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Fundamental Properties of M-convex and L-convex Functions in Continuous Variables[1]

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Abstract

The concepts of M-convexity and L-convexity, introduced by Murota (1996,1998) for functions on the integer lattice, extract combinatorial structures in well-solved nonlinear combinatorial optimization problems. These concepts are extended to polyhedral convex functions and quadratic functions on the real space by Murota–Shioura (2000, 2001). In this paper, we consider a further extension to general convex functions. The main aim of this paper is to provide rigorous proofs for fundamental properties of general M-convex and L-convex functions.

Keywords: combinatorial optimization, matroid, base polyhedron, convex function, convex analysis.

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1 Introduction

In recent years, combinatorial optimization problems with nonlinear objective functions have been dealt with, and extensive studies have been done for revealing the "well-solved" structure in nonlinear combinatorial optimization problems [2, 4, 9, 10, 11, 23, 15, 16]. The concepts of M-convexity and L-convexity, introduced by Murota [12, 13, 15, 16] for functions on the integer lattice, extract combinatorial structures in well-solved nonlinear combinatorial optimization problems; subsequently, their variants called M$^\natural$-convexity and L$^\natural$-convexity were introduced by Murota–Shioura [17] and by Fujishige–Murota [8], respectively. Applications of M-/L-convexity can be found in mathematical economics with indivisible commodities [3, 21, 22], system analysis by mixed polynomial matrices [14], etc. These concepts are extended to polyhedral convex functions and quadratic functions by Murota–Shioura [18, 19], and to general convex functions [20]. It can be easily imagined that the previous results of M-/L-convexity for polyhedral convex functions and quadratic functions naturally extend to more general M-/L-convex functions. However, the proofs cannot be extended so directly to general M-/L-convex functions, but some technical difficulties such as topological issues arise. The main aim of this paper is to provide rigorous proofs of the fundamental properties of general M-convex and L-convex functions in continuous variables. The conjugacy relationship between M-convex and L-convex functions is shown in the companion paper [20] (see Section 2.4).

The organization of this paper is as follows. Definitions and examples are provided in Section 2, where we also consider the set version of M-/L-convexity in addition to M-/L-convex functions. In Sections 3.1 and 4.1 we show that closed M-/L-convex sets and closed proper positively homogeneous M-/L-convex functions have polyhedral structures. Fundamental properties of M-/L-convex functions are shown in Sections 3.2 and 4.2, along with equivalent axioms for M-/L-convex functions and local combinatorial structure of M-/L-convex functions such as directional derivatives, subdifferentials, and minimizers.

2 Preliminaries

2.1 Notation and Definitions

Throughout this paper, we assume that n is a positive integer, and put $N = \{1, 2, \ldots, n\}$. The cardinality of a finite set X is denoted by $|X|$. The characteristic vector of a subset $X \subseteq N$ is denoted by χ_X ($\in \{0,1\}^n$), i.e., $\chi_X(i) = 1$ for $i \in X$ and $\chi_X(i) = 0$ for $i \in N \setminus X$. We denote $\chi_i = \chi_{\{i\}}$ for $i \in N$, $\mathbf{0} = \chi_\emptyset$, and $\mathbf{1} = \chi_N$. The sets of reals and nonnegative reals are denoted by $\mathbf{R}$ and by $\mathbf{R}_+$, respectively. For $x = (x(i) \mid i = 1, \ldots, n) \in \mathbf{R}^n$

and $p = (p(i) \mid i = 1, \ldots, n) \in \mathbf{R}^n$, we define

$$\|x\|_1 = \sum_{i=1}^n |x(i)|, \quad x(X) = \sum_{i \in X} x(i) \ (X \subseteq N),$$
$$\langle p, x \rangle = \sum_{i=1}^n p(i)x(i).$$

For $\alpha \in \mathbf{R} \cup \{-\infty\}$ and $\beta \in \mathbf{R} \cup \{+\infty\}$ with $\alpha \leq \beta$, we define the intervals $[\alpha, \beta] = \{\gamma \in \mathbf{R} \mid \alpha \leq \gamma \leq \beta\}$ and $(\alpha, \beta) = \{\gamma \in \mathbf{R} \mid \alpha < \gamma < \beta\}$.

A set $S \subseteq \mathbf{R}^n$ is said to be *convex* if $(1-\alpha)x + \alpha y \in S$ for all $x, y \in S$ and $\alpha \in [0, 1]$, and a *polyhedron* if there exist some $\{p_i\}_{i=1}^k$ ($\subseteq \mathbf{R}^n$) and $\{\alpha_i\}_{i=1}^k$ ($\subseteq \mathbf{R}$) ($k \geq 0$) such that $S = \{x \in \mathbf{R}^n \mid \langle p_i, x \rangle \leq \alpha_i \ (\forall i)\}$.

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ be a function. The *effective domain* and the *epigraph* are given by

$$\operatorname{dom} f = \{x \in \mathbf{R}^n \mid -\infty < f(x) < +\infty\},$$
$$\operatorname{epi} f = \{(x, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq f(x)\}.$$

We denote the set of minimizers of f by $\arg\min f = \{x \in \mathbf{R}^n \mid f(x) \leq f(y) \ (\forall y \in \mathbf{R}^n)\}$, which can be the empty set. A function f is said to be *convex* if $\operatorname{epi} f$ is a convex set. If $f > -\infty$, then f is convex if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \qquad (2.1)$$

for all $x, y \in \operatorname{dom} f$ and $\alpha \in [0, 1]$. The inequality (2.1) for $\alpha = 1/2$ is called *mid-point convexity*. For a continuous function, the mid-point convexity is equivalent to the convexity.

Remark 2.1. Mid-point convexity does not imply convexity in general. It is known that there exists a discontinuous and nonconvex function $\varphi : \mathbf{R} \to \mathbf{R}$ satisfying Jensen's equation (see, e.g., [1, pp. 43–48], [24, p. 217]):

$$\varphi(\alpha) + \varphi(\beta) = 2\varphi((\alpha + \beta)/2) \qquad (\forall \alpha, \beta \in \mathbf{R}). \qquad (2.2)$$

Such φ is mid-point convex, and not convex. □

A convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *proper* if $\operatorname{dom} f \neq \emptyset$, and *closed* if $\operatorname{epi} f$ is a closed set. For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, any level set $\{x \in \mathbf{R}^n \mid f(x) \leq \eta\}$ ($\eta \in \mathbf{R}$) is a closed set, and $\arg\min f \neq \emptyset$ if $\operatorname{dom} f$ is bounded. A convex function is said to be *polyhedral* if its epigraph is a polyhedron. A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ is said to be *positively homogeneous* if $f(\alpha x) = \alpha f(x)$ for all $x \in \mathbf{R}^n$ and $\alpha > 0$.

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ be a convex function, and $x \in \operatorname{dom} f$. For $d \in \mathbf{R}^n$, the *directional derivative* of f at x with respect to d is defined by

$$f'(x; d) = \lim_{\alpha \downarrow 0} \{f(x + \alpha d) - f(x)\}/\alpha,$$

which is a positively homogeneous convex function in d with $f'(x; \mathbf{0}) = 0$. The *subdifferential* of f at x is defined as

$$\partial f(x) = \{p \in \mathbf{R}^n \mid f(y) \geq f(x) + \langle p, y - x \rangle \ (\forall y \in \mathbf{R}^n)\}.$$

2.2 Definition of M-convex Functions

We call a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *M-convex* if it is convex and satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ satisfying

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]), \quad (2.3)$$

where $\mathrm{supp}^+(x-y) = \{i \in N \mid x(i) > y(i)\}$ and $\mathrm{supp}^-(x-y) = \{i \in N \mid x(i) < y(i)\}$. An M-convex function is said to be *closed proper M-convex* if it is closed proper convex, in addition. The effective domain of a closed proper M-convex function is contained in a hyperplane $\{x \in \mathbf{R}^n \mid x(N) = r\}$ for some $r \in \mathbf{R}$.

Proposition 2.2 ([20, Prop. 2.2]). *For a closed proper M-convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$*, we have* $x(N) = y(N)$ $(\forall x, y \in \mathrm{dom}\, f)$.

In view of this, we say that a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is *M*$^\natural$*-convex* if the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) & ((x_0, x) \in \mathbf{R}^{\widehat{N}}, x_0 = -x(N)), \\ +\infty & (\text{otherwise}) \end{cases} \quad (2.4)$$

is M-convex, where $\widehat{N} = \{0\} \cup N$; we say that f is *closed proper M*$^\natural$*-convex* if it is closed proper convex, in addition. M$^\natural$-convexity of f is characterized by the following exchange property (cf. [17, 18, 19]):

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$, $\exists \alpha_0 > 0$ satisfying (2.3),

where $\chi_0 = \mathbf{0}$ by convention.

Theorem 2.3. *A closed proper convex function f is M*$^\natural$*-convex if and only if it satisfies* (M$^\natural$-EXC).

Proof. The "only if" part is obvious from the definition. The "if" part is proven later in Section 3.3. □

By definition, closed proper M$^\natural$-convex function is essentially equivalent to closed proper M-convex function, whereas the class of closed proper M$^\natural$-convex functions contains that of closed proper M-convex functions as a proper subclass. Every property of M-convex functions can be restated in terms of M$^\natural$-convex functions, and vice versa. In this paper, we primarily work with M-convex functions, making explicit statements for M$^\natural$-convex functions when appropriate.

We also define the set version of M-/M$^\natural$-convexity. We call a set $B \subseteq \mathbf{R}^n$ *M-convex* (resp. *M$^\natural$-convex*) if its indicator function $\delta_B : \mathbf{R}^n \to \{0, +\infty\}$ defined by

$$\delta_B(x) = \begin{cases} 0 & (\text{if } x \in B), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex (resp. M$^\natural$-convex). Equivalently, an M-convex set is defined as a convex set satisfying the exchange property (B-EXC):

(B-EXC) $\forall x, y \in B$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$ satisfying $x - \alpha(\chi_i - \chi_j) \in B$ and $y + \alpha(\chi_i - \chi_j) \in B$ for all $\alpha \in [0, \alpha_0]$.

An M-convex (resp., M$^\natural$-convex) set is said to be *closed M-convex (resp., closed M$^\natural$-convex)* if it is closed, in addition. In fact, closed M-/M$^\natural$-convex sets are polyhedra as shown later in Section 3.2. Hence, these concepts coincide with those of M-/M$^\natural$-convex polyhedra introduced in [18]. In other words, closed M-convex and M$^\natural$-convex sets are nothing but the base polyhedra of submodular systems [7] and generalized polymatroids [5, 6], respectively.

Proposition 2.4 (cf. [20, Prop. 2.2]). *For a closed M-convex set B, we have $x(N) = y(N)$ $(\forall x, y \in B)$.*

Remark 2.5. The property (B-EXC) alone, without closedness, does not imply convexity nor connectivity. An example is the set

$$\{(x(1), x(2)) \in \mathbf{R}^2 \mid x(1) + x(2) = 0, x(1) < 0\}$$
$$\cup\{(x(1), x(2)) \in \mathbf{R}^2 \mid x(1) + x(2) = 1, x(1) > 1\},$$

which satisfies (B-EXC); however, it is neither convex nor connected. Even if convexity assumption is added, (B-EXC) is still independent of combinatorial properties. A typical example is $\{x \in \mathbf{R}^n \mid \sum_{i=1}^n x(i)^2 < \gamma,\ x(N) = 0\}$ $(\gamma > 0)$, which is an open ball in a hyperplane. □

Remark 2.6. The property (M-EXC) alone does not imply convexity nor continuity. Consider a discontinuous and nonconvex function $\varphi : \mathbf{R} \to \mathbf{R}$ satisfying Jensen's equation (2.2), and define $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ by $\mathrm{dom}\, f = \{x \in \mathbf{R}^2 \mid x(1) + x(2) = 0\}$ and $f(x) = \varphi(x(1))$ for $x \in \mathrm{dom}\, f$. Then, f satisfies (M-EXC). □

Remark 2.7. The effective domain of a closed M-convex function is not a closed set in general. An example is the function $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ given by $\mathrm{dom}\, f = \{x \in \mathbf{R}^2 \mid x(1) + x(2) = 0,\ x(1) > 0\}$ and $f(x) = 1/x(1)$ for $x \in \mathrm{dom}\, f$. □

2.3 Definition of L-convex Functions

We call a function $g : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ *L-convex* if g is convex and satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q)$ $(\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda\mathbf{1}) = g(p) + \lambda r$ $(\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R})$,

where $p\wedge q, p\vee q \in \mathbf{R}^n$ are given by $(p\wedge q)(i) = \min\{p(i), q(i)\}$ and $(p\vee q)(i) = \max\{p(i), q(i)\}$ $(i \in N)$. An L-convex function is said to be *closed proper L-convex* if it is closed proper convex, in addition. In view of (LF2), we call $g : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{R}^{\widehat{N}} \to \mathbf{R}\cup\{+\infty\}$ defined by

$$\widehat{g}(p_0, p) = g(p - p_0\mathbf{1}) \quad ((p_0, p) \in \mathbf{R}^{\widehat{N}})$$

is L-convex, where $\widehat{N} = \{0\} \cup N$; we say that g is *closed proper L$^\natural$-convex* if it is closed proper convex, in addition. L$^\natural$-convexity of g is characterized by the following property:

(L$^\natural$F) $g(p)+g(q) \geq g(p\vee(q-\lambda\mathbf{1}))+g((p+\lambda\mathbf{1})\wedge q)$ $(\forall p, q \in \mathrm{dom}\, g,\ \forall \lambda \geq 0)$.

Theorem 2.8 (cf. [18, Th. 4.39]). *A function g is L$^\natural$-convex if and only if it is a convex function with* (L$^\natural$F)*.*

By definition, L$^\natural$-convex function is essentially equivalent to L-convex function, whereas the class of L$^\natural$-convex functions contains that of L-convex functions as a proper subclass. Every property of L-convex functions can be restated in terms of L$^\natural$-convex functions, and vice versa. In this paper, we primarily work with L-convex functions, making explicit statements for L$^\natural$-convex functions when appropriate.

We also define the set version of L-/L$^\natural$-convexity. We call a set $D \subseteq \mathbf{R}^n$ *L-convex* (resp. *L$^\natural$-convex*) if its indicator function $\delta_D : \mathbf{R}^n \to \{0, +\infty\}$ is L-convex (resp. L$^\natural$-convex). Equivalently, an L-convex set is defined as a convex set satisfying (LS1) and (LS2):

(LS1) $p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D$,
(LS2) $p \in D \Longrightarrow p + \lambda\mathbf{1} \in D$ $(\forall \lambda \in \mathbf{R})$.

An L-convex (resp., L$^\natural$-convex) set is said to be *closed L-convex (resp., closed L$^\natural$-convex)* if it is a closed set, in addition. In fact, L-/L$^\natural$-convex sets are polyhedra as shown later in Section 4.2. Hence, these concepts coincide with those of L-/L$^\natural$-convex polyhedra introduced in [18].

The properties (LS1) and (LS2), without closedness or convexity assumption, imply convexity.

Theorem 2.9. *If $D \subseteq \mathbf{R}^n$ satisfies* (LS1) *and* (LS2)*, then D is a convex set.*

Proof. For $p, q \in D$ and $\alpha \in [0,1]$, we show $(1-\alpha)p + \alpha q \in D$. By (LS2), we may assume $p \leq q$. Then, $q = p + \sum_{h=1}^{k} \lambda_h \chi_{N_h}$ holds for some $\lambda_h > 0$ and $N_h \subseteq N$ $(h = 1, 2, \ldots, k)$ such that $\emptyset \neq N_1 \subset N_2 \subset \cdots \subset N_k$. Putting $p'_j = p + \alpha \sum_{h=1}^{j} \lambda_h \chi_{N_h}$ $(j = 0, 1, \ldots, k)$, we have

$$\begin{aligned} p'_j &= p'_{j-1} + \alpha\lambda_j \chi_{N_j} = (p'_{j-1} + \alpha\lambda_j \mathbf{1}) \wedge (p + \sum_{h=1}^{j} \lambda_h \chi_{N_h}) \\ &= (p'_{j-1} + \alpha\lambda_j \mathbf{1}) \wedge (p \vee (q - \sum_{h=j+1}^{k} \lambda_h \mathbf{1})). \end{aligned}$$

Since $p'_0 = p \in D$, the equation above and L-convexity of D imply $p'_k = (1-\alpha)p + \alpha q \in D$. □

Remark 2.10. The properties (LF1) and (LF2) imply mid-point convexity (see Theorem 4.3 below); however, they do not imply convexity nor continuity. Consider a discontinuous and nonconvex function $\psi : \mathbf{R} \to \mathbf{R}$ satisfying Jensen's equation (2.2). Then, the function $g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by $g(p) = \psi(p(1) - p(2))$ $(p \in \mathbf{R}^2)$ satisfies the submodular inequality (LF1) with equality and (LF2) with $r = 0$. □

Remark 2.11. There exists no function which is both closed proper M-convex and closed proper L-convex. (Proof: Proposition 2.2 implies $x(N) = y(N)$ for any M-convex f and $x, y \in \operatorname{dom} f$, whereas (LF2) implies that $x + \lambda\mathbf{1} \in \operatorname{dom} f$ for any $\lambda \in \mathbf{R}$.) On the other hand, the classes of M$^\natural$-convex and L$^\natural$-convex functions have nonempty intersection; see Example 2.14. □

Remark 2.12. The effective domain of a closed proper L-convex function is not a closed set in general. For example, the function $g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \begin{cases} 1/(p(1) - p(2)) & (\text{if } p(1) - p(2) > 0), \\ +\infty & (\text{otherwise}) \end{cases}$$

is L-convex, and $\operatorname{dom} g = \{p \in \mathbf{R}^2 \mid p(1) - p(2) > 0\}$ is not a closed set. □

2.4 Conjugacy

For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$, its *(convex) conjugate* $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n).$$

Quite recently, it is proven that closed proper M-convex and L-convex functions are conjugate to each other, as in the cases of polyhedral and quadratic M-/L-convex functions [18, Th. 5.1], [19, Th. 4.1].

Theorem 2.13 ([20, Th. 1.1]).
(i) *If f is a closed proper M-convex function, then $f^\bullet$ is a closed proper L-convex function with $(f^\bullet)^\bullet = f$.*
(ii) *If g is a closed proper L-convex function, then $g^\bullet$ is a closed proper M-convex function with $(g^\bullet)^\bullet = g$.*
(iii) *The mappings $f \mapsto f^\bullet$ and $g \mapsto g^\bullet$ (f : M-convex, g : L-convex) are the inverses of each other, providing a one-to-one correspondence between the classes of closed proper M-convex and L-convex functions.*

2.5 Examples

We show some examples of M-/M$^\natural$-convex and L-/L$^\natural$-convex functions. See [20] for other examples.

Example 2.14 (affine functions). For $p_0 \in \mathbf{R}^n$ and $\beta \in \mathbf{R}$, the function $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ given by $f(x) = \langle p_0, x\rangle + \beta$ ($x \in \mathrm{dom}\, f$) is M-convex or M$^\natural$-convex according as $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid x(N) = 0\}$ or $\mathrm{dom}\, f = \mathbf{R}^n$. For $x_0 \in \mathbf{R}^n$ and $\nu \in \mathbf{R}$, the function $g : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ given by $g(p) = \langle p, x_0\rangle + \nu$ ($p \in \mathbf{R}^n$) is L-convex as well as L$^\natural$-convex. □

Example 2.15 (quadratic functions). Let $A = (a(i,j))_{i,j=1}^n \in \mathbf{R}^{n\times n}$ be a symmetric matrix. Define a quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x) = (1/2)x^{\mathrm{T}}Ax$ ($x \in \mathbf{R}^n$). Then, f is M$^\natural$-convex if and only if

$$x^{\mathrm{T}}a_i \geq \min[0, \min\{x^{\mathrm{T}}a_j \mid j \in \mathrm{supp}^-(x)\}]$$

for all $x \in \mathbf{R}^n$ and $i \in \mathrm{supp}^+(x)$, where a_i denotes the i-th column of A for $i \in N$. The function f is L$^\natural$-convex if and only if $\sum_{i=1}^n a(i,j) \geq 0$ ($j \in N$) and $a(i,j) \leq 0$ ($i, j \in N,\ i \neq j$). See [19]. □

Example 2.16. M-convex and L-convex functions arise from the minimum cost flow/tension problems with nonlinear cost functions.

Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Suppose that we are given a family of univariate closed convex functions $f_a : \mathbf{R} \to \mathbf{R}\cup\{+\infty\}$ ($a \in A$), representing the cost of flow on the arc a. A vector $\xi \in \mathbf{R}^A$ is called a flow, and the boundary $\partial\xi \in \mathbf{R}^V$ of a flow ξ is given by

$$\begin{aligned}\partial\xi(v) = &\textstyle\sum\{\xi(a) \mid \text{ arc } a \text{ leaves } v\}\\ &-\textstyle\sum\{\xi(a) \mid \text{ arc } a \text{ enters } v\} \qquad (v \in V).\end{aligned}$$

Then, the minimum cost of a flow that realizes a supply/demand vector $x \in \mathbf{R}^T$ is represented by a function $f : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$f(x) = \inf_{\xi} \left\{ \sum_{a\in A} f_a(\xi(a)) \;\middle|\; \begin{array}{l} (\partial\xi)(v) = -x(v) \ (v \in T) \\ (\partial\xi)(v) = 0 \ (v \in V \setminus T) \end{array} \right\}$$

for $x \in \mathbf{R}^T$. On the other hand, suppose that we are given another family of univariate closed convex functions $g_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, representing the cost of tension on the arc a. Any vector $\widetilde{p} \in \mathbf{R}^V$ is called a potential, and the coboundary $\delta\widetilde{p} \in \mathbf{R}^A$ of a potential $\widetilde{p}$ is defined by $\delta\widetilde{p}(a) = \widetilde{p}(u) - \widetilde{p}(v)$ for $a = (u, v) \in A$. Then, the minimum cost of a tension that realizes a potential vector $p \in \mathbf{R}^T$ is represented by a function $g : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$g(p) = \inf_{\eta, \widetilde{p}} \left\{ \sum_{a \in A} g_a(\eta(a)) \;\middle|\; \begin{array}{l} \eta(a) = -\delta\widetilde{p}(a) \ (a \in A) \\ \widetilde{p}(v) = p(v) \ (v \in T) \end{array} \right\}$$

for $p \in \mathbf{R}^T$. It can be shown that both f and g are closed proper convex if $f(x_0)$ and $g(p_0)$ are finite for some $x_0 \in \mathbf{R}^T$ and $p_0 \in \mathbf{R}^T$, which is a direct extension of the results in Iri [9] and Rockafellar [23] for the case of $|T| = 2$. These functions, however, are equipped with different combinatorial structures; f is M-convex and g is L-convex. See [20, Th. 2.10]. □

3 M-convex Functions

3.1 Sets and Positively Homogeneous Functions

M-convex sets and positively homogeneous M-convex functions constitute important subclasses of M-convex functions, which appear as local structure of general M-convex functions such as minimizers and directional derivative functions (see Theorem 3.10). These objects are polyhedral, as follows.

Theorem 3.1.
(i) *A closed set with* (B-EXC) *is a polyhedron. In particular, a closed M-convex set is a polyhedron.*
(ii) *A closed proper positively homogeneous M-convex function is polyhedral convex.*

We prove the claim (i) below. The proof of (ii) is given later in Section 3.2. For a nonempty set $B \subseteq \mathbf{R}^n$, we define a set function $\rho_B : 2^N \to \mathbf{R} \cup \{+\infty\}$ by $\rho_B(X) = \sup\{x(X) \mid x \in B\}$ $(X \subseteq N)$.

Lemma 3.2. *Let $B \subseteq \mathbf{R}^n$ be a nonempty bounded closed set with* (B-EXC), *and $\{X_k\}_{k=1}^h$ be subsets of N with $X_1 \subset X_2 \subset \cdots \subset X_h$. Then, there exists $x_* \in B$ with $x_*(X_k) = \rho_B(X_k)$ $(\forall k = 1, 2, \ldots, h)$.*

Proof. The claim is shown by induction on the value h. Let $x_h \in B$ be a vector with $x_h(X_h) = \rho_B(X_h)$. By the induction hypothesis, there exists a vector $x \in B$ satisfying $x(X_k) = \rho_B(X_k)$ $(k = 1, 2, \cdots, h-1)$, and assume that x minimizes the value $\|x - x_h\|_1$ among all such vectors. Suppose that $x(X_h) < \rho_B(X_h)$. By (B-EXC) and Proposition 2.4, there exist $i \in \mathrm{supp}^+(x - x_h) \setminus X_h$ and $j \in \mathrm{supp}^-(x - x_h)$ such that $x' = x - \alpha(\chi_i - \chi_j) \in B$

for a sufficiently small $\alpha > 0$. Here $j \in N \backslash X_{h-1}$ holds since $i \in N \backslash X_{h-1}$ and $x'(X_{h-1}) \leq \rho_B(X_{h-1}) = x(X_{h-1})$. Therefore, x' satisfies $x'(X_k) = \rho_B(X_k)$ $(\forall k = 1, 2, \cdots, h-1)$ and $\|x' - x_h\|_1 = \|x - x_h\|_1 - 2\alpha$, a contradiction. Therefore, $x(X_h) = \rho_B(X_h)$. □

Let $\rho : 2^N \to \mathbf{R} \cup \{+\infty\}$ be a set function. We call ρ *submodular* if it satisfies

$$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y) \tag{3.1}$$

for all $X, Y \subseteq N$ with $\rho(X), \rho(Y) < +\infty$. We define a polyhedron $\mathrm{B}(\rho) \subseteq \mathbf{R}^n$ by

$$\mathrm{B}(\rho) = \{x \in \mathbf{R}^n \mid x(X) \leq \rho(X) \ (X \subseteq N), \ x(N) = \rho(N)\}. \tag{3.2}$$

For any convex set $S \subseteq \mathbf{R}^n$, a point $x \in S$ is called an *extreme point* of S if there exist no $y_1, y_2 \in S \setminus \{x\}$ and $\alpha \in (0,1)$ such that $x = \alpha y_1 + (1-\alpha) y_2$.

Theorem 3.3 ([7, Th. 3.22]). *Let $\rho : 2^N \to \mathbf{R} \cup \{+\infty\}$ be a submodular function with $\rho(\emptyset) = 0$ and $\rho(N) < +\infty$. Then, $x \in \mathbf{R}^n$ is an extreme point of $\mathrm{B}(\rho)$ if and only if there exists $\{X_k\}_{k=0}^n \subseteq 2^N$ such that $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = N$ and $x(X_k) = \rho(X_k) < +\infty$ for all $k = 0, 1, \ldots, n$.*

Proof of Theorem 3.1 (i). Let $B \subseteq \mathbf{R}^n$ be a nonempty closed set with (B-EXC). It suffices to show $B = \mathrm{B}(\rho_B)$ since $\mathrm{B}(\rho_B)$ is a polyhedron. The inclusion $B \subseteq \mathrm{B}(\rho_B)$ is easy to see; therefore we prove the reverse inclusion.

We first assume that B is bounded. Let $X, Y \subseteq N$. From Lemma 3.2, there exists $x_* \in B$ with $x_*(X \cap Y) = \rho_B(X \cap Y)$ and $x_*(X \cup Y) = \rho_B(X \cup Y)$, which implies

$$\begin{aligned} &\rho_B(X) + \rho_B(Y) \geq x_*(X) + x_*(Y) \\ &= x_*(X \cap Y) + x_*(X \cup Y) = \rho_B(X \cap Y) + \rho_B(X \cup Y). \end{aligned}$$

Therefore, ρ_B is a submodular function. By Lemma 3.2 and Theorem 3.3, any extreme point of $\mathrm{B}(\rho_B)$ is contained in B. Hence we have $\mathrm{B}(\rho_B) \subseteq B$.

Next, assume that B is unbounded. For a fixed $x_0 \in B$, define $B_k = \{x \in B \mid x(i) - x_0(i) \leq k \ (i \in N)\}$ and put $\rho_k = \rho_{B_k}$ for $k = 0, 1, 2, \ldots$. Since each B_k is a bounded M-convex set, we have $B_k = \mathrm{B}(\rho_k)$. To prove $\mathrm{B}(\rho_B) \subseteq B$, let y be any vector in $\mathrm{B}(\rho_B)$. Then, there exists a sequence of vectors $\{x_k\}_{k=0}^\infty$ such that $x_k \in \mathrm{B}(\rho_k) \subseteq B$ $(k = 0, 1, 2, \ldots)$ and $\lim_{k \to \infty} x_k = y$. Since B is a close set, we have $y \in B$, i.e., $\mathrm{B}(\rho_B) \subseteq B$. □

3.2 Fundamental Properties of M-convex Functions

We first present two equivalent definitions of M-convex functions. For a convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$, we denote $f'(x; j, i) = f'(x; \chi_j - \chi_i)$ $(i, j \in N)$.

(M-EXC$_\mathbf{s}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$ satisfying (2.3) with $\alpha_0 = (x(i) - y(i))/2t$, where $t = |\mathrm{supp}^-(x-y)|$.
(M-EXC$'$) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$ satisfying $f'(x; j, i) < +\infty$, $f'(y; i, j) < +\infty$, and $f'(x; j, i) + f'(y; i, j) \leq 0$.

Theorem 3.4 ([20, Th. 3.10, 3.11]). *For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$,* (M-EXC), (M-EXC$_\mathrm{s}$), *and* (M-EXC$'$) *are all equivalent.*

A closed proper M$^\natural$-convex function is supermodular on $\mathbf{R}^n$.

Theorem 3.5 ([20, Prop. 3.4]). *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a closed proper convex function satisfying the property:*

$\forall x, y \in \mathrm{dom}\, f$ *with* $x \geq y$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists \alpha_0 > 0$:
$\quad f(x) + f(y) \geq f(x - \alpha\chi_i) + f(y + \alpha\chi_i) \quad (\alpha \in [0, \alpha_0])$.

Then, f satisfies the supermodular inequality:

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y) \;\; (x, y \in \mathbf{R}^n). \tag{3.3}$$

In particular, a closed proper M$^\natural$-convex function satisfies the supermodular inequality (3.3).

Corollary 3.6 ([20, Prop. 3.12]). *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a closed proper M-convex function. For any $x, y \in \mathbf{R}^n$ and $i \in N$ we have $f(x) + f(y) \leq f(\hat{x}) + f(\check{y})$, where $\hat{x}$ and $\check{y}$ are given as*

$$\hat{x}(j) = \begin{cases} \min\{x(j), y(j)\} & (j \in N \setminus \{i\}), \\ x(N) - \sum_{k \in N \setminus \{i\}} \min\{x(k), y(k)\} & (j = i), \end{cases}$$

$$\check{y}(j) = \begin{cases} \max\{x(j), y(j)\} & (j \in N \setminus \{i\}), \\ y(N) - \sum_{k \in N \setminus \{i\}} \max\{x(k), y(k)\} & (j = i). \end{cases}$$

Global optimality of an M-convex function is characterized by local optimality in terms of a finite number of directional derivatives.

Theorem 3.7. *For a closed proper M-convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$, we have $f(x) \leq f(y)$ $(\forall y \in \mathbf{R}^n)$ if and only if $f'(x; j, i) \geq 0$ $(\forall i, j \in N)$.*

Proof. We show the "if" part by contradiction. Assume, to the contrary, that $f(x_0) < f(x)$ holds for some $x_0 \in \mathrm{dom}\, f$. Put $S = \{y \in \mathbf{R}^n \mid f(y) \leq f(x_0)\}$, which is a closed set since f is closed convex. Let $x_* \in S$ be a vector with $\|x_* - x\|_1 = \inf\{\|y - x\|_1 \mid y \in S\}$. By (M-EXC) applied to x and x_*, there exist some $i \in \mathrm{supp}^+(x - x_*)$, $j \in \mathrm{supp}^-(x - x_*)$, and a sufficiently small $\alpha > 0$ such that

$$f(x_*) - f(x_* + \alpha(\chi_i - \chi_j)) \geq f(x - \alpha(\chi_i - \chi_j)) - f(x) \geq \alpha f'(x; j, i) \geq 0.$$

Hence, we have $f(x_* + \alpha(\chi_i - \chi_j)) \leq f(x_*) \leq f(x_0)$, which contradicts the choice of x_* since $\|(x_* + \alpha(\chi_i - \chi_j)) - x\|_1 < \|x_* - x\|_1$. $\square$

The following property means that a given point can be easily separated from some minimizer of a closed proper M-convex function.

Theorem 3.8. *Let* $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ *be a closed proper M-convex function with* $\arg\min f \neq \emptyset$.
(i) *For* $x \in \mathrm{dom}\, f$ *and* $j \in N$, *let* $i \in N$ *be such that* $f'(x; j, i) = \min_{s\in N} f'(x; j, s)$. *Then, there exists* $x^* \in \arg\min f$ *with* $x^*(i) \leq x(i)$.
(ii) *For* $x \in \mathrm{dom}\, f$ *and* $i \in N$, *let* $j \in N$ *be such that* $f'(x; j, i) = \min_{t\in N} f'(x; t, i)$. *Then, there exists* $x^* \in \arg\min f$ *with* $x^*(j) \geq x(j)$.
(iii) *For* $x \in \mathrm{dom}\, f$, *let* $i, j \in N$ *be such that* $f'(x; j, i) = \min_{s,t\in N} f'(x; t, s)$. *Then, there exists* $x^* \in \arg\min f$ *with* $x^*(i) \leq x(i)$ *and* $x^*(j) \geq x(j)$.

Proof. (i) Assume, to the contrary, that there is no $x^* \in \arg\min f$ with $x^*(i) \leq x(i)$. Let x^* be an element of $\arg\min f$ with $x^*(i)$ being minimum. Then, we have $x^*(i) > x(i)$. By applying (M-EXC) to x^*, x, and i we obtain some $k \in \mathrm{supp}^-(x^* - x)$ and $\alpha > 0$ such that

$$f(x^*) + f(x) \geq f(x^*-\alpha(\chi_i-\chi_k)) + f(x+\alpha(\chi_i-\chi_k)).$$

By the choice of x^*, we have $f(x^*-\alpha(\chi_i-\chi_k)) > f(x^*)$, and hence $f(x + \alpha(\chi_i-\chi_k)) < f(x)$. This inequality and the convexity of f yield $f'(x; i, k) < 0$. By Theorem 3.10 (i) (to be shown below), it holds that $f'(x; j, k) \leq f'(x; j, i) + f'(x; i, k) < f'(x; j, i)$, a contradiction to the choice of i.

(ii) The proof is similar to that for (i).

(iii) By (i) there exists $x^* \in \arg\min f$ with $x^*(i) \leq x(i)$; we assume that x^* maximizes $x^*(j)$ among all such vectors. If $x^*(j) \geq x(j)$ is not satisfied, (M-EXC) applies to x, x^*, and j, to yield some $k \in \mathrm{supp}^-(x-x^*)$ and $\alpha > 0$ such that

$$f(x) + f(x^*) \geq f(x-\alpha(\chi_j-\chi_k)) + f(x^*+\alpha(\chi_j-\chi_k)).$$

By the choice of x^*, we have $f(x^* + \alpha(\chi_j - \chi_k)) > f(x^*)$, and hence $f(x - \alpha(\chi_j - \chi_k)) < f(x)$. This inequality and the convexity of f yield $f'(x; k, j) < 0$. From Theorem 3.10 (i) follows $f'(x; k, i) \leq f'(x; k, j) + f'(x; j, i) < f'(x; j, i)$, a contradiction to the choice of i, j. □

The directional derivative functions and subdifferentials of an M-convex function have nice combinatorial structures such as polyhedral M-/L-convexity. For $p \in \mathbf{R}^n$, we define $f[p] : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ by $f[p](x) = f(x) + \langle p, x\rangle$ $(x \in \mathbf{R}^n)$. For a function $\gamma : N \times N \to \mathbf{R}\cup\{+\infty\}$, we define a set $\mathrm{D}(\gamma) \subseteq \mathbf{R}^n$ by

$$\mathrm{D}(\gamma) = \{p \in \mathbf{R}^n \mid p(j) - p(i) \leq \gamma(i, j) \ (i, j \in N)\}, \tag{3.4}$$

and a function $f_\gamma : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_\gamma(x) = \inf\{ \sum_{(i,j)\in A} \lambda_{ij}\gamma(i, j) \mid \sum_{(i,j)\in A} \lambda_{ij}(\chi_j - \chi_i) = x,\ \lambda_{ij} \geq 0 \ ((i, j) \in A)\}$$

for $x \in \mathbf{R}^n$, where $A = \{(i, j) \mid i, j \in N,\ \gamma(i, j) < +\infty\}$.

Lemma 3.9. *Let* $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ *be a closed proper M-convex function,* $x_0, y_0 \in \operatorname{dom} f$, $i \in \operatorname{supp}^+(x_0-y_0)$, *and* $\operatorname{supp}^-(x_0-y_0) = \{j_1, j_2, \cdots, j_t\}$, *where* $t = |\operatorname{supp}^-(x_0-y_0)|$. *Then, there exist* $y_h \in \operatorname{dom} f$ *and* $\alpha_h \in \mathbf{R}_+$ $(h = 1, \ldots, t)$ *satisfying* $\sum_{h=1}^{t} \alpha_h = x_0(i)-y_0(i)$, $y_h = y_{h-1}+\alpha_h(\chi_i-\chi_{j_h})$, *and*

$$f(y_h) \leq f(y_{h-1}) - \alpha_h f'(x; j_h, i) \ (h = 1, \ldots, t). \tag{3.5}$$

Proof. For each $h = 1, 2, \cdots, t$, we recursively define $\alpha_h \in \mathbf{R}$ and $y_h \in \mathbf{R}^n$ by

$$\begin{aligned}\alpha_h \;=\; & \sup\{\alpha \mid y_{h-1} + \alpha(\chi_i - \chi_{j_h}) \in \operatorname{dom} f, \\ & \alpha \leq \min(x(i) - y_{h-1}(i), y_{h-1}(j_h) - x(j_h)), \\ & f(y_{h-1} + \alpha(\chi_i - \chi_{j_h})) \leq f(y_{h-1}) - \alpha f'(x; j_h, i)\}\end{aligned}$$

and $y_h = y_{h-1} + \alpha_h(\chi_i - \chi_{j_h})$. Then, (3.5) follows from the definition of y_h and closed convexity of f. Assume, to the contrary, that $\sum_{h=1}^{t} \alpha_h < x_0(i) - y_0(i)$. Since $i \in \operatorname{supp}^+(x_0 - y_t)$, there exist $j_h \in \operatorname{supp}^-(x_0 - y_t) \subseteq \operatorname{supp}^-(x_0 - y_0)$ and a sufficiently small $\alpha > 0$ such that

$$f(x_0) + f(y_t) \geq f(x_0 - \alpha(\chi_i - \chi_{j_h})) + f(y_t + \alpha(\chi_i - \chi_{j_h})).$$

By Corollary 3.6, we obtain

$$f(y_h + \alpha(\chi_i - \chi_{j_h})) + f(y_t) \leq f(y_t + \alpha(\chi_i - \chi_{j_h})) + f(y_h).$$

Combining the two inequalities, we have

$$f(y_h + \alpha(\chi_i - \chi_{j_h})) - f(y_h) \leq f(x_0) - f(x_0 - \alpha(\chi_i - \chi_{j_h})) \leq -\alpha f'(x_0; j_h, i).$$

However, this contradicts the definition of y_h. □

Theorem 3.10. *Let* $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ *be a closed proper M-convex function and* $x \in \operatorname{dom} f$. *Define* $\gamma_x : N \times N \to \mathbf{R}\cup\{\pm\infty\}$ *by* $\gamma_x(i, j) = f'(x; j, i)$ $(i, j \in N)$.
(i) γ_x *satisfies* $\gamma_x(i, i) = 0$ $(i \in N)$ *and the triangle inequality* $\gamma_x(i, j) + \gamma_x(j, k) \geq \gamma_x(i, k)$ $(i, j, k \in N)$.
(ii) $\partial f(x)$ *satisfies* $\partial f(x) = \mathrm{D}(\gamma_x)$, *and is a closed L-convex set if* $\gamma_x > -\infty$.
(iii) $f'(x; \cdot)$ *satisfies* $f'(x; \cdot) = f_{\gamma_x}$, *and is closed proper positively homogeneous M-convex if* $f'(x; \cdot) > -\infty$.

Proof. (i): For $i, j, k \in N$, Corollary 3.6 implies

$$\begin{aligned}\gamma_x(i, j) + \gamma_x(j, k) \;&=\; \lim_{\alpha \downarrow 0}[f(x+\alpha(\chi_j-\chi_i)) + f(x+\alpha(\chi_k-\chi_j)) - 2f(x)]/\alpha \\ &\geq\; \lim_{\alpha \downarrow 0}[f(x + \alpha(\chi_k - \chi_i)) - f(x)]/\alpha = \gamma_x(i, k).\end{aligned}$$

(ii): Since $p \in \partial f(x)$ is equivalent to $x \in \arg\min f[-p]$, the equation $\partial f(x) = \mathrm{D}(\gamma_x)$ follows from Theorem 3.7. L-convexity of the set $\partial f(x)$ follows from (i) (see [18, Th. 3.23]).

(iii): Due to the property (i), it suffices to prove $f'(x; d) = f_{\gamma_x}(d)$ for $d \in \mathbf{R}^n$ (see [18, Th. 4.19]). Since $f'(x; \cdot)$ is positively homogeneous convex, we have

$$f'(x; d) = f'(x; \sum_{i,j \in N} \lambda_{ij}(\chi_j - \chi_i)) \leq \sum_{i,j \in N} \lambda_{ij} f'(x; j, i)$$

for any $\{\lambda_{ij}\}_{i,j \in N}$ ($\subseteq \mathbf{R}_+$) satisfying $\sum_{i,j \in N} \lambda_{ij}(\chi_j - \chi_i) = d$. Hence, we have $f'(x; d) \leq f_{\gamma_x}(d)$. On the other hand, repeated application of Lemma 3.9 implies that for any $\alpha > 0$ there exists $\{\lambda_{ij}\}_{i,j \in N}$ ($\subseteq \mathbf{R}_+$) satisfying $\sum_{i,j \in N} \lambda_{ij}(\chi_j - \chi_i) = d$ and

$$f(x + \alpha d) - f(x) \geq \alpha \sum_{i,j \in N} \lambda_{ij} f'(x; j, i) \geq \alpha f_{\gamma_x}(d).$$

This implies $f'(x; d) \geq f_{\gamma_x}(d)$. □

Proof of Theorem 3.1 (ii). Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be closed proper positively homogeneous M-convex, and define $\gamma : N \times N \to \mathbf{R} \cup \{+\infty\}$ by $\gamma(i, j) = f(\chi_j - \chi_i)$ $(i, j \in N)$. Then, it suffices to show that $f(x) = f_\gamma(x)$ holds for any $x \in \mathbf{R}^n$. Since $f(y) = f'(\mathbf{0}; y)$ for $y \in \mathbf{R}^n$, this follows immediately from Theorem 3.10 (iii). □

The next theorem shows that each "face" of the epigraph of a closed proper M-convex function is an M-convex polyhedron. The proof is easy.

Theorem 3.11. *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be an M-convex (resp. closed proper M-convex) function. For $p \in \mathbf{R}^n$, $\arg\min f[p]$ is M-convex (resp. closed M-convex) if it is nonempty.*

3.3 Proof of Theorem 2.3

We give a proof of Theorem 2.3, a characterization of closed proper M$^\natural$-convex functions by (M$^\natural$-EXC).

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a closed proper convex function satisfying (M$^\natural$-EXC). We shall derive the M-convexity of the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ in (2.4), which is equivalent to the L-convexity of $\widehat{f}^\bullet = (\widehat{f})^\bullet$ by Theorem 2.13. The property (LF2) for $\widehat{f}^\bullet$ is immediate from the definition (2.4) of $\widehat{f}$. Hence, it remains to show the submodularity (LF1) for $\widehat{f}^\bullet$. We here assume that $\mathrm{dom}\, \widehat{f}$ is bounded, since the case of unbounded $\mathrm{dom}\, \widehat{f}$ can be easily reduced to the bounded case (see [20]). Since the boundedness of

$\mathrm{dom}\,\widehat{f}$ implies $\mathrm{dom}\,\widehat{f}^{\bullet} = \mathbf{R}^{\widehat{N}}$, the submodularity of $\widehat{f}^{\bullet}$ is equivalent to the local submodularity (see, e.g., [18, Th. 4.27]):

$$\widehat{f}^{\bullet}(\widehat{p}+\lambda\chi_i) + \widehat{f}^{\bullet}(\widehat{p}+\mu\chi_j) \geq \widehat{f}^{\bullet}(\widehat{p}) + \widehat{f}^{\bullet}(\widehat{p}+\lambda\chi_i+\mu\chi_j), \tag{3.6}$$

where $\widehat{p} \in \mathbf{R}^{\widehat{N}}$, $i, j \in \widehat{N}$ are distinct indices, and λ, μ are nonnegative reals. To show the local submodularity (3.6) for $\widehat{f}^{\bullet}$, we fix $\widehat{p} \in \mathbf{R}^{\widehat{N}}$ and $i, j \in \widehat{N}$, and define functions $\widehat{g}, \widehat{f} : \mathbf{R}^2 \to \mathbf{R}\cup\{+\infty\}$ by

$$\begin{aligned} \widehat{g}_{ij}(\lambda,\mu) &= \widehat{f}^{\bullet}(\widehat{p}+\lambda\chi_i+\mu\chi_j) \qquad (\lambda,\mu \in \mathbf{R}), & (3.7)\\ \widehat{f}_{ij}(\alpha,\beta) &= \inf\{\widehat{f}(\widehat{x}) - \langle \widehat{p}, \widehat{x}\rangle \mid \widehat{x} \in \mathrm{dom}\,\widehat{f},\ \widehat{x}(i)=\alpha,\ \widehat{x}(j)=\beta\} \\ &\qquad\qquad (\alpha,\beta \in \mathbf{R}). & (3.8) \end{aligned}$$

Then, $\widehat{g}_{ij} = (\widehat{f}_{ij})^{\bullet}$ (see [20, Lemma 3.2]), and (M$^\natural$-EXC) for f implies supermodularity of $\widehat{f}_{ij}$, as shown below. Supermodularity of $\widehat{f}_{ij}$ is equivalent to submodularity of $(\widehat{f}_{ij})^{\bullet}$ ([20, Prop. 3.6]), which in turn implies local submodularity (3.6).

We now show that (M$^\natural$-EXC) for f implies supermodularity of $\widehat{f}_{ij}$. By Theorem 3.5, it suffices to show that $\widehat{f}_{ij}$ has the following property:

$\forall(\alpha,\beta),(\alpha',\beta') \in \mathrm{dom}\,\widehat{f}_{ij}$ with $\alpha > \alpha'$, $\beta \geq \beta'$, $\exists\delta_0 > 0$:
$\quad \widehat{f}_{ij}(\alpha,\beta) + \widehat{f}_{ij}(\alpha',\beta') \geq \widehat{f}_{ij}(\alpha-\delta,\beta) + \widehat{f}_{ij}(\alpha'+\delta,\beta') \quad (\forall\delta \in [0,\delta_0])$.

To prove this, we may assume $\widehat{p} = \mathbf{0}$ since $\widehat{f}(\widehat{x}) - \langle \widehat{p}, \widehat{x}\rangle$ also satisfies (M$^\natural$-EXC) as a function in $\widehat{x}$. We consider the case of $i \in N$, $j = 0$ only, since the other cases can be dealt with similarly. In this case, $\widehat{f}_{ij}$ is rewritten as

$$\widehat{f}_{ij}(\alpha,\beta) = \inf\{f(x) \mid x \in \mathrm{dom}\,f,\ x(i)=\alpha,\ x(N)=-\beta\}.$$

Since f is a closed proper convex function with bounded effective domain, there exist $x, x' \in \mathrm{dom}\,f$ satisfying $x(i) = \alpha$, $x(N) = -\beta$, $\widehat{f}_{ij}(\alpha,\beta) = f(x)$, and $x'(i) = \alpha'$, $x'(N) = -\beta'$, $\widehat{f}_{ij}(\alpha',\beta') = f(x')$. We assume that x minimizes the value $||x - x'||_1$ among all such vectors. It suffices to show that

$$f(x) + f(x') \geq f(x - \delta(\chi_i - \chi_k)) + f(x' + \delta(\chi_i - \chi_k)) \quad (\forall\delta \in [0,\delta_0]) \tag{3.9}$$

for some $k \in \mathrm{supp}^-(x - x')$ and $\delta_0 > 0$ since the RHS of (3.9) is larger than or equal to $\widehat{f}_{ij}(\alpha-\delta,\beta) + \widehat{f}_{ij}(\alpha'+\delta,\beta')$. By (M$^\natural$-EXC) for x, x', and $i \in \mathrm{supp}^+(x - x')$, there exist $s \in \mathrm{supp}^-(x - x') \cup \{0\}$ and $\delta_1 > 0$ with

$$f(x) + f(x') \geq f(x - \delta(\chi_i - \chi_s)) + f(x' + \delta(\chi_i - \chi_s)) \quad (\forall\delta \in [0,\delta_1]). \tag{3.10}$$

Since $x(i) > x'(i)$ and $x(N) \leq x'(N)$, there exists some $r \in \mathrm{supp}^+(x' - x)$. By (M$^\natural$-EXC), there exist $t \in \mathrm{supp}^-(x' - x) \cup \{0\}$ and $\delta_2 > 0$ such that

$$f(x') + f(x) \geq f(x' - \delta(\chi_r - \chi_t)) + f(x + \delta(\chi_r - \chi_t)) \quad (\forall\delta \in [0,\delta_2]). \tag{3.11}$$

We consider the following four cases.

[Case 1: $s \in \mathrm{supp}^-(x - x')$] (3.10) gives the inequality (3.9) with $k = s$ and $\delta_0 = \delta_1$.

[Case 2: $t = i \in \mathrm{supp}^-(x' - x)$] (3.11) gives the inequality (3.9) with $k = r$ and $\delta_0 = \delta_2$.

[Case 3: $t \in \mathrm{supp}^-(x' - x) \setminus \{i\}$] Putting $x^\delta = x + \delta(\chi_r - \chi_t)$ with a sufficiently small $\delta > 0$, we have $x^\delta(i) = \alpha$, $x^\delta(N) = -\beta$, and $||x^\delta - x'||_1 < ||x - x'||_1$. By (3.11), the vector x^δ satisfies $\widehat{f}_{ij}(\alpha, \beta) = f(x^\delta)$, a contradiction to the choice of x.

[Case 4: $s = t = 0$] Convexity of f as well as the inequalities (3.10) and (3.11) implies

$$\begin{aligned} f(x) + f(x') &\geq [f(x - \delta\chi_i) + f(x + \delta\chi_r)]/2 + [f(x' + \delta\chi_i) + f(x' - \delta\chi_r)]/2 \\ &\geq f(x - (\delta/2)(\chi_i - \chi_r)) + f(x' + (\delta/2)(\chi_i - \chi_r)) \end{aligned}$$

for any $\delta \in [0, \min\{\delta_1, \delta_2\}]$. Hence, we have the inequality (3.9) with $k = r$ and $\delta_0 = (1/2)\min\{\delta_1, \delta_2\}$.

4 L-convex Functions

4.1 Sets and Positively Homogeneous Functions

L-convex sets and positively homogeneous L-convex functions constitute important subclasses of L-convex functions, which appear as local structure of general L-convex functions such as minimizers and directional derivative functions (see Theorem 4.6). These objects are polyhedral, as follows.

Theorem 4.1.
(i) *A closed set with* (LS1) *and* (LS2) *is a polyhedron. In particular, a closed L-convex set is a polyhedron.*
(ii) *A positively homogeneous L-convex function is a polyhedral convex function.*

We prove the claim (i) below. The proof of (ii) is given later in Section 4.2. Recall the definition of the set $\mathrm{D}(\gamma) \subseteq \mathbf{R}^n$ in (3.4).

Proof of Theorem 4.1 (i). Let $D \subseteq \mathbf{R}^n$ be a nonempty closed set with (LS1) and (LS2). We define a function $\gamma_D : N \times N \to \mathbf{R} \cup \{+\infty\}$ by $\gamma_D(i, j) = \sup_{p \in D}\{p(j) - p(i)\}$ $(i, j \in N)$. To prove (i), it suffices to show $D = \mathrm{D}(\gamma_D)$ since $\mathrm{D}(\gamma_D)$ is a polyhedron. The inclusion $D \subseteq \mathrm{D}(\gamma_D)$ is easy to see; to show the reverse inclusion, we prove that $q \in D$ holds for any $q \in \mathrm{D}(\gamma_D)$.

For any $\varepsilon > 0$ and any $i, j \in N$, there exists $p^\varepsilon_{ij} \in D$ with $p^\varepsilon_{ij}(j) - p^\varepsilon_{ij}(i) + \varepsilon > \gamma_D(i, j) \geq q(j) - q(i)$, where we may assume that $p^\varepsilon_{ij}(i) = q(i)$ and $p^\varepsilon_{ij}(j) + \varepsilon > q(j)$ by (LS2). For $i \in N$, we define $p^\varepsilon_i = \bigvee_{i \in N} p^\varepsilon_{ij}$. Then, p^ε_i satisfies $p^\varepsilon_i \in D$ by (LS1), $p^\varepsilon_i(i) = q(i)$ and $p^\varepsilon_i(j) + \varepsilon > q(j)$ for $j \in N$. We then define $p^\varepsilon \in D$ by $p^\varepsilon = \bigwedge_{i \in N} p_i^\varepsilon$, which satisfies $q(i) - \varepsilon < p^\varepsilon(i) \leq q(i)$ for $i \in N$, and hence $\lim_{\varepsilon \to 0} p^\varepsilon = q$. Hence, the closedness of D implies $q \in D$. □

4.2 Fundamental Properties of L-convex Functions

We show various properties of L-convex functions. Note that some of the properties below are implied by (LF1) and (LF2) only, and independent of closed convexity.

Lemma 4.2 (cf. [18, Lemma 4.28]). *A function* $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (LF1) *and* (LF2) *if and only if*

$$g(p) + g(q) \geq g(p \vee (q - \lambda \mathbf{1})) + g((p + \lambda \mathbf{1}) \wedge q)$$

for all $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in \mathbf{R}$. *In particular, if* g *satisfies* (LF1) *and* (LF2), *then we have*

$$g(p) + g(q) \geq g(p + \lambda \chi_X) + g(q - \lambda \chi_X) \qquad (4.1)$$

for all $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in [0, \lambda_1 - \lambda_2]$, *where*

$$\begin{aligned} \lambda_1 &= \max\{q(i) - p(i) \mid i \in N\}, \\ X &= \{i \in N \mid q(i) - p(i) = \lambda_1\}, \\ \lambda_2 &= \max\{q(i) - p(i) \mid i \in N \setminus X\}. \end{aligned}$$

Theorem 4.3. *If* $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (LF1) *and* (LF2), *then* g *is mid-point convex.*

Proof. We show the inequality (2.1) with $\alpha = 1/2$ by induction on the cardinality of $\mathrm{supp}(p - q) \equiv \mathrm{supp}^+(p - q) \cup \mathrm{supp}^-(p - q)$. We may assume $p(i) < q(i)$ for some $i \in N$. Putting $\lambda = (q(i) - p(i))/2$, $p' = p \vee (q - \lambda \mathbf{1})$ and $q' = (p + \lambda \mathbf{1}) \wedge q$, we have $g(p) + g(q) \geq g(p') + g(q')$ by Lemma 4.2. Since $\mathrm{supp}(p' - q') \subseteq \mathrm{supp}(p - q) \setminus \{i\}$, the induction hypothesis yields $g(p') + g(q') \geq 2g([p' + q']/2) = 2g([p + q]/2)$. □

For a twice continuously differentiable function on $\mathbf{R}^n$, L-convexity can be characterized by its Hessian matrix (cf. Example 2.15).

Theorem 4.4. *Let* $g : \mathbf{R}^n \to \mathbf{R}$ *be a twice continuously differentiable function defined on* $\mathbf{R}^n$. *Then,* g *is L-convex if and only if the Hessian matrix* $H(p) = (\partial^2 g(p) / \partial p(i) \partial p(j))_{i,j=1}^n \in \mathbf{R}^{n \times n}$ *satisfies the following conditions for all* $p \in \mathbf{R}^n$*:*

$$(H(p))_{ij} \leq 0 \qquad (\forall i, j \in N \text{ with } i \neq j), \qquad (4.2)$$

$$\sum_{i=1}^n (H(p))_{ij} = 0 \qquad (\forall j \in N). \qquad (4.3)$$

Proof. It is well known that the conditions (4.2) and (4.3) for $H(p)$ imply the positive semidefiniteness of $H(p)$, and hence the convexity of g. Hence, it suffices to show the following claim:

Claim: Suppose that g is a convex function.
(i) g satisfies (LF1) if and only if $H(p)$ satisfies (4.2) for all $p \in \mathbf{R}^n$.
(ii) g satisfies (LF2) if and only if $H(p)$ satisfies (4.3) for all $p \in \mathbf{R}^n$.

We first prove (i). Suppose that g satisfies (LF1), and let $p \in \mathbf{R}^n$ and $i, j \in N$ be with $i \neq j$. Then,

$$\begin{aligned}(H(p))_{ij} &= \lim_{\lambda \downarrow 0} \big[\lim_{\mu \downarrow 0}[g(p+\lambda\chi_i+\mu\chi_j) - g(p+\lambda\chi_i)]/\mu \\ &\qquad\qquad - \lim_{\mu \downarrow 0}[g(p+\mu\chi_j) - g(p)]/\mu\big]/\lambda \\ &= \lim_{\lambda \downarrow 0}\lim_{\mu \downarrow 0}[g(p+\lambda\chi_i+\mu\chi_j) - g(p+\lambda\chi_i) - g(p+\mu\chi_j) + g(p)]/\lambda\mu \\ &\leq 0,\end{aligned}$$

i.e., (4.2) holds.

We then assume that $H(p)$ satisfies (4.2) for all $p \in \mathbf{R}^n$, and show the submodularity of g, which is equivalent to the following local submodularity:

$$\begin{aligned}&g(p + \lambda\chi_i) + g(p + \mu\chi_j) \geq g(p + \lambda\chi_i + \mu\chi_j) + g(p) \\ &\qquad (p \in \mathbf{R}^n,\ i, j \in N \text{ with } i \neq j,\ \lambda, \mu > 0).\end{aligned} \tag{4.4}$$

Assume, to the contrary, that

$$g(p + \lambda_0\chi_i) + g(p + \mu_0\chi_j) < g(p + \lambda_0\chi_i + \mu_0\chi_j) + g(p) \tag{4.5}$$

holds for some $p \in \mathbf{R}^n$, $i, j \in N$ with $i \neq j$, and $\lambda_0, \mu_0 > 0$. We define a function $\varphi : \mathbf{R} \to \mathbf{R}$ by

$$\varphi(\mu) = g(p + \lambda_0\chi_i + \mu\chi_j) - g(p + \mu\chi_j) \qquad (\mu \in \mathbf{R}).$$

Since φ is continuous and differentiable on $[0, \mu_0]$, it follows from the mean value theorem that there exists some $\mu_* \in (0, \mu_0)$ such that

$$\begin{aligned}&\partial g(p + \lambda_0\chi_i + \mu_*\chi_j)/\partial p(j) - \partial g(p + \mu_*\chi_j)/\partial p(j) \\ &\quad = \varphi'(\mu_*) = [\varphi(\mu_0) - \varphi(0)]/\mu_0 > 0,\end{aligned} \tag{4.6}$$

where the inequality is by (4.5). We then define a function $\psi : [0, \lambda_0] \to \mathbf{R}$ by

$$\psi(\lambda) = \partial g(p + \lambda\chi_i + \mu_*\chi_j)/\partial p(j) \qquad (\lambda \in [0, \lambda_0]).$$

Since ψ is continuous and differentiable on $[0, \lambda_0]$, it follows from the mean value theorem that there exists some $\lambda_* \in (0, \lambda_0)$ satisfying

$$(H(p + \lambda_*\chi_i + \mu_*\chi_j))_{ij} = \psi'(\lambda_*) = [\psi(\lambda_0) - \psi(0)]/\lambda_0 > 0,$$

where the inequality is by (4.6). This, however, is a contradiction to the assumption (4.2). Hence, (4.4) holds and therefore g is a submodular function.

We then prove the claim (ii). For a fixed $p \in \mathbf{R}^n$, define a function $\omega : \mathbf{R} \to \mathbf{R}$ by $\omega(\lambda) = g(p+\lambda\mathbf{1})$ $(\lambda \in \mathbf{R})$. Since ω is also a twice continuously differentiable function, we have $\omega''(\lambda) = \mathbf{1}^{\mathrm{T}} H(p)\mathbf{1}$ $(\lambda \in \mathbf{R})$.

If g satisfies (LF2), then ω is an affine function. This implies that $\omega''(\lambda) = 0$ for all $\lambda \in \mathbf{R}$, from which follows $\mathbf{1}^{\mathrm{T}} H(p)\mathbf{1} = 0$. Since g is convex, $H(p)$ is positive semidefinite. Hence, we have (4.3).

On the other hand, suppose that the Hessian $H(p)$ satisfies the condition (4.3). Then, we have $\mathbf{1}^{\mathrm{T}} H(p)\mathbf{1} = 0$. This shows that ω is an affine function, i.e., there exist some $r_p \in \mathbf{R}$ such that $g(p + \lambda\mathbf{1}) = g(p) + \lambda r_p$ $(\lambda \in \mathbf{R})$. By the convexity of g, we have

$$\begin{aligned} -\infty < g([p+q]/2) &\le \inf_{\lambda \in \mathbf{R}} \{g(p+\lambda\mathbf{1}) + g(q-\lambda\mathbf{1})\} \\ &= g(p) + g(q) + \inf_{\lambda \in \mathbf{R}} \lambda(r_p - r_q) \end{aligned}$$

for any $p, q \in \mathbf{R}^n$, implying $r_p = r_q$. This concludes that (LF2) holds for g. □

Global optimality of an L-convex function is characterized by local optimality in terms of a finite number of directional derivatives.

Theorem 4.5 (cf. [18, Th. 4.29]). *Let $g : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ be an L-convex function. For $p \in \mathrm{dom}\, g$, we have $g(p) \le g(q)$ $(\forall q \in \mathbf{R}^n)$ if and only if $g'(p; \chi_X) \ge 0$ $(\forall X \subset N)$ and $g'(p; \mathbf{1}) = 0$.*

The directional derivative functions and subdifferentials of an L-convex function have nice combinatorial structures such as polyhedral L-/M-convexity. For $x \in \mathbf{R}^n$, we define $g[x]$: $\mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ by $g[x](p) = g(p) + \langle p, x\rangle$ $(p \in \mathbf{R}^n)$. For a set function $\rho : 2^N \to \mathbf{R}\cup\{\pm\infty\}$, we define $g_\rho : \mathbf{R}^n \to \mathbf{R}\cup\{\pm\infty\}$ by

$$g_\rho(p) = \sum_{j=1}^{k} (\lambda_j - \lambda_{j+1})\rho(N_j), \tag{4.7}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ are distinct values in $\{p(i)\}_{i \in N}$, $\lambda_{k+1} = 0$, and $N_j = \{i \in N \mid p(i) \ge \lambda_j\}$ $(j = 1, 2, \ldots, k)$. The function g_ρ is called the *Lovász extension* of ρ [7, 10]. Recall the definition of $\mathrm{B}(\rho)$ in (3.2).

Theorem 4.6. *Let $g : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ be a closed proper L-convex function and $q \in \mathrm{dom}\, g$. Define $\rho_q : 2^N \to \mathbf{R} \cup \{\pm\infty\}$ by $\rho_q(X) = g'(q; \chi_X)$ $(X \subseteq N)$.*
(i) *ρ_q satisfies $\rho_q(\emptyset) = 0$, $-\infty < \rho_q(N) < +\infty$, and the submodular inequality* (3.1).
(ii) *$\partial g(q)$ satisfies $\partial g(q) = \mathrm{B}(\rho_q)$, and is a closed M-convex set if $\rho_q > -\infty$.*

(iii) $g'(q;\cdot)$ *satisfies* (LF1), (LF2), *and* $g'(q;\cdot) = g_{\rho_q}$. *In particular,* $g'(q;\cdot)$ *is a closed proper positively homogeneous L-convex function if* $g'(q;\cdot) > -\infty$.

Proof. We first prove (iii) and then (ii); (i) is immediate from (iii).

(iii): For any $p \in \mathbf{R}^n$ and $\varepsilon > 0$, there exists some $\mu > 0$ such that $g'(q;p) > (1/\mu)\{g(q+\mu p) - g(q)\} - \varepsilon$. Hence, (LF1) and (LF2) for $g'(q;\cdot)$ follow from (LF1) and (LF2) for g.

We then prove $g'(q;p) = g_{\rho_q}(p)$ $(p \in \mathbf{R}^n)$. Since $g'(q;\cdot)$ is positively homogeneous convex, we have

$$g_{\rho_q}(p) = \sum_{j=1}^{k}(\lambda_j - \lambda_{j+1})g'(q;\chi_{N_j}) \geq g'(q;\sum_{j=1}^{k}(\lambda_j - \lambda_{j+1})\chi_{N_j}) = g'(q;p),$$

where λ_j and N_j $(j = 1,2,\ldots,k)$ are defined as in the Lovász extension (4.7) and $\lambda_{k+1} = 0$. Put $q_0 = q + p$ and $q_j = q_{j-1} - (\lambda_j - \lambda_{j+1})\chi_{N_j}$ for $j = 1,2,\cdots,k$, where $q_k = q$. By Lemma 4.2, we obtain

$$g(q_{j-1}) - g(q_j) \geq g(q + (\lambda_j - \lambda_{j+1})\chi_{N_j}) - g(q) \geq (\lambda_j - \lambda_{j+1})g'(q;\chi_{N_j})$$

for $j = 1,2,\ldots,k$, from which follows

$$g(q+p) - g(q) \geq \sum_{j=1}^{k}(\lambda_j - \lambda_{j+1})g'(q;\chi_{N_j}) = g_{\rho_q}(p).$$

Since the inequality above holds for any $p \in \mathbf{R}^n$, we have $g(q+\mu p) - g(q) \geq g_{\rho_q}(\mu p) = \mu g_{\rho_q}(p)$, implying $g'(q;p) \geq g_{\rho_q}(p)$. Thus, $g'(q;p) = g_{\rho_q}(p)$ follows.

(ii) Since $x \in \partial g(q)$ is equivalent to $q \in \arg\min g[-x]$, we have $\partial g(q) = \mathrm{B}(\rho_q)$ by Theorem 4.5. Since ρ_q is a submodular function, $\partial g(q)$ is an M-convex set (see [18, Th. 3.3]). □

Proof of Theorem 4.1 (ii). Let $g : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ be positively homogeneous L-convex, and define $\rho : 2^N \to \mathbf{R}\cup\{+\infty\}$ by $\rho_g(X) = g(\chi_X)$ $(X \subseteq N)$. It suffices to show that $g(p) = g_\rho(p)$ holds for $p \in \mathbf{R}^n$. Since $g(p) = g'(\mathbf{0};p)$ holds for $p \in \mathbf{R}^n$, this follows immediately from Theorem 4.6 (iii). □

The next theorem shows that each "face" of the epigraph of a closed proper L-convex function is an L-convex polyhedron. The proof is easy.

Theorem 4.7. *Let* $g : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ *be an L-convex (resp. closed proper L-convex) function. For* $x \in \mathbf{R}^n$, $\arg\min g[x]$ *is L-convex (resp. closed L-convex) if it is nonempty.*

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