Characterization of Rankings Generated by Linear Discriminant Analysis

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Abstract

Pairwise linear discriminant analysis can be regarded as a process to generate rankings of the populations. But in general, not all rankings are generated. We give a characterization of generated rankings. We also derive some basic properties of this model.

Key Words and Phrases: arrangement of hyperplanes, Euclidean preferences, ideal point model, linear discriminant analysis, polytope, ranking, social choice, Stirling numbers, Voronoi polyhedron.

1 Introduction

Consider the problem of linear discriminant analysis among m normal populations in $\mathbb{R}^n$ with equal covariance matrices: $N_n(\boldsymbol{\mu}_i, \Sigma),\ i = 1, 2, \ldots, m$. We assume that Σ is known, and deal with the canonical case $\Sigma = I$ (the identity matrix). In each pairwise comparison of populations, Fisher's discriminant hyperplane in this setting is just the bisector of the line segment connecting the means of the two populations in question, and so the nearer population in the Euclidean distance is selected (Muirhead (1982), Seber (1984)). Here, we are identifying the populations with their means. Thus, Fisher's linear discriminant rule among these m populations takes the following form: Allocate a test sample point $\boldsymbol{x} \in \mathbb{R}^n$ to the nearest population.

But we often have an interest not only in the most probable population but in the second most, the third most,..., and the mth most (or the least) probable populations. That is, we want to rank the m populations from the most probable to the least probable one. In such a case, Fisher's rule suggests the ranking rule according to the distances to the populations: the nearest population is ranked first, the second nearest ranked second, and so on.

In terms of the division of the sample space $\mathbb{R}^n$ by discriminant hyperplanes, this rule is equivalent to the following. The whole $\mathbb{R}^n$ is divided into regions by $m(m-1)/2$ discriminant hyperplanes, and the order of distances to the populations is the same at all points in each such region. So each region can be indexed by the ranking determined by that order. Now the rule is: Give a test sample point the ranking which indexes the region where the point lies.

In this way, pairwise discriminant analysis can be regarded as a process to generate rankings among the m populations. However, unless the dimension n is large enough compared with the number of populations m, not all the $m!$ rankings are generated. So the questions arise as to (Q-1) how many and (Q-2) what kind of rankings are generated.

Actually, the model discussed so far is also known as the ideal point model or the unfolding model, and is widely used in applied statistics such as psychometrics, marketing research, etc. (Carroll (1980), Carroll and De Soete (1991)). Furthermore, many variants and generalizations of this model are devised and utilized for practical data analysis in those fields (DeSarbo and Hoffman (1987), Takane (1987, 1989a, 1989b), Takane, Bozdogan and Shibayama (1987)). However, although a lot of effort has been put into the development of this model for practical purposes, theoretical investigation does not seem to have been much conducted.

Moreover, in social choice theory, this model has been common under the name Euclidean preferences. One can avoid voting cycles (known as Condorcet's paradox) or escape the dictatorship conclusion of Arrow's impossibility theorem by assuming some similarity of preferences across society (Mas-Colell, Whinston and Green (1995)). Caplin and Nalebuff (1988) assume Euclidean preferences as a restriction on individual preferences, and, with some restriction on the distribution of preferences, establish the existence of a super-majority winner. The essential feature of Euclidean preferences to their results is the division of $\mathbb{R}^n$ by hyperplanes, and generalization to "intermediate preferences" (Grandmont (1978)) or "linear preferences" (Caplin and Nalebuff (1991a)) is possible from this perspective; see Caplin and Nalebuff (1988, 1991a). However, in spite of its considerable significance, this model has not been studied deeply enough in the social choice literature either. For instance, concerning (Q-1) above, Caplin and Nalebuff (1988, 1991b) only state that all $m!$ rankings are generated when $m \leq n+1$ and that some of the $m!$ rankings are ruled out when $m > n+1$. (Proposition 1 of Caplin and Nalebuff (1988), Proposition 8 of Caplin and Nalebuff (1991b)).

Recently, Kamiya and Takemura (1997, 2000) began to study theoretical questions about this model, and gave a complete answer to question (Q-1) by using the theory of hyperplane arrangements. But the second question (Q-2) is much more difficult, and they gave only a partial answer.

By finding a characterization of generated rankings, the present paper provides a complete answer to (Q-2), in the sense that for each ranking we can easily determine whether it is a) generated as an unbounded region, b) generated as a bounded region, or c) not generated. Also, this characterization will be used to provide another derivation of the answer to (Q-1), i.e., the formula for the number of generated rankings. Moreover, the paper derives a number of basic properties of the model. For example, it is shown that a "neutral" population can never be ranked last.

The distinction between Kamiya and Takemura (1997, 2000) and the present paper is as follows: In Kamiya and Takemura (1997, 2000), we embedded the sample space $\mathbb{R}^n$ in $\mathbb{R}^{m-1}$—the "right" space in the sense that it is the smallest space where all $m!$ rankings are generated; in the present paper, on the other hand, we embed $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$, and this makes central arrangements serve as a building block in the study of non-central arrangements in $\mathbb{R}^n$.

Throughout the paper, we make extensive use of the theory of hyperplane arrangements. Application of hyperplane arrangements to probability and statistics may also be found in Brown and Diaconis (1998) and Bidigare, Hanlon and Rockmore (1999). For the theory of hyperplane arrangements, the reader is referred to the excellent book by Orlik and Terao (1992). We use this book as a general reference for definitions and results concerning arrangements of hyperplanes. For basic concepts about lattices and order, Davey and Priestley (2002) is a nice introduction.

The organization of this paper is as follows. In Section 2, we define several concepts used throughout the paper. In Section 3, we derive main results of this paper. Specifically, we give a complete characterization of generated rankings. In addition, using this result on the characterization of generated rankings, we give another proof of the formulae for the numbers of various rankings in Kamiya and Takemura (1997, 2000). In Section 4, we find some properties when there is a neutral population. In Section 5, we suggest a future direction of our research.

2 Preliminaries

In this section, we introduce several concepts needed in the paper.

Suppose we are given $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_M \in \mathbb{R}^N$.

We say that the vectors $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_M$ are *lumped together* iff, when regarded as points in $\mathbb{R}^N$, they are contained in an open halfspace determined by a hyperplane passing through the origin:

$$\{\boldsymbol{\nu} \in \mathbb{R}^N : \boldsymbol{\nu}^T \boldsymbol{x} > 0\}, \quad \exists \boldsymbol{x} \in \mathbb{R}^N. \tag{1}$$

That is, $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_M$ are lumped together iff $\boldsymbol{\nu}_1^T \boldsymbol{x} > 0, \dots, \boldsymbol{\nu}_M^T \boldsymbol{x} > 0$ for some $\boldsymbol{x} \in \mathbb{R}^N$. Unless otherwise stated, we agree that a halfspace always means one determined by a hyperplane going through the origin, i.e., a homogeneous halfspace. We call the $\boldsymbol{x}$ in (1) the *direction* of that open halfspace. Furthermore, when the last coordinate of $\boldsymbol{x} = (x_1, \dots, x_N)^T$ is positive $x_N > 0$ (resp. negative $x_N < 0$), we say $\boldsymbol{x}$ is *upward* (resp. *downward*); when x_N is zero, $\boldsymbol{x}$ is said to be *horizontal*.

On the other hand, we say $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_M$ are *spread out* iff there exist nonnegative $c_1, \dots, c_M \in \mathbb{R}$ such that $(c_1, \dots, c_M) \neq (0, \dots, 0)$ and

$$c_1 \boldsymbol{\nu}_1 + \dots + c_M \boldsymbol{\nu}_M = \mathbf{0}.$$

Then we know the following fact, which is an easy consequence of Farkas' Lemma (Ziegler (1995)).

Gordan's Theorem. *For any given $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_M \in \mathbb{R}^N$, one and only one of the following two alternatives holds: (i) $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_M$ are lumped together; (ii)*

$\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_M$ *are spread out.*

Next we define the concept of non-degeneracy in our context. We say that the points $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_M \in \mathbb{R}^N$ are *non-degenerate* iff for any $l \le N$, any collection of l vectors from $\{\boldsymbol{\nu}_i - \boldsymbol{\nu}_j : i < j\}$ in which more than l different points $\boldsymbol{\nu}_i$ appear is linearly independent. Note that our definition of non-degeneracy for discriminant analysis is stronger than the usual definition of general position in convex analysis (Brøndsted (1983), p.10).

3 Main results

Suppose we are given $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathbb{R}^n$. We write $\tilde{\boldsymbol{\mu}}_i = (\boldsymbol{\mu}_i^T, -\|\boldsymbol{\mu}_i\|^2/2)^T, \ i = 1, \ldots, m$.

Throughout the paper, we make the following assumptions:

Assumptions.

(A-1) Points $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathbb{R}^n$ *are non-degenerate.*

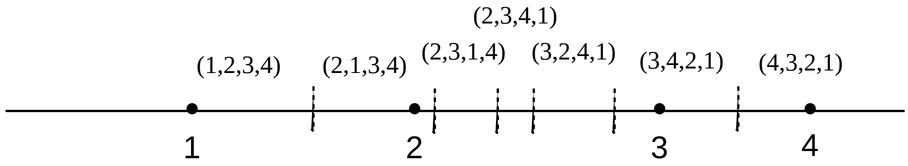
(A-2) Points $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m \in \mathbb{R}^{n+1}$ *are non-degenerate.*

We denote by $(i_1, i_2, \ldots, i_m)$ an ordering of $\{1, 2, \ldots, m\}$ in which i_1 is ranked first, i_2 is ranked second, and so on. When

$$\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{\mu}_{i_1}\| < \|\boldsymbol{x} - \boldsymbol{\mu}_{i_2}\| < \cdots < \|\boldsymbol{x} - \boldsymbol{\mu}_{i_m}\|\} \tag{2}$$

is non-empty, we index this region by $(i_1, i_2, \ldots, i_m)$, and say that *ordering* $(i_1, i_2, \ldots, m)$ *is generated.* Moreover, we identify region (2) with ordering $(i_1, i_2, \ldots, i_m)$ and say something like "Region $(i_1, i_2, \ldots, i_m)$ arises." In the so-called ideal point model, an individual with an "ideal point" $\boldsymbol{x} \in \mathbb{R}^n$ ranks given m objects $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathbb{R}^n$ as $(i_1, i_2, \ldots, i_m)$ in order of his/her preference iff $\|\boldsymbol{x} - \boldsymbol{\mu}_{i_1}\| < \|\boldsymbol{x} - \boldsymbol{\mu}_{i_2}\| < \cdots < \|\boldsymbol{x} - \boldsymbol{\mu}_{i_m}\|$. So we can say that the above indexing of regions is the one based on the ideal point model. As was explained

Figure 1: Generated rankings for a case of $n = 1,\ m = 4$



in the Introduction, Fisher's linear discriminant functions for all pairwise comparisons of m normal populations $N_n(\boldsymbol{\mu}_i, I),\ i = 1, 2, \ldots, m$, lead to the ranking rule based on this indexing of regions.

We first note the following equivalences:

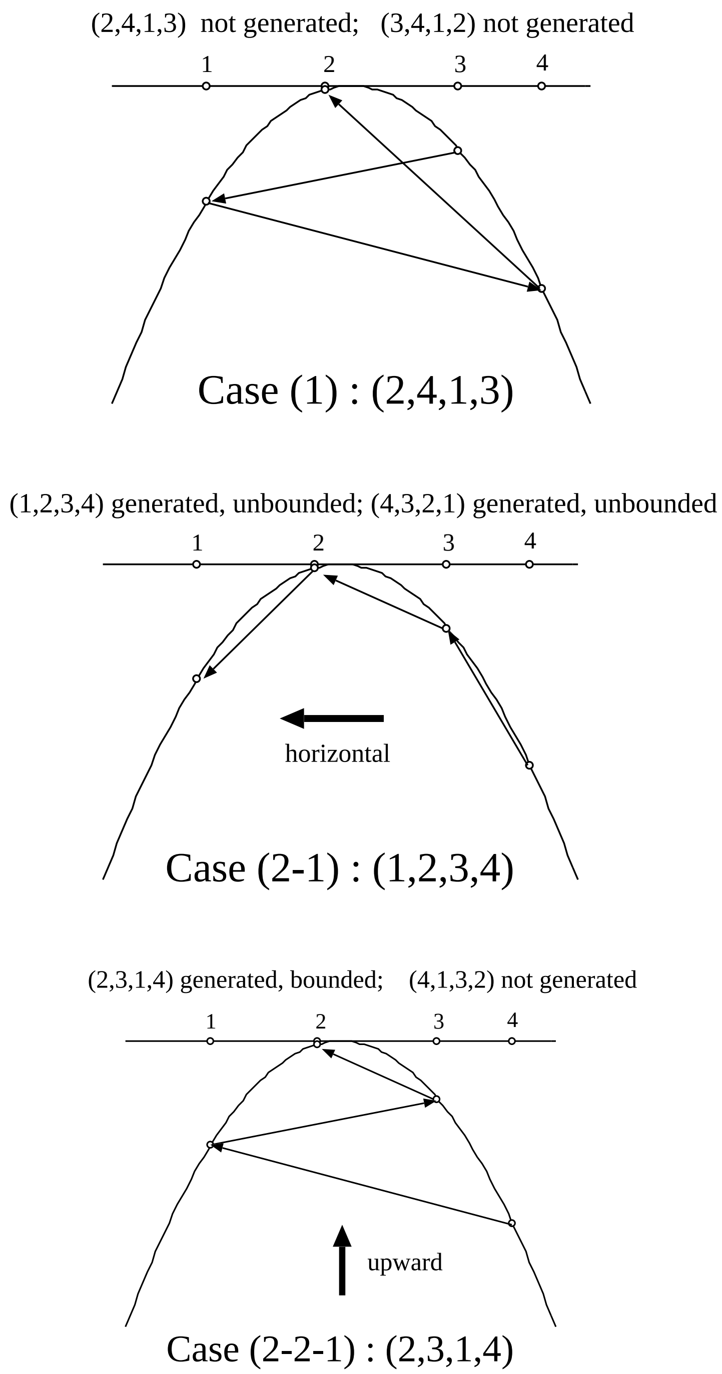
$$\begin{aligned}
&\text{Region } (i_1, i_2, \ldots, i_m) \text{ is non-empty} \\
&\iff \exists \boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{\mu}_{i_1}\| < \|\boldsymbol{x} - \boldsymbol{\mu}_{i_2}\| < \cdots < \|\boldsymbol{x} - \boldsymbol{\mu}_{i_m}\| \\
&\iff \exists \boldsymbol{x} : \boldsymbol{\mu}_{i_1}^T \boldsymbol{x} - \frac{\|\boldsymbol{\mu}_{i_1}\|^2}{2} > \boldsymbol{\mu}_{i_2}^T \boldsymbol{x} - \frac{\|\boldsymbol{\mu}_{i_2}\|^2}{2} > \cdots > \boldsymbol{\mu}_{i_m}^T \boldsymbol{x} - \frac{\|\boldsymbol{\mu}_{i_m}\|^2}{2} \\
&\iff \exists \boldsymbol{x} : (\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_{j+1}})^T \boldsymbol{x} + \left(\frac{\|\boldsymbol{\mu}_{i_j}\|^2}{-2} - \frac{\|\boldsymbol{\mu}_{i_{j+1}}\|^2}{-2} \right) > 0, \quad j = 1, 2, \ldots, m-1, \\
&\iff \exists \tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} : (\tilde{\boldsymbol{\mu}}_{i_j} - \tilde{\boldsymbol{\mu}}_{i_{j+1}})^T \tilde{\boldsymbol{x}} > 0, \quad j = 1, 2, \ldots, m-1.
\end{aligned}$$

We are now in a position to state the main results. Relabeling $i_1, i_2, \ldots, i_m$, we may only consider $(1, 2, \ldots, m)$. Illustrations of the following results for a case of $n = 1,\ m = 4$ are given in Figures 1 and 2.

(1) When $\tilde{\boldsymbol{\mu}}_1 - \tilde{\boldsymbol{\mu}}_2, \tilde{\boldsymbol{\mu}}_2 - \tilde{\boldsymbol{\mu}}_3, \ldots, \tilde{\boldsymbol{\mu}}_{m-1} - \tilde{\boldsymbol{\mu}}_m$ *(or equivalently, all* $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_j$ *with* $i < j$*) are spread out, ordering* $(1, 2, \ldots, m)$ *does not arise. In this case, neither does the reverse ordering* $(m, m-1, \ldots, 1)$.

Proof of (1). Since $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$, are spread out, there does not exist an $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^T, x_{n+1})^T \in \mathbb{R}^{n+1}$ such that $(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T \tilde{\boldsymbol{x}} > 0,\ i = 1, 2, \ldots, m-1$; still less such an $\tilde{\boldsymbol{x}}$ with $x_{n+1} = 1$. Therefore, ordering $(1, 2, \ldots, m)$

Figure 2: Characterization of rankings for $n = 1,\ m = 4$



does not arise in this case.

Next we have to show that the reverse ordering is not generated either. But this is obvious, since in this case $\tilde{\boldsymbol{\mu}}_{i+1} - \tilde{\boldsymbol{\mu}}_i = -(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1}),\ i = 1, 2, \ldots, m-1$, are also spread out.

This situation is illustrated by the upper part of Figure 2. **Q.E.D.**

(2) When $\tilde{\boldsymbol{\mu}}_1 - \tilde{\boldsymbol{\mu}}_2, \tilde{\boldsymbol{\mu}}_2 - \tilde{\boldsymbol{\mu}}_3, \ldots, \tilde{\boldsymbol{\mu}}_{m-1} - \tilde{\boldsymbol{\mu}}_m$ *(or equivalently, all* $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_j$ *with* $i < j$*) are lumped together, the situation falls into two cases:*

(2-1) There exists an open halfspace containing points $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1} \in \mathbb{R}^{n+1},\ i = 1, 2, \ldots, m-1$, *whose direction is horizontal. Or equivalently, the projections of* $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$, *onto the first* n *coordinates* $\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1},\ i = 1, 2, \ldots, m-1$, *are lumped together. In this case, ordering* $(1, 2, \ldots, m)$ *arises as an unbounded region. Furthermore, the same is true for the reverse ordering in this case.*

Proof of (2-1). Since $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2, \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3, \ldots, \boldsymbol{\mu}_{m-1} - \boldsymbol{\mu}_m$ are lumped together, there exists an $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x} > 0, \quad i = 1, 2, \ldots, m-1. \tag{3}$$

For this $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, we can take $x_{n+1} > 0$ so small that $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x} + \left(\frac{\|\boldsymbol{\mu}_i\|^2}{-2} - \frac{\|\boldsymbol{\mu}_{i+1}\|^2}{-2}\right) x_{n+1} > 0$ for all $i = 1, 2, \ldots, m-1$. Dividing both sides by x_{n+1}, we obtain $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x}' + \left(\frac{\|\boldsymbol{\mu}_i\|^2}{-2} - \frac{\|\boldsymbol{\mu}_{i+1}\|^2}{-2}\right) > 0,\ i = 1, 2, \ldots, m-1$, with $\boldsymbol{x}' = \boldsymbol{x}/x_{n+1}$. Thus we see that the region indexed by $(1, 2, \ldots, m)$ arises.

Next, since $\boldsymbol{x}'$ satisfies $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x}' > 0$, we have

$$\begin{aligned}
(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T (\boldsymbol{x}_0 + c\boldsymbol{x}') &+ \left(\frac{\|\boldsymbol{\mu}_i\|^2}{-2} - \frac{\|\boldsymbol{\mu}_{i+1}\|^2}{-2}\right) \\
&= \left\{ (\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x}_0 + \left(\frac{\|\boldsymbol{\mu}_i\|^2}{-2} - \frac{\|\boldsymbol{\mu}_{i+1}\|^2}{-2}\right) \right\} \\
&\quad + c(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x}' > 0, \qquad i = 1, 2, \ldots, m-1,
\end{aligned}$$

for any $c \geq 0$ and any $\boldsymbol{x}_0$ in the region. Hence, the region recedes in the direction of $\boldsymbol{x} \neq \mathbf{0}$, and this proves that ordering $(1, 2, \ldots, m)$ arises as an unbounded region.

Finally, to see the reverse ordering also arises as an unbounded region, just notice that $\boldsymbol{\mu}_{i+1} - \boldsymbol{\mu}_i = -(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1}),\ i = 1, 2, \ldots, m-1$, are also lumped together.

This case is illustrated by the middle part of Figure 2. In Figure 2, the thick arrow represents the horizontal direction of the halfspace. **Q.E.D.**

(2-2) There does not exist an open halfspace containing points $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1} \in \mathbb{R}^{n+1},\ i = 1, 2, \ldots, m-1$, *whose direction is horizontal. Or equivalently, the projections of* $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$, *onto the first* n *coordinates* $\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1},\ i = 1, 2, \ldots, m-1$, *are spread out. In this case, one and only one of the following two situations occurs:*

(2-2-1) The direction of any open halfspace containing $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$, *is upward, regardless of the choice of such an open halfspace.*

(2-2-2) The direction of any open halfspace containing $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$, *is downward, regardless of the choice of such an open halfspace.*

Proof of (2-2). Suppose there existed $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^T, x_{n+1})^T \in \mathbb{R}^{n+1},\ x_{n+1} > 0$, and $\tilde{\boldsymbol{y}} = (\boldsymbol{y}^T, y_{n+1})^T \in \mathbb{R}^{n+1},\ y_{n+1} < 0$, such that $(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T \tilde{\boldsymbol{x}} > 0,\ (\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T \tilde{\boldsymbol{y}} > 0,\ i = 1, 2, \ldots, m-1$. Then, $\tilde{\boldsymbol{z}} = (\boldsymbol{z}^T, z_{n+1})^T = c\tilde{\boldsymbol{x}} + (1-c)\tilde{\boldsymbol{y}}$ with $c = y_{n+1}/(y_{n+1} - x_{n+1})$ would satisfy $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{z} = (\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T \tilde{\boldsymbol{z}} = c(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T \tilde{\boldsymbol{x}} + (1-c)(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T \tilde{\boldsymbol{y}} > 0,\ i = 1, 2, \ldots, m-1$, contradicting the fact that $\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1},\ i = 1, 2, \ldots, m-1$, are spread out. The discussion above shows that the direction of an open halfspace containing points $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$, is always upward or always downward, depending only on the given $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$. **Q.E.D.**

In the case of (2-2-1), ordering $(1, 2, \ldots, m)$ *arises as a bounded region, whereas the reverse ordering does not arise.*

Proof. In this case, there exists an $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^T, x_{n+1})^T \in \mathbb{R}^{n+1},\ x_{n+1} > 0$, such that

$$(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T \tilde{\boldsymbol{x}} > 0, \quad i = 1, 2, \ldots, m-1. \tag{4}$$

Dividing both sides of this inequality by x_{n+1}, we find that ordering $(1, 2, \ldots, m)$ arises in this case.

We move on to showing that the region indexed by $(1, 2, \ldots, m)$ is bounded. Suppose to the contrary that the region in question is unbounded. Then, the closure of the region is a non-empty, unbounded polyhedral set. Thus this polyhedral set recedes in a certain direction $\boldsymbol{x} \neq \boldsymbol{0}$:

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T (\boldsymbol{x}_0 + c\boldsymbol{x}) + \left(\frac{\|\boldsymbol{\mu}_i\|^2}{-2} - \frac{\|\boldsymbol{\mu}_{i+1}\|^2}{-2} \right) \geq 0, \quad i = 1, 2, \ldots, m-1, \tag{5}$$

for all $c \geq 0$ and $\boldsymbol{x}_0$ in the polyhedral set. Now, since $\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1},\ i = 1, 2, \ldots, m-1$, are spread out, the direction of recession $\boldsymbol{x}$ cannot satisfy $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x} > 0$ for all $i = 1, 2, \ldots, m-1$; there must be some $i_1 < i_2 < \cdots < i_k$ such that $(\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_j+1})^T \boldsymbol{x} \leq 0,\ j = 1, 2, \ldots, k$. We claim that these inequalities are actually all equalities, for if there existed some i_j satisfying $(\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_j+1})^T \boldsymbol{x} < 0$, then $(\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_j+1})^T (\boldsymbol{x}_0 + c\boldsymbol{x}) + \left(\frac{\|\boldsymbol{\mu}_{i_j}\|^2}{-2} - \frac{\|\boldsymbol{\mu}_{i_j+1}\|^2}{-2} \right) < 0$ for all sufficiently large c, in contradiction to (5). Hence we have

$$(\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_j+1})^T \boldsymbol{x} = 0, \quad j = 1, 2, \ldots, k.$$

We will show that $k \leq n-1$. If $k \geq n$, then in view of Assumption (A-1), any subcollection of n vectors from $\{\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_j+1} : 1 \leq j \leq k\}$ would be linearly independent, which is impossible since these n vectors lie in the orthogonal complement of the line spanned by vector $\boldsymbol{x}$. So k cannot exceed $n-1$. Now we consider the following central arrangement of hyperplanes $\mathcal{A} = \{H_{i_j,i_j+1} : 1 \leq j \leq k\}$, where $H_{i_j,i_j+1} = \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_j+1})^T \boldsymbol{x} = 0\}$. By Assumption (A-1), it can easily be checked that the Poincaré polynomial of $\mathcal{A}$ (Orlik and Terao (1992), Definition 2.48) is $\pi(\mathcal{A}, t) = (1+t)^k$. So Zaslavsky's result (Zaslavsky (1975) or Orlik and Terao (1992), Theorem 2.68) implies that the number of regions of $\mathcal{A}$ is $\pi(\mathcal{A}, 1) = 2^k$. Therefore, we can take another point $\boldsymbol{x'} \in \mathbb{R}^n$, sufficiently close to $\boldsymbol{x}$, which satisfies $(\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_j+1})^T \boldsymbol{x'} > 0,\ j = 1, 2, \ldots, k$, while keeping the relations $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x'} > 0$ for $i \notin \{i_j : 1 \leq j \leq k\}$: that is,

$$(\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1})^T \boldsymbol{x'} > 0$$

for all $i = 1, 2, \ldots, m-1$. But this contradicts the fact that $\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i+1},\ i = 1, 2, \ldots, m-1$, are spread out. In this way, we arrive at the conclusion that the region in question is bounded.

Finally, we prove that the reverse ordering is not generated. Suppose it were generated. Then there would exist an $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^T, 1)^T \in \mathbb{R}^{n+1}$ satisfying $(\tilde{\boldsymbol{\mu}}_{i+1} - \tilde{\boldsymbol{\mu}}_i)^T \tilde{\boldsymbol{x}} > 0,\ i = 1, 2, \ldots, m-1$, and $-\tilde{\boldsymbol{x}} = (-\boldsymbol{x}^T, -1)^T$ would meet $(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1})^T (-\tilde{\boldsymbol{x}}) > 0,\ i = 1, 2, \ldots, m-1$. But this is a contradiction, since in the case we are considering, the direction of any open halfspace containing $\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_{i+1},\ i = 1, 2, \ldots, m-1$, must be upward.

This case is illustrated by the lower part of Figure 2. **Q.E.D.**

In the case of (2-2-2), ordering $(1, 2, \ldots, m)$ *does not arise, while the reverse ordering arises as a bounded region.*

Proof. In this case, there does not exist an $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^T, x_{n+1})^T \in \mathbb{R}^{n+1}$ satisfying (4) with $x_{n+1} > 0$, much less with $x_{n+1} = 1$. Hence, $(1, 2, \ldots, m)$ does not arise.

The proof of the fact that the reverse ordering arises as a bounded region is easy: Just notice that there exists an $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^T, x_{n+1})^T \in \mathbb{R}^{n+1},\ x_{n+1} < 0$, satisfying (4), and rewrite the left-hand side of (4) as $(\tilde{\boldsymbol{\mu}}_{i+1} - \tilde{\boldsymbol{\mu}}_i)^T(-\tilde{\boldsymbol{x}})$. Then we can apply case (2-2-1). **Q.E.D.**

The indeterminacy of the first n coordinates of the direction vector $\tilde{\boldsymbol{x}}$ corresponds to the translation invariance of the discriminant analysis in $\mathbb{R}^n$. When we translate the origin by $\boldsymbol{a}$ so that

$$\boldsymbol{x} \mapsto \boldsymbol{x} - \boldsymbol{a}, \qquad \boldsymbol{\mu}_i \mapsto \boldsymbol{\mu}_i - \boldsymbol{a}, \quad i = 1, \ldots, m,$$

the problem remains the same. Therefore, it is natural that only the last coordinate of $\tilde{\boldsymbol{x}}$ is related to the characterization of generation of rankings.

The considerations so far are summarized in Theorem 3.1 below. Let us say that ordering $(i_1, i_2, \ldots, i_m)$ is *consistent with* $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$ (resp. $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$) iff $\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_{j+1}},\ j = 1, \ldots, m-1$ (resp. $\tilde{\boldsymbol{\mu}}_{i_j} - \tilde{\boldsymbol{\mu}}_{i_{j+1}},\ j = 1, \ldots, m-1$) are lumped together. Note that if $\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_{j+1}},\ j = 1, \ldots, m-1$, are lumped together, then so are all $\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_k},\ j < k$. Similarly for $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$. In the so-called ideal vector model, an individual with an "ideal vector" $\boldsymbol{x}$ ranks $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$ as $(i_1, i_2, \ldots, i_m)$ iff $\boldsymbol{\mu}_{i_1}^T \boldsymbol{x} > \boldsymbol{\mu}_{i_2}^T \boldsymbol{x} > \cdots > \boldsymbol{\mu}_{i_m}^T \boldsymbol{x}$. But this condition can be written as $(\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_{j+1}})^T \boldsymbol{x} > 0,\ j = 1, \ldots, m-1$. So we can say that $(i_1, i_2, \ldots, i_m)$ is consistent with $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$ iff $(i_1, i_2, \ldots, i_m)$ is an admissible ordering in the ideal vector model in the sense that there exists an ideal vector $\boldsymbol{x} \in \mathbb{R}^n$ which yields the ordering $(i_1, i_2, \ldots, i_m)$. The same can be said about $(i_1, i_2, \ldots, i_m)$ being consistent with $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$.

Theorem 3.1. *Assume* $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathbb{R}^n$ *satisfy (A-1) and (A-2). Then we have the following:*

(1) Suppose $(i_1, i_2, \ldots, i_m)$ *is not consistent with* $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$*. In this case,*

ordering $(i_1, i_2, \ldots, i_m)$ *does not arise, and neither does the reverse ordering* $(i_m, i_{m-1}, \ldots, i_1)$.

(2) Suppose $(i_1, i_2, \ldots, i_m)$ *is consistent with* $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$.

(2-1) If $(i_1, i_2, \ldots, i_m)$ *is consistent also with* $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$*, then ordering* $(i_1, \ldots, i_m)$ *arises as an unbounded region and the same is true for the reverse ordering.*

(2-2) If $(i_1, i_2, \ldots, i_m)$ *is not consistent with* $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$*, then one and only one of the following two situations occurs:*

(2-2-1) Ordering $(i_1, i_2, \ldots, i_m)$ *arises as a bounded region, while the reverse ordering does not arise.*

(2-2-2) Ordering $(i_1, i_2, \ldots, i_m)$ *does not arise, whereas the reverse ordering arises as a bounded region.*

Case (2-2-1) occurs when the direction of an open halfspace containing $\tilde{\boldsymbol{\mu}}_{i_j} - \tilde{\boldsymbol{\mu}}_{i_{j+1}},\ j = 1, 2, \ldots, m-1$*, is upward, and case (2-2-2) occurs when such direction is downward; whether it is upward or downward is determined uniquely by the given* $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$.

Remark 3.1. *If* $(i_1, i_2, \ldots, i_m)$ *is consistent with* $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$*, it is necessarily consistent with* $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$.

Consider the pairing of orderings which are reverse to each other. In the special case $n = m - 2$, at least one ordering in each pair arises:

Corollary 3.1. *When* $n = m - 2$*, at least one of* $(i_1, i_2, \ldots, i_m)$ *and* $(i_m, i_{m-1}, \ldots, i_1)$ *arises for any ordering* $(i_1, i_2, \ldots, i_m)$.

Proof. It suffices to show that $\tilde{\boldsymbol{\mu}}_{i_j} - \tilde{\boldsymbol{\mu}}_{i_{j+1}},\ j = 1, 2, \ldots, m-1$, are always lumped together so that case (1) never happens. But by virtue of Assumption (A-2), these $m-1 = n+1$ vectors $\tilde{\boldsymbol{\mu}}_{i_1} - \tilde{\boldsymbol{\mu}}_{i_2}, \tilde{\boldsymbol{\mu}}_{i_2} - \tilde{\boldsymbol{\mu}}_{i_3}, \ldots, \tilde{\boldsymbol{\mu}}_{i_{m-1}} - \tilde{\boldsymbol{\mu}}_{i_m}$ in $\mathbb{R}^{n+1}$

are linearly independent, so they are not spread out. **Q.E.D.**

Corollary 3.1 implies that when $n = m - 2$, missing rankings are completely characterized as the reverse rankings of bounded regions.

Corollary 3.2. *When $n \geq m - 1$, all $m!$ orderings are generated as unbounded regions.*

Proof. By an argument similar to the one in the proof of Corollary 3.1, we can see that when $n \geq m-1$, vectors $\boldsymbol{\mu}_{i_j} - \boldsymbol{\mu}_{i_{j+1}}$, $j = 1, 2, \ldots, m-1$, are lumped together for any ordering $(i_1, i_2, \ldots, i_m)$. In view of Remark 3.1, we conclude that case (2-1) always happens. **Q.E.D.**

We have seen our main theorem implies, as its simple corollaries, the results in Kamiya and Takemura (1997) concerning the characterization of non-arising regions. Next we move on to seeing that the results in Kamiya and Takemura (1997) about the numbers of various kinds of regions can also be obtained from the same theorem. Specifically, thanks to Theorem 3.1, the problem of counting the numbers of regions reduces to that of counting the numbers of consistent orderings as follow.

By Theorem 3.1, we find that the number of unbounded regions is equal to the number of orderings for which case (2-1) happens. Recalling Remark 3.1, we know that case (2-1) happens when and only when the ordering in question, $(i_1, i_2, \ldots, i_m)$, is consistent with $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$. Therefore, the number of unbounded regions is the same as the number of orderings $(i_1, i_2, \ldots, i_m)$ which are consistent with $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$.

On the other hand, according to Theorem 3.1, the number of bounded regions is equal to the number of orderings for which (2-2-1) occurs, which in turn is equal to half the number of orderings for which (2-2) occurs. Now the number of orderings for which (2-2) happens is equal to the number of orderings for which (2) occurs, minus the number of orderings for which (2-1) occurs, that is, the number of orderings consistent with $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$, minus the number of orderings consistent with $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$.

Finally, the number of all arising regions is given, of course, by the sum of the number of bounded regions and the number of unbounded ones.

Thus, we can express the numbers of regions in terms of the numbers of consistent orderings, which are given with the help of the following proposition.

Proposition 3.1. *Suppose $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m \in \mathbb{R}^N$ are non-degenerate. Then the number of orderings $(i_1, i_2, \ldots, i_m)$ for which $\boldsymbol{\nu}_{i_j} - \boldsymbol{\nu}_{i_{j+1}},\ j = 1, 2, \ldots, m-1$, are lumped together is given by*

$$c(m, N) = \begin{cases} 2(\mathcal{S}^m_{m-N+1} + \mathcal{S}^m_{m-N+3} + \cdots + \mathcal{S}^m_m) & \text{when } N \text{ is odd,} \\ 2(\mathcal{S}^m_{m-N+1} + \mathcal{S}^m_{m-N+3} + \cdots + \mathcal{S}^m_{m-1}) & \text{when } N \text{ is even,} \end{cases}$$

where $\mathcal{S}^m_k$ are the signless Stirling numbers of the first kind: $t(t+1)\cdots(t+m-1) = \sum_k \mathcal{S}^m_k t^k$.

For Stirling numbers, see Pólya, Tarjan and Woods (1983) or Riordan (1978). The proof of Proposition 3.1 will be given in the Appendix.

By the argument preceding Proposition 3.1, the number of unbounded regions is

$$c(m, n).$$

Similarly, the number of bounded regions is

$$\frac{1}{2}\left\{c(m, n+1) - c(m, n)\right\}.$$

Adding these two numbers, we get the number of all regions as

$$c(m, n) + \frac{1}{2}\left\{c(m, n+1) - c(m, n)\right\} = \frac{1}{2}c(m, n) + \frac{1}{2}c(m, n+1).$$

In this way, we obtain the following corollary:

Corollary 3.3. *Suppose* $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_m \in \mathbb{R}^n$ *satisfy (A-1) and (A-2). Then the number of unbounded regions is*

$$\begin{cases} 2(\mathcal{S}^m_{m-n+1} + \mathcal{S}^m_{m-n+3} + \cdots + \mathcal{S}^m_m) & \text{when } n \text{ is odd,} \\ 2(\mathcal{S}^m_{m-n+1} + \mathcal{S}^m_{m-n+3} + \cdots + \mathcal{S}^m_{m-1}) & \text{when } n \text{ is even,} \end{cases}$$

the number of bounded regions is

$$\mathcal{S}^m_{m-n} - \mathcal{S}^m_{m-n+1} + \mathcal{S}^m_{m-n+2} - \cdots + (-1)^n \mathcal{S}^m_m,$$

and the number of all regions is

$$\mathcal{S}^m_{m-n} + \mathcal{S}^m_{m-n+1} + \mathcal{S}^m_{m-n+2} + \cdots + \mathcal{S}^m_m.$$

Remark 3.2. *The formulae in Corollary 3.3 are trivially true when* $n \geq m-1$. *This can be confirmed from Corollary 3.2 and relations* (7), (8) *and* (9) *in the Appendix.*

4 A neutral population

In this section, we examine a particular case where we have a "neutral" alternative population. By specializing to this particular situation, we can find some more specific properties than were obtained in the preceding section.

Suppose there exists an $i_0 \in \{1, 2, \dots, m\}$ such that $\boldsymbol{\mu}_{i_0}$ is contained in the polytope spanned by the other $\boldsymbol{\mu}_i,\ i = 1, 2, \dots, m,\ i \neq i_0 : \boldsymbol{\mu}_{i_0} \in \mathrm{conv}\{\boldsymbol{\mu}_i : i \neq i_0\}$. In this case, we will say $\boldsymbol{\mu}_{i_0}$ is *neutral among* $\{\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_m\}$.

Remark 4.1. *Under the assumption of non-degeneracy, it can easily be checked that* $\boldsymbol{\mu}_{i_0}$ *is neutral among* $\{\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_m\}$ *if and only if* $\boldsymbol{\mu}_i - \boldsymbol{\mu}_{i_0},\ i = 1, 2, \dots, m,\ i \neq i_0$, *are spread out.*

We begin by seeing that a neutral alternative can never be ranked last.

Theorem 4.1. *Suppose* $\boldsymbol{\mu}_{i_0}$ *is neutral among* $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\}$. *Then, for any ordering* $(j_1, \ldots, j_{m-1})$ *of* $\{1, 2, \ldots, m\} - \{i_0\}$, *ordering* $(j_1, \ldots, j_{m-1}, i_0)$ *is not generated.*

Proof. Without loss of generality, we can assume $i_0 = m$ and $(j_1, \ldots, j_{m-1}) = (1, \ldots, m-1)$. It is sufficient to show that for ordering $(1, \ldots, m-1, m)$, case (2) necessarily means case (2-2-2) so that cases other than (1) and (2-2-2) do not happen. Since the problem is invariant under translation, we may assume $\boldsymbol{\mu}_m = \mathbf{0}$.

Suppose $(1, \ldots, m-1, m)$ is consistent with $\tilde{\boldsymbol{\mu}}_1, \ldots, \tilde{\boldsymbol{\mu}}_m$. Then there exists an $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^T, x_{n+1})^T \in \mathbb{R}^{n+1}$ such that $(\tilde{\boldsymbol{\mu}}_i - \tilde{\boldsymbol{\mu}}_j)^T \tilde{\boldsymbol{x}} > 0$ for all $i < j$. In particular, the following inequality holds for each $i \leq m-1$:

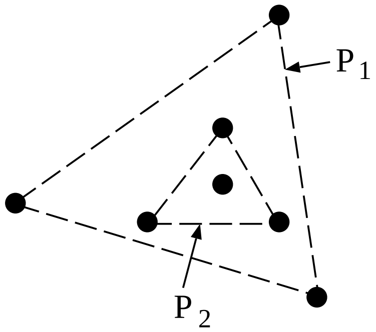
$$\boldsymbol{\mu}_i^T \boldsymbol{x} - \frac{\|\boldsymbol{\mu}_i\|^2}{2} \cdot x_{n+1} > 0. \tag{6}$$

Since $\boldsymbol{\mu}_m = \mathbf{0}$ is neutral among $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\}$, we have that $\boldsymbol{\mu}_i,\ i = 1, 2, \ldots, m-1$, are spread out by Remark 4.1. So the first term on the left-hand side of (6) is zero or negative for some $i = 1, 2, \ldots, m-1$, which implies that x_{n+1} must be negative. Therefore, case (2-2-2) occurs for $(1, \ldots, m-1, m)$, and this completes the proof. **Q.E.D.**

As a simple corollary, we obtain a property concerning boundedness of Voronoi polyhedra. Recall that the Voronoi polyhedron associated with $\boldsymbol{\mu}_{i_0}$ is the set of points from which $\boldsymbol{\mu}_{i_0}$ is not farther than any other alternative $\boldsymbol{\mu}_i,\ i \neq i_0 : \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{\mu}_{i_0}\| \leq \|\boldsymbol{x} - \boldsymbol{\mu}_i\| \text{ for all } i \neq i_0\}$ (Okabe, Boots and Sugihara (2000)).

Corollary 4.1. *Suppose* $\boldsymbol{\mu}_{i_0}$ *is neutral among* $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\}$. *Then, the Voronoi polyhedron associated with* $\boldsymbol{\mu}_{i_0}$ *is bounded.*

Figure 3: Neutral alternative



Proof. It is sufficient to verify that no ordering with i_0 in the first position arises as an unbounded region.

Consider the ordering

$$(i_0, j_1, \ldots, j_{m-1}),$$

where $(j_1, \ldots, j_{m-1})$ is an arbitrary ordering of $\{1, 2, \ldots, m\} - \{i_0\}$. Then we have by Theorem 4.1 that the reverse ordering $(j_{m-1}, \ldots, j_1, i_0)$, with i_0 ranked last, does not arise. This implies that case (1) or (2-2-2) happens for the reverse ordering, which in turn means that case (1) or (2-2-1) occurs for the original ordering. In particular, case (2-1) does not happen for the original ordering, so this ordering never arises as an unbounded region. **Q.E.D.**

We have seen in Theorem 4.1 that a neutral alternative $\boldsymbol{\mu}_{i_0}$ cannot be ranked last; we now show that when $\boldsymbol{\mu}_{i_0}$ lies "deep inside" the polytope $\mathrm{conv}\{\boldsymbol{\mu}_i : i \neq i_0\}$, it cannot be ranked even second last, third last, and so on. This situation is illustrated in Figure 3. Denote the set of all vertices of a polytope P by $\mathrm{vert}(P)$.

Theorem 4.2. *Suppose $\boldsymbol{\mu}_{i_0}$ is neutral among $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\} - \mathrm{vert}(\mathrm{conv}\{\boldsymbol{\mu}_i : i \neq i_0\})$ as well as among $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\}$. Then, orderings with i_0 in the last or* *second last position are not generated.*

Note that if $\boldsymbol{\mu}_{i_0}$ is neutral among $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\} - \mathrm{vert}(\mathrm{conv}\{\boldsymbol{\mu}_i : i \neq i_0\})$, it is also neutral among $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\}$.

Proof. Write $\mathrm{Alt}_1 = \{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\}$, $P_1 = \mathrm{conv}(\mathrm{Alt}_1 - \{\boldsymbol{\mu}_{i_0}\})$ and $V_1 = \mathrm{vert}(P_1)$.

Fix an arbitrary point $\boldsymbol{x}$ in $\mathbb{R}^n$, and consider the closed ball B centered at $\boldsymbol{x}$ with radius $\|\boldsymbol{\mu}_{i_0} - \boldsymbol{x}\|$. Since $\boldsymbol{\mu}_{i_0}$ is neutral among Alt_1, it is in P_1 and there exists a vertex of P_1, say $\boldsymbol{\mu}_{j_1}$, which is not in B. Note that $\boldsymbol{\mu}_{i_0} \notin V_1$.

This consideration implies that when $\boldsymbol{\mu}_{i_0}$ is neutral among Alt_1, at least one alternative, $\boldsymbol{\mu}_{j_1}$, is ranked lower than i_0 at an arbitrary $\boldsymbol{x} \in \mathbb{R}^n$.

Repeat the argument above with $\mathrm{Alt}_2 = \mathrm{Alt}_1 - V_1$ instead of Alt_1, noticing that $\boldsymbol{\mu}_{i_0} \in \mathrm{Alt}_2$. Then we obtain the existence of $\boldsymbol{\mu}_{j_2} \in V_2 = \mathrm{vert}(P_2)$, $P_2 = \mathrm{conv}(\mathrm{Alt}_2 - \{\boldsymbol{\mu}_{i_0}\})$, which is not in B. Here, since $\boldsymbol{\mu}_{j_2} \in V_2 \subset \mathrm{Alt}_2 - \{\boldsymbol{\mu}_{i_0}\} = (\mathrm{Alt}_1 - \{\boldsymbol{\mu}_{i_0}\}) - V_1$ and $\boldsymbol{\mu}_{j_1} \in V_1$, we can see that $\boldsymbol{\mu}_{j_2}$ and $\boldsymbol{\mu}_{j_1}$ are different. Thus, at least two alternatives are ranked lower than i_0. **Q.E.D.**

Actually, the proof of Theorem 4.2 contains the proof of Theorem 4.1, but the latter is based on the argument in the preceding section.

Evidently, continuing this process, we can obtain similar conditions which guarantee that orderings with i_0 in the last three, last four,..., positions are not generated.

In parallel with Corollary 4.1, we get the following corollary:

Corollary 4.2. *Suppose* $\boldsymbol{\mu}_{i_0}$ *is neutral among* $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\} - \mathrm{vert}(\mathrm{conv}\{\boldsymbol{\mu}_i : i \neq i_0\})$ *as well as among* $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m\}$*. Then, the union of closures of regions where* i_0 *is ranked first or second is bounded.*

5 Concluding remarks

In this paper we gave a complete characterization of generated rankings. This characterization allows us to easily determine which rankings are generated and which of the generated rankings have unbounded regions. In addition we investigated some properties of neutral alternatives.

However, there are some harder problems to be solved. One such problem is to determine all possible subsets of rankings which may be obtained from some pairwise discriminant analysis of m populations in $\mathbb{R}^n$: From Corollary 3.3 we know that $\sum_{k=m-n}^{m} \mathcal{S}_k^m$ rankings are generated from each discriminant analysis of m populations in $\mathbb{R}^n$. Suppose that we are given a particular subset of $\sum_{k=m-n}^{m} \mathcal{S}_k^m$ rankings. At the moment it seems difficult to determine whether there exist $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathbb{R}^n$ such that exactly those rankings are generated by the discriminant analysis of $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$.

6 Appendix

In this Appendix, we prove Proposition 3.1.

By the argument just before Theorem 3.1, the desired number is equal to the number of generated orderings in the ideal vector model. Hence the problem is equivalent to that of counting the number of regions of the arrangement of hyperplanes $\mathcal{A}(m, N) = \{H_{ij} : 1 \le j < i \le m\}$, where

$$H_{ij} = \left\{ \boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{\nu}_i^T \boldsymbol{x} = \boldsymbol{\nu}_j^T \boldsymbol{x} \right\}.$$

Write the intersection poset of $\mathcal{A}(m, N)$ as $L(m, N) = L(\mathcal{A}(m, N))$; the partial order on $L(m, N)$ is defined by reverse inclusion (Orlik and Terao (1992), Definition 2.1). Then each element of $L(m, N)$ can be indexed by a partition of

m indices into blocks; the element of $L(m,N)$ indexed by partition J is denoted X_J.

Lemma 6.1 (Kamiya and Takemura (2000), Lemma 3.1). *Suppose* $\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m \in \mathbb{R}^N$ *are non-degenerate with* $N \le m-1$. *Then,* $L(m,N)$ *is isomorphic to the poset* $\mathcal{I}_{m-N}$ *of partitions of* m *indices into one or* $k \ge m-N+1$ *blocks:* $L(m,N) \simeq \mathcal{I}_{m-N}$. *That is,* $\mathcal{I}_{m-N} \ni J \mapsto X_J \in L(m,N)$ *is bijective, and* J_1 *is a refinement of* J_2 *iff* $X_{J_1} \le X_{J_2}$. *Furthermore, the rank function is given by*

$$r(X_J) = \begin{cases} m-k, & J \text{ with } k \ge m-N+1 \text{ blocks}, \\ N, & J = \{\{1,2,\dots,m\}\}. \end{cases}$$

Now, it is well known that the partition lattice of m indices is isomorphic to the lattice of the braid arrangement in $\mathbb{R}^m$ (Orlik and Terao (1992), Proposition 2.9). Moreover, the Poincaré polynomial of the braid arrangement in $\mathbb{R}^m$ is known to be $(1+t)(1+2t)\cdots(1+(m-1)t)$ (Orlik and Terao (1992), Proposition 2.54), which can be written as $\sum_{l=0}^{m-1}(-1)^l \mathcal{S}_{m-l}^m(-t)^l$. Referring to this polynomial, we immediately obtain the Poincaré polynomial of $\mathcal{A}(m,N),\ N \le m-1$, as

$$\pi(\mathcal{A}(m,N),t) = \sum_{l=0}^{N-1}(-1)^l \mathcal{S}_{m-l}^m(-t)^l + \left\{-\sum_{l=0}^{N-1}(-1)^l \mathcal{S}_{m-l}^m\right\}(-t)^N.$$

Applying Zaslavsky's result, we obtain the desired result for $N \le m-1$.

The proposition is trivially true when $N > m-1$. Just notice the following two facts: (i) For any ordering $(i_1,i_2,\dots,i_m)$, vectors $\boldsymbol{\nu}_{i_j} - \boldsymbol{\nu}_{i_{j+1}},\ j = 1,2,\dots,m-1$, are linearly independent and hence lumped together. Thus, the number of consistent orderings is $m!$; (ii) The generating function $t(t+1)\cdots(t+$

$m-1) = \sum_k \mathcal{S}_k^m t^k$ of $\mathcal{S}_k^m$ implies

$$\mathcal{S}_k^m = 0, \quad k \leq 0, \tag{7}$$

$$\mathcal{S}_1^m + \mathcal{S}_2^m + \cdots + \mathcal{S}_m^m = m!, \tag{8}$$

and

$$-\mathcal{S}_1^m + \mathcal{S}_2^m - \cdots + (-1)^m \mathcal{S}_m^m = 0, \quad m \geq 2. \tag{9}$$

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