# Characterizing Feasible Pattern Sets with a Minimum Number of Breaks

Ryuhei Miyashiro, Hideya Iwasaki and Tomomi Matsui

METR 2002-14 DECEMBER 2002

# Characterizing Feasible Pattern Sets with a Minimum Number of Breaks

Ryuhei Miyashiro[1], Hideya Iwasaki[2], and Tomomi Matsui[1]

[1] Department of Mathematical Informatics,
Graduate School of Information Science and Technology,
The University of Tokyo, Tokyo 113-8656, Japan
{miya, tomomi}@misojiro.t.u-tokyo.ac.jp
[2] Department of Computer Science,
The University of Electro-Communications, Tokyo 182-8585, Japan
iwasaki@cs.uec.ac.jp

**Abstract.** In sports timetabling, finding feasible pattern sets for a round robin tournament is a significant problem. Although this problem has been tackled in several ways, good characterization of feasible pattern sets is not known yet. In this paper, we consider the feasibility of a pattern set, and propose a necessary condition for feasible pattern sets. In the case of a pattern set with a minimum number of breaks, we prove a theorem leading a polynomial-time algorithm to check whether the given pattern set satisfies the necessary condition or not. Computational experiments show that the proposed condition characterizes feasible pattern sets with a minimum number of breaks when the number of teams is less than or equal to 26.

## 1 Introduction

Constructing a timetable for a sports competition is an important task for the organizers of the competition because the timetable affects not only the results of games but also the revenue of the competition. Since creating an appropriate timetable by hand is difficult, demand for automated timetabling has been increasing.

Sports timetabling is the research region that concerns creating an optimal timetable of a sports competition, constructing timetabling algorithms, and investigating mathematical structure of the problem. Recently, a number of papers on sports timetabling have been reported [1–8, 10–20]. Most of them considered timetabling of a round robin tournament.

In this paper, we consider a round robin tournament with home and away assignment. There are mainly two approaches to construct timetables for such tournaments. One approach is to fix an opponent of each game first and then set the place of each game [14]. The other is to decide the place of each game first, i.e., fix a pattern set, then assign opponents to the pattern set [8, 13, 17]. Each approach has its advantage and disadvantage compared to the other approach. When the place of games have a great influence on a competition, the latter approach is preferred. Constructing a timetable with this approach, we often encounter the problem to determine whether it is possible to make a timetable from a fixed pattern set. We call this problem "pattern set feasibility problem." Although some enumerative methods, such as integer programming, are fairly effective for solving this problem, few theoretical results of research on this problem are known. In this paper, we investigate pattern set feasibility problem, and propose a highly efficient algorithm to solve this problem in a practical case.

This paper consists of five sections, including this introductory one. Section 2 defines pattern set feasibility problem formally. Section 3 proposes a necessary condition for feasible pattern sets. Section 4 contains three subsections, which concern a special class of pattern sets and its feasibility. Section 5 describes future work and conclusions.

## 2 Pattern Set Feasibility Problem

In this paper, we consider a single round robin tournament of $2n$ teams with $2n-1$ slots. Figure 1 is an example of a timetable of six teams. In a single round robin tournament, each team plays one game in each slot, and plays against every other team once through the tournament. Each team has its home, and each game is held at the home of one of the teams playing. In a timetable, each game with '@' means that the game is held at the home of the opponent, while each game without '@' means that the game is held at the home of the team corresponding to the row. For example, in Fig. 1, team 4 plays against team 3 at the home of team 3 in slot 5.

| | | 1 | 2 | 3 | 4 | 5 | (slot) |
|---|---|---|---|---|---|---|---|
| 1 | : | 3 | @4 | 5 | @6 | 2 | |
| 2 | : | @5 | 6 | 4 | 3 | @1 | |
| 3 | : | @1 | 5 | @6 | @2 | 4 | |
| 4 | : | @6 | 1 | @2 | 5 | @3 | |
| 5 | : | 2 | @3 | @1 | @4 | 6 | |
| 6 | : | 4 | @2 | 3 | 1 | @5 | |
| (team) | | | | | | | |

**Fig. 1.** Timetable of six teams.

*Pattern set* is a table showing only the places of the games. Figure 2 is the pattern set corresponding to the timetable of Fig. 1. In a pattern set, each 'A' means a game at away, and each 'H' means a game at home for the team corresponding to the row.

Assume that we need to construct a timetable from a given pattern set. From the pattern set of Fig. 3, we can construct two corresponding timetables (Fig. 4).

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 : | H | A | H | A | H |
| 2 : | A | H | H | H | A |
| 3 : | A | H | A | A | H |
| 4 : | A | H | A | H | A |
| 5 : | H | A | A | A | H |
| 6 : | H | A | H | H | A |

**Fig. 2.** Pattern set corresponding to Fig. 1.

If a pattern set has at least one corresponding timetable, we say that the pattern set is *feasible*. Usually, one feasible pattern set produces numerous different timetables [13].

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 : | A | A | H |
| 2 : | A | H | H |
| 3 : | H | H | A |
| 4 : | H | A | A |

**Fig. 3.** Pattern set of four teams.

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 : | @4 | @2 | 3 |
| 2 : | @3 | 1 | 4 |
| 3 : | 2 | 4 | @1 |
| 4 : | 1 | @3 | @2 |

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 : | @3 | @2 | 4 |
| 2 : | @4 | 1 | 3 |
| 3 : | 1 | 4 | @2 |
| 4 : | 2 | @3 | @1 |

**Fig. 4.** Timetables corresponding to Fig. 3.

Unfortunately, not all pattern sets can generate a timetable. Figure 5 is an example of an *infeasible* pattern set. In pattern set of Fig. 5, teams 4, 5 and 6 cannot play the games among them in slots 3, 4 and 5, because in each of those slots these teams have the same characters ('A's or 'H's). Thus, they have to play the three games in slots 1 and 2, which situation is obviously impossible.

*Pattern set feasibility problem* is to determine the feasibility of a given pattern set. For this problem, polynomial-size characterization of feasible pattern sets has not been found yet, and whether this problem is NP-complete or not is still open. Although a polynomial-time algorithm to solve this problem has not known yet, we can solve this problem with integer programming [13], and con-

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 : | H | A | H | A | H |
| 2 : | A | A | H | A | H |
| 3 : | A | H | H | A | H |
| 4 : | A | H | A | H | A |
| 5 : | H | H | A | H | A |
| 6 : | H | A | A | H | A |

**Fig. 5.** Infeasible pattern set of six teams.

straint programming [8]. Theoretical results from graph theory were described in [3–6, 17, 18].

In the rest of this paper, we consider pattern set feasibility problem. Before going to the next section, we give three trivial remarks.

*Remark 1.* Permutation of the rows (teams) of a pattern set does not change the feasibility.

*Remark 2.* Permutation of the columns (slots) of a pattern set does not change the feasibility.

*Remark 3.* Replacing the characters 'A' with 'H' and 'H' with 'A' of a pattern set does not change the feasibility.

## 3 Necessary Condition for Feasible Pattern Sets

In this section, we propose a necessary condition for feasible pattern sets.

It is easy to see that every feasible pattern set must satisfy the following two conditions:

**(i)** in each slot, the number of 'A's and that of 'H's are equivalent,
**(ii)** each team has a different row from that of the other teams.

If a given pattern set does not satisfy the above conditions, the pattern set is infeasible.

**Assumption:** In the rest of this paper, all pattern sets satisfy the conditions (i) and (ii).

Of course, there are infeasible pattern sets that meet the conditions (i) and (ii). Figure 5 is an example of such pattern sets. From observation of teams 4, 5 and 6, we find that the pattern set is infeasible. In the following, we show another infeasible pattern set and introduce a new necessary condition for feasible pattern sets.

Figure 6 is a pattern set of ten teams, showing only teams 1–5. We can judge that this pattern set is infeasible without seeing teams 6–10. In each of slots 1, 8 and 9, teams 1, 2, 3, 4 and 5 cannot play against each other, because these teams

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 : | A | A | H | A | H | A | H | A | H |
| 2 : | A | H | H | A | H | A | H | A | H |
| 3 : | A | H | A | H | H | A | H | A | H |
| 4 : | A | H | A | H | A | A | H | A | H |
| 5 : | A | H | A | H | A | H | A | A | H |
| 6 : | | | | | ··· | | | | |
| ⋮ | | | | | ⋮ | | | | |
| 10 : | | | | | ··· | | | | |
| #A : | 5 | 1 | 3 | 2 | 2 | 4 | 1 | 5 | 0 |
| #H : | 0 | 4 | 2 | 3 | 3 | 1 | 4 | 0 | 5 |
| min : | 0 | 1 | 2 | 2 | 2 | 1 | 1 | 0 | 0 |

**Fig. 6.** Infeasible pattern set of ten teams, showing only teams 1–5.

have the same characters (‘A’s or ‘H’s). In slot 2, we can hold at most one game among these five teams, at most two games in slot 3, and so on. In total, we can assign at most nine games among the five teams, however, we have to hold ten games ($=\binom{5}{2}$). Hence, this pattern set is infeasible.

We describe the above procedure in a general setting. Let $T$ be an arbitrary subset of teams of a given pattern set. In each slot, count the number of ‘A’s and that of ‘H’s in $T$, then take the minimum of them. If the sum total of the minimums is strictly less than $\binom{|T|}{2}$, the given pattern set is infeasible.

In the remainder of this paper, we denote the set of all teams by $U = \{1, 2, \dots, 2n\}$ and the set of all slots by $S = \{1, 2, \dots, 2n-1\}$. For any subset of teams $T \subseteq U$ and slot $s \in S$, let functions $A(T, s)$ and $H(T, s)$ return the number of ‘A’s and that of ‘H’s in $s$ among $T$, respectively. We define function $\alpha$ by $\alpha(T) \stackrel{\text{def.}}{=} \sum_{s\in S} \min\{A(T, s), H(T, s)\} - |T|(|T|-1)/2$. Using this function $\alpha$, we propose a necessary condition for feasible pattern sets; $\forall T \subseteq U$, $\alpha(T) \geq 0$.

*Example 1.* For the pattern set of Fig. 6,
$\alpha(\{1, 2, 3, 4, 5\}) = (0 + 1 + 2 + 2 + 2 + 1 + 1 + 0 + 0) - 5(5-1)/2 = -1$.
Since there exists $T \subseteq U$ satisfying $\alpha(T) < 0$, the pattern set is infeasible.

*Remark 4.* We already assumed that all pattern sets satisfy the following conditions:

**(i)** in each slot, the number of ‘A’s and that of ‘H’s are equivalent,
**(ii)** each team has a different row from that of the other teams.

It is clear that these conditions are described as follows:

**(i)** $\alpha(U) = 0$,
**(ii)** $\forall T \subseteq U$, $|T| = 2 \implies \alpha(T) \geq 0$.

We denote the complement of $T$ by $\bar{T}$; $\bar{T} = U \setminus T$. The following theorem shows relationship between $\alpha(T)$ and $\alpha(\bar{T})$.

**Theorem 1.** *For any pattern set,* $\alpha(T) = \alpha(\bar{T})$.

*Proof.* For arbitrary $s \in S$, $\min\{A(T,s), H(T,s)\} + \max\{A(\bar{T},s), H(\bar{T},s)\} = n$ because $A(T,s) + A(\bar{T},s) = H(T,s) + H(\bar{T},s) = n$.
Thus, $\min\{A(\bar{T},s), H(\bar{T},s)\} = |\bar{T}| - \max\{A(\bar{T},s), H(\bar{T},s)\}$
$= (2n - |T|) - \big(n - \min\{A(T,s), H(T,s)\}\big) = n - |T| + \min\{A(T,s), H(T,s)\}$.
Hence, $\alpha(\bar{T}) = \sum_{s\in S} \min\{A(\bar{T},s), H(\bar{T},s)\} - |\bar{T}|(|\bar{T}|-1)/2$
$= \sum_{s\in S}\big(n - |T| + \min\{A(T,s), H(T,s)\}\big) - (2n-|T|)(2n-|T|-1)/2$
$= \sum_{s\in S} \min\{A(T,s), H(T,s)\} + (2n-1)(n-|T|) - (2n-|T|)(2n-|T|-1)/2$
$= \sum_{s\in S} \min\{A(T,s), H(T,s)\} - |T|(|T|-1)/2 = \alpha(T)$.

In the rest of this paper, we may abbreviate $\sum_{s\in S} \min\{A(T,s), H(T,s)\}$ to $\min\{A(T), H(T)\}$.

# 4 Pattern Set with a Minimum Number of Breaks and Characterizing its Feasibility

In practical sports timetabling, the organizers often prefer a pattern set satisfying particular properties. In this section, we consider the feasibility of such pattern sets.

In Subsection 4.1, we introduce a pattern set with a minimum number of breaks and its well-known properties, represented in [3, 4, 6]. In Subsection 4.2, we propose a polynomial-time algorithm to check whether a pattern set with a minimum number of breaks satisfies the necessary condition described in Sect. 3. In Subsection 4.3, we report results of computational experiments that show the efficiency of the necessary condition.

## 4.1 Pattern Sets with a Minimum Number of Breaks

The pattern set shown in Fig. 7 is feasible, however, not desirable for most of the organizers, because team 1 plays five consecutive away games. In general, a pattern set with many consecutive away games for a team is not preferred. Similarly, many consecutive home games are not desirable, such as team 6 in Fig. 7.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 : | A | <u>A</u> | <u>A</u> | <u>A</u> | <u>A</u> |
| 2 : | A | H | A | H | A |
| 3 : | H | <u>H</u> | A | <u>A</u> | H |
| 4 : | H | A | H | A | <u>A</u> |
| 5 : | A | <u>A</u> | H | <u>H</u> | <u>H</u> |
| 6 : | H | <u>H</u> | <u>H</u> | <u>H</u> | <u>H</u> |

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 : | @6 | @2 | @4 | @5 | @3 |
| 2 : | @3 | 1 | @6 | 4 | @5 |
| 3 : | 2 | 4 | @5 | @6 | 1 |
| 4 : | 5 | @3 | 1 | @2 | @6 |
| 5 : | @4 | @6 | 3 | 1 | 2 |
| 6 : | 1 | 5 | 2 | 3 | 4 |

**Fig. 7.** Undesirable pattern set of six teams and a corresponding schedule.

If a team has both ‘A’s or both ‘H’s in slots $s$ and $s+1$, we say that the team has a break at slot $s+1$. In this paper, a break is expressed with an underline below the latter character. For instance, in the pattern set of Fig. 7, team 4 has consecutive ‘A’s in slots 4 and 5, and we say that team 4 has a break at slot 5.

The organizers do not prefer the pattern set of Fig. 7 because it has many breaks. Generally, they are interested in a pattern set with few breaks. The following theorem shows the minimum number of breaks in a feasible pattern set of $2n$ teams.

**Theorem 2.** *For any feasible pattern set, the number of breaks is greater than or equal to* $2n-2$ *[3].*

*Proof.* There are at most two teams without any breaks, “AHAH$\cdots$AHA” and “HAHA$\cdots$HAH.” Hence, the number of breaks is greater than or equal to $2n-2$.

If a pattern set of $2n$ teams has exactly $2n-2$ breaks, we call the pattern set a *pattern set with a minimum number of breaks* (PSMB). In a PSMB, two teams have no break and other $2n-2$ teams have just one break. Fig. 8 is an example of PSMB of eight teams.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 : | A | H | A | H | A | H | A |
| 2 : | H | A | $\underline{\text{A}}$ | H | A | H | A |
| 3 : | H | A | H | A | $\underline{\text{A}}$ | H | A |
| 4 : | H | A | H | A | H | $\underline{\text{H}}$ | A |
| 5 : | H | A | H | A | H | A | H |
| 6 : | A | H | $\underline{\text{H}}$ | A | H | A | H |
| 7 : | A | H | A | H | $\underline{\text{H}}$ | A | H |
| 8 : | A | H | A | H | A | $\underline{\text{A}}$ | H |

**Fig. 8.** PSMB of eight teams.

For any PSMB, the following theorem holds.

**Theorem 3.** *At any slot of each PSMB, there are exactly two breaks or no breaks [3].*

*Proof.* In a PSMB, each team has at most one break. Thus, there are at most two teams with a break at slot $s$, as “AH$\cdots$AH$\underline{\text{H}}$A$\cdots$AH” and “HA$\cdots$HA$\underline{\text{A}}$H$\cdots$HA.” Assume that only team $t$ has a break at slot $s$. Then the number of ‘A’s and that of ‘H’s are different in slot $s-1$ because all teams except $t$ have no break at slot $s$ (Fig. 9). This contradicts the property (i).

In a pattern set, if a team has the row obtained by substituting ‘A’ for ‘H’ and ‘H’ for ‘A’ of the row of team $t$, we say that the team is the *complement* team of $t$ and vice versa. We represent the complement team of $t$ as $\widetilde{t}$, and say

| | $\cdots$ | $s-1$ | $s$ | $\cdots$ |
|---|---|---|---|---|
| $t$ : | | A | $\underline{\text{A}}$ | |
| $\cdots$ : | | H | A | |
| $\cdots$ : | | H | A | |
| $\cdots$ : | | A | H | |
| $\cdots$ : | | A | H | |
| $\cdots$ : | | A | H | |

**Fig. 9.** No slot with just one break in a PSMB.

that $\{t, \widetilde{t}\}$ is a complement pair. From Theorem 3, it is easy to see that every PSMB of $2n$ teams consists of $n$ complement pairs.

When each team in a pattern set has exactly one break, we call the pattern set an *equitable pattern set* [3]. Figure 10 is an example of equitable pattern sets of eight teams. Although each PSMB is an optimal pattern set in terms of minimizing the number of breaks, organizers sometimes prefer an equitable pattern set because each team evenly has one break in an equitable pattern set.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 : | A | $\underline{\text{A}}$ | H | A | H | A | H |
| 2 : | A | H | A | $\underline{\text{A}}$ | H | A | H |
| 3 : | A | H | A | H | A | $\underline{\text{A}}$ | H |
| 4 : | A | H | A | H | A | H | $\underline{\text{H}}$ |
| 5 : | H | $\underline{\text{H}}$ | A | H | A | H | A |
| 6 : | H | A | H | $\underline{\text{H}}$ | A | H | A |
| 7 : | H | A | H | A | H | $\underline{\text{H}}$ | A |
| 8 : | H | A | H | A | H | A | $\underline{\text{A}}$ |

**Fig. 10.** Equitable pattern set of eight teams.

In fact, PSMB and equitable pattern sets are essentially equivalent, because every equitable pattern set is obtained by cyclic permutation of the slots of a PSMB. For example, the equitable pattern set of Fig. 10 can be constructed from the PSMB of Fig. 8, by letting the slot $s$ of the PSMB be slot $s+1$ ($s = 1, 2, \ldots, 6$), and slot 7 be slot 1. Thus, we may consider that the teams with no breaks have a break at slot 1, as "$\underline{\text{A}}$HAHAHA." Since cyclic permutation of the slots does not affects the feasibility of a pattern set, our results on PSMB, described in the next subsection, are also applicable to equitable pattern sets.

Considering the feasibility of a pattern set, we can freely permutate its rows (Remark 1). When we deal with a PSMB, we sort its rows first by the following procedure.

> Taking the team that has no break and 'H' in slot $2n-1$, let the team be team 1. From the remaining teams, choose the team that has a break

at the earliest slot and 'H' in slot $2n-1$, and let this team be the next team. Repeat this step until no such team remains. After that, let the complement of team $t$ be team $t+n$ for $t=1,2,\ldots,n$ (see Fig. 11).

We can sort the rows of a PSMB in $\mathrm{O}(n^2)$ steps. In the rest of this paper, we assume that the rows of a given PSMB have already been sorted by the above procedure, and regard a team with no break as a team having a break at slot 1.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 : | H | A | $\underline{\text{A}}$ | H | A | H | A |
| 2 : | H | A | H | A | H | $\underline{\text{H}}$ | A |
| 3 : | H | A | H | A | $\underline{\text{A}}$ | H | A |
| 4 : | A | H | $\underline{\text{H}}$ | A | H | A | H |
| 5 : | A | H | A | H | A | H | A |
| 6 : | H | A | H | A | H | A | H |
| 7 : | A | H | A | H | $\underline{\text{H}}$ | A | H |
| 8 : | A | H | A | H | A | $\underline{\text{A}}$ | H |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(6 \rightarrow)$ 1 : | H | A | H | A | H | A | H |
| $(4 \rightarrow)$ 2 : | A | H | $\underline{\text{H}}$ | A | H | A | H |
| $(7 \rightarrow)$ 3 : | A | H | A | H | $\underline{\text{H}}$ | A | H |
| $(8 \rightarrow)$ 4 : | A | H | A | H | A | $\underline{\text{A}}$ | H |
| $(5 \rightarrow)$ 5 : | A | H | A | H | A | H | A |
| $(1 \rightarrow)$ 6 : | H | A | $\underline{\text{A}}$ | H | A | H | A |
| $(3 \rightarrow)$ 7 : | H | A | H | A | $\underline{\text{A}}$ | H | A |
| $(2 \rightarrow)$ 8 : | H | A | H | A | H | $\underline{\text{H}}$ | A |

**Fig. 11.** PSMB before and after sorting of its rows.

### 4.2 Necessary Condition for Feasible Pattern Sets with a Minimum Number of Breaks

In Sect. 3, we proposed a necessary condition for feasible pattern sets; $\forall T \subseteq U$, $\alpha(T) \geq 0$. However, to check the inequality for all $T \subseteq U$ takes exponential steps. In this subsection, we consider the property of this necessary condition in the case of PSMB.

The objective of this subsection is to show that the following proposition holds.

**Proposition 1.** *For any PSMB, whether the PSMB satisfies the necessary condition $\forall T \subseteq U$, $\alpha(T) \geq 0$ or not can be checked in polynomial steps.*

In order to prove Prop. 1, we have to define several terms and show a number of lemmas.

If we can construct a consecutive sequence of all elements in $T \subseteq U$, we say that $T$ is *consecutive*. If $T$ is consecutive on the assumption that the next of $2n$ is 1, we say that $T$ is *cyclically consecutive*. For example, when $2n = 6$, $\{2,3,4\}$ is both consecutive and cyclically consecutive, $\{5,6,1,2\}$ is not consecutive but cyclically consecutive, and $\{1,3,4\}$ is neither consecutive nor cyclically consecutive.

For a subset of teams $T$, if there exists a subset of teams $P$ such that $T \subseteq P \subseteq U$, $|P| = n$, and $P$ is cyclically consecutive, we say that $T$ is *narrow*.

*Remark 5.* For any PSMB, a subset of teams $T$ is narrow if and only if there exists a slot at which all teams in $T$ have the same characters ('A's or 'H's). In particular, if $T$ contains a complement pair $\{t, \widetilde{t}\}$, $T$ is not narrow.

Now we define a partition $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4\}$ of $2^U$ as below:

$\mathcal{T}_1 = \{T \subseteq U : |T| \leq n$ and $T$ is cyclically consecutive$\}$,
$\mathcal{T}_2 = \{T \subseteq U : |T| \leq n$ and $T$ is not cyclically consecutive but narrow$\}$,
$\mathcal{T}_3 = \{T \subseteq U : |T| \leq n$ and $T$ is neither cyclically consecutive nor narrow$\}$, and
$\mathcal{T}_4 = \{T \subseteq U : |T| > n\}$.

Obviously, $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 = 2^U$ and $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset$ for $i \neq j$. Table 1 illustrates the partition of $2^U$.

**Table 1.** Partition of $2^U$.

| | cyclically consecutive | not cyclically consecutive | |
|---|---|---|---|
| $\lvert T\rvert \leq n$ | $\mathcal{T}_1$ | $\mathcal{T}_2$ | narrow |
| | — | $\mathcal{T}_3$ | not narrow |
| $\lvert T\rvert > n$ | $\mathcal{T}_4$ | | not narrow |

Here we denote a precise view of the proof of Prop. 1. It consists of the following proposition.

**Proposition 2.** *For any PSMB,* $\forall T \in \mathcal{T}_1,\ \alpha(T) \geq 0 \iff \forall T \subseteq U,\ \alpha(T) \geq 0.$

Clearly, for each subset of teams $T$ of any pattern set, we can check whether $T$ satisfies the inequality $\alpha(T) \geq 0$ or not in polynomial steps. Since $|\mathcal{T}_1| = \mathrm{O}(n^2)$, Prop. 2 directly implies Prop. 1. In the rest of this subsection, we prove Prop. 2.

Let $\widetilde{T}$ be the set of teams that consists of the complement teams of all teams in $T$; $\widetilde{T} \stackrel{\text{def.}}{=} \{\widetilde{t}_1, \widetilde{t}_2, \ldots, \widetilde{t}_{|T|}\}$, where $T = \{t_1, t_2, \ldots, t_{|T|}\}$. Then, the following theorem holds.

**Theorem 4.** *For any PSMB,* $\alpha(T) = \alpha(\widetilde{T})$.

*Proof.* Since $A(T, s) = H(\widetilde{T}, s)$ and $H(T, s) = A(\widetilde{T}, s)$ for an arbitrary slot $s$,
$\alpha(T) = \sum_{s \in S} \min\{A(T, s), H(T, s)\} - |T|(|T| - 1)/2$
$= \sum_{s \in S} \min\{H(\widetilde{T}, s), A(\widetilde{T}, s)\} - |\widetilde{T}|(|\widetilde{T}| - 1)/2 = \alpha(\widetilde{T})$.

**Lemma 1.** *For any PSMB,* $\forall T \in \mathcal{T}_1,\ \alpha(T) \geq 0 \implies \forall T \in \mathcal{T}_2,\ \alpha(T) \geq 0.$

*Proof.* We show that $\forall T \in \mathcal{T}_2,\ \exists T^* \in \mathcal{T}_1,\ \alpha(T^*) \leq \alpha(T)$.

For a subset of teams $T \in \mathcal{T}_2$ of a given PSMB, find a subset of teams $P$ such that $T \subseteq P \subseteq U$, $|P| = n$, and $P$ is consecutive. If there is no such $P$, consider $\widetilde{T}$ instead of $T$ in the rest of this proof. Theorem 4 justifies this replacement.

We represent $T$ as a general form $T = \{t_1, t_2, \dots, t_{|T|}\}$ where $t_1 < t_2 < \cdots < t_{|T|}$. Let function $\sigma$ take team $t$ and return the slot at which $t$ has a break. By cyclically permutating of the slots of the given PSMB, let $t_1 \in T$ be having a break at slot 1. Since $T$ is narrow, now $\sigma(t_1) < \sigma(t_2) < \cdots < \sigma(t_{|T|})$ holds. Moreover, $\sigma(t_1) < \sigma(t_1 + 1) < \sigma(t_1 + 2) < \cdots < \sigma(t_{|T|})$. By the following procedure, we construct a subset of teams $T^*$ such that $T^* \in \mathcal{T}_1$, $|T^*| = |T|$ and $\alpha(T^*) \le \alpha(T)$.

Let $m \in T$ be the element satisfying $m = t_{\lceil \frac{|T|}{2} \rceil}$. Define $T^* \subseteq U$ by $T^* = \{m - \lceil \frac{|T|}{2} \rceil + 1, m - \lceil \frac{|T|}{2} \rceil + 2, \dots, m, \dots, m + \lfloor \frac{|T|}{2} \rfloor\}$. We also represent $T^*$ as $T^* = \{t^*_1, t^*_2, \dots, t^*_{\lceil \frac{|T|}{2} \rceil}, \dots, t^*_{|T|}\}$ where $t^*_1 < t^*_2 < \cdots < t^*_{\lceil \frac{|T|}{2} \rceil} < \cdots < t^*_{|T|}$. $T^*$ belongs to $\mathcal{T}_1$ because $T^*$ is consecutive and $|T^*| = |T| \le n$. Thus, it is sufficient to show that $\min\{A(T^*), H(T^*)\} \le \min\{A(T), H(T)\}$ instead of showing $\alpha(T^*) \le \alpha(T)$ because $|T^*| = |T|$.

In order to make the proof easy, we introduce another expression of a PSMB. For a given PSMB, replace 'A' with '0' and 'H' with '1' in each odd slot, and 'A' with '1' and 'H' with '0' in each even slot, respectively. For example, the PSMB of Fig. 12 changes to Fig. 13. In this expression, counting the number of 'A's and that of 'H's in each slot corresponds to counting the number of '0's and that of '1's.

This 0-1 expression clarifies that the following equalities hold (see Figs. 14 and 15):

$\min\{A(T), H(T)\} = \sum_{i=1}^{\lceil \frac{|T|}{2} \rceil} \big(\sigma(m) - \sigma(t_i)\big) + \sum_{i=\lceil \frac{|T|}{2} \rceil + 1}^{|T|} \big(\sigma(t_i) - \sigma(m)\big)$, and

$\min\{A(T^*), H(T^*)\} = \sum_{i=1}^{\lceil \frac{|T|}{2} \rceil} \big(\sigma(m) - \sigma(t^*_i)\big) + \sum_{i=\lceil \frac{|T|}{2} \rceil + 1}^{|T|} \big(\sigma(t^*_i) - \sigma(m)\big)$.

From the definition of $T^*$, if $i \le \lceil \frac{|T|}{2} \rceil$, then $\sigma(t_i) \le \sigma(t^*_i)$, otherwise $\sigma(t_i) \ge \sigma(t^*_i)$. (Note that $t_{\lceil \frac{|T|}{2} \rceil} = m = t^*_{\lceil \frac{|T|}{2} \rceil}$.)

Hence, $\min\{A(T), H(T)\} - \min\{A(T^*), H(T^*)\}$

$$
\begin{aligned}
&= + \sum_{i=1}^{\lceil \frac{|T|}{2} \rceil} \big(\sigma(m) - \sigma(t_i)\big) + \sum_{i=\lceil \frac{|T|}{2} \rceil + 1}^{|T|} \big(\sigma(t_i) - \sigma(m)\big) \\
&\quad - \sum_{i=1}^{\lceil \frac{|T|}{2} \rceil} \big(\sigma(m) - \sigma(t^*_i)\big) - \sum_{i=\lceil \frac{|T|}{2} \rceil + 1}^{|T|} \big(\sigma(t^*_i) - \sigma(m)\big) \\
&= \sum_{i=1}^{\lceil \frac{|T|}{2} \rceil} \big(\sigma(t^*_i) - \sigma(t_i)\big) + \sum_{i=\lceil \frac{|T|}{2} \rceil + 1}^{|T|} \big(\sigma(t_i) - \sigma(t^*_i)\big) \ge 0.
\end{aligned}
$$

*Example 2.* For $T = \{1, 3, 5, 6, 8\} \in \mathcal{T}_2$ (Fig. 14), $T^* = \{3, 4, 5, 6, 7\} \in \mathcal{T}_1$ (Fig. 15), and $\alpha(T) = \alpha(\{1, 3, 5, 6, 8\}) = 9 > 1 = \alpha(\{3, 4, 5, 6, 7\}) = \alpha(T^*)$.

**Proposition 3.** *For any PSMB,*
$\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2,\ \alpha(T) \ge 0 \implies \forall T \in \mathcal{T}_3,\ \alpha(T) \ge 0.$

To prove Prop. 3, we divide $\mathcal{T}_3$ into two subsets. Let $\mathcal{T}_{3\text{nc}}$ consist of the subset of teams which contains no complement pairs, and $\mathcal{T}_{3\text{wc}} = \mathcal{T}_3 \setminus \mathcal{T}_{3\text{nc}}$. Note that if a subset of teams $T$ contains a complement pair and $|T| \le n$, $T$ belongs to $\mathcal{T}_{3\text{wc}}$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 : | $\underline{H}$ | A | H | A | H | A | H | A | H | A | H | A | H | A | H |
| 2 : | A | $\underline{A}$ | H | A | H | A | H | A | H | A | H | A | H | A | H |
| 3 : | A | H | A | $\underline{A}$ | H | A | H | A | H | A | H | A | H | A | H |
| 4 : | A | H | A | H | A | $\underline{A}$ | H | A | H | A | H | A | H | A | H |
| 5 : | A | H | A | H | A | H | A | $\underline{A}$ | H | A | H | A | H | A | H |
| 6 : | A | H | A | H | A | H | A | H | A | $\underline{A}$ | H | A | H | A | H |
| 7 : | A | H | A | H | A | H | A | H | A | H | $\underline{H}$ | A | H | A | H |
| 8 : | A | H | A | H | A | H | A | H | A | H | A | H | A | $\underline{A}$ | H |
| 9 : | $\underline{A}$ | H | A | H | A | H | A | H | A | H | A | H | A | H | A |
| 10 : | H | $\underline{H}$ | A | H | A | H | A | H | A | H | A | H | A | H | A |
| 11 : | H | A | H | $\underline{H}$ | A | H | A | H | A | H | A | H | A | H | A |
| 12 : | H | A | H | A | H | $\underline{H}$ | A | H | A | H | A | H | A | H | A |
| 13 : | H | A | H | A | H | A | H | $\underline{H}$ | A | H | A | H | A | H | A |
| 14 : | H | A | H | A | H | A | H | A | H | $\underline{H}$ | A | H | A | H | A |
| 15 : | H | A | H | A | H | A | H | A | H | A | $\underline{A}$ | H | A | H | A |
| 16 : | H | A | H | A | H | A | H | A | H | A | H | A | H | $\underline{H}$ | A |

**Fig. 12.** PSMB with A-H expression.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 : | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 : | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 : | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| 7 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |
| 8 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |
| 9 : | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 : | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 : | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 : | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 : | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 : | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 |
| 15 : | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 |
| 16 : | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 |

**Fig. 13.** PSMB with 0-1 expression.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 : | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| 8 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |

**Fig. 14.** $T = \{1, 3, 5, 6, 8\} \in \mathcal{T}_2$, $\alpha(T) = 9$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 : | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 : | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| 7 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |

**Fig. 15.** $T^* = \{3, 4, 5, 6, 7\} \in \mathcal{T}_1$, $\alpha(T^*) = 1$.

**Lemma 2.** *For any pattern set, if a subset of teams $T$ contains a complement pair $\{t, \widetilde{t}\}$ and $|T| \leq n$, then $\alpha(T \setminus \{t, \widetilde{t}\}) < \alpha(T)$.*

*Proof.* Since exactly one of $t$ and $\widetilde{t}$ contributes to the value of $\min\{A(T, s), H(T, s)\}$ in each slot $s$, $\min\bigl\{A(T \setminus \{t, \widetilde{t}\}), H(T \setminus \{t, \widetilde{t}\})\bigr\} = \min\{A(T), H(T)\} - (2n - 1)$. Thus, we have $\alpha(T) - \alpha(T \setminus \{t, \widetilde{t}\})$
$= +\min\{A(T), H(T)\} - |T|(|T| - 1)/2$
$\quad - \min\bigl\{A(T \setminus \{t, \widetilde{t}\}), H(T \setminus \{t, \widetilde{t}\})\bigr\} + (|T| - 2)(|T| - 3)/2$
$= -|T|(|T| - 1)/2 + (2n - 1) + (|T| - 2)(|T| - 3)/2$
$= 2(n - |T|) + 2 > 0.$

**Lemma 3.** *For any PSMB, $\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2$, $\alpha(T) \geq 0 \implies \forall T \in \mathcal{T}_{3\mathrm{nc}}$, $\alpha(T) \geq 0$.*

*Proof.* We show that $\forall T \in \mathcal{T}_{3\mathrm{nc}}$, $\exists T^* \in \mathcal{T}_1 \cup \mathcal{T}_2$, $\alpha(T^*) \leq \alpha(T)$. In this proof, we have much help from graphical expression of a subset of teams of a PSMB defined below.

Draw a regular $2(2n-1)$-gon, and name the vertices of the $2(2n-1)$-gon $\overline{1}$, $\overline{2}$, …, $\overline{2n-1}$, $\underline{1}$, $\underline{2}$, …, $\underline{2n-1}$ clockwise. For a subset of teams $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3\mathrm{nc}}$ of 0-1 expression, we color some vertices black or gray. Remind that $|T| \leq n$ and $T$ includes no complement pairs. If $T$ contains a team that has '1' in slot $2n - 1$ and a break at slot $s$, we color the vertex $\overline{s}$ black and $\underline{s}$ gray. If $T$ contains a team that has '0' in slot $2n - 1$ and a break at slot $s$, we color the vertex $\underline{s}$ black and $\overline{s}$ gray. Then, $2|T|$ vertices of $2(2n - 1)$ vertices are colored, while the remaining vertices are still uncolored. Note that each pair of vertices $\{\overline{v}, \underline{v}\}$ consists of a pair of black and gray vertices, or two uncolored vertices.

For any subset of teams $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3\mathrm{nc}}$, $f(T)$ denotes the corresponding set of all black vertices of the $2(2n-1)$-gon. It is easy to show that for any subset of black vertices $V' \subseteq f(T)$, there exists a subset of teams $T' \subseteq T$ satisfying $f(T') = V'$.

Next, we define a *black run*. A black run is a maximal subset of neighboring vertices of the $2(2n-1)$-gon satisfying that every vertex is colored black or uncolored, and both end vertices are colored black. We also define a *gray run* in a similar way. Black runs and gray runs are called runs for simplicity. The size of a run is the number of colored vertices in the run.

In the following, we introduce some functions that take a vertex of the $2(2n-1)$-gon or a subset of the vertices:

$$\mathit{shift}^+(\overline{v}) \overset{\text{def.}}{=} \begin{cases} \underline{1} & (\text{if } \overline{v} = \overline{2n-1}), \\ \overline{v+1} & (\text{otherwise}), \end{cases} \qquad \mathit{shift}^+(\underline{v}) \overset{\text{def.}}{=} \begin{cases} \overline{1} & (\text{if } \underline{v} = \underline{2n-1}), \\ \underline{v+1} & (\text{otherwise}), \end{cases}$$

$$\mathit{shift}^-(\overline{v}) \overset{\text{def.}}{=} \begin{cases} \underline{2n-1} & (\text{if } \overline{v} = \overline{1}), \\ \overline{v-1} & (\text{otherwise}), \end{cases} \qquad \mathit{shift}^-(\underline{v}) \overset{\text{def.}}{=} \begin{cases} \overline{2n-1} & (\text{if } \underline{v} = \underline{1}), \\ \underline{v-1} & (\text{otherwise}), \end{cases}$$

$$\mathit{vshift}^+(V) \overset{\text{def.}}{=} \{\mathit{shift}^+(v) \mid v \in V\}, \quad \mathit{vshift}^-(V) \overset{\text{def.}}{=} \{\mathit{shift}^-(v) \mid v \in V\},$$

$$\mathit{opp}(\overline{v}) \overset{\text{def.}}{=} \underline{v},\ \mathit{opp}(\underline{v}) \overset{\text{def.}}{=} \overline{v},\ \mathit{vopp}(V) \overset{\text{def.}}{=} \{\mathit{opp}(v) \mid v \in V\},$$

$$\mathit{del}(\overline{v}) \overset{\text{def.}}{=} v,\ \mathit{del}(\underline{v}) \overset{\text{def.}}{=} v.$$

For any subset of vertices $V$, we construct a 0-1 matrix $M = g(V)$ whose rows are indexed by $V$, columns are indexed by the set of slots $S$, and each element $m_{vs}$ is indexed by $(v, s) \in V \times S$. Each element of $M = (m_{vs})$ is defined by

$$m_{vs} \overset{\text{def.}}{=} \begin{cases} 0 & \big(\text{if } v \in \{\overline{1}, \overline{2}, \dots, \overline{2n-1}\} \text{ and } s < \mathit{del}(v)\big), \\ 0 & \big(\text{if } v \in \{\underline{1}, \underline{2}, \dots, \underline{2n-1}\} \text{ and } s \geq \mathit{del}(v)\big), \\ 1 & \big(\text{otherwise}\big). \end{cases}$$

For any 0-1 matrix $M = (m_{vs})$ whose rows are indexed by the subset of vertices $V$ and columns are indexed by the set of slots $S$, we define function $\gamma$ by $\gamma(M) \overset{\text{def.}}{=} \sum_{s \in S} \min\{\sum_{v \in V} m_{vs}, \sum_{v \in V}(1 - m_{vs})\} - |V|(|V|-1)/2$. Also, function $\beta$ is defined by $\beta(V) = \gamma\big(g(V)\big)$. The definition of $\gamma$ directly implies that, for any subset of teams $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3\text{nc}}$, $\alpha(T) = \beta(V)$ holds when $V = f(T)$.

In the following, we transform a given subset of teams $T \in \mathcal{T}_{3\text{nc}}$ into $T^* \in \mathcal{T}_1 \cup \mathcal{T}_2$ such that $\alpha(T^*) \leq \alpha(T)$. Input the set of black vertices $V = f(T)$ into the following procedure. The procedure consists of outermost, middle and innermost loops.

**begin** procedure

$V' := V$
let all the vertices of the $2(2n-1)$-gon uncolored
color all the vertices in $V'$ black and all the vertices in $\mathit{vopp}(V')$ gray
**repeat**
  choose a minimum size black run $R'$ from the $2(2n-1)$-gon
  put $V'_{\min}$ be the set of black vertices in $R'$
  **repeat**
    **if** $\beta\big((V' \setminus V'_{\min}) \cup \mathit{vshift}^-(V'_{\min})\big) \leq \beta(V')$ **then**
      **repeat**
        $V' := (V' \setminus V'_{\min}) \cup \mathit{vshift}^-(V'_{\min})$
        $V'_{\min} := \mathit{vshift}^-(V'_{\min})$
      **until** $\mathit{vopp}(V'_{\min}) \cap (V' \setminus V'_{\min}) \neq \emptyset$
    **else**
      **repeat**
        $V' := (V' \setminus V'_{\min}) \cup \mathit{vshift}^+(V'_{\min})$
        $V'_{\min} := \mathit{vshift}^+(V'_{\min})$
      **until** $\mathit{vopp}(V'_{\min}) \cap (V' \setminus V'_{\min}) \neq \emptyset$
    **endif**
    $V'_{\mathrm{temp}} := \mathit{vopp}(V'_{\min}) \cap (V' \setminus V'_{\min})$
    $V' := V' \setminus V'_{\mathrm{temp}}$
    $V'_{\min} := V'_{\min} \setminus \mathit{vopp}(V'_{\mathrm{temp}})$
  **until** $V'_{\min} = \emptyset$
  let all the vertices of the $2(2n-1)$-gon uncolored
  color all the vertices in $V'$ black and all the vertices in $\mathit{vopp}(V')$ gray
**until** the number of runs in $2(2n-1)$-gon is less than or equal to 2
$V^* := V'$
**output** $V^*$

**end**

It is not difficult to see that the value of $\beta(V')$ does not increase after each iteration of the middle loop. More precisely, in a similar way with the proof of Lemma 2, we can prove that removing two elements from $V'$ at the end of middle loop decreases the value of $\beta(V')$. Thus, the value of $\beta(V')$ is non-increasing throughout this procedure.

We need to show that this procedure outputs $V^*$ after a finite number of steps. The innermost loop and the middle loop obviously terminate finitely. The number of black and gray runs decreases by four at the end of each iteration of the outermost loop. Since the number of runs is finite, this procedure terminates after a finite number of steps.

At the end of each iteration of the middle loop, the current subset of vertices $V'$ is contained in $V$, because $V'_{\min}$ corresponds to a minimum size black run throughout this procedure.

From the above observations of the procedure, the obtained vertices $V^*$ is a subset of $V$. Thus, there exists the subset of teams $T^* \subseteq T$ such that $f(T^*) = V^*$.

Hence, we have $\alpha(T^*) = \beta(V^*) \leq \beta(V) = \alpha(T)$. Since the number of runs becomes two (one is black and the other is gray), $T^*$ is narrow, i.e., $T^* \in \mathcal{T}_1 \cup \mathcal{T}_2$. This completes the proof.

*Example 3.* Figures 16–21 show the process of the transformation described in Lemma 3. In Fig. 16, $T = \{1, 2, 3, 6, 7, 8, 12, 13\} \in \mathcal{T}_{3\text{nc}}$, $f(T) = V = \{\overline{1}, \overline{2}, \overline{3}, \underline{6}, \underline{7}, \overline{8}, \overline{12}, \overline{13}\}$ and $\alpha(T) = \beta(V) = 20$. Via Fig. 17–20, $V^* = f(T^*) = \{\overline{1}, \overline{10}, \overline{11}, \overline{14}\}$, $V^* \subseteq V$, $T^* = \{1, 6, 7, 8\} \in \mathcal{T}_2$ and $\beta(V^*) = \alpha(T^*) = 8$ in Fig. 21.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 : | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 : | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| 7 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |
| 8 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |
| 12 : | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 : | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Fig. 16.** $T = \{1, 2, 3, 6, 7, 8, 12, 13\} \in \mathcal{T}_{3\text{nc}}$, $\alpha(T) = 20$, $V = \{\overline{1}, \overline{2}, \overline{4}, \underline{6}, \underline{8}, \overline{10}, \overline{11}, \overline{14}\}$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{1}$ : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{2}$ : | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{4}$ : | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{10}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| $\overline{11}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |
| $\overline{14}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |
| $\underline{5}$ : | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\underline{7}$ : | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Fig. 17.** $V' = \{\overline{1}, \overline{2}, \overline{4}, \underline{5}, \underline{7}, \overline{10}, \overline{11}, \overline{14}\}$, $\beta(V') = 20$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{1}$ : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{2}$ : | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{4}$ : | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{10}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| $\overline{11}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |
| $\overline{14}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |
| $\underline{4}$ : | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\underline{6}$ : | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Fig. 18.** $V' = \{\overline{1}, \overline{2}, \overline{4}, \underline{4}, \underline{6}, \overline{10}, \overline{11}, \overline{14}\}$, $\beta(V') = 20$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{1}$ : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{2}$ : | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{10}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| $\overline{11}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |
| $\overline{14}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |
| $\underline{6}$ : | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Fig. 19.** $V' = \{\overline{1}, \overline{2}, \underline{6}, \overline{10}, \overline{11}, \overline{14}\}$, $\beta(V') = 18$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{1}$ : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{2}$ : | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\overline{10}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| $\overline{11}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |
| $\overline{14}$ : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |
| $\underline{2}$ : | 1 | $\underline{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Fig. 20.** $V' = \{\overline{1}, \overline{2}, \underline{2}, \overline{10}, \overline{11}, \overline{14}\}$, $\beta(V') = 14$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 : | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 | 1 |
| 7 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 1 | 1 | 1 |
| 8 : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 |

**Fig. 21.** $V^* = \{\overline{1}, \overline{10}, \overline{11}, \overline{14}\}$, $T^* = \{1, 6, 7, 8\} \in \mathcal{T}_2$, $\alpha(T^*) = 8$.

**Lemma 4.** *For any PSMB,*
$\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3\text{nc}},\ \alpha(T) \geq 0 \implies \forall T \in \mathcal{T}_{3\text{wc}},\ \alpha(T) \geq 0.$

*Proof.* We prove that $\forall T \in \mathcal{T}_{3\text{wc}},\ \exists T^* \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3\text{nc}},\ \alpha(T^*) \leq \alpha(T)$.

For a given $T \in \mathcal{T}_{3\text{wc}}$, we update $T$ by removing a complement pair $\{t, \widetilde{t}\}$. If $T \setminus \{t, \widetilde{t}\}$ still contains another complement pair, repeat this procedure until there exists no complement pairs. Let $T^*$ be the finally obtained set of teams. By Lemma 2, $\alpha(T) > \alpha(T \setminus \{t, \widetilde{t}\}) > \cdots > \alpha(T^*)$. Since $T^*$ does not include any complement pairs and $|T^*| < n$, $T^* \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3\text{nc}}$.

**Lemma 5.** *For any PSMB,* $\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2,\ \alpha(T) \geq 0 \implies \forall T \in \mathcal{T}_3,\ \alpha(T) \geq 0.$

*Proof.* From Lemma 3, $\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2,\ \alpha(T) \geq 0 \implies \forall T \in \mathcal{T}_{3\text{nc}},\ \alpha(T) \geq 0$. Lemma 4 shows $\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3\text{nc}},\ \alpha(T) \geq 0 \implies \forall T \in \mathcal{T}_{3\text{wc}},\ \alpha(T) \geq 0$. Since $\mathcal{T}_3 = \mathcal{T}_{3\text{nc}} \cup \mathcal{T}_{3\text{wc}}$, $\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2,\ \alpha(T) \geq 0 \implies \forall T \in \mathcal{T}_3,\ \alpha(T) \geq 0$.

**Lemma 6.** *For any PSMB,*
$\forall T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3,\ \alpha(T) \geq 0 \iff \forall T \in \mathcal{T}_4,\ \alpha(T) \geq 0.$

*Proof.* Theorem 1 gives direct proof of this lemma.

**Theorem 5.** *For any PSMB,* $\forall T \in \mathcal{T}_1,\ \alpha(T) \geq 0 \iff \forall T \subseteq U,\ \alpha(T) \geq 0.$

*Proof.* $\forall T \in \mathcal{T}_1,\ \alpha(T) \geq 0 \impliedby \forall T \subseteq U,\ \alpha(T) \geq 0$ is obvious.

$\forall T \in \mathcal{T}_1,\ \alpha(T) \geq 0 \implies \forall T \subseteq U,\ \alpha(T) \geq 0$ follows from Lemmas 1, 5, and 6.

**Theorem 6.** *For any PSMB, whether the PSMB satisfies* $\forall T \subseteq U,\ \alpha(T) \geq 0$ *or not can be tested in* $\mathrm{O}(n^4)$ *steps.*

*Proof.* For each $T \subseteq U$, we can judge whether $\alpha(T) \geq 0$ holds or not in $\mathrm{O}(n^2)$ steps. Theorem 5 suggests that we only need to see the subsets of teams belonging to $\mathcal{T}_1$. Since $|\mathcal{T}_1| = \mathrm{O}(n^2)$, we obtain the desired result.

*Remark 6.* In fact, we do not need to observe all the subsets belonging to $\mathcal{T}_1$. Since Theorem 4 shows $\alpha(T) = \alpha(\widetilde{T})$, it is sufficient to show either $\alpha(T) \geq 0$ or $\alpha(\widetilde{T}) \geq 0$ for each $T \in \mathcal{T}_1$.

*Remark 7.* We can easily check whether each $T \subseteq U$ satisfies $\alpha(T) \geq 0$ or not in $\mathrm{O}(n^2)$ steps. Furthermore, we developed a linear time algorithm to check whether each $T \in \mathcal{T}_1$ satisfies $\alpha(T) \geq 0$. This algorithm is described in [12]. Thus, whether a given PSMB satisfies the necessary condition $\forall T \subseteq U,\ \alpha(T) \geq 0$ can be examined in $\mathrm{O}(n^3)$ steps.

### 4.3 Computational Experiments

We showed, for an arbitrary PSMB, whether the PSMB satisfies our necessary condition for feasible pattern sets can be checked in polynomial steps. If we find a subset of teams $T$ such that $\alpha(T) < 0$ for a given PSMB, we can conclude that the PSMB is infeasible. However, for PSMB satisfying the necessary condition, we have no additional information about the feasibility. If there are many PSMB satisfying the necessary condition but infeasible, the proposed condition is not strong enough for finding feasible PSMB. In this subsection, we show the strength of our necessary condition by computational experiments.

First, we enumerate all PSMB, including both feasible and infeasible ones. The number of PSMB is $\binom{2n-2}{n-1}$, by choosing the slots with breaks. Next, we examine whether each PSMB satisfies the necessary condition $\forall T \subseteq U$, $\alpha(T) \geq 0$, by checking $\forall T \in \mathcal{T}_1$, $\alpha(T) \geq 0$. Then, for each PSMB satisfying the condition, we decide its feasibility by solving an integer programming problem with ILOG CPLEX 7.0 [9].

Table 2 shows results of the experiments. The column "#cand." is the number of PSMB satisfying $\forall T \subseteq U$, $\alpha(T) \geq 0$, and "#feas." is the number of feasible PSMB. Computational experiments showed that, when the number of teams is less than or equal to 26, the proposed necessary condition $\forall T \subseteq U$, $\alpha(T) \geq 0$ is also a sufficient condition for feasible PSMB.

We did not perform computational experiments for more than 26 teams, because it would take too much time to solve the integer programming problems, and a round robin tournament with more than 26 teams is rarely held in practice.

**Table 2.** Results of computational experiments.

| #teams | #PSMB | #cand. | #feas. |
|---|---|---|---|
| 4 | 2 | 2 | 2 |
| 6 | 6 | 3 | 3 |
| 8 | 20 | 8 | 8 |
| 10 | 70 | 10 | 10 |
| 12 | 252 | 30 | 30 |
| 14 | 924 | 49 | 49 |
| 16 | 3432 | 136 | 136 |
| 18 | 12870 | 216 | 216 |
| 20 | 48620 | 580 | 580 |
| 22 | 184756 | 1045 | 1045 |
| 24 | 705432 | 2772 | 2772 |
| 26 | 2704156 | 5122 | 5122 |

## 5 Future Work & Conclusions

There are many extensions of pattern set feasibility problem.

We conjecture that, for arbitrary number of teams, $\forall T \subseteq U$, $\alpha(T) \geq 0$ is a necessary and sufficient condition for feasible PSMB. Although there is a possibility of this conjecture failing, our results are practical enough in terms of the number of teams for real timetabling problems.

In this paper, we mainly considered the feasibility of PSMB. We proposed a necessary condition for feasible pattern sets; $\forall T \subseteq U$, $\alpha(T) \geq 0$. In the case of PSMB, we proved that the necessary condition can be checked in polynomial steps, and showed the strength of the condition by computational experiments. Then, how about for a general pattern set? For a general pattern set, it is conjectured that pattern set feasibility problem is NP-complete [13]. To the best of our knowledge, this conjecture is still open.

Pattern set feasibility problem with partial assignment of games is also a challenging one. In the pattern set of Fig. 22, some games have already fixed. Is the pattern set feasible? Furthermore, we can consider the feasibility problem of an incomplete pattern set with or without fixed games (Fig. 23). Such situations often appear in the field of real timetabling [13]. It is easy to see that we can also apply our necessary condition to these problems with small modification.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 : | H | 2 | H | A | A | A | H |
| 2 : | A | @1 | H | H | A | @8 | H |
| 3 : | H | A | A | A | H | H | H |
| 4 : | 7 | H | A | H | A | A | H |
| 5 : | H | A | H | @6 | H | H | A |
| 6 : | H | A | H | 5 | H | A | A |
| 7 : | @4 | H | A | A | A | H | A |
| 8 : | A | H | A | H | H | 2 | A |

**Fig. 22.** Pattern set with partial assignment of games.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 : | A | @4 | | 2 | | H | 8 |
| 2 : | H | A | | @1 | | 7 | |
| 3 : | | | 8 | A | | H | |
| 4 : | A | 1 | | A | | H | A |
| 5 : | | A | H | A | @6 | | H |
| 6 : | @8 | H | H | | 5 | | @1 |
| 7 : | | | A | | | @2 | H |
| 8 : | 6 | H | @3 | H | | A | H |

**Fig. 23.** Incomplete pattern set with partial assignment of games.

In this paper, we considered pattern set feasibility problem. We proposed a necessary condition for feasible pattern sets. For a pattern set with a minimum number of breaks (PSMB), we proved a theorem leading a polynomial-time algorithm to check our necessary condition. Computational experiments showed that the condition is a necessary and sufficient condition for feasible PSMB of up to 26 teams. We conjecture that the condition is a necessary and sufficient condition for feasible PSMB of an arbitrary number of teams. Considering the feasibility of pattern sets yields various interesting problems, and is still significant in sports timetabling.

## References

1. Armstrong, J.R., and R. J. Willis, R.J.: Scheduling the Cricket World Cup—a Case Study. Journal of the Operations Research Society **44** (1993) 1067–1072
2. Blest, D.C., Fitzgerald, D.G.: Scheduling Sports Competitions with a Given Distribution of Times. Discrete Applied Mathematics **22** (1988/89) 9–19
3. de Werra, D.: Geography, Games and Graphs. Discrete Applied Mathematics **2** (1980) 327–337
4. de Werra, D.: Minimizing Irregularities in Sports Schedules Using Graph Theory. Discrete Applied Mathematics **4** (1982) 217–226
5. de Werra, D.: On the Multiplication of Divisions: the Use of Graphs for Sports Scheduling. Networks **15** (1985) 125–136
6. de Werra, D.: Some Models of Graphs for Scheduling Sports Competitions. Discrete Applied Mathematics **21** (1988) 47–65
7. Ferland, J.A., Fleurent, C.: Computer Aided Scheduling for a Sport League. INFOR **29** (1991) 14–25
8. Henz, M.: Scheduling a Major College Basketball Conference—Revisited. Operations Research **49** (2001) 163–168
9. ILOG: ILOG CPLEX 7.0 User's Manual. ILOG, Gentilly, 2000
10. Miyashiro, R., Matsui, T.: Note on Equitable Round-Robin Tournaments. Proceedings of 6th KOREA-JAPAN Joint Workshop on Algorithms and Computation (2001) 135–140
11. Miyashiro, R., Matsui, T.: Notes on Equitable Round-Robin Tournaments. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Science **E85-A** (2002), 1006–1010
12. Miyashiro, R., Iwasaki, H., Matsui, T.: Characterizing Feasible Pattern Sets with a Minimum Number of Breaks. In: Burke, E.K., De Causmaecker, P. (eds.): Proceedings of the 4th International Conference on the Practice and Theory of Automated Timetabling (2002) 311–313
13. Nemhauser, G.L., Trick, M.A.: Scheduling a Major College Basketball Conference. Operations Research **46** (1998) 1–8
14. Trick, M.A.: A Scheduling-Then-Break Approach to Sports Timetabling. In: Burke, E.K., Erben, W. (eds.): Practice and Theory of Automated Timetabling III. Lecture Notes in Computer Science, Vol. 2079. Springer-Verlag, Berlin Heidelberg New York (2001) 242–253
15. Russell, K.G.: Balancing Carry-Over Effects in Round Robin Tournaments. Biometrika **67** (1980) 127–131
16. Russell, R.A., Leung, J.M.Y.: Devising a Cost Effective Schedule for a Baseball League. Oper. Res. **42** (1994) 614–625
17. Schreuder, J.A.M.: Combinatorial Aspects of Construction of Competition Dutch Professional Football Leagues. Discrete Applied Mathematics **35** (1992) 301–312

18. van Weert, A., Schreuder, J.A.M.: Construction of Basic Match Schedules for Sports Competitions by Using Graph Theory. In: Burke, E.K., Carter, M.W. (eds.): The Practice and Theory of Automated Timetabling II. Lecture Notes in Computer Science, Vol. 1408. Springer-Verlag, Berlin Heidelberg New York (1998) 201–210
19. Willis, R.J., Terrill, B.J.: Scheduling the Australian State Cricket Season Using Simulated Annealing. Journal of the Operations Research Society **45** (1994) 276–280
20. Wright, M.: Timetabling County Cricket Fixtures Using a Form of Tabu Search. Journal of the Operations Research Society **45** (1994) 758–770