# NUMERICAL INDEFINITE INTEGRATION BY DOUBLE EXPONENTIAL SINC METHOD

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ABSTRACT. We present a numerical method for approximating an indefinite integral by the double exponential sinc method. The approximation error of the proposed method with $N$ integrand function evaluations is

$$O(\exp(-c_1 N/\log(c_2 N)))$$

for a reasonably wide class of integrands, including those with end-point singularities. The proposed method compares favorably with the existing formulas based on the ordinary sinc method. Computational results show the accordance of the actual convergence rates with the theoretical estimate.

## 1. INTRODUCTION

A variety of numerical methods based on sinc approximations has been studied during the last three decades [5, 6, 7, 8]. The methods cover function approximation, approximation of derivatives, approximate definite and indefinite integration, approximate solution of initial and boundary value ODE problems, and so on. In particular, the sinc interpolation formula is given by

$$(1.1) \qquad f(x) \approx \sum_{k=-N}^{N} f(kh)S(k,h)(x),$$

where

$$(1.2) \qquad S(k,h)(x) = \frac{\sin\left[\pi(x-kh)/h\right]}{\pi(x-kh)/h}.$$

The methods are collectively referred to as *sinc numerical methods*.

In this paper, we present a method and experimental results for approximating an indefinite integral $F(x) = \int_{-1}^{x} f(t)\mathrm{d}t$. In the literature [2, 3, 5], formulas for numerical indefinite integration based on the sinc approximation have been proposed, often based on a single exponential ("SE") transformation[1] such as $\psi_1(\zeta) = \tanh(\zeta/2)$.

[1] In contrast to the term "double exponential", we use "single exponential" when referring to the ordinary sinc method.

To be specific, Stenger's formula [5] with this transformation is

(1.3)     $\displaystyle \int_{-1}^{x} f(t)\mathrm{d}t$

$$= \frac{\exp(A\psi_1^{-1}(x))}{2\cosh(A\psi_1^{-1}(x))} h \sum_{k=-N}^{N} f(\psi_1(kh))\psi_1'(kh)$$

$$+ h \sum_{k=-N}^{N} \left[ \sum_{l=-N}^{N} \sigma_{k-l} \left( f(\psi_1(lh))\psi_1'(lh) \right. \right.$$

$$\left. \left. - \frac{A}{2\cosh^2(Alh)} h \sum_{k=-N}^{N} f(\psi_1(kh))\psi_1'(kh) \right) \right] S(k,h)(\psi_1^{-1}(x))$$

$$+ \mathrm{O}\left( \sqrt{N} \exp\left( -\sqrt{c_f N} \right) \right),$$

where $\sigma_{k-l}$ is defined in (2.14) and $c_f$ is a constant that depends on the integrand $f$. The constant $A$ is determined appropriately according to the property of $f$. Such a formula has also been considered by Haber [2].

  In place of the single exponential transformation, we employ in this paper a double exponential ("DE") transformation. Double exponential transformations are proposed by H. Takahasi and M. Mori [9] in designing a definite integration formula. Recently, it is known that the double exponential transformations are useful for various kinds of the sinc numerical methods [4]. We employ one of the double exponential transformations such as $\psi_2(\zeta) = \tanh((\pi/2)\sinh\zeta)$ to propose a more efficient formula:

(1.4)     $\displaystyle \int_{-1}^{x} f(t)\mathrm{d}t$

$$= \frac{1}{2} \left[ \tanh\left( B\sinh(C\psi_2^{-1}(x)) \right) + 1 \right] h \sum_{k=-N}^{N} f(\psi_2(kh))\psi_2'(kh)$$

$$+ h \sum_{k=-N}^{N} \left[ \sum_{l=-N}^{N} \sigma_{k-l} \left( f(\psi_2(lh))\psi_2'(lh) \right. \right.$$

$$\left. \left. - \frac{BC\cosh(Clh)}{2\cosh^2(B\sinh(Clh))} h \sum_{k=-N}^{N} f(\psi_2(kh))\psi_2'(kh) \right) \right] S(k,h)(\psi_2^{-1}(x))$$

$$+ \mathrm{O}\left( \exp\left[ \frac{-c_f' N}{\log(c_f'' N)} \right] \right),$$

where $c_f'$ and $c_f''$ are constants that depend on the integrand $f$. The constants $B$ and $C$ are determined appropriately according to the property of $f$. The error term

$$\mathrm{O}\left( \exp\left[ \frac{-c_f' N}{\log(c_f'' N)} \right] \right)$$

in this formula is smaller in order of magnitude than the error term

$$\mathrm{O}\left( \sqrt{N} \exp\left( -\sqrt{c_f N} \right) \right)$$

in the formula (1.3). Thus the proposed formula (1.4) is often more efficient than the formula (1.3), although there exist some functions (see Example 6 in Section 4) for which this is not the case.

The organization of this paper is as follows. In Section 2, we present our numerical indefinite integration formula on the entire real line $\mathbb{R}$ together with the main error estimate theorem. With the use of a double exponential transformation, the formula is adapted to the numerical indefinite integration formula on a finite interval in Section 3. In Section 4, we present some numerical results to confirm our theoretical error estimates. In Section 5, the proofs of the theorem and lemmas are provided. Concluding remarks are made in Section 6.


## 2. Indefinite integration on the entire real line

In this section, we consider an indefinite integration on the entire real line $\mathbb{R}$ as the fundamental case of theoretical treatments. This case is essential in that the general case in Section 3 is a direct consequence of the results of this section in combination with the double exponential transformation.

Proofs of theorems are given in Section 5, unless otherwise indicated.


### 2.1. Notation. We introduce a function space as follows.

**Definition 2.1.** For a positive number $d$, a strip region $\mathcal{D}_d$ in $\mathbb{C}$ is defined as:

$$(2.1) \qquad \mathcal{D}_d = \left\{ z \in \mathbb{C} \mid |\mathrm{Im}\, z| < d \right\}.$$

Then, a function space $\mathbf{H}^1(\mathcal{D}_d)$ is defined as:

$$(2.2) \qquad \mathbf{H}^1(\mathcal{D}_d) = \left\{ g \mid g \text{ is analytic in } \mathcal{D}_d,\ N_1(g, \mathcal{D}_d) < \infty \right\},$$

where

$$(2.3) \qquad N_1(g, \mathcal{D}_d) = \lim_{\varepsilon \to 0} \int_{\partial \mathcal{D}_d(\varepsilon)} |f(z)||\mathrm{d}z|,$$

$$(2.4) \qquad \mathcal{D}_d(\varepsilon) = \{ z \in \mathbb{C} \mid |\mathrm{Re}\, z| < 1/\varepsilon, |\mathrm{Im}\, z| < d(1 - \varepsilon) \}.$$

In addition, we use operators $\mathcal{J}$, $C_{N,h}$, and $C_h$ for an indefinite integration and the sinc interpolation.

**Definition 2.2.** For a function $f$ defined on a complex region containing the real line $\mathbb{R}$, we define operators $\mathcal{J}$, $C_{N,h}$, and $C_h$ by

$$(2.5) \qquad (\mathcal{J}f)(z) = \int_{-\infty}^{z} f(t)\mathrm{d}t,$$

$$(2.6) \qquad (C_{N,h}f)(z) = \sum_{k=-N}^{N} f(kh)S(k,h)(z),$$

$$(2.7) \qquad (C_h f)(z) = \lim_{N \to \infty} (C_{N,h}f)(z),$$

where $N$ is a positive integer and $h$ is a positive real number.

2.2. **Sinc interpolation.** The sinc interpolation is a basic tool in the derivation of our formula. The interpolation error for a function of double exponential decay type is estimated as follows.

**Theorem 2.1** ([8]). *Assume that $f$ satisfies*

$$(2.8) \qquad\qquad f \in \mathbf{H}^1(\mathcal{D}_d),$$

$$(2.9) \qquad\qquad \forall x \in \mathbb{R}, \ |f(x)| \leq \alpha \exp(-\beta \exp(\gamma|x|))$$

*for some positive numbers $\alpha$, $\beta$, $\gamma$, and $d$. Then, there exists a positive number $c$, independent of $N$, such that*

$$(2.10) \qquad \sup_{-\infty < x < \infty} |f(x) - (C_{N,h}f)(x)| \leq c \exp\left[\frac{-\pi d\gamma N}{\log(\pi d\gamma N/\beta)}\right],$$

*where*

$$(2.11) \qquad\qquad h = \frac{\log(\pi d\gamma N/\beta)}{\gamma N}.$$

2.3. **Derivation of the indefinite integration formula on $\mathbb{R}$.** We describe the derivation of the proposed formula on $\mathbb{R}$ and explain the basic idea behind it.

First, we apply the sinc interpolation to $(\mathcal{J}f)(x) = \int_{-\infty}^x f(t)\mathrm{d}t$ to obtain

$$(2.12) \qquad (\mathcal{J}f)(x) \approx (C_{N,h}\mathcal{J}f)(x)$$

$$= \sum_{k=-N}^{N} (\mathcal{J}f)(kh)S(k,h)(x)$$

$$= \sum_{k=-N}^{N} \left(\int_{-\infty}^{kh} f(t)\mathrm{d}t\right) S(k,h)(x).$$

Next, we apply the sinc interpolation to $f$ in the above expression to obtain

$$(2.13) \qquad (\mathcal{J}f)(x) \approx (C_{N,h}\mathcal{J}C_{N,h}f)(x)$$

$$= \sum_{k=-N}^{N} \left[\int_{-\infty}^{kh} \sum_{l=-N}^{N} f(lh)S(l,h)(t)\mathrm{d}t\right] S(k,h)(x)$$

$$= h \sum_{k=-N}^{N} \left[\sum_{l=-N}^{N} \sigma_{k-l}f(lh)\right] S(k,h)(x),$$

where

$$(2.14) \qquad\qquad \sigma_{k-l} = \frac{1}{2} + \int_0^{k-l} \frac{\sin \pi t}{\pi t}\mathrm{d}t.$$

If $f$ and $\mathcal{J}f$ satisfy the assumptions of Theorem 2.1, then the approximations in (2.12) and (2.13), denoted "$\approx$", should work. We can apply Theorem 2.1 to obtain the following theorem. Its proof is omitted.

**Theorem 2.2.** *Assume that $f$ and $\mathcal{J}f$ satisfy*

$$(2.15) \qquad\qquad f \in \mathbf{H}^1(\mathcal{D}_d),$$

$$(2.16) \qquad\qquad \mathcal{J}f \in \mathbf{H}^1(\mathcal{D}_d),$$

$$(2.17) \qquad\qquad \forall x \in \mathbb{R}, \ |f(x)| \leq \alpha \exp(-\beta \exp(\gamma|x|)),$$

$$(2.18) \qquad\qquad \forall x \in \mathbb{R}, \ |(\mathcal{J}f)(x)| \leq \alpha \exp(-\beta \exp(\gamma|x|))$$

*for some positive numbers $\alpha$, $\beta$, $\gamma$, and $d$. Then, there exists a positive number $c$, independent of $N$, such that*

$$(2.19) \qquad \sup_{-\infty < x < \infty} |(\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}C_{N,h}f)(x)| \leq c \exp\left[\frac{-\pi d\gamma N}{\log(\pi d\gamma N/\beta)}\right],$$

*where*

$$(2.20) \qquad h = \frac{\log(\pi d\gamma N/\beta)}{\gamma N}.$$

The assumptions in the above theorem are stated in terms of both $f$ and $\mathcal{J}f$. It is, however, more natural to state the assumptions in terms of the given integrand $f$. We present the following theorem that imposes an additional condition $\lim_{x\to\infty}(\mathcal{J}f)(x) = 0$. The general case free from this additional condition is presented in Theorem 2.4.

**Theorem 2.3.** *Assume that $f$ satisfies*

$$(2.21) \qquad f \in \mathbf{H}^1(\mathcal{D}_d),$$

$$(2.22) \qquad \forall x \in \mathbb{R}, \ |f(x)| \leq \alpha \exp(-\beta \exp(\gamma|x|)),$$

$$(2.23) \qquad \int_{-\infty}^{\infty} f(t)\mathrm{d}t = 0$$

*for some positive numbers $\alpha$, $\beta$, $\gamma$, and $d$. Then, for any $\varepsilon$ with $0 < \varepsilon < d$, there exists a positive number $c_\varepsilon$, independent of $N$, such that*

$$(2.24) \qquad \sup_{-\infty < x < \infty} |(\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}C_{N,h}f)(x)| \leq c_\varepsilon \exp\left[\frac{-\pi(d-\varepsilon)\gamma N}{\log(\pi(d-\varepsilon)\gamma N/\beta)}\right],$$

*where*

$$(2.25) \qquad h = \frac{\log(\pi(d-\varepsilon)\gamma N/\beta)}{\gamma N}.$$

We next treat the general case in which (2.23) is not assumed. We consider

$$(2.26) \qquad g(z) = f(z) - \kappa(z)\int_{-\infty}^{\infty} f(t)\mathrm{d}t$$

with a function $\kappa$ such that $\int_{-\infty}^{\infty} \kappa(t)\mathrm{d}t = 1$. Then we have

$$(2.27) \qquad \int_{-\infty}^{\infty} g(t)\mathrm{d}t = 0.$$

To apply Theorem 2.3 to $g$ we must choose $\kappa$ so that (2.21) and (2.22) are satisfied for some $\alpha$, $\beta$, $\gamma$, and $d$. We consider

$$(2.28) \qquad \kappa(z) = \frac{BC\cosh(Cz)}{2\cosh^2(B\sinh(Cz))} \left(= \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{2}\tanh\left(B\sinh(Cz)\right) + \frac{1}{2}\right]\right),$$

parameterized by $B$ and $C$. We also introduce notation

$$(2.29) \qquad r(z) = \kappa(z)\int_{-\infty}^{\infty} f(t)\mathrm{d}t.$$

The following proposition provides the decay rate of $\kappa$ and the function space that contains $\kappa$. The proof is straightforward and omitted.

**Proposition 2.1.** *Let $\beta_\kappa$, $\gamma_\kappa$, and $d_\kappa$ be determined as*

$$(2.30) \qquad \begin{cases} \beta_\kappa = B - \varepsilon_\beta, \ \gamma_\kappa = C, \ d_\kappa = \frac{\pi}{2C} - \varepsilon_d, & \text{if } 0 < B < \frac{\pi}{2}, \\ \beta_\kappa = B - \varepsilon_\beta, \ \gamma_\kappa = C, \ d_\kappa = \frac{1}{C}\arcsin\left(\frac{\pi}{2B}\right) - \varepsilon_d, & \text{if } \frac{\pi}{2} \le B, \end{cases}$$

*where $\varepsilon_\beta$ and $\varepsilon_d$ are any positive numbers such that $\beta_\kappa > 0$ and $d_\kappa > 0$. Then we have*

$$(2.31) \qquad \kappa \in \mathbf{H}^1(\mathcal{D}_{d_\kappa}),$$
$$(2.32) \qquad \forall x \in \mathbb{R}, \ |\kappa(x)| \le \alpha_\kappa \exp(-\beta_\kappa \exp(\gamma_\kappa |x|)).$$

To apply Theorem 2.3 to $g$, we need to determine the decay rate of $g$ and the function space to which $g$ belongs. The following lemma provides them. Its proof is easy and omitted.

**Lemma 2.1.** *Let $\beta_f$, $\gamma_f$, and $d_f$ be constants such that*

$$(2.33) \qquad f \in \mathbf{H}^1(\mathcal{D}_{d_f}),$$
$$(2.34) \qquad \forall x \in \mathbb{R}, \ |f(x)| \le \alpha_f \exp(-\beta_f \exp(\gamma_f |x|)),$$

*and let $\beta_\kappa$, $\gamma_\kappa$, and $d_\kappa$ be constants in (2.30). Then, for*

$$(2.35) \qquad \beta_g = \begin{cases} \beta_f, & \text{if } \gamma_f < \gamma_\kappa, \\ \beta_\kappa, & \text{if } \gamma_f > \gamma_\kappa, \\ \min\{\beta_f, \beta_\kappa\}, & \text{if } \gamma_f = \gamma_\kappa, \end{cases}$$
$$(2.36) \qquad \gamma_g = \min\{\gamma_f, \gamma_\kappa\},$$
$$(2.37) \qquad d_g = \min\{d_f, d_\kappa\},$$

*we have*

$$(2.38) \qquad g \in \mathbf{H}^1(\mathcal{D}_{d_g}),$$
$$(2.39) \qquad \forall x \in \mathbb{R}, \ |g(x)| \le \alpha_g \exp(-\beta_g \exp(\gamma_g |x|)).$$

Applying Theorem 2.3 to $g$, we immediately obtain the following theorem only with the assumptions (2.21) and (2.22) for $f$.

**Theorem 2.4.** *Assume that $f$ satisfies*

$$(2.40) \qquad f \in \mathbf{H}^1(\mathcal{D}_{d_f}),$$
$$(2.41) \qquad \forall x \in \mathbb{R}, \ |f(x)| \le \alpha_f \exp(-\beta_f \exp(\gamma_f |x|))$$

*for some positive numbers $\alpha_f$, $\beta_f$, $\gamma_f$, and $d_f$. Then, for any $\varepsilon$ with $0 < \varepsilon < d_g$ there exists a positive number $c_\varepsilon$, independent of $N$, such that*

$$(2.42) \quad \sup_{-\infty < x < \infty} \left| \int_{-\infty}^{x} f(t)\mathrm{d}t \right.$$

$$- \left[ \frac{1}{2} \left[ \tanh\left(B\sinh(Cx)\right) + 1 \right] \int_{-\infty}^{\infty} f(t)\mathrm{d}t + h \sum_{k=-N}^{N} \left[ \sum_{l=-N}^{N} \sigma_{k-l} \left( f(lh) \right. \right. \right.$$

$$\left. \left. \left. - \frac{BC\cosh(Clh)}{2\cosh^2\left(B\sinh(Clh)\right)} \int_{-\infty}^{\infty} f(t)\mathrm{d}t \right) \right] S(k,h)(x) \right] \right|$$

$$= \sup_{-\infty < x < \infty} \left| (\mathcal{J}f)(x) - \left[ (\mathcal{J}r)(x) + (C_{N,h}\mathcal{J}C_{N,h}g)(x) \right] \right|$$

$$\leq c_\varepsilon \exp\left[ \frac{-\pi(d_g - \varepsilon)\gamma_g N}{\log(\pi(d_g - \varepsilon)\gamma_g N / \beta_g)} \right],$$

*where*

$$(2.43) \qquad\qquad h = \frac{\log(\pi(d_g - \varepsilon)\gamma_g N / \beta_g)}{\gamma_g N},$$

*and $\beta_g$, $\gamma_g$, and $d_g$ are taken as in (2.35)–(2.37).*

In Theorem 2.4, the formula (2.42) contains the constant $\int_{-\infty}^{\infty} f(t)\mathrm{d}t$, that should be replaced by $h\sum_{k=-N}^{N} f(kh)$ in actual computation.

**Theorem 2.5.** *Under the same assumptions as in Theorem 2.4, the following estimate holds for some $c_\varepsilon'$ :*

$$(2.44)$$

$$\sup_{-\infty < x < \infty} \left| \int_{-\infty}^{x} f(t)\mathrm{d}t \right.$$

$$- \left[ \frac{1}{2} \left[ \tanh\left(B\sinh(Cx)\right) + 1 \right] h \sum_{k=-N}^{N} f(kh) + h \sum_{k=-N}^{N} \left[ \sum_{l=-N}^{N} \sigma_{k-l} \left( f(lh) \right. \right. \right.$$

$$\left. \left. \left. - \frac{BC\cosh(Clh)}{2\cosh^2\left(B\sinh(Clh)\right)} h \sum_{k=-N}^{N} f(kh) \right) \right] S(k,h)(x) \right] \right|$$

$$\leq c_\varepsilon' \exp\left[ \frac{-\pi(d_g - \varepsilon)\gamma_g N}{\log(\pi(d_g - \varepsilon)\gamma_g N / \beta_g)} \right].$$

**2.4. Optimal parameters.** Given an integrand $f$, we are free to choose parameters $B$ and $C$ in (2.30). We want to minimize the error (2.44) with respect to the parameters $B$ and $C$ for a given integrand $f$ satisfying (2.40) and (2.41). Hence, we are to determine the set of the parameter values $(B, C)$ that give the maximum value of $\gamma_g d_g$, and then to choose a $(B, C)$, from among these maximizers, that makes $\beta_g$ as large as possible. Recall that $\beta_g$, $\gamma_g$, and $d_g$ are determined from $B$ and $C$ by Proposition 2.1 and Lemma 2.1. Here we note the following key fact.

**Proposition 2.2** ([7]). *Assume that $f$ satisfies*

$$(2.45) \qquad\qquad f \in \mathbf{H}^1(\mathcal{D}_{d_f}),$$

$$(2.46) \qquad\qquad \forall x \in \mathbb{R}, \ |f(x)| \leq \alpha_f \exp(-\beta_f \exp(\gamma_f|x|))$$

*for some positive numbers $\alpha_f$, $\beta_f$, $\gamma_f$, and $d_f$. If $f \not\equiv 0$, then $\gamma_f d_f \leq \pi/2$.*

Thus we may focus on the case $\gamma_f d_f \leq \pi/2$. Then we determine $B$ and $C$ as follows. When $\gamma_f d_f < \pi/2$,

$$(2.47) \qquad B = \frac{\pi}{2\sin(\gamma_f d_f)} - \varepsilon_B,$$

$$(2.48) \qquad C = \gamma_f$$

are desired parameters, where $\varepsilon_B$ is any positive number such that $\pi/2 < B$. Then we have

$$(2.49) \qquad \beta_g = \min\left\{\beta_f, \frac{\pi}{2\sin(\gamma_f d_f)} - \varepsilon_B - \varepsilon_\beta'\right\},$$

$$(2.50) \qquad \gamma_g = \gamma_f,$$

$$(2.51) \qquad d_g = d_f,$$

where $\varepsilon_\beta'$ is any positive number such that $\beta_g > 0$.

When $\gamma_f d_f = \pi/2$,

$$(2.52) \qquad B = \pi/2,$$

$$(2.53) \qquad C = \gamma_f$$

are desired parameters. Then we have

$$(2.54) \qquad \beta_g = \min\left\{\beta_f, \frac{\pi}{2} - \varepsilon_\beta'\right\},$$

$$(2.55) \qquad \gamma_g = \gamma_f,$$

$$(2.56) \qquad d_g = \frac{\pi}{2\gamma_f} - \varepsilon_d',$$

where $\varepsilon_\beta'$ and $\varepsilon_d'$ are any positive numbers such that $\beta_g > 0$ and $d_g > 0$.

The explanation of the above argument is shown in Subsection 5.3.

## 2.5. Discussion.

We discuss a technical difference between our setting and Stenger's [5]. Recall (2.40) and (2.41), our assumptions about the integrand $f$, in which the double exponential decay of $f$ is assumed only on the real line $\mathbb{R}$. On the other hand, the assumption in [5] is that

$$(2.57) \qquad f \in \mathbf{L}_\alpha(\mathcal{D}_d)$$

for some positive numbers $\alpha$ and $d$, where $\mathbf{L}_\alpha(\mathcal{D}_d)$ is a function space of analytic functions $f$ on $\mathcal{D}_d$ satisfying

$$(2.58) \qquad \forall z \in \mathcal{D}_d, \ |f(z)| \leq c\frac{|e^{\alpha z}|}{(1+|e^z|)^{2\alpha}}$$

for some positive number $c$. Thus the single exponential decay is assumed not only on the real line but also on the strip region $\mathcal{D}_d$.

In parallel with $\mathbf{L}_\alpha(\mathcal{D}_d)$, we could have imposed the double exponential decay in a strip region and considered a function space $\mathbf{K}_{\beta,\gamma}(\mathcal{D}_d)$ of analytic functions $f$ on $\mathcal{D}_d$ satisfying

$$(2.59) \qquad \forall z \in \mathcal{D}_d, \ |f(z)| \leq c\exp(-\beta\exp(\gamma|z|))$$

for some positive number $c$. This function space, however, is less appropriate for the following reason. Consider $\kappa$ in (2.28) as a typical function with double exponential decay. We compare the tuple $(\beta_\kappa, \gamma_\kappa, d_\kappa)$ in (2.30) and a possible tuple $(\beta_\kappa', \gamma_\kappa', d_\kappa')$

such that $\mathbf{K}_{\beta'_\kappa, \gamma'_\kappa}(\mathcal{D}_{d'_\kappa})$ contains $\kappa$ in (2.28). First, since $d'_\kappa$ depends only on the strip region where $\kappa$ is analytic, we have

$$(2.60) \qquad d'_\kappa = \begin{cases} \frac{\pi}{2C} - \varepsilon_d, & \text{if } 0 < B < \frac{\pi}{2}, \\ \frac{1}{C}\arcsin\left(\frac{\pi}{2B}\right) - \varepsilon_d, & \text{if } \frac{\pi}{2} \leq B \end{cases}$$

for some positive number $\varepsilon_d$. On the other hand, we can show

$$(2.61) \qquad\qquad\qquad \beta'_\kappa < B\sin(\varepsilon_d C),$$

$$(2.62) \qquad\qquad\qquad \gamma'_\kappa = C.$$

Thus we cannot take as large $\beta'_\kappa$ and $d'_\kappa$ as $\beta_\kappa$ and $d_\kappa$ simultaneously. This implies that the alternative setting using $\mathbf{K}_{\beta'_\kappa, \gamma'_\kappa}(\mathcal{D}_{d'_\kappa})$ would lead to a poorer theoretical error bound than the present setting. In addition, the present setting, imposing a decay rate only on the real line, seems to be more natural, although this is certainly a matter of taste.

## 3. Indefinite integration on a finite interval

Without loss of generality, we may use $[-1, 1]$ as a finite interval. For the approximation of $F(x) = \int_{-1}^{x} f(t)\mathrm{d}t$, we take a double exponential transformation $z = \psi_2(\zeta)$, where

$$(3.1) \qquad\qquad\qquad \psi_2(\zeta) = \tanh\left[\frac{\pi}{2}\sinh\zeta\right],$$

which maps $(-\infty, \infty)$ to $(-1, 1)$. The following theorem gives an error estimate in the case of an indefinite integration on $[-1, 1]$. Its proof is immediate from Theorem 2.5.

**Theorem 3.1.** *Assume that, for a variable transformation $z = \psi_2(\zeta)$, the transformed function $\hat{f}(\zeta) = f(\psi_2(\zeta))\psi'_2(\zeta)$ satisfies*

$$(3.2) \qquad\qquad \hat{f} \in \mathbf{H}^1(\mathcal{D}_{d_{\hat{f}}}),$$

$$(3.3) \qquad\qquad \forall x \in \mathbb{R}, \ |\hat{f}(x)| \leq \alpha_{\hat{f}}\exp(-\beta_{\hat{f}}\exp(\gamma_{\hat{f}}|x|))$$

*for some positive numbers $\alpha_{\hat{f}}$, $\beta_{\hat{f}}$, $\gamma_{\hat{f}}$, and $d_{\hat{f}}$. Then, for any $\varepsilon$ with $0 < \varepsilon < d_{\hat{g}}$ there exists a positive number $c_\varepsilon$, independent of $N$, such that*

$$(3.4)$$

$$\sup_{-1 < x < 1}\left| \int_{-1}^{x} f(t)\mathrm{d}t \right.$$

$$- \left[\frac{1}{2}\left[\tanh\left(B\sinh(C\psi_2^{-1}(x))\right) + 1\right] h \sum_{k=-N}^{N} f(\psi_2(kh))\psi'_2(kh)\right.$$

$$+ h \sum_{k=-N}^{N}\left[\sum_{l=-N}^{N}\sigma_{k-l}\left(f(\psi_2(lh))\psi'_2(lh)\right.\right.$$

$$\left.\left.\left. - \frac{BC\cosh(Clh)}{2\cosh^2(B\sinh(Clh))} \, h\sum_{k=-N}^{N} f(\psi_2(kh))\psi'_2(kh)\right)\right] S(k, h)(\psi_2^{-1}(x))\right]\right|$$

$$\leq c''_\varepsilon \exp\left[\frac{-\pi(d_{\hat{g}} - \varepsilon)\gamma_{\hat{g}} N}{\log(\pi(d_{\hat{g}} - \varepsilon)\gamma_{\hat{g}} N/\beta_{\hat{g}})}\right].$$

*where*

(3.5) $$\hat{g} = \hat{f} - r,$$

(3.6) $$h = \frac{\log(\pi(d_{\hat{g}} - \varepsilon)\gamma_{\hat{g}} N / \beta_{\hat{g}})}{\gamma_{\hat{g}} N},$$

*and $B$, $C$, $\beta_{\hat{g}}$, $\gamma_{\hat{g}}$, and $d_{\hat{g}}$ are taken as in (2.47)–(2.51), or (2.52)–(2.56), with $f$ and $g$ replaced by $\hat{f}$ and $\hat{g}$, respectively.*

The formula (3.4) has been presented in Introduction as (1.4).

For the comparison between "SE" formula (1.3) and "DE" formula (1.4) in Section 4, we describe here the error estimate of (1.3) based on Stenger [5]. Recall the definition of $\mathbf{L}_\alpha(\mathcal{D}_d)$ in Subsection 2.5.

**Theorem 3.2.** *Assume that, for a variable transformation $z = \psi_1(\zeta)$, the transformed function $\breve{f}(\zeta) = f(\psi_1(\zeta))\psi_1'(\zeta)$ satisfies $\breve{f} \in \mathbf{L}_{\alpha_{\breve{f}}}(\mathcal{D}_{d_{\breve{f}}})$ for some positive numbers $\alpha_{\breve{f}}$ and $d_{\breve{f}}$. Then, there exists a positive number $c$, independent of $N$, such that*

(3.7)
$$\sup_{-1 < x < 1} \left| \int_{-1}^{x} f(t)\mathrm{d}t \right.$$
$$- \left[ \frac{\exp(A\psi_1^{-1}(x))}{2\cosh(A\psi_1^{-1}(x))} \ h \sum_{k=-N}^{N} f(\psi_1(kh))\psi_1'(kh) \right.$$
$$+ h \sum_{k=-N}^{N} \left[ \sum_{l=-N}^{N} \sigma_{k-l} \left( f(\psi_1(lh))\psi_1'(lh) \right.\right.$$
$$\left.\left.\left. - \frac{A}{2\cosh^2(Alh)} \ h \sum_{k=-N}^{N} f(\psi_1(kh))\psi_1'(kh) \right) \right] S(k,h)(\psi_1^{-1}(x)) \right] \right|$$
$$\leq cN^{1/2} \exp\left( -\sqrt{\pi \alpha_{\breve{f}}' d_{\breve{f}}' N} \right),$$

*where*

(3.8) $$\alpha_{\breve{f}}' = \min\left( \alpha_{\breve{f}}, 2A \right), \ d_{\breve{f}}' = \min\left( d_{\breve{f}}, \frac{\pi}{2A} - \varepsilon_d \right), \ h = \sqrt{\frac{\pi d_{\breve{f}}'}{\alpha_{\breve{f}}' N}},$$

*and $\varepsilon_d$ is any positive number such that $d_{\breve{f}}' > 0$.*

Since

(3.9) $$\alpha_{\breve{f}}' d_{\breve{f}}' \leq \min\left( \alpha_{\breve{f}} d_{\breve{f}}, \pi - 2A\varepsilon_d \right)$$

by (3.8), the parameter $A$ as

(3.10) $$A = \frac{\alpha_{\breve{f}}}{2}$$

maximizes $\alpha_{\breve{f}}' d_{\breve{f}}'$, i.e., minimizes the error estimate in (3.7). Thus, setting $A$ as in (3.10), we obtain the best "SE" formula.

TABLE 1. Parameter values ($\varepsilon = 0.01$)

| | "SE" formula (1.3) | | "DE" formula (1.4) | | |
| --- | --- | --- | --- | --- | --- |
| | $A$ | $h$ | $B$ | $C$ | $h$ |
| Example 1 | $1/2$ | $\sqrt{\dfrac{2\pi(\pi-\varepsilon)}{N}}$ | $\pi/2 - \varepsilon$ | $1$ | $\dfrac{\log(2(\pi-4\varepsilon)N)}{N}$ |
| Example 2 | $1$ | $\sqrt{\dfrac{\pi(\pi-\varepsilon)}{(1-\varepsilon)N}}$ | $\pi/2 - \varepsilon$ | $1$ | $\dfrac{\log[\pi(\pi/2-2\varepsilon)N/(\pi/2-(\pi/2)\varepsilon)]}{N}$ |
| Example 3 | $1$ | $\sqrt{\dfrac{\pi(\pi-\varepsilon)}{N}}$ | $\pi/2 - \varepsilon$ | $1$ | $\dfrac{\log((\pi-4\varepsilon)N)}{N}$ |
| Example 4 | $3/2$ | $\sqrt{\dfrac{2\pi(2\pi/3-\varepsilon)}{3N}}$ | $\pi/2 - \varepsilon$ | $1$ | $\dfrac{\log((\pi-2\varepsilon)N)}{N}$ |
| Example 5 | $1$ | $\sqrt{\dfrac{\pi(\pi/2-\varepsilon)}{N}}$ | $\pi - \varepsilon$ | $1$ | $\dfrac{\log((\pi/3-2\varepsilon)N)}{N}$ |
| Example 6 | $1$ | $\sqrt{\dfrac{\pi(K-\varepsilon)}{N}}$ | $\pi/2 - \varepsilon$ | $1$ | $\dfrac{\log((\pi/2)N)}{N}$ |

## 4. NUMERICAL RESULTS

In this section, we show numerical results of the formulas (1.3) and (1.4) to compare the actual errors of the two. We adopt the single exponential transformation

$$(4.1) \qquad \psi_1(\zeta) = \tanh(\zeta/2)$$

in the formula (1.3). The integrands used for numerical experiments are as follows:

Example 1. $f_1(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$, $\quad \displaystyle\int_{-1}^{x} f_1(t)\mathrm{d}t = \dfrac{1}{\pi}\left(\arcsin x + \dfrac{\pi}{2}\right),$

Example 2. $f_2(x) = \dfrac{1}{4\log 2}\log\left(\dfrac{1+x}{1-x}\right),$

$$\int_{-1}^{x} f_2(t)\mathrm{d}t = \dfrac{1}{4\log 2}[\log(1+x)^{1+x} + \log(1-x)^{1-x} - 2\log 2],$$

Example 3. $f_3(x) = \dfrac{1}{2}$, $\quad \displaystyle\int_{-1}^{x} f_3(t)\mathrm{d}t = \dfrac{1}{2}(x+1),$

Example 4. $f_4(x) = \dfrac{2}{\pi}\sqrt{1-x^2}$, $\quad \displaystyle\int_{-1}^{x} f_4(t)\mathrm{d}t = \dfrac{1}{\pi}\left(\arcsin x + x\sqrt{1-x^2}\right) + \dfrac{1}{2},$

Example 5. $f_5(x) = \dfrac{2}{\pi(1+x^2)}$, $\quad \displaystyle\int_{-1}^{x} f_5(t)\mathrm{d}t = \dfrac{1}{2} + \dfrac{2}{\pi}\arctan x,$

Example 6. $f_6(x) = -2[x\,\mathrm{cn}(2\tanh^{-1} x, \sqrt{0.5})$

$\qquad\qquad + \mathrm{dn}(2\tanh^{-1} x, \sqrt{0.5})\,\mathrm{sn}(2\tanh^{-1} x, \sqrt{0.5})],$

$$\int_{-1}^{x} f_6(t)\mathrm{d}t = (1-x^2)\,\mathrm{cn}(2\tanh^{-1} x, \sqrt{0.5}).$$

The functions $f_1, \ldots, f_4$ are taken from [2]. We take $N = 1, 4, 9, 16, \ldots, 100$, and take the necessary parameters according to Table 1, where

$$(4.2) \qquad K = \int_0^{\pi/2} \dfrac{d\theta}{\sqrt{1-0.5\sin^2\theta}} = 1.85407\cdots.$$

Let us explain how the parameter values for "DE" formula (1.4) are determined with reference to the result of Subsection 2.4. For $f_1$ in Example 1, we have

(4.3)      $f_1(x) = \mathrm{O}(|1 - x^2|^{-1/2}), \ (x \to \pm 1),$

$\implies |f_1(\psi_2(x))\psi_2'(x)| = \mathrm{O}(\exp(-(\pi/4)\exp|x|)), \ (x \to \pm\infty \text{ in } \mathbb{R}),$

which implies $\beta_{\hat{f}_1} = \pi/4$ and $\gamma_{\hat{f}_1} = 1$. Moreover, by (4.10) we have $d_{\hat{f}_1} = \pi/2 - \varepsilon_d$, where $0 < \varepsilon_d < \pi/2$. Then, by (2.47)–(2.51), we have

(4.4)      $B = \dfrac{\pi}{2\sin(\pi/2 - \varepsilon_d)} - \varepsilon_B, \ C = 1, \ \beta_{\hat{g}} = \dfrac{\pi}{4}, \ \gamma_{\hat{g}} = 1, \ d_{\hat{g}} = \dfrac{\pi}{2} - \varepsilon_d,$

where $\varepsilon_B$ is any positive number such that $B > 0$. Since $B$ in (4.4) may be arbitrarily close to $\pi/2$, we can set $B = \pi/2 - \varepsilon$ for any $\varepsilon$ with $0 < \varepsilon < \pi/2$. Finally we have

(4.5)      $h = \dfrac{\log(\pi(d_{\hat{g}} - \varepsilon)\gamma_{\hat{g}}N/\beta_{\hat{g}})}{\gamma_{\hat{g}}N} = \dfrac{\log(2(\pi - 4\varepsilon)N)}{N},$

when $\varepsilon_d = \varepsilon$.

The parameters for the other integrands are determined similarly. We note that Theorem 3.1 cannot be applied to the integrand $f_6$. More precisely, there exists no $d$ such that $\hat{f}_6 \in \mathbf{H}^1(\mathcal{D}_d)$ for the transformed integrand $\hat{f}_6$ with double exponential decay. Thus the parameter values for $f_6$ have no theoretical justification.

The parameter values for "SE" formula (1.3) are based on Stenger [5]. To be more precise, for $f_1$ in Example 1, we have

(4.6)      $|f_1(\psi_1(z))\psi_1'(z)| = \mathrm{O}(\exp(-(1/2)|z|)) \ \left(|z| \to \pm\infty \text{ in } \mathcal{D}_{d_{\check{f}_1}}\right)$

by (4.9), which implies $\alpha_{\check{f}_1} = 1/2$. Moreover, by (4.9) we have $d_{\check{f}_1} = \pi - \varepsilon_d$, where $0 < \varepsilon_d < \pi$. Then, by (3.8) and (3.10) we have

(4.7)      $\alpha'_{\check{f}_1} = \dfrac{1}{2}, \ d'_{\check{f}_1} = \pi - \varepsilon_d,$

and

(4.8)      $h = \sqrt{\dfrac{\pi d'_{\check{f}}}{\alpha'_{\check{f}}N}} = \sqrt{\dfrac{2\pi(\pi - \varepsilon)}{N}},$

where $\varepsilon_d = \varepsilon$. The parameters for the other integrands are determined in a similar manner.

The values of the functions are evaluated at 379 points from $[-1, 1]$ chosen as follows:

$x = 0.00, \ \pm 0.01k \ (k = 1, 2, \dots, 90), \ \pm(1 - 0.001k) \ (k = 1, 2, \dots, 99).$

The points are denser near the endpoints of the interval. We perform double precision floating-point computation. We compute the absolute errors at the above points and show the maximum values of them in Figures 1–6. "SE" and "DE" indicate the results of the examples by (1.3) and (1.4), respectively.

The integrands $f_1$ and $f_2$ tend to infinity near the endpoints of $[-1, 1]$. To avoid cancellation of significant digits near $x = \pm 1$, we change the expressions
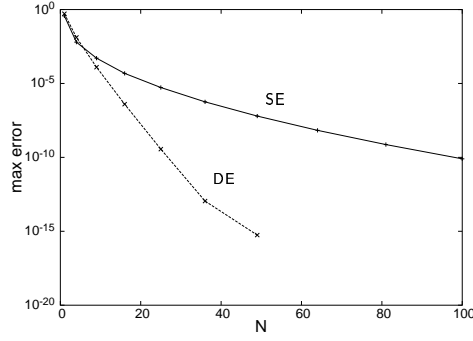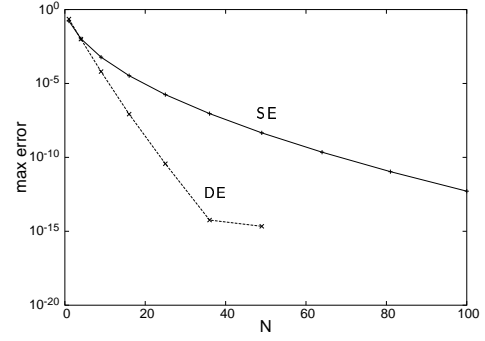
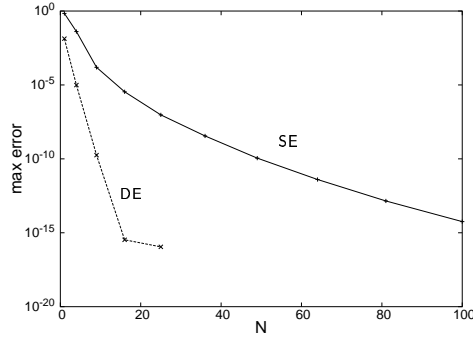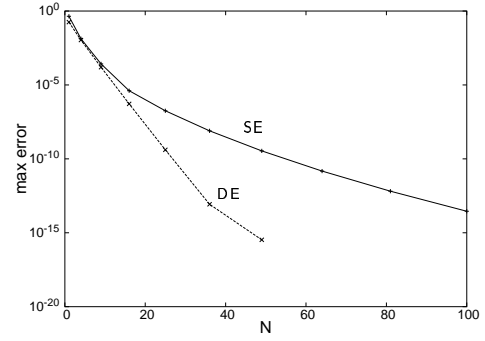FIGURE 1. Example 1



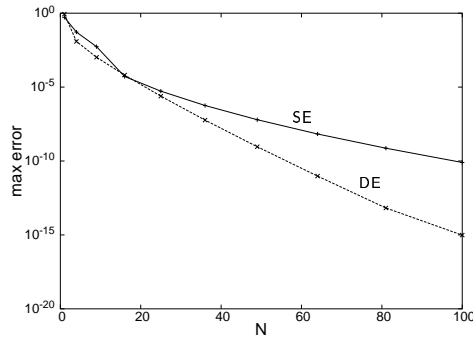FIGURE 2. Example 2



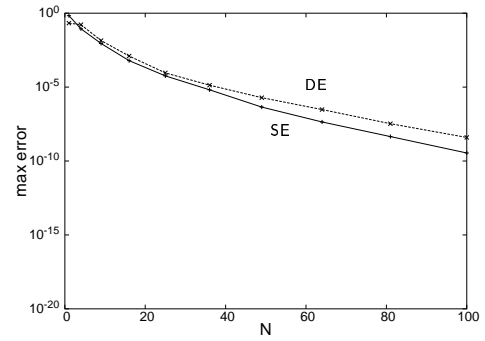FIGURE 3. Example 3



FIGURE 4. Example 4



FIGURE 5. Example 5



FIGURE 6. Example 6

of $f_i(\psi_1(z))\psi_1'(z)$ and $f_i(\psi_2(z))\psi_2'(z)$. For example,

$$(4.9) \qquad f_1(\psi_1(z))\psi_1'(z) = \frac{1}{\pi\sqrt{1 - \tanh^2(z/2)}}\frac{1}{2\cosh^2(z/2)} = \frac{1}{2\pi\cosh(z/2)},$$

$$(4.10) \qquad f_1(\psi_2(z))\psi_2'(z) = \frac{1}{\pi\sqrt{1 - \tanh^2((\pi/2)\sinh z)}}\frac{\pi\cosh z}{2\cosh^2((\pi/2)\sinh z)}$$

$$= \frac{\cosh z}{2\cosh((\pi/2)\sinh z)}.$$

The integrand $f_2$ tends to infinity more mildly than $f_1$, and the approximations of $f_2$ is better than the one of $f_1$. It is natural that the integrand $f_3$ is approximated very well because it has no singularities. Although the integrand $f_4$ has singularities at $x = \pm 1$, it is also approximated very well. It is well known that the double exponential formula is also effective for such an integrand.

¿From Examples 1–4, we can observe that the approximation errors hardly depend on the non-analyticity of the integrands at the endpoints of the interval.

Here, we note that $\gamma_{\hat{f}_i} d_{\hat{f}_i}$ is arbitrarily near $\pi/2$ for $i = 1, \dots, 4$, whereas $\gamma_{\hat{f}_5} d_{\hat{f}_5}$ is properly less than $\pi/2$ as a consequence of the singularities of $f_5$ at $z = \pm \mathrm{i}$. We can observe that the location of singularities in $\mathbb{C}$ affects the approximation error of the formula in $\mathbb{R}$.

¿From Examples 1–5, we can observe that our formula provides the results expected from the estimation (3.4) and is more accurate than the single exponential formula.

Finally, we emphasize that the estimate (1.4) is not applicable to $f_6$ but the estimate (1.3) is valid for $f_6$.

## 5. PROOFS

In this section, we prove the theorems stated in Sections 2 and 3.

### 5.1. **Proof of Theorem 2.3.**

**Lemma 5.1** ([5]). *For the operator norm of $C_{N,h}$ defined as*

$$(5.1) \qquad \|C_{N,h}\|_\infty = \sup_{\sup_{-\infty < x < \infty} |f(x)| \leq 1} \left| \sum_{k=-N}^{N} f(kh) S(k,h)(x) \right|,$$

*we have*

$$(5.2) \qquad \|C_{N,h}\|_\infty \leq \sup_{-\infty < x < \infty} \sum_{k=-N}^{N} |S(k,h)(x)| \leq \frac{2}{\pi}\{3 + \log N\}.$$

**Lemma 5.2** ([5]). *Let $f$ satisfy that $f \in \mathbf{H}^1(\mathcal{D}_d)$. Then,*

$$(5.3) \qquad |(\mathcal{J}f)(t) - (\mathcal{J}C_h f)(t)| \leq \frac{hN_1(f, \mathcal{D}_d)}{4d\sinh(\pi d/h)}.$$

**Lemma 5.3** ([5]). *For*

$$(5.4) \qquad J(k,h)(x) = \int_{-\infty}^{x} S(k,h)(t)\mathrm{d}t$$

*we have*

$$(5.5) \qquad |J(k,h)(x)| \leq 1.1h.$$

Let $\varepsilon$ be an arbitrary positive number and put $d' = d - \varepsilon/2$.

**Lemma 5.4.** *Under the conditions (2.21) and (2.22), there exists a positive number $M(d')$, depending on $d'$, such that*

$$(5.6) \qquad \forall z \in \overline{\mathcal{D}}_{d'}, \ |f(z)| \leq M(d').$$

*Proof.* For a fixed $z \in \overline{\mathcal{D}}_{d'}$, by Cauchy's integral formula we obtain the inequality

$$|f(z)| \leq \frac{2}{\pi(d(1-\delta) - |\mathrm{Im}z|)} \int_{\partial \mathcal{D}_d(\delta)} |f(\zeta)||\mathrm{d}\zeta|,$$

where $\delta$ is sufficiently small positive number and $\mathcal{D}_d(\delta)$ is defined in (2.4). Then we have

$$\begin{aligned}
|f(z)| &\leq \lim_{\delta \to 0} \frac{2}{\pi(d(1-\delta) - |\mathrm{Im}z|)} \int_{\partial \mathcal{D}_d(\delta)} |f(\zeta)||\mathrm{d}\zeta| \\
&= \frac{2}{\pi(d - |\mathrm{Im}z|)} N_1(f, \mathcal{D}_d) \\
&\leq \frac{4}{\pi(d - d')} N_1(f, \mathcal{D}_d),
\end{aligned}$$

which proves (5.6). $\qquad\square$

**Lemma 5.5.** *Assume that $f$ is analytic in $\overline{\mathcal{D}}_{d'}$. Moreover, assume (2.22) and (5.6), and set*

$$(5.7) \qquad B(\gamma, d', y) = [\cos(\gamma|y|) - \cot(\gamma d')\sin(\gamma|y|)]\beta.$$

*Then we have*

$$(5.8) \qquad |f(x + y\mathrm{i})| \leq M' \exp(-B(\gamma, d', y)\exp(\gamma|x|))$$

*for all $y$ such that $|y| \leq d'$.*

Put $d'' = d' - \varepsilon/2$.

**Lemma 5.6.** *Assume that $f$ is analytic in $\overline{\mathcal{D}}_{d'}$. Under the conditions (2.23) and (5.8), there exists a positive number $\alpha'$ such that*

$$(5.9) \qquad \mathcal{J}f \in \mathbf{H}^1(\mathcal{D}_{d''}),$$

$$(5.10) \qquad \forall x \in \mathbb{R}, \ |(\mathcal{J}f)(x)| \leq \alpha' \exp(-\beta \exp(\gamma|x|)).$$

For the proof of Lemmas 5.5 and 5.6, we need Propositions 5.1 and 5.2 below.

Let $a$ be a real number such that $0 < a < \pi/2$ and define the fan-shaped domain $\mathcal{F}_a$ as

$$\mathcal{F}_a = \left\{ z \in \mathbb{C} \ \middle| \ \frac{\pi}{2} - a < \arg z < \frac{\pi}{2} \right\}.$$

**Proposition 5.1** (Phragmén-Lindelöf [1]). *Assume that $f$ is analytic in $\mathcal{F}_a$ and continuous in $\overline{\mathcal{F}}_a$. In addition, assume*

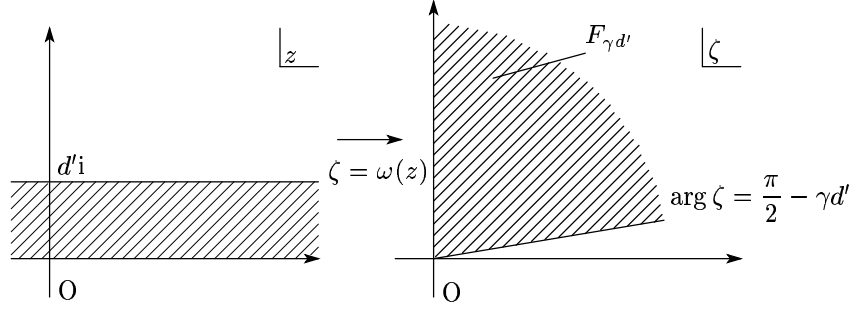$$(5.11) \qquad \forall z \in \overline{\mathcal{F}}_a, \ |f(z)| \leq M,$$

$$(5.12) \qquad \exists c < 0, \ f(r\exp(\mathrm{i}\phi_a)) = \mathrm{O}(\exp(cr)), \ (r \to \infty),$$

*where $\phi_a = \dfrac{\pi}{2} - a$. Then,*

$$\exists M' > 0, \forall z \in \mathcal{F}_a, \ |f(z)| \leq M' \exp\left( \frac{\cos(\arg z)}{\cos \phi_a} c|z| \right).$$

**Proposition 5.2** (Montel [1]). *Assume that $f$ is analytic and bounded in $\{z \in \mathbb{C} \mid \mathrm{Re}z > x_0, y_0 \leq \mathrm{Im}z \leq y_1\}$. If*

$$\lim_{\substack{\mathrm{Re}z \to \infty \\ \mathrm{Im}z = y_2}} f(z) = c$$

FIGURE 7. Correspondence of the domains by $\omega$

*for a fixed $y_2$ such that $y_0 < y_2 < y_1$, then $f(z)$ converges to $c$ as $\mathrm{Re}\, z \to \infty$ uniformly with respect to $\mathrm{Im}\, z$ such that $y_0 < \mathrm{Im}\, z < y_1$.*

*Proof of Lemma 5.5.* We consider the case where $(x, y)$ is in the first quadrant of $\overline{\mathcal{D}}_{d'}$. The proofs of the other cases are similar.

We define the conformal mapping $\omega$ as

$$\zeta = \omega(z) = \exp\left(\gamma z + \mathrm{i}\left(\frac{\pi}{2} - \gamma d'\right)\right),$$

and we set $\check{f}(\zeta) = f(\omega^{-1}(\zeta))$. Let $z = x + y\mathrm{i}$ and $\mathcal{F}_{\gamma d'}$ be the domain shown in the right of Figure 7.

We will apply Proposition 5.1 to $\check{f}$.

First, it is obvious that $\check{f}$ is analytic in $\mathcal{F}_{\gamma d'}$ and continuous in $\overline{\mathcal{F}}_{\gamma d'} - \{0\}$. Further, the fact that $f(x)$ tends to zero as $x \to \pm\infty$, the boundedness of $f$ in $\overline{\mathcal{D}}_{d'}$ (that is, (5.6)) and Proposition 5.2 guarantee the continuity of $\check{f}$ at the origin. Thus the analyticity of $\check{f}$ in $\mathcal{F}_{\gamma d'}$ and the continuity of $\check{f}$ in $\overline{\mathcal{F}}_{\gamma d'}$ are established.

Secondly, by (5.6) and the continuity of $\check{f}$ in $\overline{\mathcal{F}}_{\gamma d'}$ we have

$$(5.13) \qquad \forall \zeta \in \overline{\mathcal{F}}_{\gamma d'} \quad |\check{f}(\zeta)| = |f(z)| \leq M(d').$$

Lastly, it follows from

$$|\zeta| \geq 1 \Rightarrow |\zeta| = \exp(\gamma x) = \exp(\gamma |x|)$$
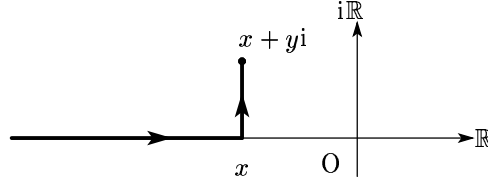
and (2.22) that

$$(5.14) \qquad \check{f}(\zeta) = \mathrm{O}(\exp(-\beta \exp(\gamma |x|))) = \mathrm{O}(\exp(-\beta |\zeta|)), \quad (|\zeta| \to \infty)$$

holds for $\zeta$ with $\arg \zeta = \dfrac{\pi}{2} - \gamma d'$.

Now all the assumptions of Proposition 5.1 for $\check{f}$ are established. Hence we have

$$(5.15) \qquad \forall \zeta \in \mathcal{F}_{\gamma d'}, \quad |\check{f}(\zeta)| \leq M' \exp\left(-\frac{\cos(\arg \zeta)}{\cos\left(\frac{\pi}{2} - \gamma d'\right)} \beta |\zeta|\right).$$

FIGURE 8. Integration path for computing $\int_{-\infty}^{z} f(\zeta) \mathrm{d}\zeta$

Transforming this result to the $z$-plane, we finally have that

$$
\begin{aligned}
|f(z)| &\leq M' \exp\left(-\frac{\cos\left(\frac{\pi}{2} - \gamma(d' - y)\right)}{\cos\left(\frac{\pi}{2} - \gamma d'\right)}\beta \exp(\gamma|x|)\right) \\
&= M' \exp\left(-\frac{\sin\left(\gamma(d' - y)\right)}{\sin\left(\gamma d'\right)}\beta \exp(\gamma|x|)\right) \\
&= M' \exp\left(-B(\gamma, d', y) \exp(\gamma|x|)\right)
\end{aligned}
$$

for all $z$ in $\overline{\mathcal{D}}_{d'} \cap \{x + y\mathrm{i} \in \mathbb{C} \mid x \geq 0, y \geq 0\}$, which is the desired inequality.    $\square$

*Proof of Lemma 5.6.*

<u>Step 1.</u> We estimate the value of $\int_{-\infty}^{z} f(\zeta)\mathrm{d}\zeta$ when $z = x + y\mathrm{i} \in \mathcal{D}_{d''}$.
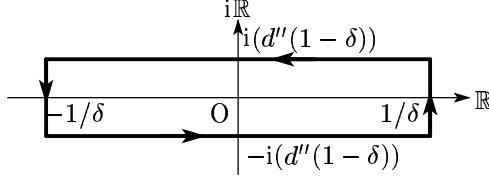   First, we consider the case where $x < 0$ and $y \geq 0$. We have

$$
\begin{aligned}
\left|\int_{-\infty}^{z} f(\zeta)\mathrm{d}\zeta\right| &= \left|\int_{-\infty}^{x} f(s)\mathrm{d}s + \int_{0}^{y} f(x + t\mathrm{i})\,\mathrm{i}\mathrm{d}t\right| \quad \text{(See Fig.8.)} \\
&\leq \int_{-\infty}^{x} |f(s)|\mathrm{d}s + \int_{0}^{y} |f(x + t\mathrm{i})|\mathrm{d}t \\
&\leq M'\left[\int_{-\infty}^{x} \exp(-\beta \exp(-\gamma s))\mathrm{d}s + \int_{0}^{y} \exp\left\{-\beta\frac{\exp(-\gamma x)}{\sin(\gamma d')}\sin(\gamma(d' - t))\right\}\mathrm{d}t\right].
\end{aligned}
$$

The first term of [ ] on the extreme right-hand side is estimated as follows:

$$
\begin{aligned}
(5.16) \qquad (\text{first term}) &\leq \int_{-\infty}^{x} \exp(-\gamma s)\exp(-\beta \exp(-\gamma s))\mathrm{d}s \\
&= \frac{1}{\beta\gamma}\int_{-\infty}^{x} \beta\gamma \exp(-\gamma s)\exp(-\beta \exp(-\gamma s))\mathrm{d}s \\
&= \frac{1}{\beta\gamma}\exp(-\beta \exp(-\gamma x)), \\
&= \frac{1}{\beta\gamma}\exp(-\beta \exp(\gamma|x|)).
\end{aligned}
$$

For the second term, we note that

$$
\gamma \leq \frac{\pi}{2d}, \ 0 \leq t \leq d'' < d' < d \Rightarrow \ \sin(\gamma(d' - t)) \geq \frac{2}{\pi}(\gamma(d' - t)),
$$

FIGURE 9. Contour for computing $N_1(\mathcal{J}f, \mathcal{D}_{d''})$

and obtain

(5.17)    (second term)

$$\leq \int_0^y \exp\left\{-\beta \frac{\exp(-\gamma x)}{\sin(\gamma d')} \frac{2}{\pi}(\gamma(d'-t))\right\} \mathrm{d}t$$

$$\leq \int_0^{d'-\varepsilon} \exp\left\{-\beta \frac{\exp(-\gamma x)}{\sin(\gamma d')} \frac{2}{\pi}(\gamma(d'-t))\right\} \mathrm{d}t$$

$$= \frac{\pi \sin(\gamma d')}{2\beta\gamma} \exp(\gamma x)$$

$$\cdot \left[\exp\left\{-\frac{\beta\gamma\varepsilon}{\pi\sin(\gamma d')}\exp(-\gamma x)\right\} - \exp\left\{-\frac{2\beta\gamma d'}{\pi\sin(\gamma d')}\exp(-\gamma x)\right\}\right]$$

$$= \frac{\pi \sin(\gamma d')}{2\beta\gamma} \exp(-\gamma|x|)$$

$$\cdot \left[\exp\left\{-\frac{\beta\gamma\varepsilon}{\pi\sin(\gamma d')}\exp(\gamma|x|)\right\} - \exp\left\{-\frac{2\beta\gamma d'}{\pi\sin(\gamma d')}\exp(\gamma|x|)\right\}\right].$$

Next, we consider the case where $x \geq 0$ and $y \geq 0$. It follows from (2.23) that

(5.18)
$$\left|\int_{-\infty}^z f(\zeta)\mathrm{d}\zeta\right| = \left|\int_{-\infty}^x f(s)\mathrm{d}s + \int_0^y f(x+t\mathrm{i})\mathrm{i}\mathrm{d}t\right|$$

$$= \left|-\int_x^\infty f(s)\mathrm{d}s + \int_0^y f(x+t\mathrm{i})\mathrm{i}\mathrm{d}t\right|$$

$$\leq \int_x^\infty |f(s)|\mathrm{d}s + \int_0^y |f(x+t\mathrm{i})|\mathrm{d}t.$$

By applying (5.8) to the terms in the extreme right-hand side, we obtain the same bounds given in (5.16) and (5.17).

Lastly, we consider the case where $y < 0$. Since the bound for $|\int_{-\infty}^z f(\zeta)\mathrm{d}\zeta|$ should be symmetric with respect to the real axis, we have the same bounds given in (5.16) and (5.17).

Step 2. First, we prove (5.10). It follows from the results in the case where $y = 0$ in Step 1 that $(\mathcal{J}f)(x) = \mathrm{O}(\exp(-\beta\exp(\gamma|x|)))$, i.e., (5.10).

Next, we prove (5.9). We take the contour shown in Fig. 9 for computing $N_1(\mathcal{J}f, \mathcal{D}_{d''})$, where $\delta$ is a sufficiently small positive number. The integral consid-

ered here is

$$(5.19) \qquad \underbrace{\int_{-d''(1-\delta)}^{d''(1-\delta)} |(\mathcal{J}f)(-1/\delta + y\mathrm{i})|\mathrm{d}y}_{I_1(\delta)} + \underbrace{\int_{-d''(1-\delta)}^{d''(1-\delta)} |(\mathcal{J}f)(1/\delta + y\mathrm{i})|\mathrm{d}y}_{I_2(\delta)}$$

$$+ \underbrace{\int_{-1/\delta}^{1/\delta} |(\mathcal{J}f)(x - d''(1-\delta)\mathrm{i})|\mathrm{d}x}_{I_3(\delta)} + \underbrace{\int_{-1/\delta}^{1/\delta} |(\mathcal{J}f)(x + d''(1-\delta)\mathrm{i})|\mathrm{d}x}_{I_4(\delta)}.$$

We set

$$M_1(x) = \text{(the extreme right-hand side of (5.16))},$$
$$M_2(x) = \text{(the extreme right-hand side of (5.17))}.$$

Note that $|(\mathcal{J}f)(z)| \le M_1(x) + M_2(x)$ holds for all $z \in \mathcal{D}_{d''}$ as shown in Step 1. Then we have

$$(5.20) \qquad I_1(\delta) + I_2(\delta) + I_3(\delta) + I_4(\delta)$$

$$\le 4d''(M_1(1/\delta) + M_2(1/\delta)) + 2\int_{x=-1/\delta}^{x=1/\delta} (M_1(x) + M_2(x))\, \mathrm{d}x.$$

Since

$$M_1(x) = \mathrm{O}(\exp(-\beta \exp(\gamma|x|))),$$
$$M_2(x) = \mathrm{o}(\exp(-\gamma|x|)),$$

the right-hand side of (5.20) is bounded as $\delta$ approaches zero. Thus we have $\mathcal{J}f \in \mathbf{H}^1(\mathcal{D}_{d''})$, i.e., (5.9). $\qquad\square$

*Proof of Theorem 2.3.* Note that $d'' = d - \varepsilon$. We have

$$|(\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}C_{N,h}f)(x)|$$
$$\le |(\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}f)(x)| + |(C_{N,h}\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}C_h f)(x)|$$
$$+ |(C_{N,h}\mathcal{J}C_h f)(x) - (C_{N,h}\mathcal{J}C_{N,h}f)(x)|.$$

We define $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ as follows:

$$\varepsilon_1(x) = (\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}f)(x),$$
$$\varepsilon_2(x) = (C_{N,h}\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}C_h f)(x),$$
$$\varepsilon_3(x) = (C_{N,h}\mathcal{J}C_h f)(x) - (C_{N,h}\mathcal{J}C_{N,h}f)(x).$$

To complete the proof, we estimate $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ in turn.

Estimation of $\varepsilon_1$ By Lemma 5.6, we have

$$\mathcal{J}f \in \mathbf{H}^1(\mathcal{D}_{d''}),$$
$$\forall x \in \mathbb{R},\ |(\mathcal{J}f)(x)| \le \alpha' \exp(-\beta \exp(\gamma|x|)).$$

Therefore it follows from Theorem 2.1 that

$$(5.21) \qquad |\varepsilon_1(x)| \le c_1 \exp\left[\frac{-\pi d''\gamma N}{\log(\pi d''\gamma N/\beta)}\right].$$

<u>Estimation of $\varepsilon_2$</u> By Lemma 5.2 we have

$$|(\mathcal{J}f)(x) - (\mathcal{J}C_h f)(x)| \leq \frac{hN_1(f, \mathcal{D}_d)}{4d\sinh(\pi d/h)} \leq c_2 h \exp(-\pi d/h),$$

where $c_2$ is a constant which is independent of $h$ (assuming, of course, that $h$ is uniformly bounded). With $h$ in (2.25), it follows from Lemma 5.1 that

$$(5.22) \qquad |\varepsilon_2(x)| = |(C_{N,h}\mathcal{J}f)(x) - (C_{N,h}\mathcal{J}C_h f)(x)|$$

$$\leq \|C_{N,h}\|_\infty \sup_{-\infty < x < \infty} |(\mathcal{J}f)(x) - (\mathcal{J}C_h f)(x)|$$

$$\leq \frac{2}{\pi}(3 + \log N) \cdot c_2 h \exp(-\pi d/h)$$

$$= \frac{2}{\pi}(3 + \log N) \cdot c_2 \frac{\log(\pi d''\gamma N/\beta)}{\gamma N} \exp\left[\frac{-\pi d\gamma N}{\log(\pi d''\gamma N/\beta)}\right]$$

$$\leq c_3 \frac{(\log N)^2}{N} \exp\left[\frac{-\pi d\gamma N}{\log(\pi d''\gamma N/\beta)}\right],$$

where $c_3$ is a constant which is independent of $N$.

<u>Estimation of $\varepsilon_3$</u> By Lemma 5.3, we have

$$|(\mathcal{J}C_h f)(x) - (\mathcal{J}C_{N,h}f)(x)|$$

$$= \left|\int_{-\infty}^x \sum_{|k|>N} f(kh)S(k,h)(t)\mathrm{d}t\right| = \left|\sum_{|k|>N} f(kh)J(k,h)(x)\right|$$

$$\leq \sup_{-\infty < x < \infty} |J(k,h)(x)| \sum_{|k|>N} |f(kh)| \leq 1.1h \sum_{|k|>N} \alpha \exp(-\beta \exp(\gamma|kh|))$$

$$= 2.2\alpha h \sum_{k=N+1}^{\infty} \exp(-\beta \exp(\gamma kh)) \leq 2.2\alpha h \int_N^\infty \exp(-\beta \exp(\gamma ht))\mathrm{d}t$$

$$\leq \frac{2.2\alpha h}{\beta\gamma h \exp(\gamma hN)} \int_N^\infty \beta\gamma h \exp(\gamma ht) \exp(-\beta \exp(\gamma ht))\mathrm{d}t$$

$$= \frac{2.2\alpha \exp(-\beta \exp(\gamma hN))}{\beta\gamma \exp(\gamma hN)}$$

$$= \frac{2.2\alpha}{\pi d''\gamma^2} \frac{1}{N} \exp(-\pi d''\gamma N),$$

where $h$ is taken as (2.25). Then, analogously to (5.22), we have

$$(5.23) \qquad |\varepsilon_3(x)| = |(C_{N,h}\mathcal{J}C_h f)(x) - (C_{N,h}\mathcal{J}C_{N,h}f)(x)|$$

$$\leq \|C_{N,h}\|_\infty \sup_{-\infty < x < \infty} |(\mathcal{J}C_h f)(x) - (\mathcal{J}C_{N,h}f)(x)|$$

$$\leq c_4 \frac{\log N}{N} \exp(-\pi d''\gamma N),$$

where $c_4$ is a constant which is independent of $N$.

Combining (5.21), (5.22), and (5.23), we get (2.24). Thus the proof of Theorem 2.3 is completed. $\qquad\square$

5.2. **Proof of Theorem 2.5.** We need the following proposition and lemma. The proof of the proposition is similar to that of Theorem 3.2 in [7], and hence omitted.

**Proposition 5.3.** *Assume that a function $f$ satisfies* (2.40) *and* (2.41) *for some* $\alpha_f$, $\beta_f$, $\gamma_f$, *and* $d_f$. *Then we have*

$$(5.24) \qquad \left| \int_{-\infty}^{\infty} f(t)\mathrm{d}t - h \sum_{k=-N}^{N} f(kh) \right| \le c \exp \left[ \frac{-2\pi d_f \gamma_g N}{\log(\pi(d_g - \varepsilon)\gamma_g N/\beta_g)} \right]$$

*for a constant $c$, independent of $N$, where we take $h$ as* (2.43), *i.e.,*

$$(5.25) \qquad h = \frac{\log(\pi(d_g - \varepsilon)\gamma_g N/\beta_g)}{\gamma_g N}.$$

**Lemma 5.7.** *Let*

$$h = \frac{\log(\pi(d_g - \varepsilon)\gamma_g N/\beta_g)}{\gamma_g N}.$$

*Then*

$$(\mathcal{J}\kappa)(x) - (C_{N,h}\mathcal{J}C_{N,h}\kappa)(x) = \mathrm{O}(1) \quad (N \to \infty).$$

*Proof.* We have

$$(5.26) \qquad |(\mathcal{J}\kappa)(x) - (C_{N,h}\mathcal{J}C_{N,h}\kappa)(x)|$$
$$\le |(\mathcal{J}\kappa)(x) - (C_{N,h}\mathcal{J}\kappa)(x)| + |(C_{N,h}\mathcal{J}\kappa)(x) - (C_{N,h}\mathcal{J}C_h\kappa)(x)|$$
$$+ |(C_{N,h}\mathcal{J}C_h\kappa)(x) - (C_{N,h}\mathcal{J}C_{N,h}\kappa)(x)|.$$

First, the second and third terms on the right-hand side of (5.26) are bounded. This is because we can apply the estimates like (5.22) and (5.23) in the proof of Theorem 2.3, respectively. Next, it follows from $\sup_{-\infty < x < \infty} |(\mathcal{J}\kappa)(z)| < \infty$ and the arguments similar to those used in (5.23) that the first term of the right-hand side of (5.26) is bounded. □

Let $\tilde{r}_N = (h \sum_{k=-N}^{N} f(kh))\kappa$ and $\tilde{g}_N = f - \tilde{r}_N$. Then the quadrature formula in Theorem 2.5 can be written as

$$(5.27) \qquad (\mathcal{J}\tilde{r}_N)(x) + (C_{N,h}\mathcal{J}C_{N,h}\tilde{g}_N)(x),$$

and the difference between it and the quadrature formula in Theorem 2.4 can be written as

$$(5.28) \qquad (\mathcal{J}(\tilde{r}_N - r))(x) + (C_{N,h}\mathcal{J}C_{N,h}(\tilde{g}_N - g))(x).$$

Hence, to prove (2.44), it suffices to show

$$(\mathcal{J}(\tilde{r}_N - r))(x) + (C_{N,h}\mathcal{J}C_{N,h}(\tilde{g}_N - g))(x) = \mathrm{o}\left( \exp \left[ \frac{-\pi(d_g - \varepsilon)\gamma_g N}{\log(\pi(d_g - \varepsilon)\gamma_g N/\beta_g)} \right] \right).$$

However it can easily be derived from Proposition 5.3 and Lemma 5.7 as follows:

$$|(\mathcal{J}(\tilde{r}_N - r))(x) + (C_{N,h}\mathcal{J}C_{N,h}(\tilde{g}_N - g))(x)|$$

$$= |(\mathcal{J}\kappa)(x) - (C_{N,h}\mathcal{J}C_{N,h}\kappa)(x)| \left| \int_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t - h \sum_{k=-N}^{N} \hat{f}(kh) \right|$$

$$\le c' \exp \left[ \frac{-2\pi d_f \gamma_g N}{\log(\pi(d_g - \varepsilon)\gamma_g N/\beta_g)} \right] = \mathrm{o}\left( \exp \left[ \frac{-\pi(d_g - \varepsilon)\gamma_g N}{\log(\pi(d_g - \varepsilon)\gamma_g N/\beta_g)} \right] \right).$$

**5.3. Explanation of the argument in Subsection 2.4.** Some details of the argument in Subsection 2.4 are provided here. First note that (2.36) and (2.37) imply

$$(5.29) \qquad \gamma_g d_g \leq \gamma_f d_f,$$

where $\gamma_f d_f \leq \pi/2$ by Proposition 2.2. We divide into two cases according to the value of $\gamma_f d_f$.

<u>Case 1 $(\gamma_f d_f < \pi/2)$.</u> We take $B$ and $C$ as (2.47) and (2.48), respectively. Then, we have

$$\gamma_g = \min\{\gamma_f, C\} = \gamma_f,$$

$$d_g = \min\left\{d_f, \frac{1}{C}\arcsin\left[\frac{\pi}{2B}\right] - \varepsilon_d\right\}$$

$$= \min\left\{d_f, \frac{1}{\gamma_f}\arcsin\left[\frac{\sin(\gamma_f d_f)}{1 - (2\varepsilon_B/\pi)\sin(\gamma_f d_f)}\right] - \varepsilon_d\right\} = d_f,$$

because $\varepsilon_B > 0$ and $\varepsilon_d$ is sufficiently small. Hence we have $\gamma_g d_g$ attains the upper bound $\gamma_f d_f$ in (5.29). Note that this upper bound is attained only when $\gamma_g = \gamma_f$ and $d_g = d_f$, and this, in turn, implies

$$(5.30) \qquad 0 < B < \frac{\pi}{2},$$

or

$$(5.31) \qquad \frac{\pi}{2} \leq B < \frac{\pi}{2\sin(\gamma_f d_f)},$$

where the latter follows from

$$\pi/2 \leq B, \ \gamma_f \leq C, \ d_f < (1/C)\arcsin(\pi/(2B)).$$

Thus $\beta_g = \min\{\beta_f, B - \varepsilon_B\}$ is nearly optimal.

<u>Case 2 $(\gamma_f d_f = \pi/2)$.</u> Since

$$\gamma_g d_g \leq \gamma_\kappa d_\kappa < \frac{\pi}{2},$$

the upper bound $\gamma_f d_f$ in (5.29) cannot be attained in the case of $\gamma_f d_f = \pi/2$. However, $\gamma_g d_g$ can be made arbitrarily close to $\pi/2$ with the choice of $B = \pi/2$ and $C = \gamma_f$ given in (2.52) and (2.53), for which we have

$$\gamma_g d_g = \min\{\gamma_f, C\}\min\left\{d_f, \frac{\pi}{2C} - \varepsilon_d\right\}$$

$$= \gamma_f \min\left\{d_f, \frac{\pi}{2\gamma_f} - \varepsilon_d\right\}$$

$$= \frac{\pi}{2} - \varepsilon_d\gamma_f.$$

To maximize $\beta_g = \min\{\beta_f, B - \varepsilon_B\}$ we are to choose $B$ as large as possible, while keeping $\gamma_g d_g$ invariant. Our choice of $B = \pi/2$ is also optimal in this respect, since, for $B > \pi/2$, we have

$$(5.32) \qquad \gamma_f d_f - \gamma_g d_g > \frac{\pi}{2} - \arcsin\left(\frac{\pi}{2B}\right) > 0.$$

## 6. Concluding remarks

The error bound of our formula depends on individual integrands. Theoretically it is more desirable to have a formula that is valid for functions in a certain function space and has a uniform error bound that depends only on the function space. A possible deeper theoretical problem is concerned with the optimality of the formula in the function space. Here, the optimality means the superiority of the formula in its accuracy over any other formula applicable to the elements of the function space. For example, such a problem has been addressed for a trapezoidal formula [7] and a sinc interpolation [8].

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