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DOUBLE ROTATIONS

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Abstract. We consider a map called a double rotation, which is composed of two rotations on a circle. Specifically, a double rotation is a map on the interval $[0, 1)$ that maps $x \in [0, c)$ to $\{x + \alpha\}$, and $x \in [c, 1)$ to $\{x + \beta\}$. Although double rotations are discontinuous and non-invertible in general, we show that almost every double rotation can be reduced to a simple rotation, and the set of parameters such that the double rotation is irreducible to a rotation has a fractal structure. We also examine a characteristic number of double rotations that is called a discharge number. The discharge number as a function of c reflects the fractal structure, and is very complicated.

1. Introduction

In this paper, we consider the family of *double rotations* $f_{(\alpha,\beta,c)} \colon [0, 1) \to [0, 1)$ defined by

$$f_{(\alpha,\beta,c)}(x) = \begin{cases} \{x + \alpha\} & \text{if } x \in [0, c) \\ \{x + \beta\} & \text{if } x \in [c, 1) \end{cases}$$

for $(\alpha, \beta, c) \in [0, 1) \times [0, 1) \times [0, 1]$. A typical graph of a double rotation is shown in Figure 1(a). Although the map is apparently discontinuous and non-invertible in general, we show that, for almost every parameter (α, β, c), the double rotation $f_{(\alpha,\beta,c)}$ can be reduced to a rotation. We also investigate a characteristic number of double rotations that is called a discharge number. For $f_{(\alpha,\beta,c)}$ and $x \in [0, 1)$, we consider the elements of the orbit starting from x, and define the discharge number as the ratio of the elements that fall within the interval $[c, 1)$. Figure 1(b) shows a graph of the discharge number as a function of c for fixed α and β. Despite the simple definition of the double rotation, the graph is complicated and like a devil's staircase. The complex appearance indicates a fractal structure in the parameter space.

In the context of studies on piecewise isometries, the class of double rotations can be considered as a special form of interval translation mappings (ITMs) introduced by Boshernitzan and Kornfeld [1]. An ITM is defined as a transformation on an interval such that there exists a partition of the interval into finitely many intervals and, on each of the intervals, it is a translation. Then every double rotation is an ITM, because a partition consisting of at most four such intervals always exists. In fact, the example ITM given by them is a double rotation with a special parameter. The class of ITMs recently investigated by Bruin and Troubetzkoy [2] is also a subclass of double rotations such that $c = 1 - \alpha$. In both studies, induced transformations (first return maps) of the maps are considered, and self-similarity in the maps are reported. We also investigate such self-similarity in double rotations by considering induced transformations in the present paper. Piecewise isometries

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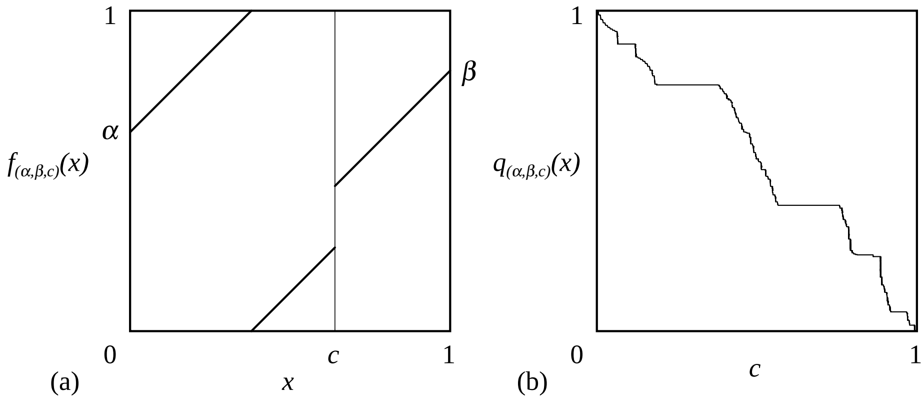

FIGURE 1. (a) Double rotation. (b) Discharge number as a function of c for $\alpha \approx 0.621$ and $\beta \approx 0.813$.

such as two-parameter piecewise rotations [3] are also closely related with double rotations.

Double rotations appear in the field of electrical engineering as well. A simple three capacitance equivalent circuit model of partial discharge phenomena [4] can be reduced to a double rotation [5]. The parameter c of the double rotation corresponds to the amplitude of the voltage applied to the model, and the discharge number corresponds to the average discharge rate of the model. Although the three capacitance model is an old model, it is still important, for most partial discharge models are based on it.

2. Partition of the parameter space

In this section, we define a partition of the parameter space and a transformation on it. The meanings of the partition and the transformation will be given in the next section.

Let $D = [0,1) \times [0,1) \times [0,1]$ be the parameter space of double rotations. Let us define two regions D_0 and D_1 in D as

$$D_0 = \{(\alpha,\beta,c) \in D \mid 0 < \alpha < \beta\}, \qquad D_1 = \{(\alpha,\beta,c) \in D \mid 0 < \beta < \alpha\},$$

and let $D_* = D_0 \cup D_1$. For each region D_0 and D_1, consider the partitions

$$\begin{aligned} D_{0,1} &= \{(\alpha,\beta,c) \in D_0 \mid c \le 1-\beta\}, & D_{1,1} &= \{(\alpha,\beta,c) \in D_1 \mid c \le \beta\}, \\ D_{0,2} &= \{(\alpha,\beta,c) \in D_0 \mid 1-\beta < c < 1-\alpha\}, & D_{1,2} &= \{(\alpha,\beta,c) \in D_1 \mid \beta < c < \alpha\}, \\ D_{0,3} &= \{(\alpha,\beta,c) \in D_0 \mid 1-\alpha \le c\}, & D_{1,3} &= \{(\alpha,\beta,c) \in D_1 \mid \alpha \le c\}, \end{aligned}$$

as shown in Figure 2. Let $D_{*,j}$ denote $D_{0,j} \cup D_{1,j}$ for each $j \in \{1,2,3\}$. The set $D \setminus D_*$ is called the *boundary*, and denoted by B. Then B can also be expressed as

$$B = \{(\alpha,\beta,c) \in D \mid \alpha = 0,\ \beta = 0 \text{ or } \alpha = \beta\}.$$

On each region $D_{e,j}$ with $e \in \{0,1\}$ and $j \in \{1,3\}$, let us define transformations

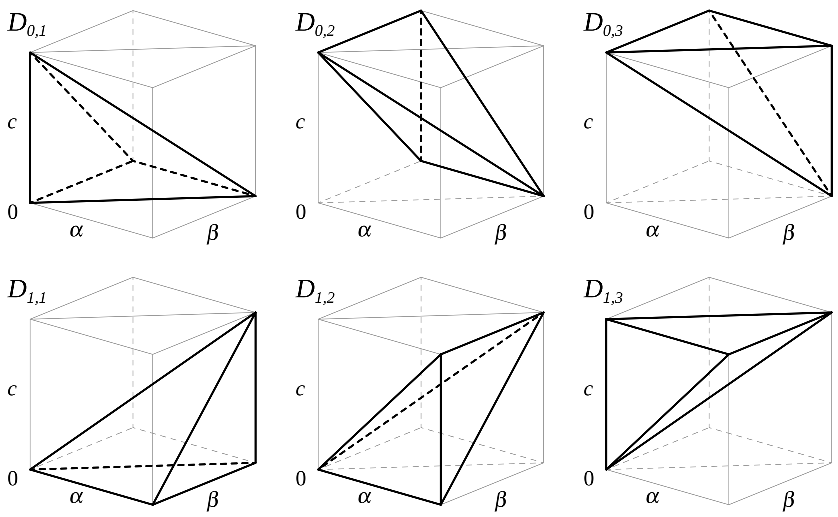

FIGURE 2. Partition of the parameter space: the parameter space D (grey cube) is composed of D_0 and D_1 (grey triangular prisms), and each D_e is composed of $D_{e,j}$ (black tetrahedra).

$T_{(e,j)}\colon D_{e,j} \to D$ by

$$\begin{aligned}
T_{(0,1)}(\alpha,\beta,c) &= \Big(\Big\{\frac{\alpha}{1-\beta}\Big\}, \Big\{\frac{\beta}{1-\beta}\Big\}, \frac{c}{1-\beta}\Big),\\
T_{(0,3)}(\alpha,\beta,c) &= \Big(\Big\{\frac{\alpha-1}{\alpha}\Big\}, \Big\{\frac{\beta-1}{\alpha}\Big\}, \frac{c+\alpha-1}{\alpha}\Big),\\
T_{(1,1)}(\alpha,\beta,c) &= \Big(\Big\{\frac{\alpha-1}{\beta}\Big\}, \Big\{\frac{\beta-1}{\beta}\Big\}, \frac{c}{\beta}\Big),\\
T_{(1,3)}(\alpha,\beta,c) &= \Big(\Big\{\frac{\alpha}{1-\alpha}\Big\}, \Big\{\frac{\beta}{1-\alpha}\Big\}, \frac{c-\alpha}{1-\alpha}\Big).
\end{aligned}$$

Then, we define a transformation $T\colon D_{*,1} \cup D_{*,3} \to D$ simply as

$$T(\alpha,\beta,c) = T_{(e,j)}(\alpha,\beta,c), \qquad \text{if } (\alpha,\beta,c) \in D_{e,j}.$$

With the transformation T, for a given initial parameter (α,β,c), the sequence of parameters $T(\alpha,\beta,c)$, $T^2(\alpha,\beta,c)$, $\ldots$ can be considered, as far as T is defined for each parameter. Once the parameter is mapped into the set $D_{*,2} \cup B$, this process terminates. When we consider such sequences, throughout this paper, we let (α_k,β_k,c_k) denote $T^k(\alpha,\beta,c)$, if $(\alpha_i,\beta_i,c_i) \in D_{*,1} \cup D_{*,3}$ for $0 \le i < k$.

3. Induced transformation of double rotation

Before starting investigation on characteristics of double rotations, it should be noted that the family of double rotations naturally has two symmetries as shown in the following proposition.

Proposition 3.1. *Let* $(\alpha,\beta,c) \in D$.

(i) *$f_{(\alpha,\beta,c)}$ is isomorphic to $f_{H(\alpha,\beta,c)}$, where H is the transformation given by $H(\alpha,\beta,c) = (\beta,\alpha,1-c)$.*

(ii) *$f_{(\alpha,\beta,c)}$ is isomorphic to $f_{(\{1-\beta\},\{1-\alpha\},1-c)}$ except for the behaviour at the discontinuity points.*

Proof. (i) Let Φ_θ denote a rotation given by $\Phi_\theta(x) = x+\theta \pmod 1$. Then, for arbitrary $x \in [0,1)$, we have

$$\Phi_c^{-1} \circ f_{(\alpha,\beta,c)} \circ \Phi_c(x) = \begin{cases} x+\alpha \pmod 1 & \text{if } x \in [1-c,1) \\ x+\beta \pmod 1 & \text{if } x \in [0,1-c) \end{cases}$$
$$= f_{(\beta,\alpha,1-c)}(x).$$

(ii) Let h be the transformation given by $h(x) = 1-x$. If $x \neq 0$ and $x \neq 1-c$, we have

$$h^{-1} \circ f_{(\alpha,\beta,c)} \circ h(x) = \begin{cases} x-\alpha \pmod 1 & \text{if } x \in (1-c,1) \\ x-\beta \pmod 1 & \text{if } x \in (0,1-c) \end{cases}$$
$$= f_{(\{1-\beta\},\{1-\alpha\},1-c)}.$$

Note that both 0 and $1-c$ are the discontinuity points of $f_{(\{1-\beta\},\{1-\alpha\},1-c)}$. □

Because of these two symmetries, $f_{(\alpha,\beta,c)}$ is also isomorphic to $f_{(\{1-\alpha\},\{1-\beta\},c)}$ except for the discontinuity points. By the first symmetry, there is a one-to-one correspondence of elements between $D_{0,1}$ and $D_{1,3}$, between $D_{0,2}$ and $D_{1,2}$, and between $D_{0,3}$ and $D_{1,1}$. By the second symmetry, domains $D_{e,1}$ and $D_{e,3}$ are symmetric to each other for $e = 0$ and 1. Furthermore, it should be noted that the definition of T is also symmetric in the sense that $H \circ T \circ H(\alpha,\beta,c) = T(\alpha,\beta,c)$ for arbitrary $(\alpha,\beta,c) \in D_{*,1} \cup D_{*,3}$.

In the case when $(\alpha,\beta,c) \in B$, the double rotation $f_{(\alpha,\beta,c)}$ is considered to be trivial in the sense that it is a simple rotation or has an identity map on $[0,c)$ or $[c,1)$. In the following, for a non-trivial double rotation with a parameter $(\alpha,\beta,c) \in D_*$, we consider the induced transformation of $f_{(\alpha,\beta,c)}$ on the set $I_{(\alpha,\beta,c)}$, which is defined as

$$I_{(\alpha,\beta,c)} = \begin{cases} [0,1-\beta) & \text{if } (\alpha,\beta,c) \in D_{0,1} \\ [c+\beta-1,c+\alpha) & \text{if } (\alpha,\beta,c) \in D_{0,2} \\ [1-\alpha,1) & \text{if } (\alpha,\beta,c) \in D_{0,3} \\ [0,c) \cup [c+1-\beta,1) & \text{if } (\alpha,\beta,c) \in D_{1,1} \\ [0,\beta) \cup [\alpha,1) & \text{if } (\alpha,\beta,c) \in D_{1,2} \\ [0,c-\alpha) \cup [c,1) & \text{if } (\alpha,\beta,c) \in D_{1,3}. \end{cases}$$

Then, we show that the induced transformation is isomorphic to another double rotation or a simple rotation, as shown in Figure 3. The definition of $I_{(\alpha,\beta,c)}$ is also symmetric in the sense that $I_{(\alpha,\beta,c)} = \Phi_c(I_{H(\alpha,\beta,c)})$ for $(\alpha,\beta,c) \in D_*$.

Lemma 3.1. *If $(\alpha,\beta,c) \in D_{0,1}$, the induced transformation of $f_{(\alpha,\beta,c)}$ on $I_{(\alpha,\beta,c)}$ is isomorphic to $f_{T(\alpha,\beta,c)}$.*

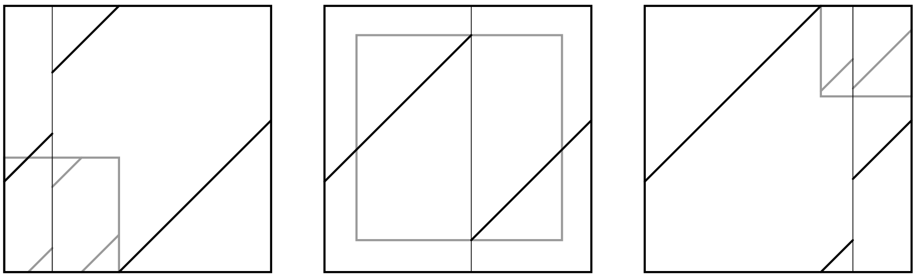

FIGURE 3. Induced transformation of double rotation: double rotations (black lines) and the induced transformations (grey lines) for parameters in (a) $D_{0,1}$, (b) $D_{0,2}$, and (c) $D_{0,3}$.

Proof. From the assumption $(\alpha, \beta, c) \in D_{0,1}$, we have $\alpha < \beta$ and $c \leq 1 - \beta$. Therefore $f_{(\alpha,\beta,c)}(x)$ can be expressed as

$$f_{(\alpha,\beta,c)}(x) = \begin{cases} x + \alpha & \text{if } x \in [0, c) \\ x + \beta & \text{if } x \in [c, 1-\beta) \\ x + \beta - 1 & \text{if } x \in [1-\beta, 1). \end{cases}$$

Then $f_{(\alpha,\beta,c)}(x) = x \pmod{1-\beta}$ for $x \in [1-\beta, 1)$. Since $\beta - 1 < 0$, any orbit starting from $x \in [1-\beta, 1)$ visits the interval $I_{(\alpha,\beta,c)} = [0, 1-\beta)$ in a finite time. Therefore, the induced transformation of $f_{(\alpha,\beta,c)}$ on $[0, 1-\beta)$ is

$$f_{(\alpha,\beta,c)}\big|_{[0,1-\beta)}(x) = \begin{cases} x + \alpha \pmod{1-\beta} & \text{if } x \in [0, c) \\ x + \beta \pmod{1-\beta} & \text{if } x \in [c, 1-\beta). \end{cases}$$

Then, with $h(x) = (1-\beta)x$, the induced transformation is isomorphic to the map

$$h^{-1} \circ f_{(\alpha,\beta,c)}\big|_{[0,1-\beta)} \circ h(x) = \begin{cases} x + \dfrac{\alpha}{1-\beta} \pmod 1 & \text{if } x \in \left[0, \dfrac{c}{1-\beta}\right) \\ x + \dfrac{\beta}{1-\beta} \pmod 1 & \text{if } x \in \left[\dfrac{c}{1-\beta}, 1\right), \end{cases}$$

which is nothing but a double rotation $f_{T(\alpha,\beta,c)} = f_{(\{\frac{\alpha}{1-\beta}\},\{\frac{\beta}{1-\beta}\},\frac{c}{1-\beta})}$. □

Lemma 3.2. *If $(\alpha, \beta, c) \in D_{0,2}$, the restriction of $f_{(\alpha,\beta,c)}$ to $I_{(\alpha,\beta,c)}$ is isomorphic to $\Phi_{\alpha/(1+\alpha-\beta)}$.*

Proof. From the assumption $(\alpha, \beta, c) \in D_{0,2}$, we have $1 - \beta < c < 1 - \alpha$. Therefore $f_{(\alpha,\beta,c)}(x)$ can be expressed as

$$f_{(\alpha,\beta,c)}(x) = \begin{cases} x + \alpha & \text{if } x \in [0, c) \\ x + \beta - 1 & \text{if } x \in [c, 1). \end{cases}$$

Let $h(x) = (1+\alpha-\beta)x + (c+\beta-1)$. Then, for $x \in [0,1)$, we obtain

$$h^{-1} \circ f_{(\alpha,\beta,c)} \circ h(x) = \begin{cases} x + \dfrac{\alpha}{1+\alpha-\beta} & \text{if } x \in \Big[0, \dfrac{1-\beta}{1+\alpha-\beta}\Big) \\ x + \dfrac{\beta-1}{1+\alpha-\beta} & \text{if } x \in \Big[\dfrac{1-\beta}{1+\alpha-\beta}, 1\Big), \end{cases}$$
$$= x + \frac{\alpha}{1+\alpha-\beta} \pmod 1,$$

which is the rotation $\Phi_{\alpha/(1+\alpha-\beta)}$. Since $I_{(\alpha,\beta,c)} = h([0,1))$, the restriction of $f_{(\alpha,\beta,c)}$ to $I_{(\alpha,\beta,c)}$ is isomorphic to $\Phi_{\alpha/(1+\alpha-\beta)}$. It should be noted that every $x \in [0,1)$ is mapped into $I_{(\alpha,\beta,c)}$ by $f_{(\alpha,\beta,c)}$ in a finite time. □

Lemma 3.3. *If $(\alpha,\beta,c) \in D_{0,3}$, the induced transformation of $f_{(\alpha,\beta,c)}$ on $I_{(\alpha,\beta,c)}$ is isomorphic to $f_{T(\alpha,\beta,c)}$.*

Proof. From the assumption $(\alpha,\beta,c) \in D_{0,3}$, we have $\alpha < \beta$ and $1-\alpha \le c$. Therefore $f_{(\alpha,\beta,c)}(x)$ can be expressed as

$$f_{(\alpha,\beta,c)}(x) = \begin{cases} x+\alpha & \text{if } x \in [0, 1-\alpha) \\ x+\alpha-1 & \text{if } x \in [1-\alpha, c) \\ x+\beta-1 & \text{if } x \in [c, 1). \end{cases}$$

Then $f_{(\alpha,\beta,c)}(x) = x \pmod \alpha$ for $x \in [0, 1-\alpha)$. Hence the induced transformation satisfies

$$f_{(\alpha,\beta,c)}\big|_{[1-\alpha,1)}(x) = \begin{cases} x+\alpha-1 \pmod \alpha & \text{if } x \in [1-\alpha, c) \\ x+\beta-1 \pmod \alpha & \text{if } x \in [c, 1). \end{cases}$$

Similarly to Lemma 3.1, we obtain $h^{-1} \circ f_{(\alpha,\beta,c)}\big|_{[1-\alpha,1)} \circ h = f_{(\{\frac{\alpha-1}{\alpha}\}, \{\frac{\beta-1}{\alpha}\}, \frac{\alpha+c-1}{\alpha})} = f_{T(\alpha,\beta,c)}$, where $h(x) = \alpha x + (1-\alpha)$. □

Proposition 3.2. *Let $(\alpha,\beta,c) \in D_{e,j}$ for $e \in \{0,1\}$ and $j \in \{1,2,3\}$.*

(i) *If $j \in \{1,3\}$, the induced transformation $f_{(\alpha,\beta,c)}\big|_{I_{(\alpha,\beta,c)}}$ is isomorphic to $f_{T(\alpha,\beta,c)}$.*
(ii) *If $j = 2$, the restriction of $f_{(\alpha,\beta,c)}$ to $I_{(\alpha,\beta,c)}$ is isomorphic to a rotation $\Phi_{\alpha/(1+\alpha-\beta)}$ and $\Phi_{\beta/(1-\alpha+\beta)}$ for $e = 0$ and 1, respectively.*

Proof. If $e = 0$, the proof is given by Lemma 3.1, 3.2 and 3.3. We prove the case for $e = 1$ by using the symmetry. Recall that the definitions of T and $I_{(\alpha,\beta,c)}$ are symmetric; $H \circ T \circ H(\alpha,\beta,c) = T(\alpha,\beta,c)$ for $(\alpha,\beta,c) \in D_{*,1} \cup D_{*,3}$, and $I_{(\alpha,\beta,c)} = \Phi_c(I_{H(\alpha,\beta,c)})$ for $(\alpha,\beta,c) \in D_*$.

(i) Since $(\alpha,\beta,c) \in D_{1,1} \cup D_{1,3}$, Lemma 3.1 or 3.3 can be applied to $H(\alpha,\beta,c) \in D_{0,1} \cup D_{0,3}$. Therefore, we obtain

$$f_{(\alpha,\beta,c)}\big|_{I_{(\alpha,\beta,c)}} = f_{(\alpha,\beta,c)}\big|_{\Phi_c(I_{H(\alpha,\beta,c)})} \cong f_{H(\alpha,\beta,c)}\big|_{I_{H(\alpha,\beta,c)}}$$
$$\cong f_{T \circ H(\alpha,\beta,c)} \cong f_{H \circ T \circ H(\alpha,\beta,c)} = f_{T(\alpha,\beta,c)}.$$

(ii) Since $(\alpha,\beta,c) \in D_{1,2}$, Lemma 3.2 can be applied to $H(\alpha,\beta,c) \in D_{0,2}$. Therefore the restriction of $f_{H(\alpha,\beta,c)}$ to $I_{H(\alpha,\beta,c)}$ is isomorphic to $\Phi_{\beta/(1-\alpha+\beta)}$. By the symmetry, the restriction of $f_{(\alpha,\beta,c)}$ to $\Phi_c(I_{H(\alpha,\beta,c)}) = I_{(\alpha,\beta,c)}$ is also isomorphic to the rotation $\Phi_{\beta/(1-\alpha+\beta)}$. □

In this way, reductions of double rotations correspond to the transformation T.

The example ITM given by Boshernitzan and Kornfeld [1] can be considered as a double rotation with the parameter $(\alpha, \alpha^2, 1-\alpha) \in D$, where $\alpha \approx 0.311108$ is the unique root of the equation $x^3 - x^2 - 3x + 1 = 0$ such that $\alpha \in [0, 1)$. In fact, $(\alpha, \alpha^2, 1-\alpha)$ is a periodic point of T with period three, because $(\alpha, \alpha^2, 1-\alpha) \in D_{1,3}$ and

$$\begin{aligned} T(\alpha, \alpha^2, 1-\alpha) &= \Big(\frac{\alpha}{1-\alpha}, \frac{\alpha^2}{1-\alpha}, 1 - \frac{\alpha}{1-\alpha}\Big) \in D_{1,3}, \\ T^2(\alpha, \alpha^2, 1-\alpha) &= \Big(\frac{\alpha}{1-2\alpha}, \frac{\alpha^2}{1-2\alpha}, 1 - \frac{\alpha}{1-2\alpha}\Big) \in D_{1,1}, \\ T^3(\alpha, \alpha^2, 1-\alpha) &= \Big(1 + \frac{3\alpha-1}{\alpha^2}, 3 + \frac{\alpha^2+2\alpha-1}{\alpha^2}, \frac{1-3\alpha}{\alpha^2}\Big) = (\alpha, \alpha^2, 1-\alpha). \end{aligned}$$

Thus $f_{(\alpha,\alpha^2,1-\alpha)}$ can be reduced to itself, as shown in [1]. The class of ITMs investigated by Bruin and Troubetzkoy [2] also corresponds to the plane in D specified by $c = 1-\alpha$. In the region D_0, this plane is located at the boundary between $D_{0,2}$ and $D_{0,3}$. Therefore, $f_{(\alpha,\beta,1-\alpha)}$ can be reduced to a rotation. In the region D_1, although the behaviour of T is complicated as shown in [2], the plane is mapped into itself by T. In this sense, the reductions are closed within the class.

4. Cantor structure

As shown in the previous section, reductions of double rotations can be described by the transformation T on the parameter space. In this section, to investigate the induced transformations successively reduced from a double rotation, we will investigate the dynamics of T.

Firstly, in the following proposition, we consider the orbits starting from two distinct initial parameters (α, β, c) and (α, β, c') that share same α and β.

Proposition 4.1. *Let $j_1, \ldots, j_n \in \{1, 3\}$. Suppose both $(\alpha_i, \beta_i, c_i) = T^i(\alpha, \beta, c)$ and $(\alpha'_i, \beta'_i, c'_i) = T^i(\alpha, \beta, c')$ are elements of $D_{*,j_{i+1}}$ for $0 \le i < n$. Then $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$ for all i such that $0 \le i \le n$.*

Proof. As for the initial parameters, we assumed $\alpha_0 = \alpha'_0$ and $\beta_0 = \beta'_0$. Now let us assume $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$. The next parameters are given by

$$(\alpha_{i+1}, \beta_{i+1}, c_{i+1}) = T_{(e,j)}(\alpha_i, \beta_i, c_i) \quad \text{and} \quad (\alpha'_{i+1}, \beta'_{i+1}, c'_{i+1}) = T_{(e,j)}(\alpha_i, \beta_i, c'_i)$$

for common e and $j = j_{i+1}$. Then, in every definition of $T_{(e,j)}$, the values of α_{i+1} (or α'_{i+1}) and β_{i+1} (or β'_{i+1}) do not depend on the value of c_i (or c'_i). Therefore $\alpha_{i+1} = \alpha'_{i+1}$ and $\beta_{i+1} = \beta'_{i+1}$. □

Let $S = [0, 1) \times [0, 1)$. For an initial parameter $(\alpha, \beta) \in S$ and a sequence $j_1, \ldots, j_n$ such that $j_1, \ldots, j_{n-1} \in \{1, 3\}$ and $j_n \in \{1, 2, 3\}$, we define the set $C_{j_1,\ldots,j_n}$ as

$$C_{j_1,\ldots,j_n} = \{c \in [0, 1] \mid T^i(\alpha, \beta, c) \in D_{*,j_{i+1}} \text{ for } 0 \le i < n\ \}.$$

Then, according to the proposition, if a sequence $j_1, \ldots, j_n \in \{1, 3\}$ is given, the sequence of (α_n, β_n) starting from given $(\alpha, \beta) \in S$ can be uniquely determined by $(\alpha_n, \beta_n, c_n) = T^n(\alpha, \beta, c)$, where c is arbitrarily chosen from $C_{j_1,\ldots,j_n}$.

Let $C = [0,1]$, because C can be considered as $C_{j_1,\dots,j_n}$ for $n = 0$. To consider the case $n = 1$, let us assume that (α, β) is in the set S_* defined by

$$S_* = \{(\alpha, \beta) \in [0,1) \times [0,1) \,|\, \alpha \neq 0,\ \beta \neq 0, \text{ and } \alpha \neq \beta\}.$$

Then, from the definitions of the regions $D_{e,j}$, the sets C_1, C_2 and C_3 turn out to be the intervals given by

$$\begin{array}{llll} C_1 = [0, 1-\beta], & C_2 = (1-\beta, 1-\alpha), & C_3 = [1-\alpha, 1], & \text{if } \alpha > \beta, \\ C_1 = [0, \beta], & C_2 = (\beta, \alpha), & C_3 = [\alpha, 1], & \text{if } \alpha < \beta. \end{array}$$

If $(\alpha, \beta) \notin S_*$, the sets C_1, C_2 and C_3 are not well-defined (empty sets), because (α, β, c) is in the boundary B for any c. We also define $\sigma_j(\alpha, \beta)$ for $j \in \{1, 2, 3\}$ and $(\alpha, \beta) \in S_*$ as the length of C_j. Specifically, $\sigma_j(\alpha, \beta)$ is given by

$$\sigma_1(\alpha, \beta) = \begin{cases} 1-\beta & \text{if } \alpha < \beta \\ \beta & \text{if } \alpha > \beta, \end{cases} \qquad \sigma_3(\alpha, \beta) = \begin{cases} \alpha & \text{if } \alpha < \beta \\ 1-\alpha & \text{if } \alpha > \beta, \end{cases}$$

and $\sigma_2(\alpha, \beta) = |\beta - \alpha|$. Then, under the assumption $(\alpha, \beta) \in S_*$, every $\sigma_j(\alpha, \beta)$ is always positive, and $\sigma_1(\alpha, \beta) + \sigma_2(\alpha, \beta) + \sigma_3(\alpha, \beta) = 1$. This definition is symmetric in the sense that $\sigma_1(\alpha, \beta) = \sigma_3(\beta, \alpha)$ and $\sigma_2(\alpha, \beta) = \sigma_2(\beta, \alpha)$

In the following proposition, we consider the structure of the sets $C_{j_1,\dots,j_n}$ in the parameter space $C = [0, 1]$ for fixed α and β.

Proposition 4.2. *Let $j_1, \dots, j_n \in \{1, 3\}$. Assume $(\alpha_i, \beta_i) \in S_*$ for $0 \leq i \leq n$. Then the following statements are true.*

(i) *$C_{j_1,\dots,j_n}$ is a closed interval. Let $C_{j_1,\dots,j_n} = [\lambda_n, \lambda_n + \delta_n]$ in the following.*
(ii) *The map from $c \in C_{j_1,\dots,j_n}$ to $c_n \in [0,1]$ is a linear homeomorphism given by $c_n = (c - \lambda_n)/\delta_n$.*
(iii) *The sets $C_{j_1,\dots,j_n,1}$ and $C_{j_1,\dots,j_n,3}$ are closed intervals, and the set $C_{j_1,\dots,j_n,2}$ is an open interval. More explicitly, these intervals are given by*

$$\begin{aligned} C_{j_1,\dots,j_n,1} &= [\lambda_n, \lambda_n + \delta_n \sigma_1(\alpha_n, \beta_n)], \\ C_{j_1,\dots,j_n,2} &= (\lambda_n + \delta_n \sigma_1(\alpha_n, \beta_n), \lambda_n + \delta_n(1 - \sigma_3(\alpha_n, \beta_n))), \\ C_{j_1,\dots,j_n,3} &= [\lambda_n + \delta_n(1 - \sigma_3(\alpha_n, \beta_n)), \lambda_n + \delta_n]. \end{aligned}$$

Thus $C_{j_1,\dots,j_n}$ is composed of the disjoint union of the intervals $C_{j_1,\dots,j_n,1}$, $C_{j_1,\dots,j_n,2}$ and $C_{j_1,\dots,j_n,3}$.
(iv) *The length of $C_{j_1,\dots,j_n,j_{n+1}}$ for $j_{n+1} \in \{1,2,3\}$ is given by*

$$\left|C_{j_1,\dots,j_n,j_{n+1}}\right| = \sigma_{j_1}(\alpha_0, \beta_0) \cdots \sigma_{j_{n+1}}(\alpha_n, \beta_n).$$

Proof. We prove the proposition by induction on n. For $n = 0$, the statements are true as follows. (i) $C = [0,1]$ is a closed interval. (ii) $c_0 = c = (c - \lambda_0)/\delta_0$, because $\lambda_0 = 0$ and $\delta_0 = 1$. (iii) By definition, $C_1 = [0, \sigma_1(\alpha_0, \beta_0)]$, $C_2 = [\sigma_1(\alpha_0, \beta_0), 1 - \sigma_3(\alpha_0, \beta_0))$, $C_3 = [1 - \sigma_3(\alpha_0, \beta_0), 1]$. (iv) Also by definition, $|C_{j_1}| = \sigma_{j_1}(\alpha_0, \beta_0)$.

Let us assume the proposition is true for n and verify it for $n+1$. (i) $C_{j_1,\dots,j_{n+1}}$ is closed, because it is already assumed in the statement (iii) of the proposition for n. Let $C_{j_1,\dots,j_{n+1}} = [\lambda_{n+1}, \lambda_{n+1} + \delta_{n+1}]$. (ii) From the definitions of $T_{(e,j)}$ and σ_j, we obtain

$$c_{n+1} = \frac{c_n}{\sigma_1(\alpha_n, \beta_n)} = \frac{c - \lambda_n}{\delta_n \sigma_1(\alpha_n, \beta_n)} \quad \text{if } j_{n+1} = 1,$$

$$c_{n+1} = \frac{c_n - (1 - \sigma_3(\alpha_n, \beta_n))}{\sigma_3(\alpha_n, \beta_n)} = \frac{c - (\lambda_n + \delta_n(1 - \sigma_3(\alpha_n, \beta_n)))}{\delta_n \sigma_3(\alpha_n, \beta_n)} \quad \text{if } j_{n+1} = 3,$$

because $c_n = (c - \lambda_n)/\delta_n$. On the other hand, in the statement (iii) for n, it is assumed that

$$\lambda_{n+1} = \lambda_n \quad \text{and} \quad \delta_{n+1} = \delta_n \sigma_1(\alpha_n, \beta_n) \quad \text{if } j_{n+1} = 1,$$

$$\lambda_{n+1} = \lambda_n + \delta_n(1 - \sigma_3(\alpha_n, \beta_n)) \quad \text{and} \quad \delta_{n+1} = \delta_n \sigma_3(\alpha_n, \beta_n) \quad \text{if } j_{n+1} = 3.$$

Therefore, we obtain $c_{n+1} = (c - \lambda_{n+1})/\delta_{n+1}$ for both $j_{n+1} = 1$ and 3. (iii) By the the map obtained in (ii), $C_{j_{n+2}}$ for $(\alpha_{n+1}, \beta_{n+1}) \in S_*$ corresponds to $C_{j_1,\dots,j_{n+1},j_{n+2}}$ in the statement. (iv) immediately follows from (iii). □

In the proposition, we assumed $(\alpha_i, \beta_i) \in S_*$ for $0 \le i \le n$. In fact, it will be proven in Proposition 6.1 that, if α and β are linearly independent over $\mathbb{Q}$, then $(\alpha_i, \beta_i) \in S_*$ for all $i \ge 0$, Therefore, we assume that (α, β) is in the set $\hat{S}$ defined by

$$\hat{S} = \{(\alpha, \beta) \in [0,1) \times [0,1) \,|\, \text{If } s, t \in \mathbb{Z} \text{ and } s\alpha + t\beta \in \mathbb{Z}, \text{ then } s = t = 0\}.$$

Under the assumption, we can consider the structure of the sets $C_{j_1,\dots,j_n}$ without taking account of the boundary. Parameters outside of the set $\hat{S}$ will be discussed in Section 6.

In the following, we consider the length of $C_{j_1,\dots,j_n}$ as n goes to infinity. For convenience, let $\sigma_{j_1,\dots,j_n}(\alpha, \beta)$ for $j_1, \dots, j_{n-1} \in \{1, 3\}$ and $j_n \in \{1, 2, 3\}$ be an abbreviation of the product $\sigma_{j_1}(\alpha_0, \beta_0) \cdots \sigma_{j_n}(\alpha_{n-1}, \beta_{n-1})$. Note that $|C_{j_1,\dots,j_n}| = \sigma_{j_1,\dots,j_n}(\alpha, \beta)$ is always positive, because $\sigma_{j_i}(\alpha_{i-1}, \beta_{i-1})$ is always positive.

Lemma 4.1. *Let* $(\alpha, \beta) \in S_*$.

(i) *Let* $n = [(1 - \sigma_1(\alpha, \beta))^{-1}]$. *If* $0 \le i < n$ *and* $j_1 = \cdots = j_i = 1$, *then the lengths of the intervals* $C_{j_1,\dots,j_i,j_{i+1}}$ *are given by* $|C_{j_1,\dots,j_i,1}| = 1 - (i + 1)(1 - \sigma_1(\alpha, \beta))$, $|C_{j_1,\dots,j_i,2}| = \sigma_2(\alpha, \beta)$, *and* $|C_{j_1,\dots,j_i,3}| = \sigma_3(\alpha, \beta)$.

(ii) *Let* $n = [(1 - \sigma_3(\alpha, \beta))^{-1}]$. *If* $0 \le i < n$ *and* $j_1 = \cdots = j_i = 3$, *then the lengths of the intervals* $C_{j_1,\dots,j_i,j_{i+1}}$ *are given by* $|C_{j_1,\dots,j_i,1}| = \sigma_1(\alpha, \beta)$, $|C_{j_1,\dots,j_i,2}| = \sigma_2(\alpha, \beta)$, *and* $|C_{j_1,\dots,j_i,3}| = 1 - (i + 1)(1 - \sigma_3(\alpha, \beta))$.

Proof. The statement (ii) is symmetric to (i). Therefore, we prove (i) by induction on i. For $i = 0$, the equations are nothing but the definitions of $\sigma_j(\alpha, \beta)$.

Let us assume the lemma is true for i and verify it for $i + 1 < n$. Since the lengths of $C_{j_1,\dots,j_i,1}$ and $C_{j_1,\dots,j_i}$ are given by the lemma for i and $i - 1$, we obtain

$$\sigma_1(\alpha_i, \beta_i) = \frac{|C_{j_1,\dots,j_i,1}|}{|C_{j_1,\dots,j_i}|} = \frac{1 - (i+1)(1 - \sigma_1(\alpha, \beta))}{1 - i(1 - \sigma_1(\alpha, \beta))} = 1 - \frac{1}{1 + (1 - \sigma_1(\alpha, \beta))^{-1} - (i + 1)}$$

where $i + 1 < n = [(1 - \sigma_1(\alpha, \beta))^{-1}]$. Hence,

$$\sigma_1(\alpha_i, \beta_i) > 1 - \frac{1}{1 + \{(1 - \sigma_1(\alpha, \beta))^{-1}\}} > \frac{1}{2}.$$

Then $\sigma_1(\alpha_i,\beta_i) > 1-\sigma_1(\alpha_i,\beta_i)$ and it follows from the definition of T that

$$(\alpha_{i+1},\beta_{i+1}) = \begin{cases} \left(\dfrac{\alpha_i}{1-\beta_i}, \dfrac{\beta_i}{1-\beta_i}\right) & \text{if } \alpha_i < \beta_i \\ \left(1+\dfrac{\alpha_i-1}{\beta_i}, 1+\dfrac{\beta_i-1}{\beta_i}\right) & \text{if } \alpha_i > \beta_i. \end{cases}$$

Therefore, we obtain

$$\sigma_2(\alpha_{i+1},\beta_{i+1}) = \frac{\sigma_2(\alpha_i,\beta_i)}{\sigma_1(\alpha_i,\beta_i)}, \quad \sigma_3(\alpha_{i+1},\beta_{i+1}) = \frac{\sigma_3(\alpha_i,\beta_i)}{\sigma_1(\alpha_i,\beta_i)}.$$

Applying this result to Proposition 4.2(iv), we obtain

$$\begin{aligned} \left|C_{j_1,\dots,j_{i+1},2}\right| &= \sigma_{j_1,\dots,j_i}(\alpha_0,\beta_0)\sigma_1(\alpha_i,\beta_i)\sigma_2(\alpha_{i+1},\beta_{i+1}) \\ &= \sigma_{j_1,\dots,j_i}(\alpha_0,\beta_0)\sigma_2(\alpha_i,\beta_i) = |C_{j_1,\dots,j_i,2}| = \sigma_2(\alpha,\beta), \\ \left|C_{j_1,\dots,j_{i+1},3}\right| &= \sigma_{j_1,\dots,j_i}(\alpha_0,\beta_0)\sigma_1(\alpha_i,\beta_i)\sigma_3(\alpha_{i+1},\beta_{i+1}) \\ &= \sigma_{j_1,\dots,j_i}(\alpha_0,\beta_0)\sigma_3(\alpha_i,\beta_i) = |C_{j_1,\dots,j_i,3}| = \sigma_3(\alpha,\beta), \\ \left|C_{j_1,\dots,j_{i+1},1}\right| &= \left|C_{j_1,\dots,j_{i+1}}\right| - \left(\left|C_{j_1,\dots,j_{i+1},2}\right| + \left|C_{j_1,\dots,j_{i+1},3}\right|\right) \\ &= \left|C_{j_1,\dots,j_{i+1}}\right| - (1-\sigma_1(\alpha,\beta)) = 1-(i+2)(1-\sigma_1(\alpha,\beta)). \end{aligned}$$

Therefore, the lemma is also true for $i+1$. □

By using this property, we show that the interval $[0,1]$ is inevitably divided into small intervals in the following lemma and proposition.

Let $E(\alpha,\beta) = (1-\max(\sigma_1(\alpha,\beta),\sigma_3(\alpha,\beta)))^{-1}$.

Lemma 4.2. *Let $(\alpha,\beta)\in\hat{S}$ and $n=[E(\alpha,\beta)]$. For any sequence $j_1,\dots,j_n\in\{1,3\}$ of length n,*

$$|C_{j_1,\dots,j_n}| = \sigma_{j_1,\dots,j_n}(\alpha,\beta) < \frac{1}{2}.$$

Proof. At least one of $\sigma_1(\alpha,\beta)$ or $\sigma_3(\alpha,\beta)$ is less than $1/2$, because $\sigma_1(\alpha,\beta)+\sigma_3(\alpha,\beta)<1$. If both $\sigma_1(\alpha,\beta)$ and $\sigma_3(\alpha,\beta)$ are less than $1/2$, then n is 1 and the statement is true. Therefore, we can assume only one of $\sigma_1(\alpha,\beta)$ or $\sigma_3(\alpha,\beta)$ is less than $1/2$. Moreover, because of the symmetry, we can assume $\sigma_3(\alpha,\beta)<1/2$ and $\sigma_1(\alpha,\beta)\geq 1/2$ without any loss of generality. Then, under the assumption, $n=[(1-\sigma_1(\alpha,\beta))^{-1}]$. If $j_1=\dots=j_n=1$, it follows from Lemma 4.1(i) that

$$|C_{j_1,\dots,j_n}| = 1-n(1-\sigma_1(\alpha,\beta)) = \{(1-\sigma_1(\alpha,\beta))^{-1}\}(1-\sigma_1(\alpha,\beta)) < \frac{1}{2}.$$

Otherwise, let $i\leq n$ be the smallest integer such that $j_i=3$. Then $j_1=\dots=j_{i-1}=1$. Hence

$$|C_{j_1,\dots,j_n}| < \left|C_{j_1,\dots,j_{i-1},3}\right| = \sigma_3(\alpha,\beta) < \frac{1}{2}$$

because of Lemma 4.1(i). □

Now it is easy to show the following proposition.

Proposition 4.3. *Let $(\alpha,\beta)\in\hat{S}$.*

$$\lim_{n\to\infty}\max_{j_1,\dots,j_n\in\{1,3\}}|C_{j_1,\dots,j_n}| = 0.$$

Proof. Let $n \geq 0$ be an integer and,

$$n' = n + \max_{j_1,\ldots,j_n \in \{1,3\}} [E(\alpha_n, \beta_n)].$$

Then, for any sequence $j_1, \ldots, j_{n'} \in \{1,3\}$, it follows from Lemma 4.2 that

$$\begin{aligned}\left|C_{j_1,\ldots,j_{n'}}\right| &= \sigma_{j_1,\ldots,j_n}(\alpha,\beta)\sigma_{j_{n+1},\ldots,j_{n'}}(\alpha_n,\beta_n)\\ &< \frac{1}{2}\left|C_{j_1,\ldots,j_n}\right| \leq \frac{1}{2}\max_{j_1,\ldots,j_n\in\{1,3\}}\left|C_{j_1,\ldots,j_n}\right|.\end{aligned}$$

Therefore, for arbitrary $\epsilon > 0$, we can find finite n such that the maximum length of $|C_{j_1,\ldots,j_n}|$ becomes less than ϵ by repeating this process recursively. □

In the following, we consider the set Γ defined by

$$\Gamma = \{c \in [0,1] \,|\, T^i(\alpha,\beta,c) \in D_{*,1} \cup D_{*,3} \text{ for } i \geq 0\ \}.$$

In fact, the previous proposition implies that Γ is a Cantor set as shown later in Theorem 4.1. For the investigation of the Lebesgue measure of Γ, we define Γ_n for $n \geq 0$ as

$$\Gamma_n = \{c \in [0,1] \,|\, T^i(\alpha,\beta,c) \in D_{*,1} \cup D_{*,3} \text{ for } 0 \leq i < n\ \}.$$

Then the sets Γ and Γ_n can be expressed by the intervals $C_{j_1,\ldots,j_n}$ as

$$\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n, \qquad \Gamma_n = \bigcup_{j_1,\ldots,j_n\in\{1,3\}} C_{j_1,\ldots,j_n}.$$

Since $C_{j_1,\ldots,j_n}$ are always closed for $j_1, \ldots, j_n \in \{1,3\}$, the sets Γ_n and Γ are also closed.

Let us consider a subset V of S_* defined by

$$V = \{(\alpha,\beta) \in S_* \,|\ \ 0 \leq 3\alpha - 2\beta \leq 1 \text{ and } 0 \leq 3\beta - 2\alpha \leq 1\}.$$

Then note that the condition of $(\alpha,\beta) \in V$ is equivalent to each of the following two inequalities:

$$\frac{\sigma_2(\alpha,\beta)}{\sigma_j(\alpha,\beta)} \leq \frac{1}{2} \quad \text{for } j = 1 \text{ and } 3, \tag{1}$$

$$\sigma_2(\alpha,\beta) \leq \frac{1}{3E(\alpha,\beta)}. \tag{2}$$

Intuitively, if $(\alpha,\beta) \in V$, the measure of Γ_1 becomes relatively large and near to one, because $\sigma_2(\alpha,\beta)$ is relatively small. Therefore, the set V plays an important role in evaluating the measure of Γ_n. The following lemma claims that it is impossible for any sequence (α_i,β_i) to stay in the set V forever.

Lemma 4.3. *Let $(\alpha,\beta) \in \hat{S}$. There exists an integer $n > 0$ such that, for any sequence $j_1, \ldots, j_n \in \{1,3\}$ of length n, a non-negative integer $i < n$ such that $(\alpha_i,\beta_i) \notin V$ can be found.*

Proof. Let us assume that there exist a parameter $(\alpha,\beta) \in \hat{S}$ and an infinite sequence of $j_i \in \{1,3\}$ such that $(\alpha_n,\beta_n) \in V$ for all $n \geq 0$. It follows from the definition of T that

$$\beta_{i+1} - \alpha_{i+1} = \frac{\beta_i - \alpha_i}{\sigma_{j_{i+1}}(\alpha_i,\beta_i)} \pmod 1.$$

Taking into account that $\sigma_2(\alpha,\beta)$ can be both $\beta-\alpha$ and $\alpha-\beta$, we obtain

$$\sigma_2(\alpha_{i+1},\beta_{i+1}) = \begin{cases} \dfrac{\sigma_2(\alpha_i,\beta_i)}{\sigma_{j_{i+1}}(\alpha_i,\beta_i)} \\ 1-\dfrac{\sigma_2(\alpha_i,\beta_i)}{\sigma_{j_{i+1}}(\alpha_i,\beta_i)} \geq \dfrac{1}{2} \geq \dfrac{\sigma_2(\alpha_i,\beta_i)}{\sigma_{j_{i+1}}(\alpha_i,\beta_i)}, \end{cases}$$

because of Inequality (1). Therefore, in both cases,

$$\sigma_{j_{i+1}}(\alpha_i,\beta_i) \geq \frac{\sigma_2(\alpha_i,\beta_i)}{\sigma_2(\alpha_{i+1},\beta_{i+1})}.$$

Applying this inequality to Proposition 4.2(iv), for arbitrary $n>0$, we obtain

$$|C_{j_1,\ldots,j_n}| = \sigma_{j_1}(\alpha_0,\beta_0)\cdots\sigma_{j_n}(\alpha_{n-1},\beta_{n-1}) \geq \frac{\sigma_2(\alpha,\beta)}{\sigma_2(\alpha_n,\beta_n)} > \sigma_2(\alpha,\beta).$$

On the other hand, according to Proposition 4.3, the limit of $|C_{j_1,\ldots,j_n}|$ as n goes to infinity must be zero. This is a contradiction and thus the first assumption turns out to be false. □

Thus, it is impossible for any sequence (α_i,β_i) to stay in the set V forever. Furthermore, if $(\alpha,\beta)\notin V$, by taking adequate n, $\mu(\Gamma_n)$ becomes relatively small, where μ denotes the Lebesgue measure. Therefore, now we can show the following lemma concerning the measure of Γ_n.

Lemma 4.4. *Let $(\alpha,\beta)\in\hat{S}$. There exists an integer $n>0$ such that*

$$\mu(\Gamma_n) = \sum_{j_1,\ldots,j_n\in\{1,3\}} |C_{j_1,\ldots,j_n}| < \frac{5}{6}.$$

Proof. Let us consider the set of sequences

$$W_1 = \{\{j_1,\ldots,j_k\} | (\alpha_i,\beta_i)\in V \text{ for } 0\leq i<k \text{ and } (\alpha_k,\beta_k)\notin V\}$$

and let $n = \max_{\{j_1,\ldots,j_k\}\in W_1}(k+[E(\alpha_k,\beta_k)])$.

Assume $\{j_1,\ldots,j_k\}\in W_1$. Then $(\alpha_k,\beta_k)\notin V$ and

$$\sigma_2(\alpha_k,\beta_k) > \frac{1}{3E(\alpha_k,\beta_k)} > \frac{1}{3}\frac{1}{[E(\alpha_k,\beta_k)]+1}.$$

Since $n\geq k+[E(\alpha_k,\beta_k)]$, it follows from Lemma 4.1 that

$$\begin{aligned} \sum_{j_{k+1},\ldots,j_n\in\{1,3\}} |C_{j_1,\ldots,j_n}| &\leq |C_{j_1,\ldots,j_k}|\left(1-[E(\alpha_k,\beta_k)]\sigma_2(\alpha_k,\beta_k)\right) \\ &< |C_{j_1,\ldots,j_k}|\left(1-\frac{1}{3}\frac{1}{1+[E(\alpha_k,\beta_k)]^{-1}}\right) \leq \frac{5}{6}|C_{j_1,\ldots,j_k}|, \end{aligned}$$

where $[E(\alpha_k,\beta_k)]^{-1}\leq 1$ is applied to the last inequality.

Therefore, for $l=1$, the set of sequences W_l satisfies the following conditions.

(i) For arbitrary $\{j_1,\ldots,j_k\}\in W_l$,

$$\sum_{j_{k+1},\ldots,j_n\in\{1,3\}} |C_{j_1,\ldots,j_n}| < \frac{5}{6}|C_{j_1,\ldots,j_k}|.$$

(ii) For arbitrary sequence $j_1,\ldots,j_n\in\{1,3\}$, there exists a unique integer $k\leq n$ such that $\{j_1,\ldots,j_k\}\in W_l$.

Now assume that W_l satisfies the conditions and contains two or more elements. From the set W_l, we construct a new set W_{l+1} in the following way. Let $\{j_1, \ldots, j_k\}$ be one of the longest sequences in W_l. Then, W_l must contain both $\{j_1, \ldots, j_{k-1}, 1\}$ and $\{j_1, \ldots, j_{k-1}, 3\}$, because if otherwise W_l cannot satisfy the condition (ii). Let us construct W_{l+1} from W_l by putting a new element $\{j_1, \ldots, j_{k-1}\}$ instead of the elements $\{j_1, \ldots, j_{k-1}, 1\}$ and $\{j_1, \ldots, j_{k-1}, 3\}$. Then, W_{l+1} satisfies the condition (ii). Moreover, the new element $\{j_1, \ldots, j_{k-1}\}$ satisfies the condition (i), because

$$\sum_{j_k,\ldots,j_n\in\{1,3\}} |C_{j_1,\ldots,j_n}| = \sum_{j_{k+1},\ldots,j_n\in\{1,3\}} \left(\left|C_{j_1,\ldots,j_{k-1},1,j_{k+1},\ldots,j_n}\right| + \left|C_{j_1,\ldots,j_{k-1},3,j_{k+1},\ldots,j_n}\right|\right)$$
$$< \frac{5}{6}\left(\left|C_{j_1,\ldots,j_{k-1},1}\right| + \left|C_{j_1,\ldots,j_{k-1},3}\right|\right) < \frac{5}{6}\left|C_{j_1,\ldots,j_{k-1}}\right|.$$

Thus W_{l+1} satsifies the conditions. The number of elements in W_{l+1} decreased by one from W_l.

By induction, there exists a finite integer l such that W_l contains only one element and still satisfies the conditions. The only element must be an empty string, because of the condition (ii). Then the condition (i) for the empty string, namely

$$\sum_{j_1,\ldots,j_n\in\{1,3\}} |C_{j_1,\ldots,j_n}| < \frac{5}{6}|C| = \frac{5}{6},$$

is nothing but the statement of the lemma. □

Now we are ready to prove the first theorem.

Theorem 4.1. *Let $(\alpha, \beta) \in \hat{S}$. Then Γ is a Cantor set and has measure zero.*

Proof. Γ is compact, because it is closed and bounded within $[0, 1]$. The connected components in Γ_n are the intervals $C_{j_1,\ldots,j_n}$ for $j_1, \ldots, j_n \in \{1, 3\}$, because they are separated by the intervals $C_{j_1,\ldots,j_{n-1},2}$. The length of each connected component as n goes to infinity is zero because of Proposition 4.3. Therefore, every connected component in Γ is a single point, which means Γ is totally disconnected. Γ is also a perfect set, because, for arbitrary $c \in \Gamma$, we can construct a sequence $\gamma_i \in \Gamma$ that never equals to c but converges to c in the following way. For a positive integer i, there exists a sequence $j_1, \ldots, j_i \in \{1, 3\}$ such that $c \in C_{j_1,\ldots,j_i}$. Then one of the intervals $C_{j_1,\ldots,j_i,1}$ and $C_{j_1,\ldots,j_i,3}$ does not contain c. Let $\gamma_i \in \Gamma$ be one of the endpoints of the interval not containing c. Then $\gamma_i \neq c$. Since $|c - \gamma_i| \leq |C_{j_1,\ldots,j_i}|$, it follows from Proposition 4.3 that $\lim_{i\to\infty} |c-\gamma_i| = 0$. Thus Γ is compact, totally disconnected and perfect. Therefore it is a Cantor set.

Let n be a positive integer. Then, by applying Lemma 4.4 for (α_n, β_n), we can find an integer $n' > n$ such that

$$\sum_{j_{n+1},\ldots,j_{n'}} \left|C_{j_1,\ldots,j_{n'}}\right| < \frac{5}{6}|C_{j_1,\ldots,j_n}|$$

for any sequence $j_1, \ldots, j_n \in \{1, 3\}$. Taking summation over all the sequences, we obtain

$$\mu(\Gamma_{n'}) = \sum_{j_1,\ldots,j_{n'}} \left|C_{j_1,\ldots,j_{n'}}\right| < \frac{5}{6}\sum_{j_1,\ldots,j_n} |C_{j_1,\ldots,j_n}| = \frac{5}{6}\mu(\Gamma_n).$$

Therefore, for arbitrary $\epsilon > 0$, we can find finite n such that the $\mu(\Gamma_n) < \epsilon$ by repeating this process recursively. Then we obtain $\mu(\Gamma) = 0$, because $\Gamma \subset \Gamma_n$ for all $n > 0$. □

Almost every $(\alpha, \beta) \in S$ satisfies the assumption $(\alpha, \beta) \in \hat{S}$, and then almost every $c \in [0, 1]$ is not in the set Γ. Therefore, we can conclude that almost every double rotation is reducible to a rotation. Besides, since $f_{(\alpha_n, \beta_n, c_n)}$ becomes a rotation if $c_n \in \{0, 1\}$, it should be noted that $f_{(\alpha,\beta,c)}$ can also be reduced to a rotation if c is an endpoint of an interval $C_{j_1,\ldots,j_n}$ for a sequence $j_1, \ldots, j_n \in \{1, 3\}$, even if $c \in \Gamma$.

5. Discharge number

In this section, we consider a characteristic number of double rotations, which is called the discharge number. For a parameter $(\alpha, \beta, c) \in D$ and an initial state $x \in [0, 1)$, the *discharge number* of $f_{(\alpha,\beta,c)}$ for x is defined as

$$q_{(\alpha,\beta,c)}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[c,1)}(f^i_{(\alpha,\beta,c)}(x)),$$

if the limit exists, where $\chi_{[c,1)}$ denotes the characteristic function of $[c, 1)$.

Figure 1(b) shows a graph of $q_{(\alpha,\beta,c)}(x)$ as a function of c, where α and β are fixed. In the graph, $q_{(\alpha,\beta,c)}(x)$ is independent of x, so that the graph takes only a single value for each c. At first glance, we notice that the graph is like a devil's staircase. It is monotonically non-increasing and seems closely related to the Cantor structure we discussed in the previous section. Each flat step in the graph seems likely to correspond to each interval $C_{j_1,\ldots,j_n,2}$.

We will assume $(\alpha, \beta, c) \in \hat{D} = \hat{S} \times [0, 1]$ and investigate such properties of the discharge number in this section.

For the first step, in the following lemma, we show that the discharge number is well-defined, independent of x, and constant on the interval C_2. It should be noted that, from the symmetry, $q_{(\alpha,\beta,c)}(x) = 1 - q_{H(\alpha,\beta,c)}(\Phi_c^{-1}(x))$.

Proposition 5.1. *Let $(\alpha, \beta, c) \in \hat{D}$. If $(\alpha, \beta, c) \in D_{*,2}$, then $q_{(\alpha,\beta,c)}(x)$ is a constant function given by*

$$q_{(\alpha,\beta,c)}(x) = \frac{\sigma_3(\alpha, \beta)}{\sigma_1(\alpha, \beta) + \sigma_3(\alpha, \beta)},$$

and the value is independent of c.

Proof. In the statement, the equation for $D_{1,2}$ is symmetric to the equation for $D_{0,2}$, because the equations satisfy $q_{(\alpha,\beta,c)}(x) = 1 - q_{H(\alpha,\beta,c)}(\Phi_c^{-1}(x))$ for $(\alpha, \beta, c) \in D_{1,2}$. Therefore, we can assume $(\alpha, \beta, c) \in D_{0,2}$ without any loss of generality. Then, for any $x \in [0, 1)$, there exists a finite integer k such that $f^i_{(\alpha,\beta,c)}(x) \in I_{(\alpha,\beta,c)}$ for all $i \geq k$. On the interval $I_{(\alpha,\beta,c)}$, it has been shown in Proposition 3.2 that $f_{(\alpha,\beta,c)}$ is isomorphic to a rotation $\Phi_{\alpha/(1+\alpha-\beta)}$. This rotation is irrational because of the assumption $(\alpha, \beta) \in \hat{S}$. Then the orbit starting from $f^k_{(\alpha,\beta,c)}(x)$ is equidistributed on the interval $I_{(\alpha,\beta,c)}$. Therefore, we obtain the discharge number

$$q_{(\alpha,\beta,c)}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[c,1)}(f^i_{(\alpha,\beta,c)}(x)) = \frac{|I_{(\alpha,\beta,c)} \cap [c, 1)|}{|I_{(\alpha,\beta,c)}|} = \frac{\alpha}{1 + \alpha - \beta},$$

where recall that $I_{(\alpha,\beta,c)} = [c + \beta - 1, c + \alpha)$. □

Let $U = \{x \in [0,c) \mid x+\alpha \geq 1\} \cup \{x \in [c,1) \mid x+\beta \geq 1\}$. For further investigation of discharge numbers, we define two other characteristic numbers $p_{(\alpha,\beta,c)}$ and $r_{(\alpha,\beta,c)}$ for the sets $[0,c)$ and U, respectively, in a similar way that we have defined $q_{(\alpha,\beta,c)}$ for the interval $[c,1)$. We also define $P_n(x)$, $Q_n(x)$ and $R_n(x)$ for a positive integer n as the number of integers $i \in \{0,\ldots,n-1\}$ such that $f^i(x)$ is in $[0,c)$, $[c,1)$ and U, respectively. Then, we have

$$P_n(x) = \sum_{i=0}^{n-1} \chi_{[0,c)}(f^i_{(\alpha,\beta,c)}(x)), \qquad p_{(\alpha,\beta,c)}(x) = \lim_{n\to\infty} \frac{1}{n} P_n(x),$$
$$Q_n(x) = \sum_{i=0}^{n-1} \chi_{[c,1)}(f^i_{(\alpha,\beta,c)}(x)), \qquad q_{(\alpha,\beta,c)}(x) = \lim_{n\to\infty} \frac{1}{n} Q_n(x),$$
$$R_n(x) = \sum_{i=0}^{n-1} \chi_{U}(f^i_{(\alpha,\beta,c)}(x)), \qquad r_{(\alpha,\beta,c)}(x) = \lim_{n\to\infty} \frac{1}{n} R_n(x).$$

It should be noted that, using these numbers, we can express $f^n_{(\alpha,\beta,c)}(x)$ as

$$f^n_{(\alpha,\beta,c)}(x) = x + \alpha P_n(x) + \beta Q_n(x) - R_n(x). \tag{3}$$

Similarly, the values $p_{(\alpha,\beta,c)}(x)$, $q_{(\alpha,\beta,c)}(x)$ and $r_{(\alpha,\beta,c)}(x)$ are closely related to each other, as shown in the following lemma.

Lemma 5.1. *If $p_{(\alpha,\beta,c)}(x)$, $q_{(\alpha,\beta,c)}(x)$ and $r_{(\alpha,\beta,c)}(x)$ have respective limits, then they satisfy the equations*

$$p_{(\alpha,\beta,c)}(x) + q_{(\alpha,\beta,c)}(x) = 1,$$
$$\alpha p_{(\alpha,\beta,c)}(x) + \beta q_{(\alpha,\beta,c)}(x) = r_{(\alpha,\beta,c)}(x).$$

Proof. Since $[0,c) \cup [c,1)$ is a disjoint union and equals to the whole state space $[0,1)$, it is clear that $P_n(x) + Q_n(x) = n$. Therefore, we obtain the first equation

$$p_{(\alpha,\beta,c)}(x) + q_{(\alpha,\beta,c)}(x) = \lim_{n\to\infty} \frac{1}{n}(P_n(x) + Q_n(x)) = 1.$$

We also obtain the second equation, by dividing Equation (3) by n and taking the limit as n goes to infinity, where note that both x and $f^n(x)$ are bounded within the interval $[0,1)$. □

This lemma claims that, if one of the values $p_{(\alpha,\beta,c)}(x)$, $q_{(\alpha,\beta,c)}(x)$ and $r_{(\alpha,\beta,c)}(x)$ is obtained, other two can be derived from the equations. In this sense, these three values can be considered as equivalent.

Now we consider dependencies between the discharge numbers of $f_{(\alpha,\beta,c)}$ and $f_{T(\alpha,\beta,c)}$, for $(\alpha,\beta,c) \in D_{*,1} \cup D_{*,3}$.

For an initial state $x \in [0,1)$, let us consider the orbit $x_i = f^i_{(\alpha,\beta,c)}(x)$. Then the subsequence that consists of the elements such that $x_i \in I_{(\alpha,\beta,c)}$ is nothing but an orbit of the induced transformation $f_{(\alpha,\beta,c)}\big|_{I_{(\alpha,\beta,c)}}$. Let x_k be the first element of the orbit. By the isomorphism between $f_{(\alpha,\beta,c)}\big|_{I_{(\alpha,\beta,c)}}$ and $f_{T(\alpha,\beta,c)}$, the element $x_k \in I_{(\alpha,\beta,c)}$ has a corresponding point x' in $[0,1)$. Let us call x' as the initial state induced from x.

In the following lemma, we will evaluate $q_{(\alpha,\beta,c)}$ in the form

$$\begin{pmatrix} p_{(\alpha,\beta,c)}(x) \\ q_{(\alpha,\beta,c)}(x) \end{pmatrix} \sim \begin{pmatrix} u \\ v \end{pmatrix},$$

where $\sim$ means that $(p_{(\alpha,\beta,c)}(x), q_{(\alpha,\beta,c)}(x))$ and (u, v) are equivalent to each other in the sense that one is a non-zero real multiple of the other. Then the value of $q_{(\alpha,\beta,c)}(x)$ is obtained by $v/(u+v)$.

Lemma 5.2. *Let $(\alpha, \beta, c) \in D_{0,1}$ and $x \in [0, 1)$. Suppose the discharge number $q_{T(\alpha,\beta,c)}(x')$ is well-defined, where x' is the initial state induced from x. Then, the discharge number $q_{(\alpha,\beta,c)}(x)$ is also well-defined and given by*

$$\begin{pmatrix} p_{(\alpha,\beta,c)}(x) \\ q_{(\alpha,\beta,c)}(x) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ \dfrac{\alpha}{1-\beta} & \dfrac{1}{1-\beta} \end{pmatrix} \begin{pmatrix} p_{T(\alpha,\beta,c)}(x') \\ q_{T(\alpha,\beta,c)}(x') \end{pmatrix}.$$

Proof. Firstly, consider the first n elements of the orbit starting from x. Let n' be the number of the elements of the orbit that is in $I_{(\alpha,\beta,c)} = [0, 1-\beta)$. Then we have $n' = n - R_n(x)$, because $U = [1-\beta, 1)$. As we have defined before, $P_n(x)$ and $Q_n(x)$ are the number of elements in the intervals $[0, c)$ and $[c, 1)$, respectively. As for the induced transformation, let $P'_{n'}(x')$ and $Q'_{n'}(x')$ be the number of elements in the intervals $[0, c)$ and $[c, 1-\beta)$, respectively. Then we obtain $P_n(x) = P'_{n'}(x')$ and $Q_n(x) = Q'_{n'}(x') + R_n(x)$. Applying these results to Equation (3), we obtain

$$(1-\beta)R_n(x) = \alpha P'_{n'}(x') + \beta Q'_{n'}(x') + (x - f^n(x)),$$

where note that $|x - f^n(x)| < 1$. Then,

$$\begin{aligned} r_{(\alpha,\beta,c)}(x) &= \lim_{n\to\infty} \frac{1}{n} R_n(x) = \lim_{n\to\infty} \frac{R_n(x)}{P'_{n'}(x') + Q'_{n'}(x') + R_n(x)} \\ &= \lim_{n\to\infty} \frac{\alpha P'_{n'}(x') + \beta Q'_{n'}(x') + (x - f^n(x))}{(1+\alpha-\beta)P'_{n'}(x') + Q'_{n'}(x') + (x - f^n(x))} \\ &= \frac{\alpha p_{T(\alpha,\beta,c)}(x') + \beta q_{T(\alpha,\beta,c)}(x')}{(1+\alpha-\beta)p_{T(\alpha,\beta,c)}(x') + q_{T(\alpha,\beta,c)}(x')}, \end{aligned}$$

because n' also goes to infinity as n goes to infinity. From Lemma 5.1, we obtain

$$\begin{aligned} p_{(\alpha,\beta,c)}(x) &= \frac{(1-\beta)p_{T(\alpha,\beta,c)}(x')}{(1+\alpha-\beta)p_{T(\alpha,\beta,c)}(x') + q_{T(\alpha,\beta,c)}(x')}, \\ q_{(\alpha,\beta,c)}(x) &= \frac{\alpha p_{T(\alpha,\beta,c)}(x') + q_{T(\alpha,\beta,c)}(x')}{(1+\alpha-\beta)p_{T(\alpha,\beta,c)}(x') + q_{T(\alpha,\beta,c)}(x')}, \end{aligned}$$

which is equivalent to the equation in the statement. □

Lemma 5.3. *Let $(\alpha, \beta, c) \in D_{0,3}$ and $x \in [0, 1)$. Suppose the discharge number $q_{T(\alpha,\beta,c)}(x')$ is well-defined, where x' is the initial state induced from x. Then, the discharge number $q_{(\alpha,\beta,c)}(x)$ is also well-defined and given by*

$$\begin{pmatrix} p_{(\alpha,\beta,c)}(x) \\ q_{(\alpha,\beta,c)}(x) \end{pmatrix} \sim \begin{pmatrix} \dfrac{1}{\alpha} & \dfrac{1-\beta}{\alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{T(\alpha,\beta,c)}(x') \\ q_{T(\alpha,\beta,c)}(x') \end{pmatrix}.$$

Proof. Similarly to the proof of Lemma 5.2, consider the first n elements of the orbit starting from x and let n' bet the number of the elements in $I_{(\alpha,\beta,c)} = [1-\alpha, 1)$. In this case we have $n' = R_n(x)$, because $U = I_{(\alpha,\beta,c)}$. Let $P'_{n'}(x')$ and $Q'_{n'}(x')$ be the number of elements in the intervals $[1-\alpha, c)$ and $[c, 1)$, respectively. Then we obtain

$P_n(x) = P'_{n'}(x') + n - R_n(x)$ and $Q_n(x) = Q'_{n'}(x')$. Since $R_n(x) = P'_{n'}(x') + Q'_{n'}(x')$, from Equation (3), we obtain

$$\alpha(n - R_n(x)) = (1-\alpha)P'_{n'}(x') + (1-\beta)Q'_{n'}(x') + (f^n(x) - x),$$

where $|f^n(x) - x| < 1$. Then

$$\begin{aligned} r_{(\alpha,\beta,c)}(x) &= \lim_{n\to\infty} \frac{1}{n} R_n(x) = \lim_{n\to\infty} \frac{P'_{n'}(x') + Q'_{n'}(x')}{P'_{n'}(x') + Q'_{n'}(x') + n - R_n(x)} \\ &= \lim_{n\to\infty} \frac{\alpha P'_{n'}(x') + \alpha Q'_{n'}(x')}{P'_{n'}(x') + (1+\alpha-\beta)Q'_{n'}(x') + (f^n(x) - x)} \\ &= \frac{\alpha p_{T(\alpha,\beta,c)}(x') + \alpha q_{T(\alpha,\beta,c)}(x')}{p_{T(\alpha,\beta,c)}(x') + (1+\alpha-\beta)q_{T(\alpha,\beta,c)}(x')}, \end{aligned}$$

because n' also goes to infinity as n goes to infinity. From Lemma 5.1, we obtain

$$\begin{aligned} p_{(\alpha,\beta,c)}(x) &= \frac{p_{T(\alpha,\beta,c)}(x') + (1-\beta)q_{T(\alpha,\beta,c)}(x')}{p_{T(\alpha,\beta,c)}(x') + (1+\alpha-\beta)q_{T(\alpha,\beta,c)}(x')}, \\ q_{(\alpha,\beta,c)}(x) &= \frac{\alpha q_{T(\alpha,\beta,c)}(x')}{p_{T(\alpha,\beta,c)}(x') + (1+\alpha-\beta)q_{T(\alpha,\beta,c)}(x')}, \end{aligned}$$

which is equivalent to the equation in the statement. □

Lemma 5.2 and 5.3 can be generalized as the following proposition.

Proposition 5.2. *Suppose* $(\alpha, \beta, c) \in D_{*,j}$ *for* $j = 1$ *or* 3.

(i) *Let* $x \in [0,1)$. *Suppose the discharge number* $q_{T(\alpha,\beta,c)}(x')$ *is well-defined, where* x' *is the initial state induced from* x. *Then, the discharge number* $q_{(\alpha,\beta,c)}(x)$ *is also well-defined and given by*

$$\begin{pmatrix} p_{(\alpha,\beta,c)}(x) \\ q_{(\alpha,\beta,c)}(x) \end{pmatrix} \sim M_j(\alpha,\beta) \begin{pmatrix} p_{T(\alpha,\beta,c)}(x') \\ q_{T(\alpha,\beta,c)}(x') \end{pmatrix},$$

where

$$M_1(\alpha,\beta) = \begin{pmatrix} 1 & 0 \\ \dfrac{\sigma_3(\alpha,\beta)}{\sigma_1(\alpha,\beta)} & \dfrac{1}{\sigma_1(\alpha,\beta)} \end{pmatrix},$$

$$M_3(\alpha,\beta) = \begin{pmatrix} \dfrac{1}{\sigma_3(\alpha,\beta)} & \dfrac{\sigma_1(\alpha,\beta)}{\sigma_3(\alpha,\beta)} \\ 0 & 1 \end{pmatrix}.$$

(ii) *In addition, let* $(\alpha, \beta, c') \in D_{*,j}$ *and* $y \in [0,1)$. *If* $q_{T(\alpha,\beta,c')}(y')$ *is well-defined and* $q_{T(\alpha,\beta,c)}(x') > q_{T(\alpha,\beta,c')}(y')$, *then* $q_{(\alpha,\beta,c)}(x) > q_{(\alpha,\beta,c')}(y)$, *where* y' *is the initial state induced from* y.

Proof. (i) If $\alpha < \beta$, the statement is equivalent to Lemma 5.2 and 5.3. Suppose $\beta > \alpha$. By the symmetry, we obtain

$$\begin{pmatrix} p_{(\alpha,\beta,c)}(x) \\ q_{(\alpha,\beta,c)}(x) \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_{j'}(\beta,\alpha) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{T(\alpha,\beta,c)}(x') \\ q_{T(\alpha,\beta,c)}(x') \end{pmatrix},$$

where $j' \in \{1,3\}$ is the one that is different from j. Then, since $M_{j'}$ and M_j satisfies the equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_{j'}(\beta,\alpha) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M_j(\alpha,\beta),$$

the statement is also true even if $\beta > \alpha$.

(ii) Let us define a matrix by

$$\begin{pmatrix} u & u' \\ v & v' \end{pmatrix} = M_j(\alpha, \beta) \begin{pmatrix} p_{T(\alpha,\beta,c)}(x') & p_{T(\alpha,\beta,c')}(y') \\ q_{T(\alpha,\beta,c)}(x') & q_{T(\alpha,\beta,c')}(y') \end{pmatrix}.$$

Since $q_{(\alpha,\beta,c)}(x) = v/(u+v)$ and $q_{(\alpha,\beta,c')}(y) = v'/(u'+v')$, we obtain

$$\begin{aligned} q_{(\alpha,\beta,c')}(y) - q_{(\alpha,\beta,c)}(x) &= \frac{1}{(u+v)(u'+v')} \det \begin{pmatrix} u & u' \\ v & v' \end{pmatrix} \\ &= \frac{\det M_j(\alpha,\beta)}{(u+v)(u'+v')} (q_{T(\alpha,\beta,c')}(y') - q_{T(\alpha,\beta,c)}(x')), \end{aligned}$$

where $(u+v)$, $(u'+v')$ and $\det M_j(\alpha,\beta) = \sigma_j(\alpha,\beta)^{-1}$ are all positive. Therefore, the sign of $q_{(\alpha,\beta,c')}(y) - q_{(\alpha,\beta,c)}(x)$ is same as $q_{T(\alpha,\beta,c')}(y') - q_{T(\alpha,\beta,c)}(x')$. □

Thus we can reduce the calculation of the discharge number of a double rotation to the calculation for its induced transformation. Accordingly, we can show the following proposition and theorem.

Proposition 5.3. *Let $(\alpha, \beta) \in S_*$ and $c \in C_{j_1,\ldots,j_n,2}$. Then the discharge number $q_{(\alpha,\beta,c)}(x)$ is independent of x and given by*

$$\begin{pmatrix} p_{(\alpha,\beta,c)}(x) \\ q_{(\alpha,\beta,c)}(x) \end{pmatrix} \sim M_{j_1}(\alpha_0, \beta_0) \cdots M_{j_n}(\alpha_{n-1}, \beta_{n-1}) \begin{pmatrix} \sigma_1(\alpha_n, \beta_n) \\ \sigma_3(\alpha_n, \beta_n) \end{pmatrix}$$

Proof. By applying Proposition 5.1 to $(\alpha_n, \beta_n, c_n) \in D_{*,2}$, we obtain

$$\begin{pmatrix} p_{(\alpha_n,\beta_n,c_n)}(x) \\ q_{(\alpha_n,\beta_n,c_n)}(x) \end{pmatrix} \sim \begin{pmatrix} \sigma_1(\alpha_n, \beta_n) \\ \sigma_3(\alpha_n, \beta_n) \end{pmatrix}$$

for arbitrary x. Then, applying Proposition 5.2 recursively, we obtain the equation in the statement. □

Theorem 5.1. *Let $(\alpha, \beta) \in \hat{S}$ and $c \in [0,1] \setminus \Gamma$. Then $q_{(\alpha,\beta,c)}(x)$ is well-defined for all $x \in [0,1)$ and the value is independent of x. Furthermore, on each connected component in $[0,1] \setminus \Gamma$, the value of $q_{(\alpha,\beta,c)}(x)$ is constant. If $c < c'$ and c' belongs to another connected component of $[0,1] \setminus \Gamma$, then $q_{(\alpha,\beta,c)}(x) > q_{(\alpha,\beta,c')}(x)$.*

Proof. Since $c \in [0,1] \setminus \Gamma$, there exists a sequence $j_1, \ldots, j_n \in \{1,3\}$ such that $c \in C_{j_1,\ldots,j_n,2}$. From Proposition 5.3, it is shown that $q_{(\alpha,\beta,c)}(x)$ is well-defined for all $x \in [0,1)$ and independent of x.

Every connected component in $[0,1] \setminus \Gamma$ equals to an open interval $C_{j_1,\ldots,j_n,2}$ for a certain sequence $j_1, \ldots, j_n \in \{1,3\}$, because the two endpoints of $C_{j_1,\ldots,j_n,2}$ belongs to Γ. Therefore, $q_{(\alpha,\beta,c)}(x)$ is constant on the component.

Let $c \in C_{j_1,\ldots,j_n,2}$ and $c' \in C_{j'_1,\ldots,j'_{n'},2}$, and assume $c < c'$. Let k be the smallest integer such that $j_k \neq j'_k$. From Proposition 5.2(ii), we can assume $k = 1$, i.e. $j_1 \neq j'_1$, to prove $q_{(\alpha,\beta,c)}(x) > q_{(\alpha,\beta,c')}(x)$ without any loss of generality. Furthermore, j_1 must be less than j'_1 because of the assumption $c < c'$. Then, possible combinations of (j_1, j'_1) are limited to $(1,2)$, $(2,3)$, and $(1,3)$. The last case can be derived from the first two. The first two cases are symmetric to each other. Therefore, we only have to examine the case $(j_1, j'_1) = (1,2)$. Assume $c \in C_1$ and $c' \in C_2$. Then we have

$$\begin{pmatrix} p_{(\alpha,\beta,c)}(x) \\ q_{(\alpha,\beta,c)}(x) \end{pmatrix} \sim \begin{pmatrix} \sigma_1(\alpha,\beta) & 0 \\ \sigma_3(\alpha,\beta) & 1 \end{pmatrix} \begin{pmatrix} p_{(\alpha_1,\beta_1,c_1)}(x) \\ q_{(\alpha_1,\beta_1,c_1)}(x) \end{pmatrix},$$

$$\begin{pmatrix} p_{(\alpha,\beta,c')}(x) \\ q_{(\alpha,\beta,c')}(x) \end{pmatrix} \sim \begin{pmatrix} \sigma_1(\alpha,\beta) \\ \sigma_3(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} \sigma_1(\alpha,\beta) & 0 \\ \sigma_3(\alpha,\beta) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since both $p_{(\alpha_1,\beta_1,c_1)}(x)$ and $q_{(\alpha_1,\beta_1,c_1)}(x)$ are non-negative, we have $q_{(\alpha,\beta,c)}(x) \geq q_{(\alpha,\beta,c')}(x)$. If $q_{(\alpha,\beta,c)}(x) = q_{(\alpha,\beta,c')}(x)$, then $q_{(\alpha_1,\beta_1,c_1)}(x) = 0$, and recursively, all of $q_{(\alpha_1,\beta_1,c_1)}(x), \ldots, q_{(\alpha_n,\beta_n,c_n)}(x)$ must be zero. On the other hand, $q_{(\alpha_n,\beta_n,c_n)}(x)$ cannot be zero, because

$$\begin{pmatrix} p_{(\alpha_n,\beta_n,c_n)}(x) \\ q_{(\alpha_n,\beta_n,c_n)}(x) \end{pmatrix} \sim \begin{pmatrix} \sigma_1(\alpha_n,\beta_n) \\ \sigma_3(\alpha_n,\beta_n) \end{pmatrix},$$

and $\sigma_3(\alpha_n,\beta_n)$ is positive. Therefore, $q_{(\alpha,\beta,c)}(x) > q_{(\alpha,\beta,c')}(x)$. □

6. Classification of double rotations

In the previous sections, we ignored the boundary B by assuming that α and β are linearly independent over $\mathbb{Q}$, namely $(\alpha,\beta,c) \in \hat{D} = \hat{S} \times [0,1]$. To justify the discussion, we have to confirm that (α_i,β_i,c_i) is never mapped into B by the transformation T under the assumption $(\alpha,\beta,c) \in \hat{D}$. For the purpose, we will show that the sequence (α_i,β_i,c_i) is always in the set $\hat{D}$, for the set $\hat{D}$ has no intersection with the boundary B. Let $L = \mathbb{Z}^2 \setminus \{(0,0)\}$.

Proposition 6.1. *Let $(\alpha,\beta,c) \in D_{*,1} \cup D_{*,3}$ and $(\alpha_1,\beta_1,c_1) = T(\alpha,\beta,c)$. Assume (s,t) and (s_1,t_1) satisfy the equation*

$$\begin{pmatrix} s_1 \\ t_1 \end{pmatrix} = M_j(\alpha,\beta)^{-1} \begin{pmatrix} s \\ t \end{pmatrix}. \tag{4}$$

Then, $(s_1,t_1) \in L$ and $s_1\alpha_1 + t_1\beta_1 \in \mathbb{Z}$ if and only if $(s,t) \in L$ and $s\alpha + t\beta \in \mathbb{Z}$. Therefore, $(\alpha_1,\beta_1,c_1) \in \hat{D}$ if and only if $(\alpha,\beta,c) \in \hat{D}$.

Proof. Firstly, let us assume (s,t) and (s_1,t_1) satisfy Equation (4), and show the following statements.

(i) Assume $s\alpha + t\beta \in \mathbb{Z}$. Then, $(s,t) \in L$ if and only if $(s_1,t_1) \in L$.
(ii) If $(s_1,t_1) \in L$ and $s_1\alpha_1 + t_1\beta_1 \in \mathbb{Z}$, then $s\alpha + t\beta \in \mathbb{Z}$.
(iii) If $(s,t) \in L$, $(s_1,t_1) \in L$, and $s\alpha + t\beta \in \mathbb{Z}$, then $s_1\alpha_1 + t_1\beta_1 \in \mathbb{Z}$.

By the symmetry, we can assume $(\alpha,\beta,c) \in D_{0,1} \cup D_{0,3}$ without any loss of generality. (i) is true because (s_1,t_1) is given by

$$\begin{pmatrix} s_1 \\ t_1 \end{pmatrix} = M_1(\alpha,\beta)^{-1} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s \\ t - (s\alpha + t\beta) \end{pmatrix} \qquad \text{if } (\alpha,\beta,c) \in D_{0,1},$$

$$\begin{pmatrix} s_1 \\ t_1 \end{pmatrix} = M_3(\alpha,\beta)^{-1} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} (s\alpha + t\beta) - t \\ t \end{pmatrix} \qquad \text{if } (\alpha,\beta,c) \in D_{0,3}.$$

Then, (ii) is also true for $(\alpha,\beta,c) \in D_{0,3}$, because $s_1 + t_1 = s\alpha + t\beta$. For $(\alpha,\beta,c) \in D_{0,1}$, it follows from the definition of T that

$$s\alpha + t\beta = (s_1\alpha_1 + t_1\beta_1) + s_1 \left[\frac{\alpha}{1-\beta}\right] + t_1 \left[\frac{\beta}{1-\beta}\right].$$

Therefore (ii) and (iii) for $(\alpha, \beta, c) \in D_{0,1}$ are true. Similarly, (iii) for $(\alpha, \beta, c) \in D_{0,3}$ is also true, because

$$(s\alpha + t\beta) - (s + t) = (s_1\alpha_1 + t_1\beta_1) + s_1 \left[\frac{\alpha - 1}{\alpha}\right] + t_1 \left[\frac{\beta - 1}{\alpha}\right].$$

Now, using these statements, we can show the proposition as follows. If we assume $(s, t) \in L$ and $s\alpha + t\beta \in \mathbb{Z}$, we have $(s_1, t_1) \in L$ from (i), and then $s_1\alpha_1 + t_1\beta_1 \in \mathbb{Z}$ from (iii). If we assume $(s_1, t_1) \in L$ and $s_1\alpha_1 + t_1\beta_1 \in \mathbb{Z}$, we have $s\alpha + t\beta \in \mathbb{Z}$ from (ii), and then $(s, t) \in L$ from (i). □

In addition, for $(\alpha, \beta, c) \in D_{*,1} \cup D_{*,3}$ and $(\alpha_1, \beta_1, c_1) = T(\alpha, \beta, c)$, it is clear that $(\alpha, \beta) \in \mathbb{Q}^2$ if and only if $(\alpha_1, \beta_1) \in \mathbb{Q}^2$. Then, we notice that there are three classes of parameters: (i) $(\alpha, \beta, c) \in D \cap (\mathbb{Q}^2 \times [0, 1])$, (ii) $(\alpha, \beta, c) \in D \setminus (\hat{D} \cup (\mathbb{Q}^2 \times [0, 1]))$, (iii) $(\alpha, \beta, c) \in \hat{D}$. A parameter in a certain class is never mapped into other classes by the transformation T.

For $(\alpha, \beta, c) \in D$, let us define the *rotation rank* of the double rotation $f_{(\alpha,\beta,c)}$ as the dimension of the vector space over $\mathbb{Q}$ spanned by α, β and 1. The rotation rank differs from the rank of ITMs defined in [1], for the rank of $f_{(\alpha,\beta,c)}$ as an ITM is the dimension of the space spanned by α, β, c and 1. For each possible rotation rank $d = 1$, 2 and 3, let Ω_d denote the set of parameters $(\alpha, \beta, c) \in D$ such that the rotation rank of $f_{(\alpha,\beta,c)}$ is d. Then the three classes can be denoted by (i) $\Omega_1 = D \cap (\mathbb{Q}^2 \times [0, 1])$, (ii) $\Omega_2 = D \setminus (\hat{D} \cup (\mathbb{Q}^2 \times [0, 1]))$, (iii) $\Omega_3 = \hat{D}$. We have investigated the class Ω_3 in the previous sections, and we will consider the classes Ω_1 and Ω_2 in the following.

Firstly, we assume the parameter (α, β, c) is in the second class Ω_2. In other words, we assume that α and β are linearly dependent over $\mathbb{Q}$ and that at least one of α and β is irrational. Then there exists $(s, t) \in L$ that satisfies $s\alpha + t\beta \in \mathbb{Z}$, and the pair (s, t) is unique except for its rational multiples. Moreover, if $(\alpha, \beta, c) \in D_{*,1} \cup D_{*,3}$, the pair (s_1, t_1) obtained from Equation (4) is the unique solution such that $s_1\alpha_1 + t_1\beta_1 \in \mathbb{Z}$. Recursively, for $n \geq 0$, (s_n, t_n) that satisfies $s_n\alpha_n + t_n\beta_n \in \mathbb{Z}$ can also be obtained uniquely by

$$\begin{pmatrix} s_n \\ t_n \end{pmatrix} = M_{j_n}(\alpha_{n-1}, \beta_{n-1})^{-1} \cdots M_{j_1}(\alpha_0, \beta_0)^{-1} \begin{pmatrix} s \\ t \end{pmatrix},$$

if $(\alpha_i, \beta_i, c_i) \in D_{*,j_{i+1}}$ for $0 \leq i < n$.

Let us divide L into two disjoint regions $L_+ = \{(s, t) \in L \mid st \geq 0\}$ and $L_- = \{(s, t) \in L \mid st < 0\}$. Then the following lemma shows that, intuitively speaking, (s_i, t_i) flows from L_+ to L_- and toward the origin $(0, 0)$ as i increases.

Lemma 6.1. *Let $(\alpha, \beta, c) \in D_{*,1} \cup D_{*,3}$. Suppose (s, t) and $(s_1, t_1) \in L$ satisfies Equation (4).*

(i) *If $(s, t) \in L_+$ and $(s_1, t_1) \in L_+$, then $|s_1 + t_1| < |s + t|$.*
(ii) *If $(s, t) \in L_-$, then $(s_1, t_1) \in L_-$.*
(iii) *If $(s, t) \in L_-$ and $(\alpha, \beta, c) \in D_{*,1}$, then $|t_1| < \max(|s|, |t|)$.*
(iv) *If $(s, t) \in L_-$ and $(\alpha, \beta, c) \in D_{*,3}$, then $|s_1| < \max(|s|, |t|)$.*
(v) *If $(s, t) \in L_-$, then $\max(|s_1|, |t_1|) \leq \max(|s|, |t|)$.*

Proof. By reversing the sign of (s, t) if necessary, we only have to consider the case that s is positive if s is non-zero, and that t is positive if s is zero. Furthermore, we can assume $(\alpha, \beta, c) \in D_{*,1}$ without any loss of generality, because all the statements are symmetric. Then, (s_1, t_1) is given by $s_1 = s$ and

$$t_1 = -s\sigma_3(\alpha, \beta) + t\sigma_1(\alpha, \beta).$$

(i) From the assumption, t, t_1 and $s = s_1$ are all non-negative. Besides, $s + t > 0$ and $s_1 + t_1 > 0$. Then $|s + t| - |s_1 + t_1|$ is given by

$$|s + t| - |s_1 + t_1| = t - t_1 = (s + t)\sigma_3(\alpha, \beta) + t\sigma_2(\alpha, \beta) > 0.$$

Hence $|s_1 + t_1| < |s + t|$. (ii) From the assumption, $st < 0$. Then $s_1 t_1$ is given by

$$s_1 t_1 = -s^2\sigma_3(\alpha, \beta) + st\sigma_1(\alpha, \beta) < 0.$$

Hence $(s_1, t_1) \in L_-$. (iii) From the assumption and (ii), both t and t_1 are negative, and $s = s_1$ are positive. Firstly, consider the case $s + t \geq 0$, where $\max(|s|, |t|) = \max(s, -t) = s$. Then we obtain $s = s_1 > -t_1 = |t_1|$, because

$$s_1 + t_1 = (s + t)(1 - \sigma_3(\alpha, \beta)) - t\sigma_2(\alpha, \beta) > 0.$$

Now, assume the other case $s + t < 0$, where $\max(|s|, |t|) = \max(s, -t) = -t$. Then we also obtain $-t > -t_1 = |t_1|$, because

$$t - t_1 = (s + t)\sigma_3(\alpha, \beta) + t\sigma_2(\alpha, \beta) < 0.$$

(v) In addition to (iii), $s_1 = s \leq \max(|s|, |t|)$. □

By using these results on the behaviour of (s_i, t_i), we can examine the behaviour of (α_i, β_i, c_i) in the following proposition.

Proposition 6.2. *Let $(\alpha, \beta, c) \in \Omega_2$. There exists a finite integer n such that $(\alpha_n, \beta_n, c_n) \in D_{*,2} \cup B$ or $c_n \in \{0, 1\}$.*

Proof. Suppose that $(\alpha_i, \beta_i, c_i) \in D_{*,1} \cup D_{*,3}$ for all $i \geq 0$.

In addition, if we suppose $(s_i, t_i) \in L_+$ for all $i \geq 0$, from Lemma 6.1(i), there must be an infinitely descending sequence of positive integers

$$|s + t| > |s_1 + t_1| > \cdots > |s_i + t_i| > \cdots > 0,$$

but this is impossible. Hence, there must be a finite integer n such that $(s_n, t_n) \in L_-$.

Therefore, we only have to consider the case $(s, t) \in L_-$. Then, from Lemma 6.1(ii), $(s_i, t_i) \in L_-$ for all $i \geq 0$. From Lemma 6.1(v), there exists a non-ascending infinite sequence of positive integers

$$\max(|s|, |t|) \geq \max(|s_1|, |t_1|) \geq \cdots \geq \max(|s_i|, |t_i|) \geq \cdots \geq 1.$$

Therefore, there exist integers n and K such that

$$\max(|s_n|, |t_n|) = \max(|s_{n+1}|, |t_{n+1}|) = \cdots = K \geq 1.$$

For this n, assume $(\alpha_n, \beta_n, c_n) \in D_{*,1}$. Then, because of Lemma 6.1(iii), $|t_{n+1}| < K$. Therefore $|s_{n+1}| = K$. Furthermore, the parameter $(\alpha_{n+1}, \beta_{n+1}, c_{n+1})$ must be also in $D_{*,1}$, because, if $(\alpha_{n+1}, \beta_{n+1}, c_{n+1}) \in D_{*,3}$, Lemma 6.1(iv) claims $|s_{n+2}| < K$ and $|t_{n+2}| = |t_{n+1}| < K$. Recursively, we obtain $(\alpha_i, \beta_i, c_i) \in D_{*,1}$ for all $i \geq n$. Similarly, we can show that, if $(\alpha_n, \beta_n, c_n) \in D_{*,3}$, then $(\alpha_i, \beta_i, c_i) \in D_{*,3}$ for all $i \geq n$. Therefore, if we consider $C_{j_{n+1}, j_{n+2}, \ldots}$ for (α_n, β_n), then we obtain $c_n \in C_{1,1,\ldots} \cup C_{3,3,\ldots}$. Besides, since (α_i, β_i, c_i) does not fall into the boundary, Lemma 4.2 can be applied even if $(\alpha_i, \beta_i) \notin \hat{S}$, and, in a similar way to Proposition

4.3, it can be shown that the limit of $|C_{j_{n+1},\dots,j_{n'}}|$ as n' goes to infinity is zero. This implies that $c_n \in \{0, 1\}$. □

For a double rotation $f_{(\alpha,\beta,c)}$, let us consider the sequence

$$[0,1) \supset f_{(\alpha,\beta,c)}([0,1)) \supset f^2_{(\alpha,\beta,c)}([0,1)) \supset \cdots .$$

If there exists an integer n such that

$$f^n_{(\alpha,\beta,c)}([0,1)) = f^{n+1}_{(\alpha,\beta,c)}([0,1)) = f^{n+2}_{(\alpha,\beta,c)}([0,1)) = \cdots ,$$

$f_{(\alpha,\beta,c)}$ is called a double rotation *of finite type* [1]. If otherwise, it is called a double rotation *of infinite type*.

As shown in the following theorem, Proposition 6.2 implies that every double rotation $f_{(\alpha,\beta,c)}$ for $(\alpha,\beta,c) \in \Omega_2$ is of finite type. For the class $(\alpha,\beta,c) \in \Omega_1$, the double rotation $f_{(\alpha,\beta,c)}$ is known to be of finite type, because the rank of $f_{(\alpha,\beta,c)}$ is at most 2, and it has already been shown in [1] that every ITM with the rank not more than 2 is of finite type. However, in the following proposition, we give another short proof specific to double rotations.

Proposition 6.3. *Let $(\alpha,\beta,c) \in \Omega_1$. There exists a finite integer n such that $(\alpha_n, \beta_n, c_n) \in D_{*,2} \cup B$.*

Proof. Suppose $(\alpha_i, \beta_i, c_i) \in D_{*,1} \cup D_{*,3}$ for all $i \geq 0$. Let m_0 be a positive integer such that $m_0\alpha_0$, $m_0\beta_0 \in \mathbb{Z}$. From the integer m_0, we construct a sequence of positive integers m_1, m_2, ... in the following way. For a positive integer m_i such that $m_i\alpha_i$, $m_i\beta_i \in \mathbb{Z}$, let $m_{i+1} = m_i\sigma_{j_{i+1}}(\alpha_i, \beta_i)$ if $(\alpha_i, \beta_i, c_i) \in D_{*,j_{i+1}}$. Then m_{i+1}, $m_{i+1}\alpha_{i+1}$ and $m_{i+1}\beta_{i+1}$ are also all positive integers, because $\sigma_{j_{i+1}}(\alpha_i, \beta_i)$, $\sigma_{j_{i+1}}(\alpha_i, \beta_i)\alpha_{i+1}$ and $\sigma_{j_{i+1}}(\alpha_i, \beta_i)\beta_{i+1}$ are all $\mathbb{Z}$-linear combinations of α_i, β_i and 1. Furthermore $m_{i+1} < m_i$. By induction, there must be an infinitely descending sequence of positive integers $m_0 > m_1 > \cdots > m_i > \cdots > 0$, but this is impossible. Hence, there exists a finite integer n such that $(\alpha_n, \beta_n, c_n) \in D_{*,2} \cup B$. □

Thus, for $(\alpha,\beta,c) \in \Omega_1$, the double rotation can be reduced to a rational rotation. Since a rational rotation may have two discharge numbers depending on the initial state, the double rotation also may have two discharge numbers.

The results for Ω_1 and Ω_2 can be summarized as follows.

Theorem 6.1. *Every double rotation with rotation rank $d \leq 2$ is of finite type.*

Proof. Let $(\alpha,\beta,c) \in D$. If an induced transformation of $f_{(\alpha,\beta,c)}$ is of finite type, then $f_{(\alpha,\beta,c)}$ is also of finite type. Therefore, from Proposition 6.2 and 6.3, we only have to show that $f_{(\alpha,\beta,c)}$ is of finite type if $(\alpha,\beta,c) \in D_{*,2} \cup B$ or $c \in \{0,1\}$. In the cases when $f_{(\alpha,\beta,c)}$ is a rotation, it is of finite type. Therefore, we can assume either α or β is zero, and $0 < c < 1$. Furthermore, by the symmetry, we only consider the case $\alpha = 0$. Then $f_{(\alpha,\beta,c)}$ is identity on $[0,c)$. If $\beta \notin \mathbb{Q}$, every $x \in [c,1)$ is mapped into $[0,c)$ in a finite time. If $\beta \in \mathbb{Q}$, by the map $f^n_{(\alpha,\beta,c)}$ for any n such that $n\beta \in \mathbb{Z}$, every $x \in [c,1)$ is mapped into $[0,c)$ or to x itself. Therefore, in either case, $f_{(\alpha,\beta,c)}$ is of finite type. □

Therefore, double rotations of infinite type only exist in Ω_3. For $(\alpha,\beta) \in \hat{S}$, let us construct the set Γ' from Γ by eliminating the endpoints of all the intervals $C_{j_1,\dots,j_n}$ for all finite sequences $j_1,\dots,j_n \in \{1,3\}$. Then, if $c \in \Gamma'$, the double rotation $f_{(\alpha,\beta,c)}$ is of infinite type. Otherwise, $f_{(\alpha,\beta,c)}$ is of finite type.

7. Concluding remarks

We have shown that, for every non-trivial double rotation with the parameter $(\alpha, \beta, c) \in D_*$, there exists an interval such that the induced transformation on it becomes a simple rotation or another double rotation. Although this approach does not seem directly applicable to general ITMs, it is a convenient tool at least for double rotations. In the sense that this approach is effective, double rotations seem to be an important class of ITMs.

By considering the induced transformations, we have investigated the fractal structure in the parameter space, and shown that almost every double rotation can be reduced to a rotation. The discharge number as a function of c for fixed α and β turned out to reflect the fractal structure. Continuity of the discharge number as a function of c for fixed $(\alpha, \beta) \in \hat{S}$ is an interesting open problem.

Besides, there may be some intriguing relation between double rotations and injective discontinuous piecewise-linear functions such as the function reduced from Caianiello's equation [6], for similar fractal structures in the parameter spaces are observed in both maps.

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