(\chi_i - \chi_{j_h})$, we have

$$x_h(k) = \min\{\widetilde{x}_h(k), x_t(k)\}, \qquad \widetilde{x}_t(k) = \max\{\widetilde{x}_h(k), x_t(k)\} \qquad (\forall k \in N \setminus \{i\}).$$

Therefore, Theorem 3.11 implies

$$f(x_h - \alpha(\chi_i - \chi_{j_h})) + f(x_t) \leq f(x_h) + f(x_t - \alpha(\chi_i - \chi_{j_h})). \tag{3.10}$$

Similarly, we have

$$f(y_h + \alpha(\chi_i - \chi_{j_h})) + f(y_t) \leq f(y_h) + f(y_t + \alpha(\chi_i - \chi_{j_h})). \tag{3.11}$$

From (3.9), (3.10), and (3.11) follows $f(x_h - \alpha(\chi_i - \chi_{j_h})) + f(y_h + \alpha(\chi_i - \chi_{j_h})) \leq f(x_h) + f(y_h)$, a contradiction to the definition of x_h and y_h. Let s be the index with $\alpha_s = \max\{\alpha_h \mid 1 \leq h \leq t\}$. For $\alpha \in [0, \alpha_s]$, we have

$$\begin{aligned} &\{f(x - \alpha(\chi_i - \chi_{j_s})) - f(x)\} + \{f(y + \alpha(\chi_i - \chi_{j_s})) - f(y)\} \\ &\quad\leq \{f(x_{s-1} - \alpha(\chi_i - \chi_{j_s})) - f(x_{s-1})\} + \{f(y_{s-1} + \alpha(\chi_i - \chi_{j_s})) - f(y_{s-1})\} \leq 0, \end{aligned}$$

where the first inequality is by Theorem 3.11 and the second by (3.8) and convexity of f. This shows (M-EXC$_\mathrm{s}$) for f since $\alpha_s \geq \{x(i) - y(i)\}/2t$. □

We can rewrite the exchange property (M-EXC) in terms of directional derivative.

(M-EXC′) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$:

$$f'(x; j, i) < +\infty, \quad f'(y; i, j) < +\infty, \text{ and } f'(x; j, i) + f'(y; i, j) \leq 0.$$

Note that (M-EXC′) for f yields (B-EXC) for $\mathrm{dom}\, f$ since $f'(x; j, i) < +\infty$ implies $x + \alpha(\chi_j - \chi_i) \in \mathrm{dom}\, f$ $(\forall \alpha \in [0, \alpha_0])$ for some $\alpha_0 > 0$.

Theorem 3.13. *For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$,* (M-EXC) $\iff$ (M-EXC′).

Proof. It suffices to show (M-EXC′) $\Longrightarrow$ (M-EXC). Assume (M-EXC′) for f. Let $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x - y)$. We prove that there exist some $j \in \mathrm{supp}^-(x - y)$ and $\alpha_0 > 0$ satisfying

$$f'(x - \alpha(\chi_i - \chi_j); j, i) + f'(y + \alpha(\chi_i - \chi_j); i, j) \leq 0 \qquad (\forall \alpha \in [0, \alpha_0]), \tag{3.12}$$

which, together with convexity of f, yields the desired inequality (3.1).

Put $x_* = x - |J|\beta\chi_i + \beta\chi_J$ and $y_* = y + |J|\beta\chi_i - \beta\chi_J$ with a sufficiently small $\beta > 0$ and

$$J = \{j \in \mathrm{supp}^-(x - y) \mid f'(x; j, i) < +\infty,\ f'(y; i, j) < +\infty\}.$$

By the convexity of $\mathrm{dom}\, f$, we have $x_*,\ y_* \in \mathrm{dom}\, f$. By (M-EXC′) applied to x_*, y_* and $i \in \mathrm{supp}^+(x_* - y_*)$, there exists $j_0 \in \mathrm{supp}^-(x_* - y_*)$ with $f'(x_*; j_0, i) + f'(y_*; i, j_0) \leq 0$. Since $f'(x_*; j_0, i) < +\infty$, we have $x' = x_* + \alpha(\chi_{j_0} - \chi_i) \in \mathrm{dom}\, f$ for some $\alpha > 0$. Since $j_0 \in \mathrm{supp}^+(x' - x)$ and $\mathrm{supp}^-(x' - x) = \{i\}$, the property (B-EXC) for $\mathrm{dom}\, f$ implies $x + \alpha'(\chi_{j_0} - \chi_i) \in \mathrm{dom}\, f$ for a sufficiently small $\alpha' > 0$, from which $f'(x; j_0, i) < +\infty$ follows. Similarly, we have $f'(y; i, j_0) < +\infty$. Hence, $j_0 \in J$.

The inequality $f'(x_*; j_0, i) + f'(y_*; i, j_0) \leq 0$, together with the convexity of f, implies

$$f'(x_*; i, j_0) + f'(y_*; j_0, i) \geq 0. \tag{3.13}$$

For $\alpha \in [0, \beta/2]$, we put $x_\alpha = x - \alpha(\chi_i - \chi_{j_0}) \in \mathrm{dom}\, f$ and $y_\alpha = y + \alpha(\chi_i - \chi_{j_0}) \in \mathrm{dom}\, f$. The property (M-EXC$'$) implies

$$f'(x_*; i, j_0) + f'(x_\alpha; j_0, i) \leq 0 \tag{3.14}$$

since $j_0 \in \mathrm{supp}^+(x_* - x_\alpha)$ and $\mathrm{supp}^-(x_* - x_\alpha) = \{i\}$. Similarly, we have

$$f'(y_\alpha; i, j_0) + f'(y_*; j_0, i) \leq 0. \tag{3.15}$$

Combining (3.13), (3.14), and (3.15), we have (3.12) with $j = j_0$ and $\alpha_0 = \beta/2$. □

Global optimality of an M-convex function is characterized by local optimality in terms of finite number of directional derivatives.

Theorem 3.14. *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be an M-convex function. For* $x \in \mathrm{dom}\, f$, *we have* $f(x) \leq f(y)$ $(\forall y \in \mathbf{R}^n)$ *if and only if* $f'(x; j, i) \geq 0$ $(\forall i, j \in N)$.

Proof. We show the "if" part by contradiction. Assume, to the contrary, that $f(x_0) < f(x)$ holds for some $x_0 \in \mathrm{dom}\, f$. Put $S = \{y \in \mathbf{R}^n \mid f(y) \leq f(x_0)\}$, which is a closed set since f is closed convex. Let $x_* \in S$ be a vector with $\|x_* - x\|_1 = \inf\{\|y - x\|_1 \mid y \in S\}$. By (M-EXC) applied to x and x_*, there exist some $i \in \mathrm{supp}^+(x - x_*)$, $j \in \mathrm{supp}^-(x - x_*)$, and a sufficiently small $\alpha > 0$ such that

$$f(x_*) - f(x_* + \alpha(\chi_i - \chi_j)) \geq f(x - \alpha(\chi_i - \chi_j)) - f(x) \geq f'(x; j, i) \geq 0.$$

Hence, we have $f(x_* + \alpha(\chi_i - \chi_j)) \leq f(x_*) \leq f(x_0)$, which contradicts the choice of x_* since $\|(x_* + \alpha(\chi_i - \chi_j)) - x\|_1 < \|x_* - x\|_1$. □

Although an M-convex function is not assumed to be polyhedral convex, its directional derivative functions and subdifferentials have nice polyhedral structure such as M-/L-convexity. For $p \in \mathbf{R}^n$, we define $f[p] : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ by $f[p](x) = f(x) + \langle p, x\rangle$ $(x \in \mathbf{R}^n)$. For a function $\gamma : N \times N \to \mathbf{R} \cup \{+\infty\}$, we define a set $\mathrm{D}(\gamma) \subseteq \mathbf{R}^n$ by

$$\mathrm{D}(\gamma) = \{p \in \mathbf{R}^n \mid p(j) - p(i) \leq \gamma(i, j)\ (i, j \in N)\}. \tag{3.16}$$

Recall the definition of f_γ in (3.5).

Lemma 3.15. *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be an M-convex function,* $x_0, y_0 \in \mathrm{dom}\, f$, $i \in \mathrm{supp}^+(x_0 - y_0)$, *and* $\mathrm{supp}^-(x_0 - y_0) = \{j_1, j_2, \cdots, j_t\}$, *where* $t = |\mathrm{supp}^-(x_0 - y_0)|$. *Then, there exist* $y_h \in \mathrm{dom}\, f$ *and* $\alpha_h \in \mathbf{R}_+$ $(h = 1, 2, \ldots, t)$ *satisfying* $\sum_{h=1}^{t} \alpha_h = x_0(i) - y_0(i)$ *and*

$$y_h = y_{h-1} + \alpha_h(\chi_i - \chi_{j_h}), \quad f(y_h) - f(y_{h-1}) + \alpha_h f'(x; j_h, i) \leq 0 \quad (h = 1, 2, \ldots, t).$$

Proof. Proof is quite similar to that for Theorem 3.12 and therefore omitted. □

Theorem 3.16. *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function and $x \in \mathrm{dom}\, f$. Define a function $\gamma_x : N \times N \to \mathbf{R} \cup \{\pm\infty\}$ by $\gamma_x(i,j) = f'(x; j, i)$ $(i, j \in N)$.*
(i) *γ_x satisfies $\gamma_x(i,i) = 0$ $(i \in N)$ and the triangle inequality $\gamma_x(i,j) + \gamma_x(j,k) \geq \gamma_x(i,k)$ $(i,j,k \in N)$.*
(ii) *The subdifferential $\partial f(x)$ is represented as*

$$\partial f(x) = \mathrm{D}(\gamma_x) = \{p \in \mathbf{R}^n \mid p(j) - p(i) \leq f'(x; j, i) \ (i, j \in N)\}.$$

In particular, $\partial f(x)$ is an L-convex set if $\gamma_x > -\infty$.
(iii) *We have $f'(x; \cdot) = f_{\gamma_x}$. In particular, $f'(x; \cdot)$ is positively homogeneous M-convex if $f'(x; \cdot) > -\infty$.*

Proof. (i): For $i, j, k \in N$, Lemma 3.10 implies

$$\begin{aligned} \gamma_x(i,j) + \gamma_x(j,k) &= \lim_{\alpha \downarrow 0} (1/\alpha)\{f(x + \alpha(\chi_j - \chi_i)) + f(x + \alpha(\chi_k - \chi_j)) - 2f(x)\} \\ &\geq \lim_{\alpha \downarrow 0} (1/\alpha)\{f(x + \alpha(\chi_k - \chi_i)) - f(x)\} \ = \ \gamma_x(i,k). \end{aligned}$$

(ii): Since $p \in \partial g(x)$ is equivalent to $x \in \arg\min f[-p]$, the equation $\partial f(x) = \mathrm{D}(\gamma_x)$ follows from Theorem 3.14. L-convexity of the set $\partial f(x)$ follows from (i) (see [24, Th. 3.23]).

(iii): Due to the property (i), it suffices to prove $f'(x; d) = f_{\gamma_x}(d)$ for $d \in \mathbf{R}^n$ (see [24, Th. 4.19]). Since $f'(x; \cdot)$ is a positively homogeneous convex function, we have

$$f'(x; d) = f'(x; \sum_{i,j \in N} \lambda_{ij}(\chi_j - \chi_i)) \leq \sum_{i,j \in N} \lambda_{ij} f'(x; j, i)$$

for any $\{\lambda_{ij}\}_{i,j \in N}$ $(\subseteq \mathbf{R}_+)$ satisfying $\sum_{i,j \in N} \lambda_{ij}(\chi_j - \chi_i) = d$. Hence, we have $f'(x; d) \leq f_{\gamma_x}(d)$.

On the other hand, repeated application of Lemma 3.15 implies that for any $\alpha > 0$ there exists $\{\lambda_{ij}\}_{i,j \in N}$ $(\subseteq \mathbf{R}_+)$ satisfying $\sum_{i,j \in N} \lambda_{ij}(\chi_j - \chi_i) = d$ and

$$f(x + \alpha d) - f(x) \geq \alpha \sum_{i,j \in N} \lambda_{ij} f'(x; j, i) \geq \alpha f_{\gamma_x}(d).$$

This implies $f'(x; d) \geq f_{\gamma_x}(d)$. Hence, we have $f'(x; d) = f_{\gamma_x}(d)$. □

The next theorem shows that each "face" of the epigraph of an M-convex function is a polyhedron given by an M-convex set. The proof is obvious and therefore omitted.

Theorem 3.17. *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function. For $p \in \mathbf{R}^n$, $\arg\min f[p]$ is M-convex if it is nonempty.*

4 L-convex Functions over the Real Space

4.1 Definitions of L-convex and L$^\natural$-convex Functions

We call a function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *L-convex* if g is a closed proper convex function satisfying (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \qquad (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R})$.

Due to the property (LF2), an L-convex function loses no information other than r when restricted to a hyperplane $\{p \in \mathbf{R}^n \mid p(i) = 0\}$ for any $i \in N$. We call a function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p_0, p) = g(p - p_0 \mathbf{1}) \quad ((p_0, p) \in \mathbf{R}^{\widehat{N}})$$

is L-convex, where $\widehat{N} = \{0\} \cup N$. L$^\natural$-convexity of g is characterized by the following property:

(L$^\natural$F) $g(p) + g(q) \geq g(p \vee (q - \lambda \mathbf{1})) + g((p + \lambda \mathbf{1}) \wedge q) \qquad (\forall p, q \in \mathrm{dom}\, g,\ \forall \lambda \geq 0)$.

Theorem 4.1 ([24, Th. 4.39]). *A closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is L$^\natural$-convex if and only if it satisfies* (L$^\natural$F)*.*

We denote by $\mathcal{L}_n$ (resp. $\mathcal{L}_n^\natural$) the class of L-convex (resp. L$^\natural$-convex) functions in n variables, i.e.,

$$\mathcal{L}_n = \{g \mid g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{L-convex}\}, \qquad \mathcal{L}_n^\natural = \{g \mid g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{L}^\natural\text{-convex}\}.$$

As is obvious from the definitions, L$^\natural$-convex function is essentially equivalent to L-convex function, whereas the class of L$^\natural$-convex functions contains that of L-convex functions as a proper subclass. This relationship between L-convexity and L$^\natural$-convexity can be summarized as

$$\mathcal{L}_n \subset \mathcal{L}_n^\natural \simeq \mathcal{L}_{n+1}.$$

From the definition of L$^\natural$-convex functions, every property of L-convex functions can be restated in terms of L$^\natural$-convex functions, and vice versa. In this paper, we primarily work with L-convex functions, making explicit statements for L$^\natural$-convex functions when appropriate.

We also define the set version of L-convexity and L$^\natural$-convexity. We call a set $D \subseteq \mathbf{R}^n$ *L-convex* (resp. *L$^\natural$-convex*) if its indicator function $\delta_D : \mathbf{R}^n \to \{0, +\infty\}$ is an L-convex (resp. L$^\natural$-convex) function. Equivalently, an L-convex set is defined as a nonempty closed set satisfying (LS1) and (LS2):

(LS1) $p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D$, $\qquad$ **(LS2)** $p \in D \Longrightarrow p + \lambda \mathbf{1} \in D\ (\forall \lambda \in \mathbf{R})$.

In fact, L-convex and L$^\natural$-convex sets are polyhedral as shown later in Section 4.2. Hence, these concepts coincide with those of L-convex and L$^\natural$-convex polyhedra introduced in [24].

Proposition 4.2. *For an L-convex function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$,* $\mathrm{dom}\, f$ *satisfies* (LS1) *and* (LS2)*, and is an L-convex set if it is a closed set.*

The properties (LS1) and (LS2), without closedness, imply convexity for sets.

Theorem 4.3. *If $D \subseteq \mathbf{R}^n$ satisfies* (LS1) *and* (LS2)*, then D is a convex set.*

Proof. We show that $(1-\alpha)p + \alpha q \in D$ holds for any $p, q \in D$ and $\alpha \in [0,1]$. By the property (LS2), we may assume $p \leq q$. Then, we have $q = p + \sum_{h=1}^{k} \lambda_h \chi_{N_h}$ for some $\lambda_h > 0$ and $N_h \subseteq N$ ($h = 1, 2, \ldots, k$) such that $\emptyset \neq N_1 \subset N_2 \subset \cdots \subset N_k$. In the following, we prove by induction that $p'_j \equiv p + \alpha \sum_{h=1}^{j} \lambda_h \chi_{N_h} \in D$ holds for $j = 0, 1, \ldots, k$. We have $p'_0 = p \in D$. Suppose that $p'_{j-1} \in D$ for $j - 1 < k$. Since $p + \sum_{h=1}^{j} \lambda_h \chi_{N_h} = p \vee (q - \sum_{h=j+1}^{k} \lambda_h \mathbf{1}) \in D$, we have

$$p'_j = p'_{j-1} + \alpha \lambda_j \chi_{N_j} = (p'_{j-1} + \alpha \lambda_j \mathbf{1}) \wedge (p + \sum_{h=1}^{j} \lambda_h \chi_{N_h}) \in D.$$

Hence, $p'_k = (1-\alpha)p + \alpha q \in D$ follows. □

Remark 4.4. The properties (LF1) and (LF2) imply mid-point convexity (see Theorem 4.9 below); however, they are independent of convexity and continuity.

Consider the function $\psi : \mathbf{R} \to \mathbf{R}$ satisfying Jensen's equation (2.3) such that ψ is neither continuous nor convex. Define a function $g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ as in (2.5). Then, g satisfies the submodular inequality (LF1) with equality and (LF2) with $r = 0$, and is neither convex nor continuous. □

Remark 4.5. There exists no function which is both M-convex and L-convex, i.e., $\mathcal{M}_n \cap \mathcal{L}_n = \emptyset$ (Proof: Proposition 3.2 (ii) implies $x(N) = y(N)$ for any $f \in \mathcal{M}_n$ and $x, y \in \mathrm{dom}\, f$, whereas (LF2) implies that $x + \lambda \mathbf{1} \in \mathrm{dom}\, f$ for any $\lambda \in \mathbf{R}$.) On the other hand, the classes of M$^\natural$-convex and L$^\natural$-convex functions have nonempty intersection, as shown in Example 2.4. □

Remark 4.6. The effective domain of an L-convex function is not a closed set in general. For example, the function $g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p_1, p_2) = \begin{cases} 1/(p_1 - p_2) & ((p_1, p_2) \in \mathbf{R}^2,\ p_1 - p_2 > 0), \\ +\infty & (\text{otherwise}) \end{cases}$$

is L-convex, and $\mathrm{dom}\, g = \{(p_1, p_2) \mid p_1 - p_2 > 0\}$ is not a closed set. □

4.2 L-convex Sets and Positively Homogeneous L-convex Functions

L-convex sets and positively homogeneous L-convex functions constitute important subclasses of L-convex functions, which appear as local structure of general L-convex functions such as minimizers and directional derivative functions (see Theorem 4.11). We show that L-convex sets and positively homogeneous L-convex functions are polyhedral sets and polyhedral convex functions, respectively.

Theorem 4.7.
(i) *Any L-convex set is a polyhedron.*
(ii) *Any positively homogeneous L-convex function is a polyhedral convex function.*

In the following, we prove Theorem 4.7. Recall the definition of a set $\mathrm{D}(\gamma) \subseteq \mathbf{R}^n$ in (3.16).

Proof of Theorem 4.7 (i). Let $D \subseteq \mathbf{R}^n$ be a nonempty closed set with satisfying (LS1) and (LS2). We define a function $\gamma_D : N \times N \to \mathbf{R} \cup \{+\infty\}$ by $\gamma_D(i,j) = \sup_{p\in D}\{p(j) - p(i)\}$ $(i, j \in N)$. To prove (i), it suffices to show $D = \mathrm{D}(\gamma_D)$ since $\mathrm{D}(\gamma_D)$ is a polyhedron. The inclusion $D \subseteq \mathrm{D}(\gamma_D)$ is easy to see; to show the reverse inclusion, we prove that $q \in D$ holds for any $q \in \mathrm{D}(\gamma_D)$.

For any $\varepsilon > 0$ and any $i, j \in N$ there exists $p^\varepsilon_{ij} \in D$ with $p^\varepsilon_{ij}(j) - p^\varepsilon_{ij}(i) + \varepsilon > \gamma_D(i,j) \geq q(j) - q(i)$, where we may assume that $p^\varepsilon_{ij}(i) = q(i)$ and $p^\varepsilon_{ij}(j) + \varepsilon > q(j)$ by (LS2). For $i \in N$, we define $p^\varepsilon_i = \bigvee_{i\in N} p^\varepsilon_{ij}$, which is contained in D by (LS1). The vector p^ε_i satisfies $p^\varepsilon_i(i) = q(i)$ and $p^\varepsilon_i(j)+\varepsilon > q(j)$ for $j \in N$. We then define $p^\varepsilon \in D$ by $p^\varepsilon = \bigwedge_{i\in N} p^\varepsilon_i$, which satisfies $q(i) - \varepsilon < p^\varepsilon(i) \leq q(i)$ for $i \in N$, and hence $\lim_{\varepsilon\to 0} p^\varepsilon = q$. Hence, the closedness of D implies $q \in D$. □

We then prove Theorem 4.7 (ii).

Lemma 4.8 ([24, Lemma 4.28]). *Let $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$. Then, g satisfies* (LF1) *and* (LF2) *if and only it satisfies*

$$g(p) + g(q) \geq g(p \vee (q - \lambda\mathbf{1})) + g((p + \lambda\mathbf{1}) \wedge q) \qquad (\forall p, q \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R}).$$

In particular, if g satisfies (LF1) *and* (LF2)*, then we have*

$$g(p) + g(q) \geq g(p + \lambda\chi_X) + g(q - \lambda\chi_X) \qquad (\forall p, q \in \mathrm{dom}\, g,\ \forall \lambda \in [0, \lambda_1 - \lambda_2]), \tag{4.1}$$

where $\lambda_1 = \max\{q(i) - p(i) \mid i \in N\}$, $X = \{i \in N \mid q(i) - p(i) = \lambda_1\}$, and $\lambda_2 = \max\{q(i) - p(i) \mid i \in N \setminus X\}$.

For a set function $\rho : 2^N \to \mathbf{R} \cup \{\pm\infty\}$, we define $g_\rho : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ by

$$g_\rho(p) = \sum_{j=1}^{k} (\lambda_j - \lambda_{j+1})\rho(N_j), \tag{4.2}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ are distinct values in $\{p(i)\}_{i\in N}$, $\lambda_{k+1} = 0$, and $N_j = \{i \in N \mid p(i) \geq \lambda_j\}$ $(j = 1, 2, \ldots, k)$. The function g_ρ is called the *Lovász extension* of ρ [9, 16].

Proof of Theorem 4.7 (ii). Let $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a positively homogeneous L-convex function, and define $\rho : 2^N \to \mathbf{R} \cup \{+\infty\}$ by $\rho_g(X) = g(\chi_X)$ $(X \subseteq N)$. We prove that $g(p) = g_\rho(p)$ holds for $p \in \mathbf{R}^n$, which shows that g is a polyhedral convex function.

We may assume that $p \geq \mathbf{0}$ since (LF2) for g implies $g(p) - g_\rho(p) = g(p+\lambda\mathbf{1}) - g_\rho(p+\lambda\mathbf{1})$ $(\lambda \in \mathbf{R})$. Define λ_j and N_j $(j = 1, 2, \ldots, k)$ as in the Lovász extension (4.2), and put $\lambda_{k+1} = 0$. Since g is positively homogeneous convex, we have $g(p) \leq \sum_{j=1}^{k}(\lambda_j - \lambda_{j+1})g(\chi_{N_j}) = g_\rho(p)$. On the other hand, it follows from Lemma 4.8 that

$$\begin{aligned} g(\sum_{j=h}^{k}(\lambda_j - \lambda_{j+1})\chi_{N_j}) - g(\sum_{j=h+1}^{k}(\lambda_j - \lambda_{j+1})\chi_{N_j}) &\geq g((\lambda_h - \lambda_{h+1})\chi_{N_h}) - g(\mathbf{0}) \\ &= (\lambda_h - \lambda_{h+1})\rho(N_h) \qquad (h = 1, 2, \ldots, k). \end{aligned}$$

This implies $g(p) \geq \sum_{j=1}^{k}(\lambda_h - \lambda_{h+1})\rho(N_h) = g_\rho(p)$. Hence, we have $g(p) = g_\rho(p)$. □

4.3 Fundamental Properties of L-convex Functions

We show various properties of L-convex functions. Note that some of the properties below are implied by (LF1) and (LF2) only, and independent of closed convexity of functions.

Theorem 4.9. *If* $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (LF1) *and* (LF2)*, then* g *is mid-point convex.*

Proof. We show the inequality (2.2) by induction on the cardinality of $\mathrm{supp}(p - q) \equiv \mathrm{supp}^+(p - q) \cup \mathrm{supp}^-(p - q)$. We may assume $p(i) < q(i)$ for some $i \in N$. Putting $\lambda = \{q(i) - p(i)\}/2$, $p' = p \vee (q - \lambda\mathbf{1})$ and $q' = (p + \lambda\mathbf{1}) \wedge q$, we have $g(p) + g(q) \geq g(p') + g(q')$ by Lemma 4.8. Since $\mathrm{supp}(p' - q') \subseteq \mathrm{supp}(p - q) \setminus \{i\}$, the induction hypothesis yields $g(p') + g(q') \geq 2g(\{p' + q'\}/2) = 2g(\{p + q\}/2)$. □

Global optimality of an L-convex function is characterized by local optimality in terms of finite number of directional derivatives.

Theorem 4.10 ([24, Th. 4.29]). *Let* $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be a convex function with* (LF1) *and* (LF2)*. For* $p \in \mathrm{dom}\, g$*, we have* $g(p) \leq g(q)$ $(\forall q \in \mathbf{R}^n)$ *if and only if* $g'(p; \chi_X) \geq 0$ $(\forall X \subset N)$ *and* $g'(p; \mathbf{1}) = 0$*.*

Proof. We show the sufficiency by contradiction. Suppose that there exists some $q \in \mathbf{R}^n$ with $g(q) < g(p)$, and assume that q minimizes the number of distinct values in $\{q(i) - p(i)\}_{i=1}^n$ of all such vectors. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the distinct values in $\{q(i) - p(i)\}_{i=1}^n$. From (LF2) and $g'(p; \mathbf{1}) = 0$, we have $k \geq 2$. By the inequality (4.1) with $\lambda = \lambda_1 - \lambda_2$ and $g'(p; \chi_X) \geq 0$, we have $g(q') \leq g(q) < g(p)$ with $q' = q - (\lambda_1 - \lambda_2)\chi_X$. This inequality, however, is a contradiction since the number of distinct values in $\{q'(i) - p(i)\}_{i=1}^n$ is $k - 1$. □

Although an L-convex function is not assumed to be polyhedral convex, its directional derivative functions and subdifferentials have nice polyhedral structure such as L-/M-convexity. For $x \in \mathbf{R}^n$, we define $g[x] : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ by $g[x](p) = g(p) + \langle p, x\rangle$ $(p \in \mathbf{R}^n)$. Recall the definitions of $\mathrm{B}(\rho)$ and g_ρ in (3.4) and in (4.2), respectively.

Theorem 4.11. *Let* $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be an L-convex function and* $q \in \mathrm{dom}\, g$*. Define a function* $\rho_q : 2^N \to \mathbf{R} \cup \{\pm\infty\}$ *by* $\rho_q(X) = g'(q; \chi_X)$ $(X \subseteq N)$*.*
(i) ρ_q *satisfies* $\rho_q(\emptyset) = 0$*,* $-\infty < \rho_q(N) < +\infty$*, and the submodular inequality (3.3).*
(ii) *The subdifferential* $\partial g(q)$ *is represented as*

$$\partial g(q) = \mathrm{B}(\rho_q) = \{x \in \mathbf{R}^n \mid x(X) \leq g'(q; \chi_X)\ (\forall X \subset N),\ x(N) = g'(q; \mathbf{1})\}.$$

In particular, $\partial g(q)$ *is an M-convex set if* $\rho_q > -\infty$*.*
(iii) *The function* $g'(q; \cdot)$ *satisfies* (LF1), (LF2)*, and* $g'(q; \cdot) = g_{\rho_q}$*. In particular,* $g'(q; \cdot)$ *is a positively homogeneous L-convex function if* $g'(q; \cdot) > -\infty$*.*

Proof. We first prove (iii) and then (ii); (i) is immediate from (iii).

(iii): For any $p \in \mathbf{R}^n$ and $\varepsilon > 0$, there exists some $\mu > 0$ such that $g'(q; p) > (1/\mu)\{g(q + \mu p) - g(q)\} - \varepsilon$. Hence, (LF1) and (LF2) for $g'(q; \cdot)$ follow from (LF1) and (LF2) for g.

We then prove $g'(q;p) = g_{\rho_q}(p)$ ($p \in \mathbf{R}^n$). Since $g'(q;\cdot)$ is a positively homogeneous convex function, we have

$$g_{\rho_q}(p) = \sum_{j=1}^{k}(\lambda_j - \lambda_{j+1})g'(q;\chi_{N_j}) \geq g'(q;\sum_{j=1}^{k}(\lambda_j - \lambda_{j+1})\chi_{N_j}) = g'(q;p),$$

where λ_j and N_j ($j = 1, 2, \ldots, k$) are defined as in the Lovász extension (4.2) and $\lambda_{k+1} = 0$. Put

$$q_0 = q + p,\ q_j = q_{j-1} - (\lambda_j - \lambda_{j+1})\chi_{N_j}\ (j = 1, 2, \cdots, k),$$

where $q_k = q$. By Lemma 4.8, we obtain

$$g(q_{j-1}) - g(q_j) \geq g(q + (\lambda_j - \lambda_{j+1})\chi_{N_j}) - g(q) \geq (\lambda_j - \lambda_{j+1})g'(q;\chi_{N_j}) \quad (j = 1, 2, \ldots, k),$$

from which follows

$$g(q+p) - g(q) \geq \sum_{k=1}^{k}(\lambda_j - \lambda_{j+1})g'(q;\chi_{N_j}) = g_{\rho_q}(p).$$

Since the inequality above holds for any $p \in \mathbf{R}^n$, we have $g(q + \mu p) - g(q) \geq g_{\rho_q}(\mu p) = \mu g_{\rho_q}(p)$, implying $g'(q;p) \geq g_{\rho_q}(p)$. Thus, $g'(q;p) = g_{\rho_q}(p)$ follows.

(ii) Since $x \in \partial g(q)$ is equivalent to $q \in \arg\min g[-x]$, we have $\partial g(q) = \mathrm{B}(\rho_q)$ by Theorem 4.10. Since ρ_q is a submodular function, $\partial g(q)$ is an M-convex set (see [24, Th. 3.3]). □

The next theorem shows that each "face" of the epigraph of an L-convex function is a polyhedron given by an L-convex set. The proof is obvious and therefore omitted.

Theorem 4.12. *Let $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be an L-convex function. For $x \in \mathbf{R}^n$, $\arg\min g[x]$ is L-convex if it is nonempty.*

5 Conjugacy

For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, its *(convex) conjugate* $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n).$$

Theorem 5.1 ([28, Theorem 12.2]). *For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, the conjugate function $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is also closed proper convex, and $f^{\bullet\bullet} = f$.*

Hence, the conjugacy operation $f \mapsto f^\bullet$ induces a symmetric one-to-one correspondence in the class of all closed proper convex functions on $\mathbf{R}^n$. In this section, we show that M-convex and L-convex functions are conjugate to each other.

Theorem 5.2.
(i) *For $f \in \mathcal{M}_n$, we have $f^\bullet \in \mathcal{L}_n$ and $f^{\bullet\bullet} = f$.* (ii) *For $g \in \mathcal{L}_n$, we have $g^\bullet \in \mathcal{M}_n$ and $g^{\bullet\bullet} = g$.*
(iii) *The mappings $f \mapsto f^\bullet$ ($f \in \mathcal{M}_n$) and $g \mapsto g^\bullet$ ($g \in \mathcal{L}_n$) provide a one-to-one correspondence between $\mathcal{M}_n$ and $\mathcal{L}_n$, and are the inverse of each other.*

In the following, we first prove "$f \in \mathcal{M}_n \Longrightarrow f^\bullet \in \mathcal{L}_n$" and then "$g \in \mathcal{L}_n \Longrightarrow g^\bullet \in \mathcal{M}_n$." The other claims are immediate from these and Theorem 5.1.

Lemma 5.3. *If a function $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ in two variables is supermodular, then its conjugate $f^\bullet : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ is submodular.*

Proof. It suffices to show

$$f^\bullet(\lambda, \mu) + f^\bullet(\lambda', \mu') \leq f^\bullet(\lambda, \mu') + f^\bullet(\lambda', \mu) \tag{5.1}$$

for $(\lambda, \mu), (\lambda', \mu') \in \mathbf{R}^2$ with $\lambda \geq \lambda'$ and $\mu \geq \mu'$. We claim that

$$[\lambda\alpha + \mu\beta - f(\alpha, \beta)] + [\lambda'\alpha' + \mu'\beta' - f(\alpha', \beta')] \leq f^\bullet(\lambda, \mu') + f^\bullet(\lambda', \mu) \tag{5.2}$$

holds for any $(\alpha, \beta), (\alpha', \beta') \in \mathbf{R}^2$. The inequality (5.1) is immediate from (5.2), since the supremum of the left-hand side of (5.2) over (α, β) and (α', β') coincides with the left-hand side of (5.1).

We now prove (5.2). If $\alpha \geq \alpha'$ and $\beta \geq \beta'$, we have $f(\alpha, \beta) + f(\alpha', \beta') \geq f(\alpha, \beta') + f(\alpha', \beta)$ by the supermodularity of f, and therefore

$$\text{LHS of (5.2)} \leq [\lambda\alpha + \mu'\beta' - f(\alpha, \beta')] + [\lambda'\alpha' + \mu\beta - f(\alpha', \beta)] \leq \text{RHS of (5.2)}.$$

If $\alpha \leq \alpha'$, we have $\lambda\alpha + \lambda'\alpha' \leq \lambda\alpha' + \lambda'\alpha$ and therefore

$$\text{LHS of (5.2)} \leq [\lambda\alpha' + \mu'\beta' - f(\alpha', \beta')] + [\lambda'\alpha + \mu\beta - f(\alpha, \beta)] \leq \text{RHS of (5.2)}.$$

We can prove (5.2) similarly for the case $\beta \leq \beta'$. □

Lemma 5.4. *For $f \in \mathcal{M}_n$, we have $f^\bullet \in \mathcal{L}_n$.*

Proof. Let $f \in \mathcal{M}_n$. By Theorem 5.1, the conjugate $f^\bullet$ is a proper closed convex function with $f^\bullet > -\infty$. Hence, we have only to show (LF1) and (LF2) for $f^\bullet$.

We firstly show (LF2) for $f^\bullet$. Put $r = x(N)$ by some $x \in \mathrm{dom}\, f$, which is independent of the choice of x by Proposition 3.2. For $p \in \mathrm{dom}\, f^\bullet$ and $\lambda \in \mathbf{R}$, we have

$$\begin{aligned} f^\bullet(p + \lambda \mathbf{1}) &= \sup\{\langle p + \lambda \mathbf{1}, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\} \\ &= \sup\{\langle p, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\} + \lambda x(N) \;=\; f^\bullet(p) + \lambda r. \end{aligned}$$

To prove the submodularity (LF1) for $f^\bullet$, we first assume that $\mathrm{dom}\, f$ is bounded. Since $\mathrm{dom}\, f^\bullet = \mathbf{R}^n$, the submodularity of $f^\bullet$ is equivalent to the local submodularity (see, e.g., [24, Theorem 4.27]):

$$f^\bullet(p + \lambda\chi_i) + f^\bullet(p + \mu\chi_j) \geq f^\bullet(p) + f^\bullet(p + \lambda\chi_i + \mu\chi_j), \tag{5.3}$$

where $p \in \mathbf{R}^n$, $i, j \in N$ are distinct indices, and $\lambda, \mu \in \mathbf{R}_+$. We fix $p \in \mathbf{R}^n$, and define functions $\widetilde{g} : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ and $\widetilde{f} : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$\begin{aligned} &\widetilde{g}(\lambda, \mu) = f^\bullet(p + \lambda\chi_i + \mu\chi_j) \qquad (\lambda, \mu \in \mathbf{R}), \\ &\widetilde{f}(\alpha, \beta) = \inf\{f(x) - \langle p, x\rangle \mid x \in \mathrm{dom}\, f,\ x(i) = \alpha,\ x(j) = \beta\} \qquad (\alpha, \beta \in \mathbf{R}). \end{aligned}$$

Then, $\widetilde{f}$ is a supermodular function, as shown below, and $\widetilde{g} = (\widetilde{f})^\bullet$ holds. Hence, the function $\widetilde{g}$ is a submodular function by Lemma 5.3, implying the inequality (5.3).

We now prove that $\widetilde{f}$ is a supermodular function.

Claim. For any $(\alpha, \beta), (\alpha', \beta') \in \mathrm{dom}\, \widetilde{f}$ with $\alpha > \alpha'$ and $\beta \geq \beta'$, there exists $\delta_0 > 0$ satisfying

$$\widetilde{f}(\alpha, \beta) + \widetilde{f}(\alpha', \beta') \geq \widetilde{f}(\alpha - \delta, \beta) + \widetilde{f}(\alpha' + \delta, \beta') \qquad (\forall \delta \in [0, \delta_0]).$$

[Proof of Claim] We may assume $p = \mathbf{0}$ since $f(x) - \langle p, x\rangle$ is also M-convex as a function in x. Since f is a closed proper convex function with bounded effective domain, there exist $x, x' \in \mathrm{dom}\, f$ satisfying $x(i) = \alpha$, $x(j) = \beta$, $\widetilde{f}(\alpha, \beta) = f(x)$, and $x'(i) = \alpha'$, $x'(j) = \beta'$, $\widetilde{f}(\alpha', \beta') = f(x')$, respectively. We have $i \in \mathrm{supp}^+(x - x')$, and therefore (M-EXC) for f implies that there exist $k \in \mathrm{supp}^-(x - x')$ and $\delta_0 > 0$ satisfying

$$\begin{aligned} \widetilde{f}(\alpha, \beta) + \widetilde{f}(\alpha', \beta') \;=\; f(x) + f(x') \;&\geq\; f(x - \delta(\chi_i - \chi_k)) + f(x' + \delta(\chi_i - \chi_k)) \\ &\geq\; \widetilde{f}(\alpha - \delta, \beta) + \widetilde{f}(\alpha' + \delta, \beta') \qquad (\forall \delta \in [0, \delta_0]), \end{aligned}$$

where it is noted that $k \neq j$ since $j \notin \mathrm{supp}^-(x - x')$. [End of Claim]

Using Claim above as well as the fact that $\widetilde{f}$ is closed proper convex, we can show the supermodularity of $\widetilde{f}$ in the same way as in Lemma 3.10. This concludes the proof of (LF1) for $f^\bullet$ when $\mathrm{dom}\, f$ is bounded.

Finally, we consider the case when $\mathrm{dom}\, f$ is unbounded. For a fixed vector $x_0 \in \mathrm{dom}\, f$, we define $f_k : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ by

$$f_k(x) = \begin{cases} f(x) & (x \in \mathbf{R}^n,\ |x(i) - x_0(i)| \leq k \text{ for all } i \in N), \\ +\infty & (\text{otherwise}). \end{cases}$$

Since $f_k \in \mathcal{M}_n$ and $\mathrm{dom}\, f_k$ is bounded, $f_k^\bullet$ fulfills (LF1). Hence, for any $p, q \in \mathrm{dom}\, f^\bullet$ we have

$$f^\bullet(p) + f^\bullet(q) = \lim_{k\to\infty}\{f_k^\bullet(p) + f_k^\bullet(q)\} \geq \lim_{k\to\infty}\{f_k^\bullet(p\wedge q) + f_k^\bullet(p\vee q)\} = f^\bullet(p\wedge q) + f^\bullet(p\vee q).$$

□

We then prove the claim "$g \in \mathcal{L}_n \Longrightarrow g^\bullet \in \mathcal{M}_n$." Recall that for $x \in \mathbf{R}^n$ the function $g[-x] : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ is defined by $g[-x](p) = g(p) - \langle p, x\rangle$ $(p \in \mathbf{R}^n)$.

Lemma 5.5. *Let $g \in \mathcal{L}_n$, and $x, y \in \mathbf{R}^n$ be vectors with $\arg\min g[-x] \neq \emptyset$ and $\arg\min g[-y] \neq \emptyset$. Then, for any $i \in \mathrm{supp}^+(x-y)$, there exists $j \in \mathrm{supp}^-(x-y)$ such that*

$$p(j) - p(i) \leq q(j) - q(i) \quad (\forall p \in \arg\min g[-x], \forall q \in \arg\min g[-y]). \tag{5.4}$$

Proof. First we note that $x(N) = y(N) = r$, where $r \in \mathbf{R}$ is the value in (LF2) for g. Since $\arg\min g[-x]$ and $\arg\min g[-y]$ are L-convex sets by Theorem 4.12, the inequality (5.4) can be rewritten as $p(j) \leq q(j)$ $(\forall p \in D_x,\ \forall q \in D_y)$, where

$$D_x = \{p \in \mathbf{R}^n \mid p \in \arg\min g[-x],\ p(i) = 0\}, \quad D_y = \{p \in \mathbf{R}^n \mid p \in \arg\min g[-y],\ p(i) = 0\}.$$

Assume, to the contrary, that for any $j \in \mathrm{supp}^-(x-y)$, there exists a pair of vectors $p_j \in D_x$, $q_j \in D_y$ such that $p_j(j) > q_j(j)$. Putting

$$p_x = \bigvee\{p_j \mid j \in \mathrm{supp}^-(x-y)\}, \qquad q_y = \bigwedge\{q_j \mid j \in \mathrm{supp}^-(x-y)\},$$

we have $p_x \in D_x$, $q_y \in D_y$, and $\mathrm{supp}^-(x-y) \subseteq \mathrm{supp}^+(p_x - q_y)$. We also put $S^+ = \mathrm{supp}^+(p_x - q_y)$, $\lambda = \min\{p_x(j) - q_y(j) \mid j \in S^+\}$ (> 0). Then, Lemma 4.8 implies

$$g(p_x) + g(q_y) \geq g((p_x - \lambda\mathbf{1})\vee q_y) + g(p_x \wedge (q_y + \lambda\mathbf{1})). \tag{5.5}$$

Since

$$((p_x - \lambda\mathbf{1})\vee q_y)(j) = \begin{cases} p_x(j) - \lambda & (j \in S^+), \\ q_y(j) & (j \in N\setminus S^+), \end{cases} \qquad (p_x \wedge (q_y + \lambda\mathbf{1}))(j) = \begin{cases} q_y(j) + \lambda & (j \in S^+), \\ p_x(j) & (j \in N\setminus S^+), \end{cases}$$

we have

$$\begin{aligned}
&\langle (p_x - \lambda\mathbf{1})\vee q_y, x\rangle + \langle p_x \wedge (q_y + \lambda\mathbf{1}), y\rangle - \langle p_x, x\rangle - \langle q_y, y\rangle \\
&\quad = \lambda\sum_{j\in S^+}\{y(j) - x(j)\} + \sum_{j\in N\setminus S^+}\{q_y(j) - p_x(j)\}\{x(j) - y(j)\} \\
&\quad \geq \lambda\sum_{j\in S^+}\{y(j) - x(j)\} \ \geq\ \lambda\sum_{v\in N\setminus\{i\}}\{y(j) - x(j)\} \ =\ \lambda\{x(i) - y(i)\} \ >\ 0,
\end{aligned} \tag{5.6}$$

where the inequalities follow from $\mathrm{supp}^-(x-y) \subseteq S^+$. From (5.5) and (5.6) follows

$$g[-x]((p_x - \lambda\mathbf{1})\vee q_y) + g[-y](p_x \wedge (q_y + \lambda\mathbf{1})) < g[-x](p_x) + g[-y](q_y),$$

which is a contradiction to the fact that $p_x \in \arg\min g[-x]$, $q_y \in \arg\min g[-y]$. □

Proposition 5.6 (cf. [28, Th. 23.4]). *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be a convex function with* $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid x(N) = r\}$ *for some* $r \in \mathbf{R}$*. Then, for any* $x \in \mathrm{dom}\, f$ *we have* $\partial f(x) \neq \emptyset$ *and* $f'(x; y) = \sup\{\langle p, y\rangle \mid p \in \partial f(x)\}$ $(y \in \mathbf{R}^n)$*.*

Lemma 5.7. *For* $g \in \mathcal{L}_n$*, we have* $g^\bullet \in \mathcal{M}_n$*.*

Proof. It is easy to see that the conjugate function $g^\bullet$ satisfies $\mathrm{dom}\, g^\bullet \subseteq \{x \in \mathbf{R}^n \mid x(N) = r\}$, where $r \in \mathbf{R}$ is the value in (LF2) for g. We firstly consider the case when $\mathrm{dom}\, g^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$. Let $x, y \in \mathrm{dom}\, g^\bullet$ and $i \in \mathrm{supp}^+(x-y)$. From Proposition 5.6 follows $\arg\min g[-x] = \partial g^\bullet(x) \neq \emptyset$ and $\arg\min g[-y] = \partial g^\bullet(y) \neq \emptyset$. By Lemma 5.5, there exists $j \in \mathrm{supp}^-(x-y)$ satisfying (5.4), implying

$$\begin{aligned}&(g^\bullet)'(x; j, i) + (g^\bullet)'(y; i, j)\\ &\quad= \sup\{p(j) - p(i) \mid p \in \arg\min g[-x]\} + \sup\{q(i) - q(j) \mid q \in \arg\min g[-y]\} \leq 0,\end{aligned}$$

where the equality is by Proposition 5.6. This shows (M-EXC′) for $g^\bullet$, which, together with Theorem 3.13, yields M-convexity of $g^\bullet$.

We then consider the general case. For fixed $j_0 \in N$ and $q \in \mathrm{dom}\, g$ with $q(j_0) = 0$, we define $g_k : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ $(k = 1, 2, \cdots)$ by

$$g_k(p) = \begin{cases} g(p) & (p \in \mathbf{R}^n,\ |p(i) - p(j_0) - q(i)| \leq k \text{ for all } i \in N), \\ +\infty & (\text{otherwise}). \end{cases}$$

It can be easily shown that each g_k is an L-convex function with $\mathrm{dom}\, g_k^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$. Therefore, the discussion above shows that each $g_k^\bullet$ is M-convex, and therefore satisfies (M-EXC$_\mathrm{s}$) by Theorem 3.12. For $x, y \in \mathrm{dom}\, g^\bullet$ $(\subseteq \mathrm{dom}\, g_k^\bullet)$ and $i \in \mathrm{supp}^+(x-y)$, there exists some $j_k \in \mathrm{supp}^-(x-y)$ such that

$$g_k^\bullet(x) + g_k^\bullet(y) \geq g_k^\bullet(x - \alpha(\chi_i - \chi_{j_k})) + g_k^\bullet(y + \alpha(\chi_i - \chi_{j_k})) \qquad (\forall \alpha \in [0, \{x(i) - y(i)\}/2t])$$

with $t = |\mathrm{supp}^-(x-y)|$. Since $\mathrm{supp}^-(x-y)$ is a finite set, we may assume that $j_k = j_*$ $(k = 1, 2, \ldots)$ for some $j_* \in \mathrm{supp}^-(x-y)$. Then, for any $\alpha \in [0, \{x(i) - y(i)\}/2t]$ we have

$$\begin{aligned}g^\bullet(x) + g^\bullet(y) = \lim_{k\to\infty}\{g_k^\bullet(x) + g_k^\bullet(y)\} &\geq \lim_{k\to\infty}\{g_k^\bullet(x - \alpha(\chi_i - \chi_{j_*})) + g_k^\bullet(y + \alpha(\chi_i - \chi_{j_*}))\}\\ &= g^\bullet(x - \alpha(\chi_i - \chi_{j_*})) + g^\bullet(y + \alpha(\chi_i - \chi_{j_*})).\end{aligned}$$

Thus, (M-EXC$_\mathrm{s}$) holds for $g^\bullet$, which shows M-convexity of $g^\bullet$ by Theorem 3.12. □

This concludes the proof of Theorem 5.2.

As an application of Theorem 5.2, we give the proof for the characterization of M$^\natural$-convex functions by the property (M$^\natural$-EXC).

Proof of Theorem 3.1. Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a closed proper convex function satisfying (M$^\natural$-EXC). We shall show the M-convexity of the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ in (3.2), which is equivalent to the L-convexity of $\widehat{f}^\bullet = (\widehat{f})^\bullet$ by Theorem 5.2. We prove the L-convexity of $\widehat{f}^\bullet$ in a similar way as in

Lemma 5.4. In particular, the proof of Lemma 5.4 shows that it suffices to prove the following claim. For $i \in N$ and $j \in \widehat{N}$ with $i \neq j$, define a function $\widehat{f}_{ij} : \mathbf{R}^2 \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\begin{aligned}\widehat{f}_{ij}(\alpha,\beta) &= \inf\{\widehat{f}(\widehat{x}) - \langle \widehat{p}, \widehat{x}\rangle \mid \widehat{x} \in \mathrm{dom}\,\widehat{f},\ \widehat{x}(i)=\alpha,\ \widehat{x}(j)=\beta\} \\ &= \begin{cases} \inf\{f(x) - \langle p, x\rangle + p_0 x(N) \mid x \in \mathrm{dom}\,f,\ x(i)=\alpha,\ x(j)=\beta\} & (j \in N), \\ \inf\{f(x) - \langle p, x\rangle + p_0 x(N) \mid x \in \mathrm{dom}\,f,\ x(i)=\alpha,\ x(N)=-\beta\} & (j = 0), \end{cases}\end{aligned}$$

where $\widehat{p} = (p_0, p) \in \mathbf{R}^{\widehat{N}}$.

Claim. Suppose that $\mathrm{dom}\,f$ is bounded. Then, each $\widehat{f}_{ij}$ satisfies the following property:

$\forall(\alpha,\beta),(\alpha',\beta') \in \mathrm{dom}\,\widehat{f}_{ij}$ with $\alpha > \alpha'$ and $\beta \geq \beta'$, $\exists\delta_0 > 0$:

$$\widehat{f}_{ij}(\alpha,\beta) + \widehat{f}_{ij}(\alpha',\beta') \geq \widehat{f}_{ij}(\alpha-\delta,\beta) + \widehat{f}_{ij}(\alpha'+\delta,\beta') \qquad (\forall\delta \in [0,\delta_0]). \tag{5.7}$$

We now prove the claim above. We may assume that $p = \mathbf{0}$ and $p_0 = 0$ since $f_{(p_0,p)}(z) = f(z) - \langle p, z\rangle + p_0 z(N)$ ($z \in \mathbf{R}^n$) also satisfies (M$^\natural$-EXC). We consider the case $j = 0$ only, since the case of $j \in N$ can be dealt with similarly and more easily.

Since f is a closed proper convex function with bounded effective domain, there exist $x, x' \in \mathrm{dom}\,f$ satisfying

$$\begin{aligned} &x(i) = \alpha, \quad x(N) = -\beta, \quad \widehat{f}_{ij}(\alpha,\beta) = f(x), \\ &x'(i) = \alpha', \quad x'(N) = -\beta', \quad \widehat{f}_{ij}(\alpha',\beta') = f(x'). \end{aligned}$$

We assume that x minimizes the value $||x - x'||_1$ among all such vectors. It suffices to show that

$$f(x) + f(x') \geq f(x - \delta(\chi_i - \chi_k)) + f(x' + \delta(\chi_i - \chi_k)) \qquad (\forall\delta \in [0,\delta_0]) \tag{5.8}$$

for some $k \in \mathrm{supp}^-(x - x')$ and $\delta_0 > 0$ since the RHS of (5.8) is larger than or equal to $\widehat{f}_{ij}(\alpha - \delta,\beta) + \widehat{f}_{ij}(\alpha' + \delta,\beta')$. By the property (M$^\natural$-EXC) for x, x', and $i \in \mathrm{supp}^+(x - x')$, there exist $s \in \mathrm{supp}^-(x - x') \cup \{0\}$ and $\delta_1 > 0$ such that

$$f(x) + f(x') \geq f(x - \delta(\chi_i - \chi_s)) + f(x' + \delta(\chi_i - \chi_s)) \qquad (\forall\delta \in [0,\delta_1]). \tag{5.9}$$

Since $x(i) > x'(i)$ and $x(N) \leq x'(N)$, there exists some $r \in \mathrm{supp}^+(x' - x)$. By (M$^\natural$-EXC), there exist $t \in \mathrm{supp}^-(x' - x) \cup \{0\}$ and $\delta_2 > 0$ such that

$$f(x') + f(x) \geq f(x' - \delta(\chi_r - \chi_t)) + f(x + \delta(\chi_r - \chi_t)) \qquad (\forall\delta \in [0,\delta_2]). \tag{5.10}$$

We consider the following four cases.

[Case 1: $s \in \mathrm{supp}^-(x - x')$] (5.9) gives the desired inequality (5.8) with $k = s$ and $\delta_0 = \delta_1$.

[Case 2: $t = i \in \mathrm{supp}^-(x' - x)$] (5.10) gives the desired inequality (5.8) with $k = r$ and $\delta_0 = \delta_2$.

[Case 3: $t \in \mathrm{supp}^-(x' - x) \setminus \{i\}$] Putting $x^\delta = x + \delta(\chi_r - \chi_t)$ with a sufficiently small $\delta > 0$, we have $x^\delta(i) = \alpha$, $x^\delta(N) = -\beta$, and $||x^\delta - x'||_1 < ||x - x'||_1$. By (5.10), the vector x^δ satisfies $\widehat{f}_{ij}(\alpha,\beta) = f(x^\delta)$, a contradiction to the choice of x.

[Case 4: $s = t = 0$] Convexity of f as well as the inequalities (5.9) and (5.10) imply

$$\begin{aligned} f(x) + f(x') &\geq (1/2)\{f(x - \delta\chi_i) + f(x + \delta\chi_r)\} + (1/2)\{f(x' + \delta\chi_i) + f(x' - \delta\chi_r)\} \\ &\geq f(x - (1/2)\delta(\chi_i - \chi_r)) + f(x' + (1/2)\delta(\chi_i - \chi_r)) \qquad (0 \leq \forall\delta \leq \min\{\delta_1, \delta_2\}). \end{aligned}$$

Hence, we have the desired inequality (5.8) with $k = r$ and $\delta_0 = (1/2)\min\{\delta_1, \delta_2\}$. □

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