MATHEMATICAL ENGINEERING TECHNICAL REPORTS

Static Gain Feedback Control Synthesis with General Frequency Domain Specifications

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(Communicated by Kazuo Murota)

METR 2003–37 November 2003

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Static Gain Feedback Control Synthesis with General Frequency Domain Specifications

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November 4, 2003

Abstract

This paper considers a robust control synthesis problem for uncertain linear systems to meet design specifications given in terms of multiple frequency domain inequalities in (semi)finite ranges. In this paper, we restrict our attention to static (gain feedback) controllers. We will develop a new multiplier method that allows for reduction of synthesis conditions to linear matrix inequality problems. We study conditions under which the reduction is exact (nonconservative) in the single-objective nominal setting. Although the multiplier method is conservative in the general setting of multi-objective robust control, numerical design examples demonstrate the utility of the method for the state feedback case.

1 Introduction

Design specifications for practical control synthesis are often given in terms of frequency domain inequalities (FDIs). Most state space approaches to such design problems rely on the Kalman-Yakubovich-Popov (KYP) lemma [1], [2] that converts an FDI to a linear matrix inequality (LMI) which is numerically tractable. While the standard KYP lemma characterizes FDIs in the entire frequency range, practical requirements are usually described by multiple FDIs in (semi)finite ranges; e.g., small sensitivity in a low frequency range and control roll-off in a high frequency range. Hence some sort of "adaptors," such as the weighting functions, have been used to fit the requirements into the KYP framework. However, the design iterations to search for the right weighting functions can be tedious and time consuming, and the controller complexity (order) tends to increase with the complexity of the weighting functions.

The objective of this paper is to develop a state space design theory that is capable of directly treating multiple FDI specifications in various frequency ranges without introducing weighting functions. To our knowledge, this problem has not been addressed in the literature. Our approach is based on the generalized Kalman-Yakubovich-Popov (GKYP) lemma [3]–[5], recently developed by the authors, that provides an LMI characterization of FDIs in (semi)finite frequency ranges. We will first give a dual version of the GKYP lemma which is more suitable than the primal for feedback synthesis. A multiplier method is then developed to render the synthesis conditions convex through a simple change of variable, in the static gain feedback setting. We discuss cases where the multiplier method is nonconservative for the single-objective nominal design. The method is extended, with some conservatism, for the multi-objective robust control synthesis for systems with polytopic uncertainties. Design examples will demonstrate applicability of our results.

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We use the following notation. For a matrix M, its transpose, complex conjugate transpose, the Moore-Penrose inverse, and the null space are denoted by M^{T}, M^*, $M^\dagger$, and $\mathcal{N}(M)$, respectively. The Hermitian part of a square matrix M is denoted by $\mathrm{He}(M) := M + M^*$. For a Hermitian matrix, $M > (\geq) 0$ and $M < (\leq) 0$ denote positive (semi)definiteness and negative (semi)definiteness. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. The subscript n will be omitted if $n = 2$. For matrices Φ and P, $\Phi \otimes P$ means their Kronecker product. For $G \in \mathbb{C}^{n\times m}$ and $\Pi \in \mathbf{H}_{n+m}$, a function $\sigma :\ \mathbb{C}^{n\times m} \times \mathbf{H}_{n+m} \to \mathbf{H}_m$ is defined by

$$\sigma(G, \Pi) := \begin{bmatrix} G \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I_m \end{bmatrix}.$$

Given a positive integer q, let $\mathbf{Z}_q$ be the set of positive integers up to q, i.e., $\mathbf{Z}_q := \{ 1,\ 2,\ \ldots,\ q \}$.

2 Problem statement and formulation

2.1 Problem statement

Consider the plant $G(\lambda)$ described by

$$\begin{bmatrix} \lambda x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{1}$$

with a static gain feedback control $u = Ky$ where λ is the frequency variable (s for continuous-time and z for discrete-time cases), and $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, and $y(t) \in \mathbb{R}^{n_y}$. Denote by $G(\lambda) \star K$ the closed-loop transfer function from w to z. The control synthesis problem of our interest is, given $\Pi \in \mathbf{H}_{n_w+n_z}$ and $\Phi, \Psi \in \mathbf{H}$, find a stabilizing feedback gain K such that

$$\sigma((G(\lambda) \star K)^*, \Pi) < 0 \quad \forall\, \lambda \in \bar{\mathbf{\Lambda}}(\Phi^{\mathsf{T}}, \Psi^{\mathsf{T}}). \tag{2}$$

where

$$\mathbf{\Lambda}(\Phi, \Psi) := \{ \lambda \in \mathbb{C} \mid \sigma(\lambda, \Phi) = 0,\ \sigma(\lambda, \Psi) \geq 0 \} \tag{3}$$

and $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda}$ if $\mathbf{\Lambda}$ is bounded and $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda} \cup \{\infty\}$ if unbounded.

For clarity of exposition, we shall restrict our attention to this single-objective nominal control problem in the main body of our theoretical developments. However, we will later discuss extensions to a more general problem where there are multiple FDI constraints of the above form as well as some uncertainty in the plant model.

2.2 Problem formulation via a dual GKYP lemma

Consider a transfer function

$$G(\lambda) = C(\lambda I - A)^{-1}B + D, \tag{4}$$

where $A \in \mathbb{C}^{n\times n}$, $D \in \mathbb{C}^{n_z\times n_w}$. The GKYP lemma in [3] provides a characterization of the FDI: $\sigma(G(\lambda), \Pi) < 0$ for all $\lambda \in \bar{\mathbf{\Lambda}}(\Phi, \Psi)$. The following result provides a dual version of the GKYP lemma.

Theorem 1 Let $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, and $G(\lambda)$ in (4) be given and consider $\mathbf{\Lambda}(\Phi^{\mathsf{T}}, \Psi^{\mathsf{T}})$ defined by (3). Suppose $\mathbf{\Lambda}$ represents curves on the complex plane and A has no eigenvalues in $\mathbf{\Lambda}$. The following statements are equivalent.

(i) $\sigma(\mathsf{G}(\lambda)^*, \Pi) < 0$ holds for all $\lambda \in \bar{\mathbf{\Lambda}}(\Phi^\mathsf{T}, \Psi^\mathsf{T})$.

(ii) There exist $P = P^*$ and $Q = Q^* > 0$ such that

$$N \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} N^* < 0, \quad N := \begin{bmatrix} \mathcal{M} & I \end{bmatrix} T, \quad \mathcal{M} := \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{bmatrix} \tag{5}$$

where T is the permutation matrix such that

$$[\, M_1 \; M_2 \; M_3 \; M_4 \,]T = [\, M_1 \; M_3 \; M_2 \; M_4 \,] \tag{6}$$

for arbitrary matrices M_1, M_2, M_3, and M_4 with column dimensions n, n_w, n, and n_z, respectively.

Proof. Define

$$\mathsf{G}_*(\lambda) := \mathsf{G}(\lambda^*)^* = \mathsf{B}^*(\lambda I - \mathsf{A}^*)^{-1}\mathsf{C}^* + \mathsf{D}^*.$$

Noting that the identity

$$\sigma(\lambda, \Phi) = \sigma(\lambda^*, \Phi^\mathsf{T})$$

holds for any $\lambda \in \mathbb{C}$ and $\Phi \in \mathbf{H}$, we see that

$$\begin{aligned} & \sigma(\mathsf{G}(\lambda)^*, \Pi) < 0 \quad \forall \lambda \in \mathbf{\Lambda}(\Phi^\mathsf{T}, \Psi^\mathsf{T}) \\ \Leftrightarrow \;& \sigma(\mathsf{G}_*(\lambda^*), \Pi) < 0 \quad \forall \lambda \in \mathbf{\Lambda}(\Phi^\mathsf{T}, \Psi^\mathsf{T}) \\ \Leftrightarrow \;& \sigma(\mathsf{G}_*(\lambda), \Pi) < 0 \quad \forall \lambda^* \in \mathbf{\Lambda}(\Phi^\mathsf{T}, \Psi^\mathsf{T}) \\ \Leftrightarrow \;& \sigma(\mathsf{G}_*(\lambda), \Pi) < 0 \quad \forall \lambda \in \mathbf{\Lambda}(\Phi, \Psi). \end{aligned}$$

From the result in [3], we readily obtain the result. ■

With the result of Theorem 1, and ignoring the stability requirement for the moment, a synthesis problem may be formulated as the search for the parameters $Q > 0$, P, and K satisfying (5) with $\mathcal{M}$ defined to be the state space matrices of $G(\lambda) \star K$ as follows:

$$\begin{aligned} \mathcal{M} &:= \mathcal{A} + \mathcal{B}K\mathcal{C} \\ &= \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} K \begin{bmatrix} C_2 & D_{21} \end{bmatrix}. \end{aligned} \tag{7}$$

The resulting condition is not convex due to the product terms between the parameters P, Q, and K. In the next section, we shall develop a multiplier method to re-parametrize the condition so that the problem becomes convex. Throughout the paper, we will assume that $\mathcal{C}$ has full row rank without loss of generality.

3 Synthesis with nullspace filling multiplier

3.1 Basic idea

By the Finsler's theorem [6], condition (5) is equivalent to the existence of a multiplier $\mathcal{W}$ such that

$$T \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} T^* < \mathrm{He} \begin{bmatrix} -I \\ \mathcal{M} \end{bmatrix} \mathcal{W}. \tag{8}$$

Note that (5) holds if and only if the left hand side (LHS) of (8) is negative definite on the range space of $[\,\mathcal{M}\ \ I\,]^*$. The role of the multiplier $\mathcal{W}$ is to fill the orthogonal subspace, the nullspace of $[\,\mathcal{M}\ \ I\,]$, so that the LHS with this modification becomes negative definite as in (8). The synthesis problem now is to compute $Q > 0$, P, $\mathcal{W}$ and K satisfying condition (8). This is still a nonconvex problem due to the product term between K and $\mathcal{W}$.

To make the problem tractable, we shall restrict the class of multipliers $\mathcal{W}$ to be

$$\mathbf{W}(\mathcal{C}, R) := \{\, \mathcal{C}^\dagger W R + (I - \mathcal{C}^\dagger \mathcal{C})V \mid W \in \mathbb{C}^{n_y \times n_y},\ \det(W) \neq 0,\ V \in \mathbb{C}^{(n+n_w)\times(2n+n_w+n_z)} \,\} \tag{9}$$

where $R \in \mathbb{C}^{n_y \times (2n+n_w+n_z)}$ is a matrix to be specified later. In this case, the product term can be made linear in terms of the new variable $\mathcal{K} := KW$ as follows:

$$\mathcal{M}\mathcal{W} = (\mathcal{A} + \mathcal{B}K\mathcal{C})(\mathcal{C}^\dagger W R + (I - \mathcal{C}^\dagger \mathcal{C})V) = \mathcal{A}\mathcal{W} + \mathcal{B}\mathcal{K}R.$$

Thus, the synthesis equation (8) becomes an LMI in terms of the parameters Q, P, $\mathcal{W}$, and $\mathcal{K}$. Moreover, the above change of variable is invertible and the feedback gain can be found by $K = \mathcal{K}W^{-1}$.

We now turn our attention to the choice of R. We would like to specify R so that the restriction $\mathcal{W} \in \mathbf{W}(\mathcal{C}, R)$ can be made without introducing conservatism. The following theorem gives a characterization of such R and summarizes the synthesis LMI.

Theorem 2 Let $R \in \mathbb{C}^{n_y \times (2n+n_w+n_z)}$, $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, $P, Q \in \mathbf{H}_n$, and the system in (1) be given. The following statements are equivalent.

(i) There exist a feedback gain K and a real scalar $\mu > 0$ such that conditions (5) and

$$S(TXT^* - \mu R^* R)S^* < 0 \tag{10}$$

$$S := \begin{bmatrix} \mathcal{M} & I \\ \mathcal{C} & 0 \end{bmatrix}, \quad X := \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix}$$

are satisfied where $\mathcal{M}$ is defined in (7).

(ii) There exist matrices $\mathcal{W} \in \mathbf{W}(\mathcal{C}, R)$, and $\mathcal{K}$ such that

$$T \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} T^* < \mathrm{He} \begin{bmatrix} -\mathcal{W} \\ \mathcal{A}\mathcal{W} + \mathcal{B}\mathcal{K}R \end{bmatrix}.$$

If (ii) holds, a gain in (i) can be given by $K := \mathcal{K}W^{-1}$.

Proof. Consider (8) with $\mathcal{W} \in \mathbf{W}(\mathcal{C}, R)$. By the projection lemma, there exists V satisfying this condition if and only if

$$STX(ST)^* < \mathrm{He}\, JWRS^*, \quad J := \begin{bmatrix} 0 \\ -I \end{bmatrix},$$

where we noted that the null space of S is equal to the range space of

$$\begin{bmatrix} -I \\ \mathcal{M} \end{bmatrix} (I - \mathcal{C}^\dagger \mathcal{C}).$$

Again by the projection lemma, there exists W satisfying the above inequality if and only if two conditions

$$\mathcal{N}(J^*)^*(ST)X(ST)^*\mathcal{N}(J^*) < 0, \quad \mathcal{N}(RS^*)^*(ST)X(ST)^*\mathcal{N}(RS^*) < 0$$

hold. The first condition is directly equivalent to (5) and the second condition is equivalent, through the Finsler's theorem, to the existence of a scalar μ such that

$$(ST)X(ST)^* < \mu(RS^*)^*(RS^*)$$

which is exactly the same as (10). ■

The nullspace filling multiplier introduced above may be considered as a generalization of the multipliers developed by de Oliveira, Bernussou, Geromel, and others, for robust stability analysis of systems with polytopic uncertainties [7]–[12]. The main advantage of this type of formulation is that there is no product term between the system parameters $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and the "Lyapunov" parameters (P, Q). An implication is that one can easily obtain vertex-type results for robust control analysis as well as synthesis. Moreover, the formulation is also useful for multi-objective control as we discuss later.

The condition in statement (ii) of Theorem 2 is given in terms of LMIs and hence can be numerically solved by semidefinite programming. Here, we note that the nonsingularity constraint on $W \in \mathbf{W}(\mathcal{C}, R)$ can be ignored when solving the LMIs because a perturbation argument applies due to the strictness of the LMI.

We see that statement (ii) gives a *sufficient* condition for existence of K satisfying (5), regardless of the choice of R. Moreover, the condition is also *necessary* if R is chosen to satisfy (10). It can be verified that a particular choice:

$$R = \mathcal{C} \left[\begin{array}{cc} I & -\mathcal{M}^* \end{array} \right],$$

satisfies (10) for a sufficiently large $\mu > 0$, provided P, Q, and K solve (5). This means that an appropriate R exists whenever the original synthesis problem is feasible. However, the above choice of R is not practical because it depends on the unknown controller parameter. It would be useful if we could specify R that satisfies (10) for *arbitrary choices of* $Q > 0$, P, and K. This may not always be possible, but can be done for certain cases as shown in the next section.

3.2 Specific cases for exact synthesis

We shall specialize Theorem 2 for some specific cases of $\mathcal{C}$ and give particular choices of R that lead to LMI synthesis conditions which are nonconservative. To this end, let us introduce the following:

Assumption 1

(a) $\mathbf{\Lambda}$ represents curves on the complex plane, and Ψ is active in $\mathbf{\Lambda}$ in the sense that $\mathbf{\Lambda}(\Phi^{\mathsf{T}}, \Psi^{\mathsf{T}}) \neq \mathbf{\Lambda}(\Phi^{\mathsf{T}}, 0)$.

(b) At least n_z eigenvalues of Π are negative.

The first part of Item (a) is a natural condition within the framework of our control specifications expressed in terms of restricted frequency inequalities. The second part means that the frequency range $\mathbf{\Lambda}$ is not the entire $j\omega$ axis nor the unit circle, but a partial segment (or segments) of it. (See [4] for details.) Item (b) is a necessary condition for feasibility of the control synthesis problem and hence can be imposed without loss of generality.

Corollary 1 (Full Information) Let $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, $P, Q \in \mathbf{H}_n$, and the system in (1) be given. Suppose $Q > 0$ and Assumption 1 hold, and consider the full information case

$$\mathcal{C} = \left[\begin{array}{cc} C_2 & D_{21} \end{array} \right] = I.$$

Then there exists a feedback gain K satisfying (5) if and only if statement (ii) in Theorem 2 holds, provided R is chosen as follows. Let $N \in \mathbb{C}^{(2n+n_w+n_z)\times(n+n_z)}$ be a full column rank matrix such that $N^*XN < 0$ for all P

and $Q > 0$. Then define R to be a full row rank matrix with n_y rows such that $RTN = 0$. In particular, one such N is given by

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad N_1 = \begin{bmatrix} pI_n \\ qI_n \end{bmatrix}, \quad N_2^* \Pi N_2 < 0$$

where $r := [\, p \;\; q \,]^* \in \mathbb{C}^2$ is such that $r^* \Phi r = 0$ and $r^* \Psi r < 0$, and the column dimension of N_2 is n_z. Existence of such r and N_2 is guaranteed by Assumption 1.

Proof. In the full information case ($\mathcal{C} = I$), the matrix S is nonsingular and hence condition (10) becomes $TXT^* < \mu R^* R$, or equivalently, $X < \mu(RT)^*(RT)$. There exists μ satisfying this inequality if and only if $N^* X N < 0$ where N is the null space of RT. Finally, to verify that the given N satisfies $N^* X N < 0$, it suffices to note that

$$\begin{bmatrix} pI \\ qI \end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} pI \\ qI \end{bmatrix} = (r^* \Phi r) P + (r^* \Psi r) Q < 0$$

holds for any P and $Q = Q^* > 0$. ∎

Corollary 1 gives an exact solution to the full information synthesis problem with an FDI in a bounded frequency range. If the FDI specification is given for the entire frequency range (i.e., the second condition in Assumption 1(a) is violated), Corollary 1 can be modified as follows. First note that the parameter Q can be set to zero without loss of generality, and the parameter P may be required to be positive definite to enforce a stability constraint. We can then specify an appropriate R by choosing r so that $r^* \Phi r$ is negative and modifying N accordingly.

Corollary 2 (State Feedback) Let $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w + n_z}$, $P, Q \in \mathbf{H}_n$, and the system in (1) be given. Suppose $Q > 0$ and Assumption 1 hold, and consider the state feedback case

$$\begin{bmatrix} C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix}. \tag{11}$$

Suppose further that:

$$\Phi_{11} = 0, \quad \Psi_{11} < 0, \quad \sigma(D_{11}^*, \Pi) < 0. \tag{12}$$

Then there exists a feedback gain K satisfying (5) if and only if statement (ii) in Theorem 2 holds, provided we choose

$$R := \begin{bmatrix} 0_n & 0 & I_n & 0 \end{bmatrix}$$

where R is partitioned so that the numbers of columns are n, n_w, n, and n_z from left to right.

Proof. Note that the null space of RS^* is given by $[\, 0 \;\; I_{n+n_z} \,]^*$. Hence there exists μ satisfying (10) if and only if the lower right $(n + n_z) \times (n + n_z)$ block matrix of $(ST)X(ST)^*$ is negative definite. It can be verified using the Schur complement that this condition is satisfied if and only if

$$\Phi_{11} P + \Psi_{11} Q < 0, \quad \sigma(\mathsf{D}^*, \Pi) < 0.$$

The result now follows from Theorem 2. ∎

The conditions in (12) are satisfied when the control specifications are given in terms of a continuous-time ($\Phi_{11} = 0$), bounded frequency ($\Psi_{11} < 0$) inequality condition that holds at infinite frequency ($\sigma(\mathsf{D}^*, \Pi) < 0$). The last condition is met if, for instance, $D_{11} = 0$ and the origin is included in the feasible domain defined by the set of G such that $\sigma(\mathsf{G}^*, \Pi) < 0$.

3.3 Heuristics for general state feedback synthesis

In this section, we will consider the general state feedback case and develop some potentially conservative but reasonable choices for R. First note that, for the state feedback case where (11) holds, condition (10) is equivalent to

$$S_o(TXT^* - \mu R^* R)S_o^* < 0, \quad S_o := \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & B_1 & I_n & 0 \\ 0 & D_{11} & 0 & I_{n_z} \end{bmatrix}$$

because S^* and S_o^* share the same range space. Let L be the null space of RS_o^* and partition it as follows:

$$\mathcal{N}(RS_o^*) = L = \begin{bmatrix} L_1^* & L_2^* & L_3^* \end{bmatrix}^*,$$

where the row dimensions of L_1, L_2, and L_3 are n, n, and n_z, respectively, and the column dimension is greater than or equal to $n + n_z$. Then condition (10) can be written as

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} + \begin{bmatrix} L_2 \\ L_3 \end{bmatrix}^* \begin{bmatrix} B_1 & 0 \\ D_{11} & I \end{bmatrix} \Pi \begin{bmatrix} B_1 & 0 \\ D_{11} & I \end{bmatrix}^* \begin{bmatrix} L_2 \\ L_3 \end{bmatrix} < 0. \tag{13}$$

If we can choose a full column-rank matrix L satisfying this condition, R can always be found so that $\mathcal{N}(RS_o^*) = L$, and for such R there exists μ such that (10) holds.

However, as noted above, it is difficult in general to find L such that the condition holds for all P and $Q > 0$. A heuristic, but reasonable choice of L may be given by

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \\ 0 & L_{32} \end{bmatrix}, \qquad \begin{array}{ll} L_{11} \in \mathbb{C}^{n \times n}, & L_{22} \in \mathbb{C}^{n \times n_z}, \\ L_{21} \in \mathbb{C}^{n \times n}, & L_{32} \in \mathbb{C}^{n_z \times n_z}, \end{array}$$

where the submatrices of L are chosen to satisfy

$$\begin{bmatrix} L_{11} \\ L_{21} \end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} L_{11} \\ L_{21} \end{bmatrix} < 0, \quad \begin{bmatrix} L_{22} \\ L_{32} \end{bmatrix}^* \begin{bmatrix} B_1 & 0 \\ D_{11} & I \end{bmatrix} \Pi \begin{bmatrix} B_1 & 0 \\ D_{11} & I \end{bmatrix}^* \begin{bmatrix} L_{22} \\ L_{32} \end{bmatrix} < 0$$

for all P and $Q > 0$. We claim that this is "reasonable" because the the upper left $n \times n$ and lower right $n_z \times n_z$ block matrices of the left hand side of (13) are both negative definite, which are necessary conditions for existence of μ satisfying (10). Although these are not sufficient in general, this approach allows us to choose L that is independent of P and Q.

To be specific, let us impose Assumption 1. Then we can choose

$$\begin{bmatrix} L_{11} \\ L_{21} \end{bmatrix} = \begin{bmatrix} pI \\ qI \end{bmatrix}, \quad r := \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \begin{array}{l} r^* \Phi r = 0, \\ r^* \Psi r < 0, \end{array} \tag{14}$$

so that the first inequality is satisfied for any P and $Q > 0$. The matrices L_{22} and L_{32} satisfying the second inequality exist as Π has n_z negative eigenvalues, provided B_1 has full column rank. In particular, let $\tilde{L}_{22}$ and $\tilde{L}_{32}$ be such that

$$\begin{bmatrix} \tilde{L}_{22} \\ \tilde{L}_{32} \end{bmatrix}^* \Pi \begin{bmatrix} \tilde{L}_{22} \\ \tilde{L}_{32} \end{bmatrix} < 0, \qquad \begin{array}{l} \tilde{L}_{22} \in \mathbb{C}^{n_w \times n_z}, \\ \tilde{L}_{32} \in \mathbb{C}^{n_z \times n_z}. \end{array}$$

Then we have

$$L_{22} := (B_1^\dagger)^* (\tilde{L}_{22} - D_{11}^* \tilde{L}_{32}), \quad L_{32} := \tilde{L}_{32}. \tag{15}$$

To compute R, first determine L through (14) and (15). Then find R from the null space of L^*. We now have a systematic method for determining a reasonable R.

As an example, consider the case of continuous-time, small gain condition in the low frequency range. In this case, L and the corresponding R can be chosen as

$$L = \begin{bmatrix} I_n & 0 \\ 0 & -(D_{11}B_1^\dagger)^* \\ 0 & I_{n_z} \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & I_n & (D_{11}B_1^\dagger)^* \end{bmatrix}. \tag{16}$$

For the continuous-time, small gain condition in the high frequency range, we have

$$L = \begin{bmatrix} 0 & 0 \\ I_n & -(D_{11}B_1^\dagger)^* \\ 0 & I_{n_z} \end{bmatrix}, \quad R = \begin{bmatrix} I_n & 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

We will later illustrate applicability of the heuristic choices of R presented here through numerical design examples.

4 Extensions

4.1 Multi-objective, robust control

We consider the following problem: Find K such that

$$\sigma((G_k(\lambda) \star K)^*, \Pi_k) < 0 \quad \forall\, \lambda \in \bar{\mathbf{\Lambda}}(\Phi_k^\intercal, \Psi_k^\intercal) \tag{18}$$

holds for all $k \in \mathbf{Z}_q$ where each $G_k(\lambda)$ is a given plant, and (Φ_k, Ψ_k, Π_k) defines a frequency domain specification to be achieved for the closed-loop system $G_k(\lambda) \star K$. The plant $G_k(\lambda)$ may represent a vertex of a set of uncertain systems for robust control, or a plant with a selected disturbance-performance (i.e., w-z) channel for multi-objective control. The following result can be obtained from Theorem 2 in a straightforward manner, and hence its proof is omitted.

Corollary 3 Let $R_k \in \mathbb{C}^{n_y \times (2n+n_w+n_z)}$, $\Phi_k, \Psi_k \in \mathbf{H}$, $\Pi_k \in \mathbf{H}_{n_w+n_z}$, and systems $G_k(\lambda)$ as in (1) be given where $k \in \mathbf{Z}_q$. There exists a static feedback gain K such that the frequency domain specifications (18) are satisfied for all $k \in \mathbf{Z}_q$ if there exist scalars $\alpha_k > 0$ and matrices $P_k = P_k^*$, $Q_k = Q_k^* > 0$, $\mathcal{W}_k \in \mathbf{W}(\mathcal{C}_k, R_k)$, and $\mathcal{K}$ such that all $\mathcal{W}_k$ have a common W and

$$T_k \begin{bmatrix} \Phi_k \otimes P_k + \Psi_k \otimes Q_k & 0 \\ 0 & \alpha_k \Pi_k \end{bmatrix} T_k^* < \mathrm{He} \begin{bmatrix} -\mathcal{W}_k \\ \mathcal{A}_k \mathcal{W}_k + \mathcal{B}_k \mathcal{K} R_k \end{bmatrix} \tag{19}$$

holds for all $k \in \mathbf{Z}_q$, where T_k and $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k)$ are defined as in (6) and (7) using the input/state/output dimensions and the state space matrices of $G_k(\lambda)$. In this case, one such gain is given by $K := \mathcal{K}W^{-1}$.

Corollary 3 gives a sufficient condition for the existence of static feedback gain that achieves the multiple FDI specifications in (18). The condition is given in terms of LMIs and can be solved numerically. The associated degree of conservatism is dependent upon the choices of R_k. In the full information or state feedback case, some reasonable choices have been proposed in the previous section. It should be noted that this formulation does not assume common "Lyapunov matrices" (P, Q) as in the quadratic stability literature [13] or in the more recent multi-objective control [14], [15], but rather, (P, Q) can be interpreted as "parameter-dependent" as discussed in [7]–[9], [16]. Thus we expect reduced conservatism when compared with these existing techniques for multi-objective robust control. It should be emphasized, however, that the main contribution of this paper is not the conservatism reduction but the synthesis method to meet FDI specifications in (semi)finite frequency ranges, which have not been addressed in the literature.

The above formulation naturally captures the multi-objective control in the sense that the control gain K is designed so that each specification defined by (Φ_k, Ψ_k, Π_k) is met for the corresponding plant $G_k(\lambda)$. However, the formulation is not suitable for robust control synthesis in its present form. To elaborate on this point, let $\mathcal{I}$ be a subset of indices $\mathbf{Z}_q$ corresponding to a robust performance specification (Φ_o, Ψ_o, Π_o) to be satisfied by a family of plants defined by the convex hull of the state space matrices of $G_k(\lambda)$ with $k \in \mathcal{I}$. In this case, we have $(\Phi_k, \Psi_k, \Pi_k) = (\Phi_o, \Psi_o, \Pi_o)$, $T_k = T_o$, and $R_k = R_o$ for all $k \in \mathcal{I}$. We shall assume that the family of plants share a common measured output signal, i.e., $\mathcal{C}_k = \mathcal{C}_o$ for all $k \in \mathcal{I}$. The sufficient condition in Corollary 3 guarantees the performance (Φ_o, Ψ_o, Π_o) for each vertex plant $G_k(\lambda)$, but not for every plants in the convex hull. This deficiency can be overcome, with some additional conservatism, by using a common $\mathcal{W}_o = \mathcal{W}_k$ for all $k \in \mathcal{I}$ (see the example in Section 5 for a relaxed version of this idea). In this case, every coefficients of $\mathcal{A}_k$ and $\mathcal{B}_k$ for $k \in \mathcal{I}$ become independent of k, allowing for an arbitrary convex combination of (19) to be taken to conclude the robust performance.

4.2 Regional pole constraints

The design specifications in (18) encompass frequency domain shaping of closed-loop transfer functions. However, the closed-loop stability has not been captured, and hence one may wish to include a stability constraint, or more generally, regional pole constraints, as an additional design specification. The following lemma gives a basic result for an eigenvalue characterization. The result has appeared earlier [17], [18] but a simple proof is given here for completeness.

Lemma 1 Let $A \in \mathbb{C}^{n\times n}$ and $\Phi \in \mathbf{H}$ be given. Suppose $\det(\Phi) < 0$. Then the following statements are equivalent.

(i) Each eigenvalue λ of A satisfies $\sigma(\lambda, \Phi) < 0$.

(ii) There exists $P = P^* > 0$ such that $\sigma(A, \Phi \otimes P) < 0$.

Proof. Suppose (ii) holds. Let λ be an eigenvalue of A with the corresponding eigenvector e. Then

$$e^*\sigma(A, \Phi \otimes P)e = \sigma(\lambda, \Phi)(e^*Pe) < 0.$$

Since $P > 0$, we conclude that (i) holds. To show the converse, suppose (i) holds. Since $\det(\Phi) < 0$, there exists a nonsingular matrix M such that

$$\Phi = M^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M, \quad M := \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{20}$$

It can be shown that $\det(M) \neq 0$ and $\det(\lambda I - A) \neq 0$ for any λ such that $\sigma(\lambda, \Phi) \neq 0$ imply $\det(dI + cA) \neq 0$. Define

$$A_o := (bI + aA)(dI + cA)^{-1}$$

and let λ_o be an eigenvalue of A_o. Then it can be verified that $c\lambda_o \neq a$ holds and $\lambda := (b - \lambda_o d)/(c\lambda_o - a)$ is an eigenvalue of A. Hence

$$\sigma(\lambda, \Phi) = \left|\frac{\det(M)}{c\lambda_o - a}\right|^2 (\lambda_o + \lambda_o^*) < 0,$$

implying that A_o is Hurwitz. It then follows that there exists $P = P^* > 0$ satisfying

$$0 > (dI + cA)^*(PA_o + A_o^*P)(dI + cA) = \sigma(A, \Phi \otimes P).$$

Thus we have (ii). ∎

Corollary 4 Let $A \in \mathbb{C}^{n\times n}$ and $\Phi \in \mathbf{H}$ be given. Suppose $\det(\Phi) < 0$. Then the following are equivalent.

(i) Each eigenvalue λ of A satisfies $\sigma(\lambda, \Phi^{\mathsf{T}}) < 0$.

(ii) There exists $P = P^* > 0$ such that $\sigma(A^*, \Phi \otimes P) < 0$.

(iii) There exist W and $P = P^* > 0$ such that

$$\Phi \otimes P < \mathrm{He} \begin{bmatrix} -I \\ A \end{bmatrix} W \begin{bmatrix} -qI & pI \end{bmatrix}$$

where $r := [\, p \;\; q \,]^* \in \mathbb{C}^2$ is an arbitrary fixed vector satisfying $r^*\Phi r < 0$.

Proof. The equivalence (i) ⇔ (ii) simply follows from Lemma 1 using the identity $\sigma(\lambda^*, \Phi) = \sigma(\lambda, \Phi^{\mathsf{T}})$. The equivalence (ii) ⇔ (iii) follows from the projection lemma. ∎

The condition in (iii) can be used to give additional constraints in the design equations discussed in the previous sections. In particular, we replace A with the closed-loop matrix $A + B_2K$ in the state feedback case, and introduce the change of variable $\mathcal{K} := KW$. As a result, we add the following constraint to the design:

$$\Phi \otimes P < \mathrm{He} \begin{bmatrix} -W \\ AW + B_2\mathcal{K} \end{bmatrix} \begin{bmatrix} -qI & pI \end{bmatrix}$$

Clearly, multiple inequalities of the same form can be added to enforce (robust) regional pole constraints expressed as the intersection of half planes and circles. In this case, as in Corollary 3, different Φ, A, B_2, and P may be used for each inequality but W and $\mathcal{K}$ have to be common for all inequality constraints.

5 Design examples

5.1 Disturbance attenuation problem

We consider the classical ACC benchmark problem of cart-spring system. The plant is described by

$$\dot{x} = Ax + B_1w + B_2u, \quad z = Cx,$$

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

where $k = 1$ is the spring constant and the unit mass is assumed for each cart. Our objective is to design a stabilizing state feedback controller $u = Kx$ such that

$$|T_{zw}(j\omega)| < \gamma, \quad \forall\, |\omega| \le \varpi_\ell$$

$$|T_{uw}(j\omega)| < \rho, \quad \forall\, |\omega| \geq \varpi_h$$

hold, where T_{zw} and T_{uw} are the closed-loop transfer functions from w to z and u, respectively, γ and ρ are the performance bounds, and ϖ_ℓ and ϖ_h are the cut-off frequencies in the low/high ranges. From Corollary 4 and Theorem 2, the synthesis conditions are given by

$$\begin{bmatrix} 0 & P_s \\ P_s & 0 \end{bmatrix} < \text{He} \begin{bmatrix} -W \\ AW + B\mathcal{K} \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} -Q_\ell & 0 & P_\ell & 0 \\ 0 & \alpha & 0 & 0 \\ P_\ell & 0 & \varpi_\ell^2 Q_\ell & 0 \\ 0 & 0 & 0 & -\alpha\gamma^2 \end{bmatrix} < \text{He} \begin{bmatrix} -I & 0 & 0 \\ 0 & -1 & 0 \\ A & B_1 & B_2 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} WR_\ell \\ V_\ell \\ \mathcal{K}R_\ell \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} Q_h & 0 & P_h & 0 \\ 0 & \beta & 0 & 0 \\ P_\ell & 0 & -\varpi_h^2 Q_h & 0 \\ 0 & 0 & 0 & -\beta\rho^2 \end{bmatrix} < \text{He} \begin{bmatrix} -I & 0 & 0 \\ 0 & -1 & 0 \\ A & B_1 & B_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} WR_h \\ V_h \\ \mathcal{K}R_h \end{bmatrix} \tag{23}$$

where W, $\mathcal{K}$, $P_s = P_s^* > 0$, $P_\ell = P_\ell^*$, $Q_\ell = Q_\ell^* > 0$, V_ℓ, $P_h = P_h^*$, $Q_h = Q_h^* > 0$, V_h, and $\alpha, \beta > 0$ are the (real) variables and R_ℓ and R_h are given by R in (16) and (17), respectively. If these equations admit a solution, a feasible state feedback gain is given by $K = \mathcal{K}W^{-1}$.

We fixed the values ϖ_ℓ, ϖ_h, and γ as

$$\varpi_\ell = 2, \quad \varpi_h = 3, \quad \gamma = 2,$$

and then minimized ρ. The optimal value of ρ and the corresponding feedback gain K are found to be

$$\rho_{\min} = 0.52,$$
$$K = \begin{bmatrix} -1.4414 & 0.0802 & -1.7213 & -0.8622 \end{bmatrix}.$$

The resulting closed-loop transfer functions are shown in Fig 1 where the shaded regions indicate the bounds on the gain of the transfer functions. We see that the upper bounds are relatively tight, showing that the associated conservatism is not significant.

For the sake of illustration, we have changed the frequency interval of the constraint $|T_{zw}| < \gamma$ from $\varpi_\ell = 2$ to $\varpi_\ell = 1$. By this change, the natural frequency of the cart-spring system ($\sqrt{2}$ [rad/s]) is now outside of the frequency range. All the other parameters are fixed as before and the feasibility problem is solved. The resulting design is shown in Figs. 3 and 4. We see that the large peak in $|T_{zw}|$ is now allowed and the time response is lightly damped. If we minimize ρ, then it can get as low as 0.27 at the expense of a larger peak value $\|T_{zw}\|_\infty = 6.8$.

The results shown in Figs. 1 and 2 are compared with a mixed H_2/H_∞ multi-objective design [14]. In particular, the conditions

$$\|T_{zw}\|_\infty < \gamma, \quad \|T_{uw}\|_2 < \rho$$

can be conservatively reduced to the search for matrices $X = X^\mathsf{T}$ and $\mathcal{K}$ satisfying

$$\begin{bmatrix} \text{He}(AX + B_2\mathcal{K}) & XC^* & B_1 \\ CX & -\gamma I & 0 \\ B_1^* & 0 & -\gamma I \end{bmatrix} < 0,$$

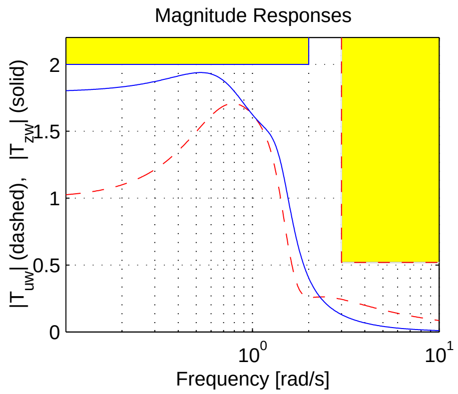

Figure 1: Transfer functions (Nominal design)

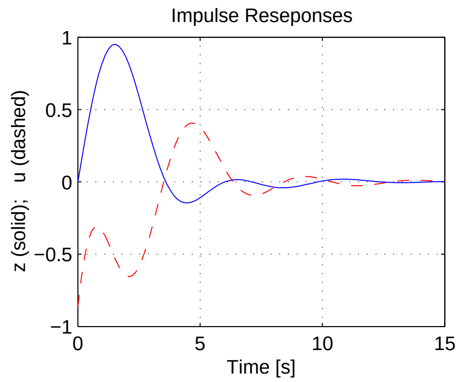

Figure 2: Impulse responses (Nominal design)

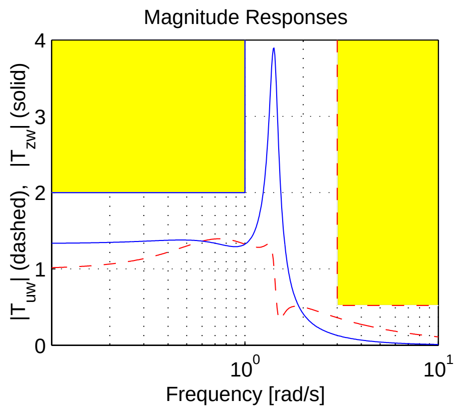

Figure 3: Transfer functions (Nominal bad design)

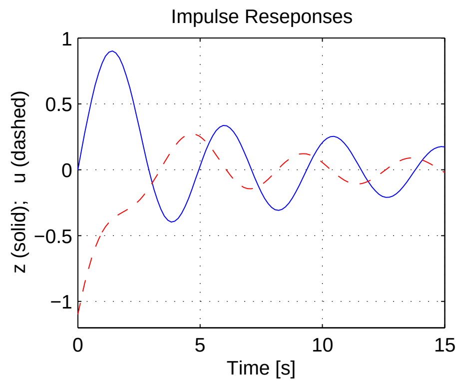

Figure 4: Impulse responses (Nominal bad design)

$$\mathrm{He}(AX + B_2\mathcal{K}) + B_1 B_1^* < 0, \qquad \left[\begin{array}{cc} \rho^2 & \mathcal{K} \\ \mathcal{K}^* & X \end{array}\right] > 0.$$

Once these LMIs are solved for X and $\mathcal{K}$, a state feedback gain is found by $K = \mathcal{K}X^{-1}$.

We have fixed the value of γ to be the same as that in our design, i.e., $\gamma = 2$, and then minimized ρ. This problem is meant to find the minimum energy control that achieves the same regulation performance as before. As a result, we obtained $\rho_{\min} = 2.00$ and

$$K = \left[\begin{array}{cccc} -2.8835 & 0.3727 & -2.4005 & -2.7767 \end{array}\right].$$

The corresponding closed-loop responses are shown in Figs. 5 and 6. We remark two things. First, the bound on the H_∞ norm of the transfer function T_{zw} is not very tight and the gap between the H_∞ norm bound and the actual norm shows the degree of conservatism for this design. Second, the transfer function T_{uw} does not roll off as much as in the previous design so that the peak value of $|u(t)|$ is three times larger, showing a limitation of the H_2 norm as a measure for the "control effort."

Next we consider the case where the spring constant k is uncertain but is known to lie in the interval $[1, 2]$. In this case, we will have the synthesis equations (21)-(23) for $k = 1$, and in addition, copies of these equations for $k = 2$ where the variables W and $\mathcal{K}$ are common but the others are not (e.g., we have two different P_h's for $k = 1$ and $k = 2$). Minimizing γ subject to these 6 LMIs and computing the gain by $K = \mathcal{K}W^{-1}$, we have

$$\varpi_\ell = 2, \quad \varpi_h = 4, \quad \rho = 0.5, \quad \gamma = 2.66,$$

$$K = \left[\begin{array}{cccc} -2.6736 & 0.8938 & -2.2190 & -1.1612 \end{array} \right].$$

The resulting closed-loop responses shown in Figs. 7 and 8, where the two solid curves in each figure show responses for cases $k = 1$ and 2, and similarly for the dashed curves. We see that conservatism is still moderate even for this robust design, suggesting a potential for practical applications.

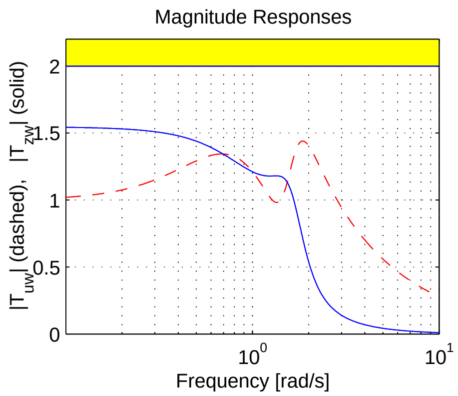

Figure 5: Transfer functions (H_2/H_∞ design)

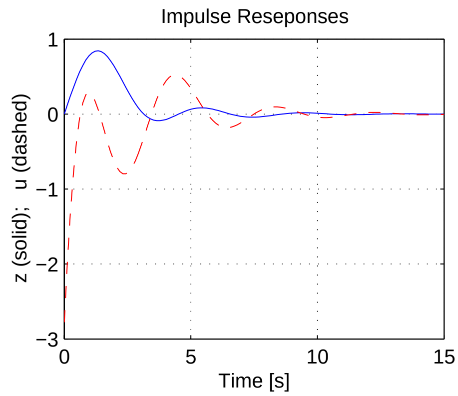

Figure 6: Impulse responses (H_2/H_∞ design)

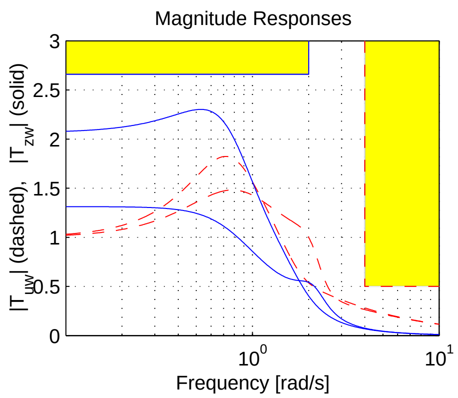

Figure 7: Transfer functions (Robust design)

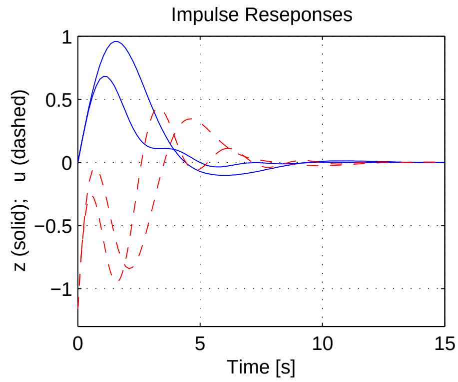

Figure 8: Impulse Responses (Robust design)

5.2 Sensitivity reduction problem

We consider the same cart-spring system as before but design state feedback controllers to achieve small sensitivity and complementary sensitivity within certain frequency ranges. In particular, our objective is to design a stabilizing state feedback controller $u = Kx$ such that

$$\left| \frac{1}{1 - KP} \right| < \gamma, \quad \forall\, |\omega| \leq \varpi_\ell$$

$$\left| \frac{KP}{1 - KP} \right| < \rho, \quad \forall\, |\omega| \geq \varpi_h$$

where $P(s) := (sI - A)^{-1}B$ is the plant transfer function from u to x with B defined by $B := B_2$. From Corollary 4 and Theorem 2, the synthesis condition is given by ahoaho

$$\left[\begin{array}{cc} 0 & P_s \\ P_s & 0 \end{array} \right] < \mathrm{He} \left[\begin{array}{c} -W \\ AW + B\mathcal{K} \end{array} \right] \left[\begin{array}{cc} I & I \end{array} \right] \tag{24}$$

$$
\begin{bmatrix} -Q_\ell & 0 & P_\ell & 0 \\ 0 & \alpha & 0 & 0 \\ P_\ell & 0 & \varpi_\ell^2 Q_\ell & 0 \\ 0 & 0 & 0 & -\alpha\gamma^2 \end{bmatrix} < \mathrm{He} \begin{bmatrix} -I & 0 & 0 \\ 0 & -1 & 0 \\ A & B & B \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} WR_\ell \\ V_\ell \\ \mathcal{K}R_\ell \end{bmatrix} \tag{25}
$$

$$
\begin{bmatrix} Q_h & 0 & P_h & 0 \\ 0 & \beta & 0 & 0 \\ P_\ell & 0 & -\varpi_h^2 Q_h & 0 \\ 0 & 0 & 0 & -\beta\rho^2 \end{bmatrix} < \mathrm{He} \begin{bmatrix} -I & 0 & 0 \\ 0 & -1 & 0 \\ A & B & B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} WR_h \\ V_h \\ \mathcal{K}R_h \end{bmatrix} \tag{26}
$$

where W, $\mathcal{K}$, $P_s = P_s^* > 0$, $P_\ell = P_\ell^*$, $Q_\ell = Q_\ell^* > 0$, V_ℓ, $P_h = P_h^*$, $Q_h = Q_h^* > 0$, V_h, $\alpha, \beta > 0$ are the (real) variables and R_ℓ and R_h are given by R in (16) and (17), respectively. If these equations admit a solution, a feasible state feedback gain is given by $K = \mathcal{K}W^{-1}$.

We fixed the values ϖ_ℓ, ϖ_h, and ρ, and then minimized γ, and the parameter values are

$$
\varpi_\ell = 1, \quad \varpi_h = 3, \quad \rho = 0.5, \quad \gamma_{\min} = 0.61.
$$

The feedback gain is found to be

$$
K = \begin{bmatrix} -0.1079 & 0.5825 & 0.9212 & 0.0845 \end{bmatrix}.
$$

and the resulting sensitivity functions are shown in Fig 9. We see that the upper bounds on the sensitivity functions are tight, showing that the associated conservatism is not significant.

Next we consider the case where the spring constant k is uncertain but is known to lie in the interval $[1, 2]$. In this case, we will have the synthesis equations (24)-(26) for $k = 1$, and in addition, copies of these equations for $k = 2$ where the variables W and $\mathcal{K}$ are common but the others are not (e.g., we have two different P_h's for $k = 1$ and $k = 2$). Solving these 6 LMIs to minimize γ, and computing the gain by $K = \mathcal{K}W^{-1}$, we have

$$
\gamma_{\min} = 0.78, \quad K = \begin{bmatrix} 0.6299 & 20.0484 & 34.0517 & 17.6313 \end{bmatrix}
$$

and the resulting sensitivity functions shown in Fig. 10.

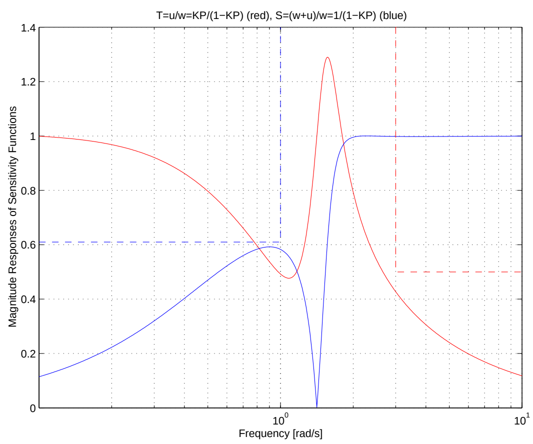

Figure 9: Sensitivity functions (Nominal design)

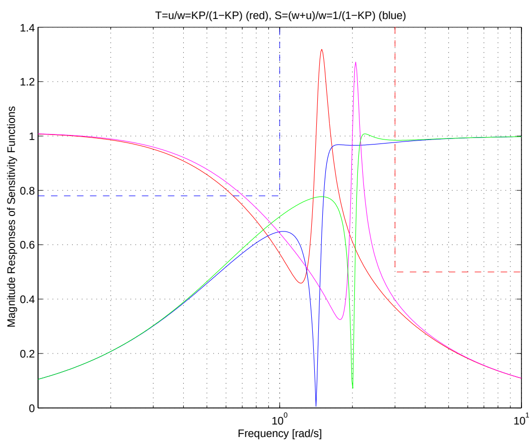

Figure 10: Sensitivity functions (Robust design)

Finally, we replace the stability requirement by a regional pole constraint. In the original nominal design, the closed-loop eigenvalues are

Poles	Damping	Frequency [rad/s]
-0.207 $\pm$ 1.51i	0.136	1.52
-0.00867	1.000	0.00867
-0.747	1.000	0.747

and we see that the system is lightly damped. The damping ratio can be increased by imposing a regional pole constraint. Although it is possible to set an exact bound on the damping ratio by two straight lines in the complex plane, we choose to use an approximation for simplicity. In particular, we require the closed-loop poles to lie in the circle with center at $-c$ and radius r. Then the first synthesis equation (24) is replaced by

$$\begin{bmatrix} P_s & cP_s \\ cP_s & (c^2 - r^2)P_s \end{bmatrix} < \mathrm{He} \begin{bmatrix} -W \\ AW + B\mathcal{K} \end{bmatrix} \begin{bmatrix} I & cI \end{bmatrix}. \tag{27}$$

Choosing $c = 4$, $r = 3.9$, and

$$\varpi_\ell = 1, \quad \varpi_h = 3, \quad \rho = 0.7,$$

we have found

$$\gamma_{\min} = 0.68, \quad K = \begin{bmatrix} -1.1494 & 0.9067 & -1.8851 & -0.1226 \end{bmatrix}$$

with the closed-loop poles

Poles	Damping	Frequency [rad/s]
-0.502 $\pm$ 1.38i	0.342	1.47
-0.725	1.000	0.725
-0.156	1.000	0.156

as illustrated in Fig. 11 and the sensitivity functions shown in Fig. 12.

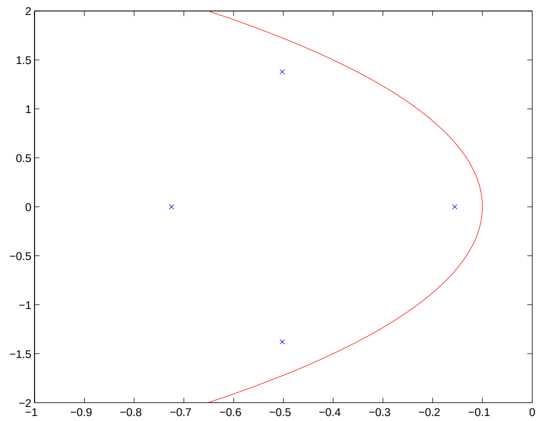

Figure 11: Closed-loop poles (Nominal design with pole constraint)

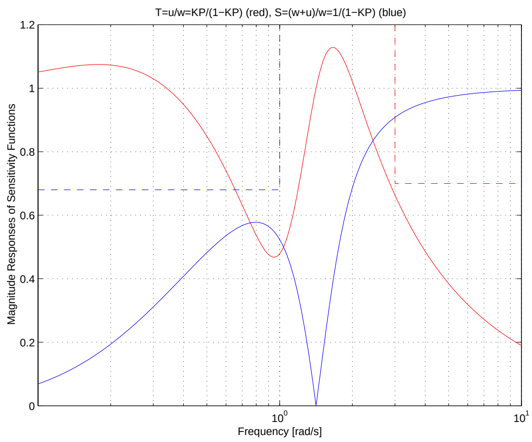

Figure 12: Sensitivity functions (Nominal design with pole constraint)

6 Conclusion

We have developed methods for synthesizing static feedback controllers to achieve, for a family of plants, multiple FDI specifications in (semi)finite frequency ranges. Sufficient conditions for existence of feasible controllers are given in terms of LMIs, and some special cases, where the conditions become also necessary, are discussed. Utility of our result for the general state feedback design is demonstrated through numerical examples.

Acknowledgments: This research is supported in part by The Ministry of Education, Science, Sport and Culture, Japan, under Grant 14550439, and by the 21st Century COE Program on Information Science and Technology Strategic Core.

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