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Multi-Object Auctions with a Single Bundle Bidding for Perfect Complements *

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Abstract

In this paper, we consider multi-object auctions in which each bidder submits a pair of a subset of objects and his purchase price for the set. The seller solves the assignment problem of objects to maximize his revenue, and decides the winning bidders who can purchase their reporting subset for the prices given. We analyze this auction on the assumption that each bidder has one special subset of objects which are perfect complements for him, and that no object out of the subset is valuable for him. We show that this auction leads to an efficient allocation through a Nash equilibrium, if it exists. We also show that when the bid-grid size is sufficiently small, the equilibrium exists. *Journal of Economic Literature* Classification Numbers: C72, D44, D51.

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1 Introduction

In recent studies, a lot of attention has been paid to multi-object auctions, in which participants can bid on some combination of objects, rather than on each object separately. These auctions are expected to play an important role when significant synergies or complementarities exist among the objects being auctioned. The sales of radio spectrum licenses are practical examples of auctions with synergies, and more examples can be found in the development of electronic commerce, such as selling delivery routes in logistics and airport time slots.

Combinatorial auctions in which bidders submit their valuation for all subsets of objects are the most typical formulation of this type of multi-object auctions. De Vries and Vohra [7] and Pekeč and Rothkopf [17] comprehensively surveyed and discussed the design of combinatorial auctions, and many studies have also been conducted on these types of auctions in the last few years. Combinatorial auctions have many desirable properties for selling objects with synergies, but they also have some defects. The computational difficulty of winner determination is one of the most outstanding problems, and many studies in computer science have been undertaken to develop computational methodology. (Andersson, Tenhunen and Ygge [1] and Rothkopf, Pekeč and Harstad [19])

Another disadvantage of combinatorial auctions is an assumption that each bidder must submit his valuations for all combinations of objects. In this assumption, each bidder must evaluate and compute all subsets of objects without costing him anything, and even if this is possible, he must write out all valuations to submit his bid. This assumption is also impractical to employ because the number of subsets increases exponentially with the increase in objects. This disadvantage is crucial in generalized Vickrey auctions which achieve efficient allocations through dominant strategies. Ausubel and Milgrom [3] pointed to it as a defect of generalized Vickrey auctions, and suggested ascending auctions with package bidding.

In this paper, we examine simultaneous and sealed bid auctions in which each bidder is allowed to submit only *one* pair of a subset of objects and his valuation for the set. The seller solves the assignment problem of objects to maximize his surplus in submitted bids and decides who the winning bidders are. Each winning bidder can purchase his reporting subset of objects at his reporting prices. For the analysis, we assume that each bidder has a positive valuation for only one special subset of objects, called an essential bundle.[1] For each bidder, all objects in his essential bundle are perfect complements, and no object outside the subset has any value for him. On this assumption of the preferences, we show that it leads to an efficient allocation through a Nash equilibrium under complete information. We also show the existence of the Nash equilibrium when the bid-grid size is sufficiently small.

In game theoretical aspects, it is difficult to analyze auctions allowing combinatorial bids, and most studies focus on limited models. For examples, Gale [9], Krishna and Rosenthal [11], Levin [12] and Rosenthal and Wang [18] examined auctions with synergies that each bidder would demand at most two objects. The research on equilibrium strategies for auctions with sets or package bidding still remains to be developed in auction theory in a combinatorial framework. Very few studies on auctions with combinatorial biddings have been made at analysis by Nash equilibrium strategies. Our aim in this paper is to show the stability of combinatorial auctions for strategic behavior through a standard analysis of equilibrium strategies, by restricting the model to a simple bidding rule and by assuming a simple preference. [2] In the point of analysis by equilibrium strategies, our approach is related to the results of Bikhchandani [4]. Bikhchandani [4] showed that a sealed bid first price auction, in which each object is sold independently, implements Walrasian equilibrium allocations by pure strategy Nash equilibria, when the bid grid-size in the auction is sufficiently small. Therefore, efficient allocations can be achieved by auctions in which each object is sold independently, if Walrasian equilibrium allocations exist. The conditions on the existence of Walrasian equilibrium allocations in this setting were investigated by Kelso and Crawford [10] and Bikhchandani and Mamer [5].

However, if no Walrasian equilibrium exists, efficient allocations may not be achieved by auctions for each object. Auctions with combinatorial biddings are important when no Walrasian equilibrium exists. The allocation problems in this paper include this case. We show an example in which a Walrasian equilibrium does not exist in our setting, that is, each player has an essential bundle. Thus, our proposed auctions can achieve efficient outcomes, even if a Walrasian equilibrium does not exist.

[1] Recently, we found that a bidder with this type of valuation function is called a *single-minded bidder* in the paper [16] by O'Callaghan and Shoham. Single-minded bidders are also dicussed in the papers [2] by Archer, Papadimitriou, Talwar, and Tardos, and [14] by Mu'alem and Nisan.

[2] Recently some variations of GVA mechanisms are proposed which attain both truthfulness and algorithmic eficiency under some assumptions. See the papers [2, 14, 16] for detail.

Multi-object auctions and combinatorial auctions have recently been brought to much attention because there are many applications in deregulation of public service and on-line auctions in Internet. De Vries and Vohra [7] and Pekeč and Rothkopf [17] assert their possibilities and report some practical examples. Auctions with a single bundle bidding on preferences with essential bundles appears in some important applications of them. In this paper, we describe some applications in sale of spectrum licenses and assignment of delivery routes.

The paper is organized as follows. In Section 2 we describe our model and auctions with essential bundles. In Section 3, we describe some examples in which auctions with essential bundles can be applied to auctions for radio-frequency spectrum and delivery routes. We also show an example in which a Walrasian equilibrium does not exist on preferences with essential bundles in Section 3. In Section 4, we show that a pair of essential bundles and a minimal bid vector which leads to an efficient allocation is a Nash equilibrium, and that a Nash equilibrium exists if the bid grid-size is small enough. In Section 5, we present the conclusion.

2 The Model

2.1 Notation and Definitions

Through the paper, an n dimensional vector in which each ith component is y_i is denoted by $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ and an $n-1$ dimensional vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ is denoted by $\boldsymbol{y}_{-i}$. In convenient, an n dimensional vector $(y_1, \ldots, y_{i-1}, z_i, y_{i+1}, \ldots, y_n)$ is written as $(z_i, \boldsymbol{y}_{-i})$.

There are n bidders and m indivisible objects. Let $N = \{1, 2, \ldots, n\}$ be the set of bidders, and $M = \{1, 2, \ldots, m\}$ the set of objects. Each subset of objects is called a *bundle*. Each bidder i has a nonnegative valuation $V_i(S)$ for each bundle $S \subseteq M$. We assume that the valuation to the empty set is zero for any bidder i, i.e., $V_i(\emptyset) \equiv 0$ for any $i \in N$. We also assume that for any bundle, the valuation of the seller is zero.

In this paper, we assume that each bidder i has a positive valuation only for one special bundle. This bundle is called the *essential bundle* of bidder i. If he does not gain some objects in the bundle, any other object in the bundle is not valuable to him at all. We also assume that the objects in the bundle are also sufficient to him, so he has no value for any object out of the bundle. The essential bundle of bidder i is denoted by T_i and his value for T_i is $v_i > 0$. Hence,

$V_i(S)$ is written as follows:

$$V_i(S) = \begin{cases} v_i & (T \subseteq S), \\ 0 & (\text{otherwise}). \end{cases}$$

We denote the unit of valuations by δ, and assume that for any $i \in N$, v_i is a multiple of δ, i.e., $\boldsymbol{v}_i \in \{\delta, 2\delta, 3\delta, \ldots\}$.

2.2 Auctions with Essential Bundles

We propose the following sealed bid simultaneous auctions with a single bundle bidding. At the beginning of the auction, each bidder $i \in N$ submits a bid (B_i, b_i) where B_i is a bundle and the nonnegative real number b_i is the amount she is willing to pay for the bundle B_i. We assume that there exists a positive integer I and all bids must be multiples of $\frac{\delta}{I}$. We write the bid unit as ϵ, i.e., $\epsilon = \frac{\delta}{I}$. The set of integer multiples of the bid unit is denoted by $Z_\varepsilon = \{\varepsilon k \mid k \text{ is a nonnegative integer}\}$. Then, $b_i \in Z_\varepsilon$ for each $i \in N$. In the following, we write a profile of bids $((B_1, b_1), (B_2, b_2), \ldots, (B_n, b_n))$ as $(\boldsymbol{B}, \boldsymbol{b})$ by changing the order of components.

The auctioneer solves an integer programming problem, called the Bundle Assignment Problem (BAP), which maximizes the surplus. This problem is defined as follows:

$$\begin{array}{lll} \text{BAP}(\boldsymbol{B}, \boldsymbol{b}): & \text{maximize} & \sum_{i \in N} b_i x_i = \boldsymbol{b} \cdot \boldsymbol{x} \\ & \text{subject to} & \sum_{i : B_i \ni j} x_i \leq 1 \quad (\forall j \in M), \\ & & x_i \in \{0, 1\} \quad (\forall i \in N), \end{array}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. We denote the set of all the optimal solutions of BAP$(\boldsymbol{B}, \boldsymbol{b})$ by $\Omega(\boldsymbol{B}, \boldsymbol{b})$.

The auctioneer solves the problem BAP$(\boldsymbol{B}, \boldsymbol{b})$ and obtains an optimal solution $\boldsymbol{x}^*$. If the problem BAP$(\boldsymbol{B}, \boldsymbol{b})$ has multiple optimal solutions, the auctioneer chooses an optimal solution $\boldsymbol{x}^* \in \Omega(\boldsymbol{B}, \boldsymbol{b})$ at random. The bid by bidder i is *accepted* if and only if $x_i^* = 1$.

If the bidder i is accepted, he can get the bundle B_i by paying the price b_i. Given the problem BAP$(\boldsymbol{B}, \boldsymbol{b})$, we partition the set of bidders by $P(\boldsymbol{B}, \boldsymbol{b}), Q(\boldsymbol{B}, \boldsymbol{b})$ and $R(\boldsymbol{B}, \boldsymbol{b})$ as follows,

$$\begin{aligned} P(\boldsymbol{B}, \boldsymbol{b}) &= \{i \in N \mid \forall \boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}),\ x_i = 1\}, \\ R(\boldsymbol{B}, \boldsymbol{b}) &= \{i \in N \mid \forall \boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}),\ x_i = 0\}, \end{aligned}$$

$$Q(\boldsymbol{B}, \boldsymbol{b}) \quad = \quad N \setminus (P(\boldsymbol{B}, \boldsymbol{b}) \cup R(\boldsymbol{B}, \boldsymbol{b})),$$

where each member in $P(\boldsymbol{B}, \boldsymbol{b})$, $Q(\boldsymbol{B}, \boldsymbol{b})$ and $R(\boldsymbol{B}, \boldsymbol{b})$ is called a *passed* bidder, a *questionable* bidder, and a *rejected* bidder, respectively.

Given a profile $(\boldsymbol{B}, \boldsymbol{b})$, the expected utility of bidder i denoted by $U_i(\boldsymbol{B}, \boldsymbol{b})$ is defined as follows,

$$U_i(\boldsymbol{B}, \boldsymbol{b}) = (V_i(B_i) - b_i)\frac{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}|}{|\Omega(\boldsymbol{B}, \boldsymbol{b})|}$$

Here, we note that $|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}|/|\Omega(\boldsymbol{B}, \boldsymbol{b})|$ is the probability that the auctioneer accepts the bidder i and this probability of the rejected bidder is equal to 0.

3 Examples

Multi-object auctions have recently been received attention because of the importance of their applications in deregulation of public service and progress of e-commerce. We consider that preferences with essential bundles are approximately appeared in some remarkable environments of them. Moreover, preferences with essential bundles include the case where a Walrasian equilibrium does not exist.

In the following Subsection 3.1, we show that auctions with a single bundle bidding on preferences with essential bundles can be applied to selling spectrum licenses and assignment of delivering routes. In Subsection 3.2, we provide an example in which a Walrasian equilibrium does not exist.

3.1 Applications

Example 1 (Spectrum Auctions:) An auctioneer wants to sell licenses for radio spectrums $M = \{1, 2, \cdots, m\}$. Each spectrum j is adjacent to $j-1$ on the right. In spectrum auctions, it is known that each agent wants to purchase spectrums neighboring each other. Krishna and Rosenthal [11] and Rosenthal and Wang [18] analyzed models in which each object has two neighboring objects on the right and the left, and "global" bidders want to purchase the object together with its neighbor on the left or the right. We assume the extreme case of the preferences in which each bidder i requires any spectrum j satisfying $L_i \geq j \geq H_i$ but no spectrum outside of it. This setting is applicable to our model where agent i's essential bundle is $T_i = \{j \in M \mid L_i \geq j \geq H_i\}$.

Example 2 (Auctions for Delivery Routes:) Suppose that a logistic consulting firm wants to sell carrier services of commodities between cities, which are provided by several transporters to customers. The logistic consulting firm holds an auction to sell the services and the customers participate it as bidders. One transport facility is located for each city and each carrier service provided a transporter is represented as a segment from one city to another city. Each customer wants to buy a delivery routes denoted as a path of connecting a pair of cities. Let W be the set of transit facilities or terminals and $E \subseteq W \times W$ be the set of segments provided by transporters. Each customer i wants to convey his commodities from one city $w_i^s \in V$ to another city $w_i^t \in V$. We assume an extreme case where there exists only one combination of segments which realizes the path from w_i^s to w_i^t and the customer has no value if he cannot convey the commodities between the cities.

This case corresponds to our model and the set of objects is E and the essential bundle of bidder i is the set of the segments which realizes the path from w_i^s to w_i^t.

3.2 Non-existence of a Walrasian Equilibrium

Bikhchandani [4] showed an important result that efficient allocations can be achieved by auctions in which each object is sold independently, if Walrasian equilibrium allocations exist. In the following, we show an example of a profile in which a Walrasian equilibrium does not exist with essential bundles. Here, we briefly review the definition of a Walrasian equilibrium in our setting. A *feasible allocation* $(S_1, \ldots, S_n)$ is a family of mutually disjoint subsets of objects M. A feasible allocation $(S_1, \ldots, S_n)$ is said to be *a Walrasian allocation*, if there exist non-negative prices $(p_1, \ldots, p_n)$ such that

$$V_i(S_i) - \sum_{k \in S_i} p_k \geq V_i(S) - \sum_{k \in S} p_k \ (\forall S \subseteq M,\ \forall i \in \{1, 2, \ldots, n\}).$$

The vector $(p_1, p_2, \ldots, p_n)$ is called a *Walrasian price vector*. A *Walrasian equilibrium* is a pair of a Walrasian allocation and a Walrasian price vector.

Example 3 (Non-existence of a Walrasian Equilibrium:) Consider the case that there are 4 bidders and 3 objects. The essential bundles of the bidders are given by $T_1 = \{1, 2\}, T_2 = \{2, 3\}, T_3 = \{1, 3\}$ and $T_4 = \{3\}$. Valuations of the bidders for the essential bundles are given by $v_1 = 5, v_2 = v_3 = 4, v_4 = 1$. We show that there does not exists any Walrasian Equilibrium in this case.

Assume that there exists a Walrasian equilibrium with the Walrasian allocation (S_1, S_2, S_3, S_4) and the Walrasian prices p_1, p_2, p_3, and p_4. Since a Walrasian allocation is efficient, we find that $S_1 = \{1, 2\}$, $S_4 = \{3\}$ and $S_2 = S_3 = \emptyset$. From the definition of a Walrasian equilibrium, we have the following four inequalities:

$$p_1 + p_2 \leq 5, \tag{1}$$

$$p_2 + p_3 \geq 4, \tag{2}$$

$$p_1 + p_3 \geq 4, \tag{3}$$

$$p_3 \leq 1. \tag{4}$$

Inequalities (2) and (4) imply that $p_2 \geq 3$, and (3) and (4) imply $p_1 \geq 3$. These inequalities imply $p_1 + p_2 \geq 6$. However, this contradicts with (1). Hence, a Walrasian equilibrium does not exist in this case.

4 Nash Equilibria

4.1 Pure Strategy Nash Equilibria

Given the bid unit ε and a profile $(\boldsymbol{B}, \boldsymbol{v})$, we consider the following set:

$$\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v}) = \left\{ \boldsymbol{b} \in Z_\varepsilon^N \;\middle|\; \begin{array}{ll} \Omega(\boldsymbol{B}, \boldsymbol{b}) = \Omega(\boldsymbol{B}, \boldsymbol{v}), & \\ b_i = v_i & (\forall i \in R(\boldsymbol{B}, \boldsymbol{v}) \cup Q(\boldsymbol{B}, \boldsymbol{v})), \\ b_i \leq v_i - k\varepsilon & (\forall i \in P(\boldsymbol{B}, \boldsymbol{v})) \end{array} \right\}$$

where k is a positive integer satisfying $k \geq |\Omega(\boldsymbol{B}, \boldsymbol{v})|$. The set $\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v})$ is a subset of bid price vectors $\boldsymbol{b}$ such that the set of optimal solutions for BAP$(\boldsymbol{B}, \boldsymbol{b})$ is equivalent to that for BAP$(\boldsymbol{B}, \boldsymbol{v})$. For any subset of bid price vectors $X \subseteq Z_\varepsilon^N$, a vector $\boldsymbol{b} \in X$ is called a *minimal vector in* X if and only if for any $\boldsymbol{b}' \in Z_\varepsilon^N$, the condition $[\boldsymbol{b}' \leq \boldsymbol{b}$ and $\boldsymbol{b}' \neq \boldsymbol{b}]$ implies $\boldsymbol{b}' \notin X$.

Theorem 1 *If $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty, then for any minimal vector $\boldsymbol{b}^*$ in $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$, the profile $(\boldsymbol{T}, \boldsymbol{b}^*)$ is a Nash equilibrium.*

To show this theorem we need following two lemmas. Their proofs appear in Appendix.

Lemma 1 *Let $\boldsymbol{B}$ be any bundle vector satisfying $T_i \subseteq B_i$ for any $i \in N$. Suppose $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$ is non-empty and let $\boldsymbol{b}^*$ be a minimal vector in $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$. Then, on the profile of bid $(\boldsymbol{B},\boldsymbol{b}^*)$, any bidder $i \in N$ can not improve his payoff by changing his price b_i^* to any price $b_i \in Z_\varepsilon$, i.e., $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$, for any $b_i \in Z_\varepsilon$.*

Lemma 2 *Let $(\boldsymbol{B},\boldsymbol{b})$ be any profile of bidding. For any bidder $i \in N$ satisfying $b_i \leq v_i$, his expected utility of reporting (T_i, b_i) is greater than or equal to reporting (B_i, b_i) for any $B_i \subseteq M$, i.e., $U_i((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b}) \geq U_i(\boldsymbol{B},\boldsymbol{b})$.*

Proof of Theorem 1

We need to show that for each bidder $i \in N$, the inequality $U_i(\boldsymbol{T},\boldsymbol{b}^*) \geq U_i((B_i,\boldsymbol{T}_{-i}),(b_i,\boldsymbol{b}^*_{-i}))$ holds for any bundle B_i and any price b_i. If $T_i \setminus B_i \neq \emptyset$, then it is clear, since $U_i(\boldsymbol{T},\boldsymbol{b}^*) \geq 0 \geq U_i((B_i,\boldsymbol{T}_{-i}),(b_i,\boldsymbol{b}_{-i}))$.

Suppose $T_i \subseteq B_i$ for any $i \in N$. Lemma 2 and the fact that $\boldsymbol{b}^* \leq \boldsymbol{v}$ imply that $U_i(\boldsymbol{T},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},\boldsymbol{b}^*)$. Then, Lemma 1 implies $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ for any $b_i \in Z_\varepsilon$. Hence, we can conclude that $(\boldsymbol{T},\boldsymbol{b}^*)$ is a Nash equilibrium. ‖

The Nash equilibrium characterized in Theorem 1 depends on the set $\mathcal{F}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$, so that it depends on the number k satisfying $k \geq |\Omega(\boldsymbol{B},\boldsymbol{v})|$. However, since $|\Omega(B,v)|$ is less than or equal to 2^n, it is enough to set $k = 2^n$ to satisfy the inequality $k \geq |\Omega(\boldsymbol{B},\boldsymbol{v})|$. We can state this formally by defining $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$ as:

$$\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T},\boldsymbol{v}) = \left\{ \boldsymbol{b} \in Z_\varepsilon^N \;\middle|\; \begin{array}{ll} \Omega(\boldsymbol{B},\boldsymbol{b}) = \Omega(\boldsymbol{B},\boldsymbol{v}), & \\ b_i = v_i & (\forall i \in R(\boldsymbol{B},\boldsymbol{v}) \cup Q(\boldsymbol{B},\boldsymbol{v})), \\ b_i \leq v_i - 2^n\varepsilon & (\forall i \in P(\boldsymbol{B},\boldsymbol{v})) \end{array} \right\}$$

Hence, we have the following corollary.

Corollary 1 *If $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$ is non-empty, then for any minimal vector $\boldsymbol{b}^*$ in $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$ the profile $(\boldsymbol{T},\boldsymbol{b}^*)$ is a Nash equilibrium.*

However, as k increases, the set $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$ becomes smaller and the possibility for emptiness of $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$ becomes larger. Conversely, we can set $k = 1$, if the optimal allocations is unique.

Corollary 2 *Suppose that the optimal allocation is unique. If the set*

$$\hat{\mathcal{F}}_\varepsilon(\boldsymbol{T}, \boldsymbol{v}) = \left\{ \boldsymbol{b} \in Z_\varepsilon^N \;\middle|\; \begin{array}{ll} \Omega(\boldsymbol{B}, \boldsymbol{b}) = \Omega(\boldsymbol{B}, \boldsymbol{v}), & \\ b_i = v_i & (\forall i \in R(\boldsymbol{B}, \boldsymbol{v}) \cup Q(\boldsymbol{B}, \boldsymbol{v})), \\ b_i \leq v_i - \varepsilon & (\forall i \in P(\boldsymbol{B}, \boldsymbol{v})) \end{array} \right\}$$

is non-empty, then for any minimal vector $\boldsymbol{b}^*$ in $\hat{\mathcal{F}}_\varepsilon$, the profile $(\boldsymbol{T}, \boldsymbol{b}^*)$ is a Nash equilibrium.

In this way, the existence of a Nash equilibrium which ensures an efficient allocation depends on the non-emptiness of $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$, so the number of optimal solutions. However, the following theorem shows $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$is non-empty when the bid-grid size is sufficiently small.

Theorem 2 *If ε is a sufficiently small positive number, $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty. Hence, there exists at least one Nash equilibrium leading to an efficient allocation.*

Proof. See Appendix. ||

Thus, multi-object auctions with essential bundles achieve an efficient allocation via a Nash equilibrium if the size of the bid unit is sufficiently small.

5 Concluding Remarks

In this paper, we proposed a multi-object auction with a single package bidding when each player has one essential bundle, in which all objects are perfect complements. We showed that the auctions lead to an efficient allocation through a Nash equilibrium, and that a Nash equilibrium exists if the bid-grid is sufficiently small.

Recently, combinatorial auctions have received a great deal of attention in multi-object auctions. Finding the optimal solutions and the determination of winners and an allocation are difficult from computational point of view. So there are several studies on combinatorial auctions focused on computational aspects. Moreover, when a bundle assignment problem has multiple optimal solutions, we choose one optimal solution randomly as a tiebreaker, but it seems more difficult because we have to enumerate all optimal solutions.

Unfortunately, in our auctions, finding the optimal solution of the related bundle assignment problem is also difficult. Rothkopf, Pekeč and Harstad [19] proposed a special case that the related bundle assignment problem is polynomially solvable. In that case, objects are totally ordered and each bidder's essential bundle corresponds to an interval. Their restriction on preferences of bidders can be applied to spectrum auctions and auctions for delivery routings corresponding to Examples 1 and 2 in this paper. Matsui and Watanabe [13] proposed a polynomial time sampling method from optimal solutions. They also described a polynomial size linear inequality system that includes a Nash equilibrium as a minimal feasible interior lattice point.

Appendix

Proof of Lemma 1.

We have $P(\boldsymbol{B},\boldsymbol{v}) = P(\boldsymbol{B},\boldsymbol{b}^*)$, $Q(\boldsymbol{B},\boldsymbol{v}) = Q(\boldsymbol{B},\boldsymbol{b}^*)$, and $R(\boldsymbol{B},\boldsymbol{v}) = R(\boldsymbol{B},\boldsymbol{b}^*)$ from the definition of a minimal vector. We also remark that any feasible solution $\boldsymbol{x}$ of BAP$(\boldsymbol{B},\boldsymbol{b}^*)$ is also feasible to BAP$(\boldsymbol{B},\boldsymbol{b})$ for any $\boldsymbol{b}$.

First, we consider the case that i is a rejected bidder of BAP$(\boldsymbol{B},\boldsymbol{v})$, i.e., $i \in R(\boldsymbol{B},\boldsymbol{v})$. From the definition of $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$, $b_i^* = v_i$. If bidder $i \in R(\boldsymbol{B},\boldsymbol{v})$ submit a bid price b_i satisfying $b_i < b_i^* = v_i$ instead of b_i^*, bidder i will not be accepted and so the payoff $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ is equal to 0. Let us consider the case that bidder $i \in R(\boldsymbol{B},\boldsymbol{v})$ submits a bid price b_i satisfying $b_i > b_i^* = v_i$. If $i \in R(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$, $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ is also equal to zero. If $i \in P(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \cup Q(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$, then

$$U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = (v_i - b_i)|\{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}|/|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))| < 0 = U_i(\boldsymbol{B},\boldsymbol{b}^*).$$

From the above, if i is a rejected bidder, then we have $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$.

Next, we consider a questionable bidder $i \in Q(\boldsymbol{B},\boldsymbol{v})$. From the definition of $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$, $b_i^* = v_i$. If $b_i > v_i$, then $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},\boldsymbol{b}^*) \mid x_i = 1\}$ and $i \in P(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. Hence, we have $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = v_i - b_i < 0 \leq U_i(\boldsymbol{B},\boldsymbol{b}^*)$. Now consider the case that $b_i < v_i$, then $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B},\boldsymbol{b}^*) \mid x_i = 0\}$ and $i \in R(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. Hence, we have $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = 0 \leq U_i(\boldsymbol{B},\boldsymbol{b}^*)$. From the above, if i is a questionable bidder, $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$.

Lastly, we consider the case that $i \in P(\boldsymbol{B},\boldsymbol{v})$. If $b_i > b_i^*$, then $\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = \Omega(\boldsymbol{B},\boldsymbol{b}^*)$. This implies $i \in P(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ and so $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))_i = v_i - b_i < v_i - b_i^* = U_i(\boldsymbol{B},\boldsymbol{b}^*)$. Now consider the case that $b_i < b_i^*$. The minimality of $\boldsymbol{b}^*$ and the definition of $\mathcal{F}_\varepsilon(\boldsymbol{B},\boldsymbol{v})$ implies that $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \neq \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. If $i \in R(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$, then $U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) = 0 \leq U_i(\boldsymbol{B},\boldsymbol{b}^*)$.

In the rest of this proof, we consider the remained case that $i \in P(\boldsymbol{B}, \boldsymbol{v}) = P(\boldsymbol{B}, \boldsymbol{b}^*)$, $b_i < b_i^*$, and $i \in P(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \cup Q(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$. First, we show that $\Omega(\boldsymbol{B}, \boldsymbol{b}^*) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i = 1\}$. Since $i \in P(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \cup Q(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$, there exists an optimal solution $\boldsymbol{x}' \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$ satisfying that $x'_i = 1$. For any optimal solution $\boldsymbol{x}^* \in \Omega(\boldsymbol{B}, \boldsymbol{b}^*)$,

$$(b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x}^* = \boldsymbol{b}^*\boldsymbol{x}^* - (b_i^* - b_i) \geq \boldsymbol{b}^*\boldsymbol{x}' - (b_i^* - b_i) = (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x}',$$

and so $\boldsymbol{x}^*$ is optimal to $\mathrm{BAP}(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$. Since $i \in P(\boldsymbol{B}, \boldsymbol{b}^*)$, $\boldsymbol{x}^* \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$ implies that $\Omega(\boldsymbol{B}, \boldsymbol{b}^*) \subseteq \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i = 1\}$. Conversely, for any $\boldsymbol{x}' \in \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i = 1\}$,

$$b^*\boldsymbol{x}' = (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x}' + (b_i^* - b_i) \geq (b_i, \boldsymbol{b}^*_{-i})x^* + (b_i^* - b_i) = \boldsymbol{b}^*\boldsymbol{x}^*,$$

and so $\boldsymbol{x}'$ is also optimal to $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b}^*)$. It implies

$$\Omega(\boldsymbol{B}, \boldsymbol{b}^*) \supseteq \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i = 1\}$$

and consequently, $\Omega(\boldsymbol{B}, \boldsymbol{b}^*) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i = 1\}$. The property $\Omega(\boldsymbol{B}, \boldsymbol{b}^*) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i = 1\}$ implies $i \notin P(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$. It is because, if $i \in P(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$, then $\Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) = \{\boldsymbol{x} \in \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) \mid x_i = 1\}$ and so $\Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i})) = \Omega(\boldsymbol{B}, \boldsymbol{b}^*)$. It contradicts with the minimality of $\boldsymbol{b}^*$. In the following, we consider the case $i \in Q(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$.

Next, we show that $b_i = b_i^* - \varepsilon$. Let $\overline{\boldsymbol{b}}$ be the vector obtained from $\boldsymbol{b}^*$ by substituting b_i^* by $b_i^* - \varepsilon$. The definition of $\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v})$ and the minimality of $\boldsymbol{b}^*$ implies that $\Omega(\boldsymbol{B}, \overline{\boldsymbol{b}}) \neq \Omega(\boldsymbol{B}, \boldsymbol{b}^*)$. In the following, we show that the assumption that $b_i < b_i^* - \varepsilon$ implies $\Omega(\boldsymbol{B}, \overline{\boldsymbol{b}}) = \Omega(\boldsymbol{B}, \boldsymbol{b}^*)$. Let $\boldsymbol{x}^*$ be a solution in $\Omega(\boldsymbol{B}, \boldsymbol{b}^*)$ and $\boldsymbol{x}$ be any feasible solution of $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b}^*)$. Clearly, $\boldsymbol{x}$ is also feasible to $\mathrm{BAP}(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$. If $x_i = 0$, then the property $\boldsymbol{x}^* \in \Omega(\boldsymbol{B}, \boldsymbol{b}^*) \subseteq \Omega(\boldsymbol{B}, (b_i, \boldsymbol{b}^*_{-i}))$ implies that

$$\overline{\boldsymbol{b}}\boldsymbol{x} = (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x} \leq (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x}^* \leq \overline{\boldsymbol{b}}\boldsymbol{x}^*.$$

If $x_i = 1$, then

$$\overline{\boldsymbol{b}}\boldsymbol{x} = \boldsymbol{b}^*\boldsymbol{x} - \varepsilon \leq \boldsymbol{b}^*\boldsymbol{x}^* - \varepsilon = \overline{\boldsymbol{b}}\boldsymbol{x}^*.$$

From the above inequalities, $\boldsymbol{x}^*$ is optimal to $\mathrm{BAP}(\boldsymbol{B}, \overline{\boldsymbol{b}})$ and so $\Omega(\boldsymbol{B}, \boldsymbol{b}^*) \subseteq \Omega(\boldsymbol{B}, \overline{\boldsymbol{b}})$.

Next, we prove that $\Omega(\boldsymbol{B}, \boldsymbol{b}^*) \supseteq \Omega(\boldsymbol{B}, \overline{\boldsymbol{b}})$ by showing that for any feasible solution $\boldsymbol{x}$ of $\mathrm{BAP}(\boldsymbol{B})$, $\boldsymbol{x} \notin \Omega(\boldsymbol{B}, \boldsymbol{b}^*)$ implies $\boldsymbol{x} \notin \Omega(\boldsymbol{B}, \overline{\boldsymbol{b}})$. When $x_i = 0$, $\boldsymbol{x}$ satisfies that

$$\overline{\boldsymbol{b}}\boldsymbol{x} = (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x} \leq (b_i, \boldsymbol{b}^*_{-i})\boldsymbol{x}^* = \overline{\boldsymbol{b}}\boldsymbol{x}^* - \varepsilon < \overline{\boldsymbol{b}}\boldsymbol{x}^*$$

and so $\boldsymbol{x}$ is not optimal to BAP$(\boldsymbol{B},\overline{\boldsymbol{b}})$. If $\boldsymbol{x}$ satisfies that $x_i = 1$, then

$$\overline{\boldsymbol{b}}\boldsymbol{x} = \boldsymbol{b}^*\boldsymbol{x} - \varepsilon < \boldsymbol{b}^*\boldsymbol{x}^* - \varepsilon = \overline{\boldsymbol{b}}\boldsymbol{x}^*,$$

and so $\boldsymbol{x}$ is not optimal to BAP$(\boldsymbol{B},\overline{\boldsymbol{b}})$. It implies that $\Omega(\boldsymbol{B},\boldsymbol{b}^*) \supseteq \Omega(\boldsymbol{B},\overline{\boldsymbol{b}})$. From the above $\Omega(\boldsymbol{B},\boldsymbol{b}^*) = \Omega(\boldsymbol{B},\overline{\boldsymbol{b}})$ and it is a contradiction.

From the above discussions, we have that $b_i = b_i^* - \varepsilon$ and $i \in Q(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. Then, by noting $\{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\} = \Omega(\boldsymbol{B},\boldsymbol{v}) = \Omega(\boldsymbol{B},\boldsymbol{b}^*)$, we have

$$\begin{aligned}
U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) &= (v_i - b_i)\frac{\left|\{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}\right|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} \\
&= (v_i - b_i^* + \epsilon)\frac{|\{\Omega(\boldsymbol{B},\boldsymbol{v})|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} \\
&\leq (v_i - b_i^* + \frac{(v_i - b_i^*)}{k})\frac{|\{\Omega(\boldsymbol{B},\boldsymbol{v})|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} \\
&= (v_i - b_i^*)\frac{|\{\Omega(\boldsymbol{B},\boldsymbol{v})|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|}(1 + \frac{1}{k}) \\
&\leq (v_i - b_i^*)\frac{|\{\Omega(\boldsymbol{B},\boldsymbol{v})|}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|}(1 + \frac{1}{|\{\Omega(\boldsymbol{B},\boldsymbol{v})|}) \\
&= (v_i - b_i^*)\frac{|\{\in \Omega(\boldsymbol{B},\boldsymbol{v})| + 1}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} = (v_i - b_i^*)\frac{\left|\{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}\right| + 1}{\left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|} \\
&\leq (v_i - b_i^*) = U_i(\boldsymbol{B},\boldsymbol{b}^*).
\end{aligned}$$

Hence, we note that the last inequality comes from the fact that $i \in Q(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$ and so

$$\left|\{\boldsymbol{x} \in \Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i})) \mid x_i = 1\}\right| < \left|\Omega(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))\right|.$$

From the above, b_i^* is a best reply with respect to $\boldsymbol{b}^*$ and so $U_i(\boldsymbol{B},\boldsymbol{b}^*) \geq U_i(\boldsymbol{B},(b_i,\boldsymbol{b}^*_{-i}))$. $\|$

Proof of Lemma 2.

Since $b_i \leq v_i$, it is clear that $U_i((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$ is nonnegative. If $T_i \setminus B_i \neq \emptyset$, then $U_i(\boldsymbol{B},\boldsymbol{b}) \leq 0 \leq U_i((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$. Thus, we only need to consider the case $T_i \subseteq B_i$. In this case, each feasible solution for BAP$(\boldsymbol{B},\boldsymbol{b})$ is also feasible to BAP$((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$.

If $i \in P((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$, we have $U_i((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b}) = v_i - b_i$ and it is always greater than or equal to $U_i(\boldsymbol{B},\boldsymbol{b})$. Hence, in the rest of the proof, we consider the case where $T_i \subseteq B_i$ and $i \in Q((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b}) \cup R((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$. In this case, there exists $\boldsymbol{x}^* \in \Omega((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$ such that $x_i^* = 0$. Let $\overline{\boldsymbol{x}}$ be any optimal solution of BAP$(\boldsymbol{B},\boldsymbol{b})$. Since $T_i \subseteq B_i$, $\overline{\boldsymbol{x}}$ is also feasible to BAP$((T_i,\boldsymbol{B}_{-i}),\boldsymbol{b})$ and so $\boldsymbol{b}\boldsymbol{x}^* \geq \boldsymbol{b}\overline{\boldsymbol{x}}$. Conversely, since $x_i^* = 0$, $\boldsymbol{x}^*$ is also feasible for BAP$(\boldsymbol{B},\boldsymbol{b})$.

Hence, we have $\boldsymbol{b}\overline{\boldsymbol{x}} \geq \boldsymbol{b}\boldsymbol{x}^*$ and so $\boldsymbol{b}\overline{\boldsymbol{x}} = \boldsymbol{b}\boldsymbol{x}^*$. From the above discussion, we find that the following three properties hold:

(i) $\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 0\} = \{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 0\} \neq \emptyset$,

(ii) $\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\} \subseteq \{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 1\}$,

(iii) $\Omega(\boldsymbol{B}, \boldsymbol{b}) \subseteq \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$.

First, suppose that $i \in R((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. Then, we can assert that $i \in R(\boldsymbol{B}, \boldsymbol{b})$. Otherwise, there exists $\overline{\boldsymbol{x}} \in \Omega(\boldsymbol{B}, \boldsymbol{b})$ such that $\overline{x}_i = 1$. From (iii), $\overline{\boldsymbol{x}}$ is also optimal to BAP$((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. It contradicts with $i \in R((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. Hence, we have $i \in R(\boldsymbol{B}, \boldsymbol{b})$, and so $U_i((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) = U_i(\boldsymbol{B}, \boldsymbol{b}) = 0$.

Lastly, suppose that $i \in Q((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})$. From (i) and (ii), we have

$$
\begin{aligned}
U_i((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) &= (v_i - b_i)\frac{|\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 1\}|}{|\Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b})|} \\
&= (v_i - b_i)\frac{|\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 1\}|}{|\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 1\}| + |\{\boldsymbol{x} \in \Omega((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \mid x_i = 0\}|} \\
&\geq (v_i - b_i)\frac{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}|}{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}| + |\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 0\}|} \\
&= (v_i - b_i)\frac{|\{\boldsymbol{x} \in \Omega(\boldsymbol{B}, \boldsymbol{b}) \mid x_i = 1\}|}{|\Omega(\boldsymbol{B}, \boldsymbol{b})|} \\
&= U_i(\boldsymbol{B}, \boldsymbol{b}).
\end{aligned}
$$

Hence, we can conclude that $U_i((T_i, \boldsymbol{B}_{-i}), \boldsymbol{b}) \geq U_i(\boldsymbol{B}, \boldsymbol{b})$ for any B_i. ‖

Proof of Theorem 2.

If $P(\boldsymbol{T}, \boldsymbol{v}) = \emptyset$, then $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T}, \boldsymbol{v}) = \{\boldsymbol{v}\}$ and so $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty. Thus, we need only to consider the case that $P(\boldsymbol{T}, \boldsymbol{v}) \neq \emptyset$. Now assume that $\varepsilon \leq \delta/(n2^n + 1)$. Let $\boldsymbol{b}$ be the vector defined by,

$$
b_i = \begin{cases} v_i & (i \in Q(\boldsymbol{T}, \boldsymbol{v}) \cup R(\boldsymbol{T}, \boldsymbol{v})), \\ v_i - k\varepsilon & (i \in P(\boldsymbol{T}, \boldsymbol{v})), \end{cases}
$$

where $k = |\Omega(\boldsymbol{B}, \boldsymbol{v})|$. Let $\boldsymbol{x}^*$ be a solution in $\Omega(\boldsymbol{T}, \boldsymbol{v})$. It is clear that the equality $\boldsymbol{b}\boldsymbol{x}^* = \boldsymbol{v}\boldsymbol{x}^* - k\varepsilon|P(\boldsymbol{T}, \boldsymbol{v})|$ holds.

In the rest of this proof, we show that $\Omega(\boldsymbol{T}, \boldsymbol{b}) = \Omega(\boldsymbol{T}, \boldsymbol{v})$. If $\boldsymbol{x}'$ is a solution satisfying $\boldsymbol{x}' \notin \Omega(\boldsymbol{T}, \boldsymbol{v})$, then $\boldsymbol{v}\boldsymbol{x}' \leq \boldsymbol{v}\boldsymbol{x}^* - \delta$ and so

$$
\boldsymbol{b}\boldsymbol{x}' \;\leq\; \boldsymbol{v}\boldsymbol{x}' \leq \boldsymbol{v}\boldsymbol{x}^* - \delta \leq \boldsymbol{v}\boldsymbol{x}^* - (n2^n + 1)\varepsilon
$$

$$\leq \quad \boldsymbol{v}\boldsymbol{x}^* - (|P(\boldsymbol{T},\boldsymbol{v})|k+1)\varepsilon < \boldsymbol{v}\boldsymbol{x}^* - k\varepsilon|P(\boldsymbol{T},\boldsymbol{v})| = \boldsymbol{b}\boldsymbol{x}^*,$$

and so $\boldsymbol{x}' \notin \Omega(\boldsymbol{T},\boldsymbol{b})$. It implies that $\Omega(\boldsymbol{T},\boldsymbol{b}) \subseteq \Omega(\boldsymbol{T},\boldsymbol{v})$.

For any solution $\boldsymbol{x}^* \in \Omega(\boldsymbol{T},\boldsymbol{v})$, the objective value with respect to BAP$(\boldsymbol{T},\boldsymbol{b})$, which is denoted by $\boldsymbol{b}\boldsymbol{x}^*$, is equivalent to the constant $z^* - k\varepsilon|P(\boldsymbol{T},\boldsymbol{v})|$ where z^* is the optimal value of BAP$(\boldsymbol{T},\boldsymbol{b})$. Thus, the property $\Omega(\boldsymbol{T},\boldsymbol{b}) \subseteq \Omega(\boldsymbol{T},\boldsymbol{v})$ implies that $\Omega(\boldsymbol{T},\boldsymbol{b}) = \Omega(\boldsymbol{T},\boldsymbol{v})$.

From the above, $\boldsymbol{b} \in \bar{\mathcal{F}}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$ and $\bar{\mathcal{F}}_\varepsilon(\boldsymbol{T},\boldsymbol{v})$ is non-empty. $\|$

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