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SUMMARY

In this paper we present indispensable moves of Markov bases for connected Markov chains over three-way contingency tables with fixed two-dimensional marginals. In Aoki and Takemura (2003a) we proved that there exists a unique minimal basis for $3 \times 3 \times K$ contingency tables consisting of four types of indispensable moves. Generalizing this result, we present a list of indispensable moves of the unique minimal Markov basis for $3 \times 4 \times K$ and $4 \times 4 \times 4$ contingency tables. This list allows us to actually perform exact tests of no three-factor interaction in three-way tables of these sizes. There are 21 types of indispensable moves for the $3 \times 4 \times K$ case and 14 types of indispensable moves for the $4 \times 4 \times 4$ case. A proof of the fact that these indispensable moves form the unique minimal basis along the lines of Aoki and Takemura (2003a) is unfortunately too long and omitted. In addition we give a (non-exhaustive) list of indispensable moves for larger three-way tables. In this paper we prove some results on constructing indispensable moves from other indispensable moves. Our indispensable moves for larger tables were found by using these results combined with some computer searches. Closely connected notions to indispensability are the notions of fundamental moves and circuits discussed in Ohsugi and Hibi (1999, 2003). We also indicate whether our indispensable moves are fundamental or circuits.

1 Introduction

In this paper we consider how to construct a connected Markov chain over three-way contingency tables with fixed two-dimensional marginals. If such a Markov chain is constructed, we can perform various conditional tests for the hypothesis that there is no three-factor interaction in the log-linear model, i.e., $p_{ijk} = \psi_{ij}\phi_{ik}\omega_{jk}$, where p_{ijk} is a cell probability. As many previous works have suggested, construction of a connected Markov chain is surprisingly difficult.

Similarly to Section 1 of Aoki and Takemura (2003a), we begin with a simple case of two-way contingency tables again to illustrate the difficulty of this problem. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let $\boldsymbol{x} = \{x_{ij}\}$ denote an $I \times J$ contingency table with entries $x_{ij} \in \mathbb{N}, i \in [I], j \in [J]$, where we define $[m] = \{1, \ldots, m\}$ for positive integer m. In the analysis of two-way contingency tables, a hypothesis of statistical independence, i.e., $p_{ij} = p_{i\cdot}p_{\cdot j}$, is the most familiar one. For testing this null hypothesis, a common approach is to base the inference on the conditional distribution given a sufficient statistic for the nuisance parameters. In this case, the sufficient statistic for the nuisance parameters are the row and column sums, $x_{i\cdot}, x_{\cdot j}, i \in [I], j \in [J]$, and the conditional distribution of $\boldsymbol{x}$ for fixed marginals

$\{x_{i\cdot}^o\}, \{x_{\cdot j}^o\}$ is the hypergeometric distribution

$$h(\boldsymbol{x}) = \begin{cases} \prod_{j=1}^{J} \binom{x_{\cdot j}}{x_{1j}, \ldots, x_{Ij}} \Big/ \binom{x_{\cdot\cdot}}{x_{1\cdot}, \ldots, x_{I\cdot}}, & \text{if } \boldsymbol{x} \in \mathcal{F}(\{x_{i\cdot}^o\}, \{x_{\cdot j}^o\}), \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mathcal{F}(\{x_{i\cdot}^o\}, \{x_{\cdot j}^o\})$ is the reference set of all $I \times J$ contingency tables having the same marginal totals to $\boldsymbol{x}^o = \{x_{ij}^o\}$ given by

$$\mathcal{F}(\{x_{i\cdot}^o\}, \{x_{\cdot j}^o\}) = \{\boldsymbol{x} \mid x_{i\cdot} = x_{i\cdot}^o,\ x_{\cdot j} = x_{\cdot j}^o,\ x_{ij} \in \mathbb{N}\}.$$

In performing the exact conditional tests, we consider Monte Carlo approach in this paper, which generates random samples from $h(\boldsymbol{x})$ and evaluates p values of various test statistics. Note that the Monte Carlo approach is a valuable tool especially for sparse data sets where enumeration of the reference set is infeasible and at the same time large sample approximations of p values is not sufficiently accurate.

In the hypergeometric case (1), random samples can be directly generated using the obvious urn scheme in the definition of the hypergeometric distribution. In decomposable log-linear models in multi-way contingency tables (e.g., Section 4.4 of Lauritzen, 1996), random samples can be directly generated by an urn scheme exploiting the nesting structure of the conditional independence.

As another approach, we can employ a Markov chain Monte Carlo method. It is known that, if a connected Markov chain over $\mathcal{F}(\{x_{i\cdot}^o\}, \{x_{\cdot j}^o\})$ is constructed, then the chain can be modified to give a connected and aperiodic Markov chain with the stationary distribution $h(\boldsymbol{x})$ by the usual Metropolis procedure (e.g., Hastings, 1970). In the case of two-way contingency tables, a connected Markov chain over $\mathcal{F}(\{x_{i\cdot}^o\}, \{x_{\cdot j}^o\})$ can be easily constructed as follows. Let $\boldsymbol{x}$ be the current state in $\mathcal{F}(\{x_{i\cdot}^o\}, \{x_{\cdot j}^o\})$. To select the next state, choose a pair of rows $i_1, i_2 \in [I]$ and a pair of columns $j_1, j_2 \in [J]$ at random, and modify $\boldsymbol{x}$ at four entries where the selected rows and columns intersect as

$$\begin{array}{c|cc|} & j_1 & j_2 \\ \cline{2-3} i_1 & +1 & -1 \\ i_2 & -1 & +1 \\ \cline{2-3} \end{array} \quad \text{or} \quad \begin{array}{c|cc|} & j_1 & j_2 \\ \cline{2-3} i_1 & -1 & +1 \\ i_2 & +1 & -1 \\ \cline{2-3} \end{array} \quad \text{with probability } \tfrac{1}{2}, \text{ respectively.} \tag{2}$$

Note that the above four-entries modifications keep the row and column sums fixed. If the modification forces negative entries, discard it and continue by choosing new pairs of rows and columns.

Now we turn our attention to the original problem of the three-way contingency tables. Let $\boldsymbol{x} = \{x_{ijk}\}$ denote an $I \times J \times K$ contingency table with entries $x_{ijk} \in \mathbb{N}, i \in [I], j \in [J], k \in [K]$. Under the model of no three-factor interaction, the sufficient statistic for the nuisance parameters are the two-dimensional marginals $\{x_{ij\cdot}\}, \{x_{i\cdot k}\}, \{x_{\cdot jk}\}$, and our aim is to generate a random sample from the conditional distribution on

$$\mathcal{F}(\{x_{ij\cdot}^o\}, \{x_{i\cdot k}^o\}, \{x_{\cdot jk}^o\}) = \{\boldsymbol{x} \mid x_{ij\cdot} = x_{ij\cdot}^o,\ x_{i\cdot k} = x_{i\cdot k}^o,\ x_{\cdot jk} = x_{\cdot jk}^o,\ x_{ijk} \in \mathbb{N}\}.$$

The conditional distribution has cell probability $h(\boldsymbol{x})$ proportional to $\left(\prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{k=1}^{K} x_{ijk}!\right)^{-1}$ (e.g., Section 4 of Agresti, 1990). However, a direct generation of random samples is difficult since this is a non-decomposable hierarchical log-linear model. On the other hand, a construction of an appropriate connected Markov chain over $\mathcal{F}(\{x_{ij\cdot}^o\}, \{x_{i\cdot k}^o\}, \{x_{\cdot jk}^o\})$ is also difficult. To see this, consider a simple analogue of four-entries modification in (2). In this three-way setting, the simplest modification is the eight-entries modification given by

$$\begin{array}{cc} i=i_1 & \\ j\backslash k & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} j_1 \\ j_2 \end{array} & \begin{array}{|cc|} \hline +1 & -1 \\ -1 & +1 \\ \hline \end{array} \end{array} \qquad \begin{array}{cc} i=i_2 & \\ j\backslash k & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} j_1 \\ j_2 \end{array} & \begin{array}{|cc|} \hline -1 & +1 \\ +1 & -1 \\ \hline \end{array} \end{array} . \tag{3}$$

However, a chain constructed from the above type of modification may *not* be connected. A simple counter-example is given by the following $3 \times 3 \times 3$ contingency table.

$$\begin{array}{|ccc|} \hline m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \\ \hline \end{array} \qquad \begin{array}{|ccc|} \hline 0 & m & 0 \\ 0 & 0 & m \\ m & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|ccc|} \hline 0 & 0 & m \\ m & 0 & 0 \\ 0 & m & 0 \\ \hline \end{array}$$

For this table, the two-dimensional marginals have the same value, i.e., $x_{ij\cdot} = x_{i\cdot k} = x_{\cdot jk} = m$ for $i \in [I], j \in [J], k \in [K]$. It is clear that this state is not connected to any other states in $\mathcal{F}(\{x^o_{ij\cdot}\}, \{x^o_{i\cdot k}\}, \{x^o_{\cdot jk}\})$ by the eight-entries modification described in (3) for any m, i.e., we cannot modify any set of eight entries of the position described as (3) without causing negative entries. In this paper we study the problem of constructing a connected Markov chain over the reference set $\mathcal{F}(\{x^o_{ij\cdot}\}, \{x^o_{i\cdot k}\}, \{x^o_{\cdot jk}\})$.

The Markov chain Monte Carlo approach is extensively used in various two-way settings, for example, Smith, Forster & McDonald (1996) for tests of independence, quasi-independence and quasi-symmetry for square two-way contingency tables; Guo & Thompson (1992) for exact tests of Hardy-Weinberg proportions (triangular two-way contingency tables). In Aoki and Takemura (2002) we give an explicit form of the unique minimal Markov basis for $I \times J$ tables with arbitrary pattern of structural zeros. There is also an extensive literature on the rate of the convergence of the chain, for example, Diaconis & Saloff-Coste (1995) for two-way contingency tables; Hernek (1998), Dyer & Greenhill (2000) for $2 \times J$ contingency tables. On the other hand, there are only a few works dealing with high dimensional tables. For example see Besag & Clifford (1989) for the Ising model and Forster, McDonald & Smith (1996) for general 2^d tables.

Diaconis and Sturmfels (1998) presented a general algorithm for computing a Markov basis in the setting of a general discrete exponential family of distribution. Their approach relies on the existence of a Gröbner basis of a well specified polynomial ideal. For the above setting of three-way contingency tables, the argument is summarized as follows. Let $\mathbb{K}$ be a field and consider the map of polynomial rings

$$\begin{aligned} f : \mathbb{K}[x_{ijk}, i \in [I], j \in [J], k \in [K]] &\to \mathbb{K}[a_{ij}, b_{ik}, c_{jk}, i \in [I], j \in [J], k \in [K]], \\ x_{ijk} &\mapsto a_{ij}b_{ik}c_{jk}, \ i \in [I], j \in [J], k \in [K]]. \end{aligned}$$

Then the Markov basis for this problem corresponds to generators for the kernel of f. Despite its generality, unfortunately, their algorithm has not been largely used because a Gröbner basis could very difficult to compute even for problems of moderate size. Although there are some works including Sturmfels (1995) and Boffi and Rossi (2001), which improve the efficiency of Gröbner bases computation, it is still difficult to obtain a Gröbner basis by standard packages even for problems of moderate sizes with more than three dimensions.

It should be noted that a Gröbner basis is in general not symmetric because it depends on the particular term order. For example, Diaconis and Sturmfels (1998) reports that the reduced Gröbner basis for the $3 \times 3 \times 3$ case contains basis elements of

$$\text{28 relations of the form } \begin{array}{|ccc|} \hline 0 & 0 & 0 \\ 0 & -1 & +1 \\ 0 & +1 & -1 \\ \hline \end{array} \qquad \begin{array}{|ccc|} \hline +1 & 0 & -1 \\ -1 & +1 & 0 \\ 0 & -1 & +1 \\ \hline \end{array} \qquad \begin{array}{|ccc|} \hline -1 & 0 & +1 \\ +1 & 0 & -1 \\ 0 & 0 & 0 \\ \hline \end{array} \tag{4}$$

and

$$\text{1 relation of the form } \begin{array}{|ccc|}\hline -2 & +1 & +1 \\ +1 & 0 & -1 \\ +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline +1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline +1 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -1 \\ \hline\end{array} \ . \tag{5}$$

However, as remarked by Diaconis & Sturmfels, the basis elements of the above types are not necessarily needed to construct a connected chain. This interesting example is investigated thoroughly in Aoki and Takemura (2003a). We have shown that a minimal Markov basis for this case is uniquely determined, and hence symmetric.

We focus our attention on a minimality of Markov basis in this paper. Here we give definitions of a Markov basis and its minimality for the three-way contingency tables with fixed two-dimensional marginals. Let $\mathcal{F}_0$ be a set of $I \times J \times K$ integer arrays with zero two-way marginal totals

$$\mathcal{F}_0 = \{ \ \boldsymbol{z} \mid z_{ij\cdot} = z_{i\cdot k} = z_{\cdot jk} = 0, \ z_{ijk} \in \mathbb{Z} \ \},$$

where $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Elements of $\mathcal{F}_0$ are called *moves*.

Definition 1 *A Markov basis is a set of moves $\mathcal{B} = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_L\}$, $\boldsymbol{z}_i \in \mathcal{F}_0$, $i = 1 \ldots, L$, such that, for any $\{x_{ij\cdot}\}, \{x_{i\cdot k}\}, \{x_{\cdot jk}\}$ and $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{F}(\{x_{ij\cdot}\}, \{x_{i\cdot k}\}, \{x_{\cdot jk}\})$, there exist $A > 0$, $(\varepsilon_1, \boldsymbol{z}_{t_1}), \ldots, (\varepsilon_A, \boldsymbol{z}_{t_A})$ with $\varepsilon_s = \pm 1$, such that*

$$\boldsymbol{x}' = \boldsymbol{x} + \sum_{s=1}^{A} \varepsilon_s \boldsymbol{z}_{t_s} \quad \textit{and} \quad \boldsymbol{x} + \sum_{s=1}^{a} \varepsilon_s \boldsymbol{z}_{t_s} \in \mathcal{F}(\{x_{ij\cdot}\}, \{x_{i\cdot k}\}, \{x_{\cdot jk}\}) \textit{ for } 1 \le a \le A.$$

A Markov basis $\mathcal{B}$ is minimal if no proper subset of $\mathcal{B}$ is a Markov basis. A minimal Markov basis is unique if there exists only one minimal Markov basis, except for the sign changes of the basis elements.

As is already stated, if a Markov basis is obtained, a connected Markov chain over $\mathcal{F}(\{x_{ij\cdot}\}, \{x_{i\cdot k}\}, \{x_{\cdot jk}\})$ is easily constructed. Logically important point here is the existence of a finite Markov basis, which is guaranteed by the Hilbert basis theorem (see Section 3.1 of Diaconis and Sturmfels, 1998). A minimal Markov basis always exists, because from any Markov basis, we can remove redundant elements one by one, until none of the remaining elements can be removed any further. In Takemura and Aoki (2003), we give some basic characterizations of a minimal Markov basis, and a necessary and sufficient condition for uniqueness of a minimal Markov basis. We show in Takemura and Aoki (2003) that for some problems a minimal Markov basis is unique and for other problems it is not unique. For the general $I \times J \times K$ case, closed form expressions of a Markov basis and a minimal Markov basis are very complicated, except when $\min(I, J, K) = 2$ (see Section 4 of Diaconis and Sturmfels, 1998).

In Aoki and Takemura (2003a), we give an explicit form of the unique minimal Markov basis for the special case of $3 \times 3 \times K$ contingency tables. Our approach in Aoki and Takemura (2003a) does not employ algebraic algorithms. The approach is very simple and summarized as follows. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ denote three-dimensional contingency tables of the same size with the same two-dimensional marginal totals. Note that all the marginal totals of $\boldsymbol{x} - \boldsymbol{y}$ are zero. Define $|\boldsymbol{z}| = \sum_{i,j,k} |z_{ijk}|$ for three-dimensional integer array $\boldsymbol{z}$. Suppose that a set of moves $\mathcal{B} = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_L\}$ is given. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are made as close as possible, in other words, $|\boldsymbol{x} - \boldsymbol{y}|$ is made as small as possible, by applying moves $\boldsymbol{z}_{i_1}, \boldsymbol{z}_{i_2}, \ldots \in \mathcal{B}$ without causing negative entries on the way, it follows that

$$|\boldsymbol{x} - \boldsymbol{y}| \text{ can be decreased to 0 for any } \boldsymbol{x} \text{ and } \boldsymbol{y} \iff \mathcal{B} \text{ is a Markov basis.}$$

This shows that we only need to consider patterns of $\boldsymbol{x} - \boldsymbol{y}$, after making $|\boldsymbol{x} - \boldsymbol{y}|$ as small as possible by applying moves from $\mathcal{B}$. Based on the above simple observations, we consider exhaustive investigations of possible sign patterns for the $3 \times 3 \times K$ contingency tables. After laborious derivation we found that the minimal Markov basis for the $3 \times 3 \times K$ case consists of four types of moves, say, moves of degree 4, 6, 8 and 10. The move of degree 10 is the largest and its size is $3 \times 3 \times 5$. See Section 2 of Aoki and Takemura (2003a). Moreover in this case, uniqueness of the minimal Markov basis is assured from the results of Takemura and Aoki (2003). Here we give an important definition.

Definition 2 *An indispensable move is a move $\boldsymbol{z} \in \mathcal{F}_0$ which is written as $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ constitute a two elements reference set $\mathcal{F}(\{x_{ij\cdot}\}, \{x_{i\cdot k}\}, \{x_{\cdot jk}\}) = \{\boldsymbol{x}, \boldsymbol{y}\}$.*

Here we use the term *indispensable* for the following reason.

Lemma 1 *(Lemma 2.3 of Takemura and Aoki, 2003) Every indispensable move belongs to each Markov basis.*

As is stated in Takemura and Aoki (2003), this lemma suggests that indispensable moves play an important role in uniqueness of minimal Markov bases. Relations between the indispensable moves and uniqueness of a minimal Markov basis is summarized as follows.

Lemma 2 *(Corollary 2.2 of Takemura and Aoki, 2003) The unique minimal Markov basis exists if and only if the set of indispensable moves forms a Markov basis. In this case, the set of indispensable moves is the unique minimal Markov basis.*

From this lemma, we see that the minimal Markov basis for the $3 \times 3 \times K$ contingency tables is the unique minimal Markov basis since all the basis elements described in Aoki and Takemura (2003a) are indispensable moves. This result leads to a general problem for the three-dimensional contingency tables.

> **Problem** For any positive integers I, J and K, does there exist a unique minimal Markov basis for the three-dimensional $I \times J \times K$ contingency tables with fixed two-dimensional marginals?

Aoki and Takemura (2003a) showed that the answer for the problem is yes when $I = 3, J = 3$ and K is any positive integer. It is also shown that the answer is also yes when $\min(I, J, K) = 2$ as stated in Section 3 of Takemura and Aoki (2003). But we do not know whether the answer is always yes for general I, J, K. It seems very difficult and is an open problem at present.

One of contributions of Aoki and Takemura (2003a) is that it provides a general method to obtain a Markov basis, though it is a laborious one as seen in Appendix of Aoki and Takemura (2003a). But no other moves are needed to construct a connected Markov chain regardless of the value of K as seen in Theorem 4 of Aoki and Takemura (2003a). This result is attractive since it may not be derived by performing algebraic algorithms, which leads to a general problem mentioned in Discussion of Aoki and Takemura (2003a).

> **Problem** Does there exist explicit upper bounds $\mu_{I,J}$ and $\rho_{I,J}$ depending on I and J, such that for any K the corresponding minimal Markov basis for the three-dimensional $I \times J \times K$ contingency tables with fixed two-dimensional marginals consists of moves whose sizes are less than or equal to $I \times J \times \mu_{I,J}$ and whose degrees are less than or equal to $\rho_{I,J}$?

From the paper by Diaconis & Sturmfels (1998) it is known that $\mu_{2,J} = J, \rho_{2,J} = 2J$ and Aoki and Takemura (2003a) showed that $\mu_{3,3} = 5, \rho_{3,3} = 10$. Recently Santos and Sturmfels (2002) give an upper bound for $\mu_{I,J}$ using the theory of Graves basis.

In this paper, we consider the next simpler case than that considered in Aoki and Takemura (2003a), $3 \times 4 \times K$ contingency tables, and $4 \times 4 \times 4$ contingency tables. For the $3 \times 4 \times K$ and $4 \times 4 \times 4$ cases, we performed a similar laborious and exhaustive investigations of possible sign patterns, and found a minimal Markov basis. We also observed that the minimal Markov bases for these problems are unique minimal Markov bases, since the obtained minimal Markov bases consist only of indispensable moves. Therefore we see that the answer to the above two problems are both yes for these problems, and as shown in Section 2.1, $\mu_{3,4} = 8, \rho_{3,4} = 16$.

This paper is an extension of Aoki and Takemura (2003a). The results of this paper are obtained by the elementary argument of Aoki and Takemura (2003a). We omit proofs of these results since it may take hundreds of pages to show all the details of the proofs. Therefore our main contribution in this paper is for actually performing the Markov chain Monte Carlo method for the $3 \times 4 \times K$ and $4 \times 4 \times 4$ problems, rather than in new theoretical developments, except for some basic findings concerning the combination of two indispensable moves preserving the indispensability described in Section 3 and the separation and combination of the two-dimensional slices described in Section 4.

The organization of the rest of the paper is as follows. In Section 2, we describe the main results, i.e., a list of all indispensable moves for the $3 \times 4 \times K$ and $4 \times 4 \times 4$ cases. In Section 3 we prove a result on combination of two indispensable moves preserving the indispensability and in Section 4 we prove a result on separation and combination of the two-dimensional slices preserving the indispensability. Some discussion is given in Section 5. In Appendix A we give a (non-exhaustive) list indispensable moves for larger tables.

2 List of indispensable moves of the unique minimal Markov basis for $3 \times 4 \times K$ and $4 \times 4 \times 4$ tables

In this section, we give a list of all indispensable moves for the $3 \times 4 \times K$ and $4 \times 4 \times 4$ cases. After laborious derivation, we found that these indispensable moves, together with the permutations of indices for each axis and the permutations of axes of these moves, constitute the unique minimal Markov basis for each case. Our approach is similar to that of Aoki and Takemura (2003a). Note that the permutation of indices for each axis of moves can be considered as an action of a direct product of symmetric groups to the moves. See Aoki and Takemura (2003b) for detail.

Let $\boldsymbol{z} \in \mathcal{F}_0$ be an indispensable move for our problems. Write $\boldsymbol{z} = \boldsymbol{z}^+ - \boldsymbol{z}^-$ where $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ are the positive and the negative parts of $\boldsymbol{z}$ having the elements $z^+_{ijk} = \max(z_{ijk}, 0)$ and $z^-_{ijk} = \max(-z_{ijk}, 0)$, respectively. We define the i-slice (or $i = i_0$-slice) of $\boldsymbol{z}$ as the two-dimensional slice $\boldsymbol{z}_{i=i_0} = \{z_{i_0jk}\}_{j\in[J],k\in[K]}$, where $i_0 \in [I]$ is fixed. We similarly define j-slice and k-slice of $\boldsymbol{z}$. To display $I \times J \times K$ moves $\boldsymbol{z}$, we write I i-slices of size $J \times K$ as follows:

$$
\begin{array}{cc}
i=1 & \\
j\backslash k & \begin{array}{ccc} 1 & \cdots & K \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ J \end{array} & \begin{array}{|ccc|} \hline z_{111} & \cdots & z_{11K} \\ \vdots & & \vdots \\ z_{1J1} & \cdots & z_{1JK} \\ \hline \end{array}
\end{array}
\quad \cdots \quad
\begin{array}{cc}
i=I & \\
j\backslash k & \begin{array}{ccc} 1 & \cdots & K \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ J \end{array} & \begin{array}{|ccc|} \hline z_{I11} & \cdots & z_{I1K} \\ \vdots & & \vdots \\ z_{IJ1} & \cdots & z_{IJK} \\ \hline \end{array}
\end{array}
\tag{6}
$$

We also define the i, j-line (or $i = i_0, j = j_0$-line) of $\boldsymbol{z}$ as the one-dimensional line $\boldsymbol{z}_{i=i_0,j=j_0} = \{z_{i_0j_0k}\}_{k\in[K]}$, where $i_0 \in [I]$ and $j_0 \in [J]$ are fixed. We similarly define i, k-line and j, k-line of $\boldsymbol{z}$. We label each indispensable move by its *size*, *degree* and *slice degree* defined as follows. The size of $I \times J \times K$ contingency table $\boldsymbol{x}$ is defined as the size of the smallest 3-way subtable containing the support of $\boldsymbol{x}$, defined by

$$\mathrm{supp}(\boldsymbol{x}) = \{(i, j, k) \mid x_{ijk} > 0\} .$$

We call this subtable the *supporting subtable* of $\boldsymbol{x}$ and denote it by

$$S(\boldsymbol{x}) = I_{\boldsymbol{x}} \times J_{\boldsymbol{x}} \times K_{\boldsymbol{x}} \subset [I] \times [J] \times [K],$$

where

$$I_{\boldsymbol{x}} = \{i \in [I] \mid x_{ijk} > 0 \text{ for some } j \in [J], k \in [K]\},$$

and so on. We also define the supporting subtable of a move $\boldsymbol{z}$ as the supporting subtable of its positive and negative parts, i.e., $S(\boldsymbol{z}) = S(\boldsymbol{z}^+) = S(\boldsymbol{z}^-)$. Note that $S(\boldsymbol{z}^+)$ and $S(\boldsymbol{z}^-)$ are equal since $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ have the same marginal totals. The size of $\boldsymbol{z}$ is defined as the size of $S(\boldsymbol{z})$. We denote the size of $\boldsymbol{z}$ by $s_i \times s_j \times s_k$. We assume that $s_i \le s_j \le s_k$ without loss of generality since other moves can be produced by permutations of axes of these moves. For example, the following $2 \times 3 \times 3$ move is an indispensable move, which we found in Section 2.2 of Aoki and Takemura (2003a).

$$\begin{array}{|ccc|}\hline +1 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -1 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline -1 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +1 \\ \hline\end{array} \tag{7}$$

By permuting axes of this move, we have other indispensable moves, i.e., $3 \times 2 \times 3$ move

$$\begin{array}{|ccc|}\hline +1 & -1 & 0 \\ -1 & +1 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline -1 & 0 & +1 \\ +1 & 0 & -1 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline 0 & +1 & -1 \\ 0 & -1 & +1 \\ \hline\end{array}$$

and $3 \times 3 \times 2$ move

$$\begin{array}{|cc|}\hline +1 & -1 \\ -1 & +1 \\ 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|cc|}\hline -1 & +1 \\ 0 & 0 \\ +1 & -1 \\ \hline\end{array} \qquad \begin{array}{|cc|}\hline 0 & 0 \\ +1 & -1 \\ -1 & +1 \\ \hline\end{array}.$$

In this case, only $2 \times 3 \times 3$ move is included in our list.

The degree of move $\boldsymbol{z}$ is defined as the total frequency of $\boldsymbol{z}^+$ or $\boldsymbol{z}^-$, i.e.,

$$\deg(\boldsymbol{z}) = \sum_{i,j,k} z^+_{ijk} = \sum_{i,j,k} z^-_{ijk} = \frac{1}{2} \sum_{i,j,k} |z_{ijk}|.$$

The slice degree of $\boldsymbol{z}$ (with the size $s_i \times s_j \times s_k$) is the degrees of each slices having the form $\{d^i_1, \ldots, d^i_{s_i}\} \times \{d^j_1, \ldots, d^j_{s_j}\} \times \{d^k_1, \ldots, d^k_{s_k}\}$, where

$$d^i_{i_0} = \deg(\boldsymbol{z}_{i=i_0}) = \sum_{j,k} z^+_{i_0 jk} = \sum_{j,k} z^-_{i_0 jk} = \frac{1}{2} \sum_{j,k} |z_{i_0 jk}|$$

and so on. For example, the $2 \times 3 \times 3$ move displayed in (7) has the degree 6 and the slice degree $\{3,3\} \times \{2,2,2\} \times \{2,2,2\}$. We label this move as

$$2 \times 3 \times 3 \text{ move of degree 6 with slice degree } \{3,3\} \times \{2,2,2\} \times \{2,2,2\}$$

in this paper. We also assume that

$$d^i_1 \le d^i_2 \le \cdots \le d^i_{s_i},\ d^j_1 \le d^j_2 \le \cdots \le d^j_{s_j},\ d^k_1 \le d^k_2 \le \cdots \le d^k_{s_k}$$

without loss of generality since we take account of the permutations of indices for each axis of moves. Therefore our display of the form (6) is according to this order of the slice degree. It should be noted that the size, degree and slice degree are examples of invariants for the permutations of indices for each axis of moves and the permutations of axes of moves. Unfortunately, we cannot completely distinguish all indispensable moves by these invariants only. We consider this point in Section 5.

Closely related notions to indispensability are the notions of fundamental moves and circuits discussed in Ohsugi and Hibi (1999, 2003). For a move $\boldsymbol{z}$ its support is defined by $\mathrm{supp}(\boldsymbol{z}) = \mathrm{supp}(\boldsymbol{z}^+) \cup \mathrm{supp}(\boldsymbol{z}^-)$. $\boldsymbol{z}$ is called a *circuit* if $\boldsymbol{z}'$ is a move such that $\mathrm{supp}(\boldsymbol{z}') \subset \mathrm{supp}(\boldsymbol{z})$ then $\boldsymbol{z}' = c\boldsymbol{z}$ for some integer c. For a three-way contingency table $\boldsymbol{x}$, let

$$\boldsymbol{t}(\boldsymbol{x}) = \{\{x_{ij\cdot}\}, \{x_{i\cdot k}\}, \{x_{\cdot jk}\}\}$$

denote the marginal frequencies and let $\mathrm{supp}(\boldsymbol{t}(\boldsymbol{x}))$ denote the set of positive marginal cells for $\boldsymbol{x}$. For a move $\boldsymbol{z}$ define $\mathrm{supp}(\boldsymbol{t}(\boldsymbol{z})) = \mathrm{supp}(\boldsymbol{t}(\boldsymbol{z}^+)) = \mathrm{supp}(\boldsymbol{t}(\boldsymbol{z}^-))$. A move $\boldsymbol{z}$ is called *fundamental* if $\boldsymbol{z}'$ is a move such that $\mathrm{supp}(\boldsymbol{t}(\boldsymbol{z}')) \subset \mathrm{supp}(\boldsymbol{t}(\boldsymbol{z}))$ then $\boldsymbol{z}' = c\boldsymbol{z}$ for some integer c. In Ohsugi and Hibi (2003) the following two facts are proved. i) Fundamental moves are indispensable and circuits. ii) There is in general no implications between the notions of indispensability and circuits. Therefore it is of theoretical interest to investigate whether our indispensable moves are fundamental or circuits. We also define a *hidden zero cell* for a move $\boldsymbol{z}$ as

$$\{(i,j,k) \mid z_{ijk} = 0,\ z^+_{ij\cdot} z^+_{i\cdot k} z^+_{\cdot jk} \neq 0\}.$$

Note that a non-fundamental move which has no hidden zero cell is also a non-circuit move by definition.

Now we list all indispensable moves for $3 \times 4 \times K$ tables in Section 2.1 and all indispensable moves of size $4 \times 4 \times 4$ in Section 2.2 by their degrees. We found that most of the indispensable moves are at the same time fundamental moves and circuits. Therefore we only give a verbal description if an indispensable move is not fundamental or not a circuit in the list. We also give a more compact information of indispensable moves in the form:

$$((\mathrm{SIZE}), (\mathrm{DEGREE}), (\mathrm{SLICE\_DEGREE}), (\mathrm{PROPERTY}), (\mathrm{HIDDEN\_ZERO}), (\mathrm{CELLS})), \qquad (8)$$

where PROPERTY means

- f: fundamental, F: not fundamental
- c: circuit, C: not circuit
- s: square free (i.e., consists only of $0, \pm 1$), S: not square free,

HIDDEN_ZERO means the multi-indices of hidden zero cells of $\boldsymbol{z}$ and CELLS means the multi-indices of $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$. In the i-slices display of $\boldsymbol{z}$, we write (0) for a hidden zero cell. All these informations are available from the author's web page:

`http://www.stat.t.u-tokyo.ac.jp/~aoki/list-of-indispensable-moves.html` .

2.1 List of indispensable moves for $3 \times 4 \times K$ tables

- $2 \times 2 \times 2$ basic move of degree 4 with slice degree $\{2,2\} \times \{2,2\} \times \{2,2\}$
$((2,2,2), (4), ((2,2), (2,2), (2,2)), (fcs), \emptyset, ((111, 122, 212, 221), (112, 121, 211, 222)))$

$$\begin{array}{|cc|}\hline +1 & -1 \\ -1 & +1 \\ \hline\end{array} \qquad \begin{array}{|cc|}\hline -1 & +1 \\ +1 & -1 \\ \hline\end{array}$$

- $2 \times 3 \times 3$ move of degree 6 with slice degree $\{3,3\} \times \{2,2,2\} \times \{2,2,2\}$
  $((2,3,3),(6),((3,3),(2,2,2),(2,2,2)),(fcs),\emptyset,((111,123,132,212,221,233),$
  $(112,121,133,211,223,232)))$

$$\begin{array}{|ccc|}\hline +1 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -1 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline -1 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +1 \\ \hline\end{array}$$

- $2 \times 4 \times 4$ move of degree 8 with slice degree $\{4,4\} \times \{2,2,2,2\} \times \{2,2,2,2\}$
  $((2,4,4),(8),((4,4),(2,2,2,2),(2,2,2,2)),(fcs),\emptyset,((111,122,133,144,212,223,234,241),$
  $(112,123,134,141,211,222,233,244)))$

$$\begin{array}{|cccc|}\hline +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & 0 & 0 & +1 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \\ \hline\end{array}$$

- $3 \times 3 \times 4$ move of degree 8 with slice degree $\{2,3,3\} \times \{2,3,3\} \times \{2,2,2,2\}$
  $((3,3,4),(8),((2,3,3),(2,3,3),(2,2,2,2)),(fcs),\emptyset,((121,132,214,223,231,313,322,334),$
  $(122,131,213,221,234,314,323,332)))$

$$\begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline 0 & 0 & -1 & +1 \\ -1 & 0 & +1 & 0 \\ +1 & 0 & 0 & -1 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline 0 & 0 & +1 & -1 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & +1 \\ \hline\end{array}$$

- $3 \times 4 \times 4$ move of degree 9 with slice degree $\{3,3,3\} \times \{2,2,2,3\} \times \{2,2,2,3\}$
  $((3,4,4),(9),((3,3,3),(2,2,2,3),(2,2,2,3)),(fcs),\emptyset,((111,124,142,214,233,241,322,334,343),$
  $(114,122,141,211,234,243,324,333,342)))$

$$\begin{array}{|cccc|}\hline +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ +1 & 0 & -1 & 0 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 \\ 0 & 0 & -1 & +1 \\ 0 & -1 & +1 & 0 \\ \hline\end{array}$$

- $3 \times 3 \times 5$ move of degree 10 with slice degree $\{3,3,4\} \times \{3,3,4\} \times \{2,2,2,2,2\}$
  $((3,3,5),(10),((3,3,4),(3,3,4),(2,2,2,2,2)),(fcs),\emptyset,$
  $((113,125,131,212,223,234,311,324,332,335),(111,123,135,213,224,232,312,325,331,334)))$

$$\begin{array}{|ccccc|}\hline -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ \hline\end{array} \qquad \begin{array}{|ccccc|}\hline 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccccc|}\hline +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ -1 & +1 & 0 & -1 & +1 \\ \hline\end{array}$$

- $3 \times 4 \times 4$ move of degree 10 with slice degree $\{3,3,4\} \times \{2,2,3,3\} \times \{2,2,3,3\}$
  (not fundamental, circuit)
  $((3,4,4),(10),((3,3,4),(2,2,3,3),(2,2,3,3)),(Fcs),(333,344),$
  $((113,131,144,224,233,242,314,323,332,341),(114,133,141,223,232,244,313,324,331,342)))$

$$\begin{array}{|cccc|}\hline 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ -1 & 0 & 0 & +1 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 \\ 0 & -1 & +1 & 0 \\ 0 & +1 & 0 & -1 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 \\ -1 & +1 & (0) & 0 \\ +1 & -1 & 0 & (0) \\ \hline\end{array}$$

- $3 \times 4 \times 5$ move of degree 10 with slice degree $\{2,4,4\} \times \{2,2,3,3\} \times \{2,2,2,2,2\}$
  $((3,4,5),(10),((2,4,4),(2,2,3,3),(2,2,2,2,2)),(fcs),\emptyset,$
  $((131,142,213,225,234,241,314,323,332,345),(132,141,214,223,231,245,313,325,334,342)))$

$$\boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & -1 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & +1 \end{array}}$$

- $3 \times 4 \times 5$ move of degree 10 with slice degree $\{3,3,4\} \times \{2,2,3,3\} \times \{2,2,2,2,2\}$
  $((3,4,5),(10),((3,3,4),(2,2,3,3),(2,2,2,2,2)),(fcs),\emptyset,$
  $((111,133,142,224,235,243,312,325,331,344),(112,131,143,225,233,244,311,324,335,342)))$

$$\boxed{\begin{array}{ccccc} +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & +1 & -1 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & +1 & 0 \end{array}}$$

- $3 \times 4 \times 5$ move of degree 12 with slice degree $\{4,4,4\} \times \{3,3,3,3\} \times \{2,2,2,2,4\}$
  $((3,4,5),(12),((4,4,4),(3,3,3,3),(2,2,2,2,4)),(fcs),\emptyset,$
  $((111,123,132,144,215,221,234,245,312,325,335,343),$
  $(112,121,134,143,211,225,235,244,315,323,332,345)))$

$$\boxed{\begin{array}{ccccc} +1 & -1 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & +1 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & -1 \end{array}}$$

- $3 \times 4 \times 6$ move of degree 12 with slice degree $\{3,4,5\} \times \{2,3,3,4\} \times \{2,2,2,2,2,2\}$
  $((3,4,6),(12),((3,4,5),(2,3,3,4),(2,2,2,2,2,2)),(fcs),\emptyset,$
  $((121,133,142,214,226,231,245,315,322,334,343,346),$
  $(122,131,143,215,221,234,246,314,326,333,342,345)))$

$$\boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & 0 & +1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 & -1 & +1 \end{array}}$$

- $3 \times 4 \times 6$ move of degree 12 with slice degree $\{4,4,4\} \times \{2,3,3,4\} \times \{2,2,2,2,2,2\}$
  $((3,4,6),(12),((4,4,4),(2,3,3,4),(2,2,2,2,2,2)),(fcs),\emptyset,$
  $((111,123,132,144,212,221,236,245,325,334,343,346),$
  $(112,121,134,143,211,225,232,246,323,336,344,345)))$

$$\boxed{\begin{array}{cccccc} +1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} -1 & +1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & +1 & -1 & -1 & +1 \end{array}}$$

- $3 \times 4 \times 6$ move of degree 12 with slice degree $\{4,4,4\} \times \{3,3,3,3\} \times \{2,2,2,2,2,2\}$
  $((3,4,6),(12),((4,4,4),(3,3,3,3),(2,2,2,2,2,2)),(fcs),\emptyset,$
  $((111,123,132,144,215,221,234,246,312,326,335,343),$
  $(112,121,134,143,211,226,235,244,315,323,332,336)))$

$$\boxed{\begin{array}{cccccc} +1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} -1 & 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \end{array}}$$

- $3 \times 4 \times 6$ move of degree 14 with slice degree $\{4,4,6\} \times \{3,3,4,4\} \times \{2,2,2,2,2,4\}$
  $((3,4,6),(14),((4,4,6),(3,3,4,4),(2,2,2,2,2,4)),(fcS),\emptyset,$
  $((111,122,133,146,212,225,234,246,314,323,336,336,341,345),$
  $(112,123,136,141,214,222,236,245,311,325,333,334,346,346)))$

$$\boxed{\begin{array}{cccccc} +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & +1 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \end{array}} \quad \boxed{\begin{array}{cccccc} -1 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & +2 \\ +1 & 0 & 0 & 0 & +1 & -2 \end{array}}$$

- $3 \times 4 \times 6$ move of degree 14 with slice degree $\{4,5,5\} \times \{3,3,3,5\} \times \{2,2,2,2,2,4\}$
  $((3,4,6),(14),((4,5,5),(3,3,3,5),(2,2,2,2,2,4)),(fcS),\emptyset,$
  $((111,122,133,144,216,223,236,241,245,312,325,334,346,346),$
  $(112,123,134,141,211,225,233,246,246,316,322,336,344,345)))$

$$\boxed{\begin{array}{cccccc} +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & +1 & -2 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & +2 \end{array}}$$

- $3 \times 4 \times 7$ move(1) of degree 14 with slice degree $\{4,4,6\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
  $((3,4,7),(14),((4,4,6),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,$
  $((111,123,132,144,215,221,237,246,312,327,334,336,343,345),$
  $(112,121,134,143,211,227,236,245,315,323,332,337,344,346)))$

$$\boxed{\begin{array}{ccccccc} +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccccc} -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \end{array}} \quad \boxed{\begin{array}{ccccccc} 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & -1 & 0 & +1 & 0 & +1 & -1 \\ 0 & 0 & +1 & -1 & +1 & -1 & 0 \end{array}}$$

- $3 \times 4 \times 7$ move(2) of degree 14 with slice degree $\{4,4,6\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
  $((3,4,7),(14),((4,4,6),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,$
  $((111,123,132,144,215,221,236,247,312,327,334,335,343,346),$
  $(112,121,134,143,211,227,235,246,315,323,332,336,344,347)))$

$$\boxed{\begin{array}{ccccccc} +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccccc} -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \end{array}} \quad \boxed{\begin{array}{ccccccc} 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & -1 & 0 & +1 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 & 0 & +1 & -1 \end{array}}$$

- $3 \times 4 \times 7$ move of degree 14 with slice degree $\{4,5,5\} \times \{3,3,3,5\} \times \{2,2,2,2,2,2,2\}$
  $((3,4,7),(14),((4,5,5),(3,3,3,5),(2,2,2,2,2,2,2)),(fcs),\emptyset,$
  $((111,122,133,144,215,226,232,241,247,313,324,337,345,346),$
  $(113,124,132,141,211,222,237,245,246,315,326,333,344,347)))$

$$\begin{array}{|ccccccc|}\hline +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 & -1 & -1 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & -1 & +1 & +1 & -1 \\ \hline \end{array}$$

- $3 \times 4 \times 7$ move of degree 14 with slice degree $\{4,5,5\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
  $((3,4,7),(14),((4,5,5),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,$
  $((111,122,133,144,215,224,237,241,246,313,326,332,335,347),$
  $(113,124,132,141,211,226,235,244,247,315,322,333,337,346)))$

$$\begin{array}{|ccccccc|}\hline +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & 0 & -1 & 0 & +1 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & +1 & -1 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ \hline \end{array}$$

- $3 \times 4 \times 7$ move of degree 16 with slice degree $\{4,6,6\} \times \{3,3,5,5\} \times \{2,2,2,2,2,2,4\}$
  $((3,4,7),(16),((4,6,6),(3,3,5,5),(2,2,2,2,2,2,4)),(fcS),\emptyset,$
  $((111,123,132,144,215,221,234,236,247,247,312,326,337,337,343,345),$
  $(112,121,134,143,211,226,237,237,244,245,315,323,332,336,347,347)))$

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & 0 & +1 & -2 \\ 0 & 0 & 0 & -1 & -1 & 0 & +2 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & +2 \\ 0 & 0 & +1 & 0 & +1 & 0 & -2 \\ \hline \end{array}$$

- $3 \times 4 \times 8$ move of degree 16 with slice degree $\{4,6,6\} \times \{3,3,5,5\} \times \{2,2,2,2,2,2,2,2\}$
  (not fundamental, not circuit)
  $((3,4,8),(16),((4,6,6),(3,3,5,5),(2,2,2,2,2,2,2,2)),(FCs),\emptyset,$
  $((111,123,132,144,218,221,234,237,245,246,312,327,335,336,343,348),$
  $(112,121,134,143,211,227,235,236,244,248,318,323,332,337,345,346)))$

$$\begin{array}{|cccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & -1 & -1 & +1 & 0 \\ 0 & 0 & 0 & -1 & +1 & +1 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & +1 & +1 & -1 & 0 \\ 0 & 0 & +1 & 0 & -1 & -1 & 0 & +1 \\ \hline \end{array}$$

2.2 List of indispensable moves for $4 \times 4 \times 4$ tables

- $4 \times 4 \times 4$ move(1) of degree 10 with slice degree $\{2,2,3,3\} \times \{2,2,3,3\} \times \{2,2,3,3\}$
  $((4,4,4),(10),((2,2,3,3),(2,2,3,3),(2,2,3,3)),(fcs),\emptyset,$
  $((113,124,231,242,314,333,341,423,432,444),(114,123,232,241,313,331,344,424,433,442)))$

$$\begin{array}{|cccc|}\hline 0 & 0 & +1 & -1 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ +1 & 0 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & +1 \\ \hline \end{array}$$

- $4 \times 4 \times 4$ move(2) of degree 10 with slice degree $\{2,2,3,3\} \times \{2,2,3,3\} \times \{2,2,3,3\}$
  $((4,4,4),(10),((2,2,3,3),(2,2,3,3),(2,2,3,3)),(fcs),\emptyset,$
  $((114,133,231,243,313,324,342,422,434,441),(113,134,233,241,314,322,343,424,431,442)))$

$$\begin{array}{|cccc|}\hline 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ -1 & 0 & +1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & +1 & -1 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 \\ -1 & 0 & 0 & +1 \\ +1 & -1 & 0 & 0 \\ \hline\end{array}$$

- $4 \times 4 \times 4$ move of degree 12 with slice degree $\{2,2,4,4\} \times \{2,2,4,4\} \times \{3,3,3,3\}$
  $((4,4,4),(12),((2,2,4,4),(2,2,4,4),(3,3,3,3)),(fcs),\emptyset,$
  $((131,142,234,243,311,324,332,333,413,422,441,444),$
  $(132,141,233,244,313,322,331,334,411,424,442,443)))$

$$\begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline +1 & 0 & -1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & +1 & +1 & -1 \\ 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline -1 & 0 & +1 & 0 \\ 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ +1 & -1 & -1 & +1 \\ \hline\end{array}$$

- $4 \times 4 \times 4$ move of degree 12 with slice degree $\{2,3,3,4\} \times \{2,3,3,4\} \times \{3,3,3,3\}$
  (not fundamental, circuit)
  $((4,4,4),(12),((2,3,3,4),(2,3,3,4),(3,3,3,3)),(Fcs),(242,421),$
  $((134,142,221,232,243,311,324,333,413,422,441,444),$
  $(132,144,222,233,241,313,321,334,411,424,442,443)))$

$$\begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & +1 & 0 & -1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ -1 & (0) & +1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline +1 & 0 & -1 & 0 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline -1 & 0 & +1 & 0 \\ (0) & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ +1 & -1 & -1 & +1 \\ \hline\end{array}$$

- $4 \times 4 \times 4$ move(1) of degree 12 with slice degree $\{3,3,3,3\} \times \{3,3,3,3\} \times \{3,3,3,3\}$
  (not fundamental, circuit)
  $((4,4,4),(12),((3,3,3,3),(3,3,3,3),(3,3,3,3)),(Fcs),(122,244,311,433),$
  $((111,123,132,214,233,241,312,321,344,422,434,443),$
  $(112,121,133,211,234,243,314,322,341,423,432,444)))$

$$\begin{array}{|cccc|}\hline +1 & -1 & 0 & 0 \\ -1 & (0) & +1 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ +1 & 0 & -1 & (0) \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline (0) & +1 & 0 & -1 \\ +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & (0) & +1 \\ 0 & 0 & +1 & -1 \\ \hline\end{array}$$

- $4 \times 4 \times 4$ move(2) of degree 12 with slice degree $\{3,3,3,3\} \times \{3,3,3,3\} \times \{3,3,3,3\}$
  $((4,4,4),(12),((3,3,3,3),(3,3,3,3),(3,3,3,3)),(fcs),\emptyset,$
  $((111,123,132,214,233,241,312,324,343,421,434,442),$
  $(112,121,133,211,234,243,314,323,342,424,432,441)))$

$$\begin{array}{|cccc|}\hline +1 & -1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ +1 & 0 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & +1 & 0 & -1 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & +1 \\ -1 & +1 & 0 & 0 \\ \hline\end{array}$$

- $4 \times 4 \times 4$ move(1) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{3,3,4,4\}$
  (not fundamental, circuit)
  $((4,4,4),(14),((3,3,3,5),(3,3,3,5),(3,3,4,4)),(FcS),(113,234,442),$
  $((111,123,142,221,232,244,313,331,344,412,424,434,443,443),$
  $(112,121,143,224,231,242,311,334,343,413,423,432,444,444)))$

$$\begin{array}{|cccc|}\hline +1 & -1 & (0) & 0 \\ -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 \\ -1 & +1 & 0 & (0) \\ 0 & -1 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 \\ 0 & 0 & -1 & +1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & +1 & -1 & 0 \\ 0 & 0 & -1 & +1 \\ 0 & -1 & 0 & +1 \\ 0 & (0) & +2 & -2 \\ \hline\end{array}$$

- $4 \times 4 \times 4$ move(2) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{3,3,4,4\}$
  (not fundamental, circuit)
  $((4,4,4),(14),((3,3,3,5),(3,3,3,5),(3,3,4,4)),(FcS),(342,421),$
  $((111,123,142,221,232,244,312,333,344,414,424,431,443,443),$
  $(112,121,143,224,231,242,314,332,343,411,423,433,444,444)))$

$$\begin{array}{|cccc|}\hline +1 & -1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 \\ -1 & +1 & 0 & 0 \\ 0 & -1 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & (0) & -1 & +1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline -1 & 0 & 0 & +1 \\ (0) & 0 & -1 & +1 \\ +1 & 0 & -1 & 0 \\ 0 & 0 & +2 & -2 \\ \hline\end{array}$$

3 Sufficient condition for type-2 combination of indispensable moves which preserves the indispensability

As is stated in Section 1, it is important to find the indispensable moves especially when the unique minimal Markov basis exists. However, our approach seems to be difficult to generalize to larger tables. Then, how can we find indispensable moves of larger sizes, i.e., $3 \times 5 \times 5, 4 \times 4 \times 5$ and so on? In this and the next sections, we give some basic features of indispensable moves, which we can make use of for finding larger indispensable moves.

As the first basic feature of the indispensable moves, in this section we consider indispensable moves having the structure that they are separated to two indispensable moves. Important findings are obtained by comparing $2 \times 3 \times 3$ indispensable move of degree 6 with slice degree $\{3,3\} \times \{2,2,2\} \times \{2,2,2\}$,

$$\begin{array}{|ccc|}\hline +1 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -1 \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline -1 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +1 \\ \hline\end{array}$$

and the following $3 \times 3 \times 3$ move of degree 7 with slice degree $\{2,2,3\} \times \{2,2,3\} \times \{2,2,3\}$,

$$\begin{array}{|ccc|}\hline +1 & -1 & 0 \\ -1 & +1 & 0 \\ 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline -1 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +1 \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline 0 & 0 & 0 \\ 0 & -1 & +1 \\ 0 & +1 & -1 \\ \hline\end{array}.$$

These two moves have the common structure that they are represented as a combination of two basic moves. The difference between these two moves lies in the number of overlapping cells, i.e., the move of degree 6 is made from two basic moves that overlap at two non-zero entries, while the move of degree 7 is made from two basic moves that overlap at one non-zero entry. In Aoki and Takemura (2003a), we called the former combination as a *type-2 combination* and the latter combination as a *type-1* *combination*. In Aoki and Takemura (2003a), it is shown that the move of degree 7 is dispensable because two basic moves can be applied one by one in an appropriate order without causing negative entries on the way instead of applying the move of degree 7. On the other hand, because the type-2 combination has two overlapped cells, one of these cells necessarily becomes negative in adding two basic moves one by one. For this reason, the type-2 combination is essential. In fact, all indispensable moves of the $3 \times 3 \times K$ case are made by the type-2 combinations of some basic moves (see Aoki and Takemura, 2003a).

From these considerations, we are interested in a relationship between the type-2 combination and the indispensability. We give a sufficient condition for type-2 combination of indispensable moves which preserves the indispensability in Theorem 1 below. Note that this is only a *sufficient condition* for obtaining an indispensable move from combining some smaller indispensable moves. We discuss this point in Section 5.

Before stating the theorem, we consider some additional constraints to the type-2 combination and consider a simple situation. Let $\boldsymbol{z}$ and $\boldsymbol{z}'$ be moves satisfying $S(\boldsymbol{z}), S(\boldsymbol{z}') \subset [I] \times [J] \times [K]$. We assume that $S(\boldsymbol{z}) \cap S(\boldsymbol{z}') (\neq \emptyset)$ is included in a one-dimensional line. Without loss of generality we write $S(\boldsymbol{z}) \cap S(\boldsymbol{z}') \subset \{(i_0, j_0, k) \mid k \in [K]\}$. The two nonzero elements of $\boldsymbol{z}$ or $\boldsymbol{z}'$ where they overlap and cancel signs are on this line. We write the $i = i_0, j = j_0$-line of $\boldsymbol{z}, \boldsymbol{z}'$ as

$$z_{i_0 j_0 k} = \begin{cases} +1, & \text{if } k = k_2 \\ -1, & \text{if } k = k_1 \\ 0, & \text{otherwise,} \end{cases} \qquad z'_{i_0 j_0 k} = \begin{cases} +1, & \text{if } k = k_1 \\ -1, & \text{if } k = k_2 \\ 0, & \text{otherwise} \end{cases}$$

without loss of generality. It should be noted that the following five lines

$$\begin{array}{ll} \{(i_0, j_0, k) \mid k \in [K]\}, & \\ \{(i, j_0, k_1) \mid i \in [I]\}, & \{(i, j_0, k_2) \mid i \in [I]\}, \\ \{(i_0, j, k_1) \mid j \in [J]\}, & \{(i_0, j, k_2) \mid j \in [J]\} \end{array} \tag{9}$$

intersect both $S(\boldsymbol{z})$ and $S(\boldsymbol{z}')$. We assume that there does not exist one-dimensional line other than the above five lines that intersect both $S(\boldsymbol{z})$ and $S(\boldsymbol{z}')$. In addition, we assume that the (i_0, k_1)-,(i_0, k_2)-,(j_0, k_1)-,(j_0, k_2)-marginals of $\boldsymbol{z}^+, \boldsymbol{z}^-, \boldsymbol{z}'^+, \boldsymbol{z}'^-$ are all one, i.e.,

$$\begin{aligned} 1 &= z^+_{i_0 \cdot k_1} = z^-_{i_0 \cdot k_1} = z'^+_{i_0 \cdot k_1} = z'^-_{i_0 \cdot k_1} = z^+_{i_0 \cdot k_2} = z^-_{i_0 \cdot k_2} = z'^+_{i_0 \cdot k_2} = z'^-_{i_0 \cdot k_2} \\ &= z^+_{\cdot j_0 k_1} = z^-_{\cdot j_0 k_1} = z'^+_{\cdot j_0 k_1} = z'^-_{\cdot j_0 k_1} = z^+_{\cdot j_0 k_2} = z^-_{\cdot j_0 k_2} = z'^+_{\cdot j_0 k_2} = z'^-_{\cdot j_0 k_2}. \end{aligned}$$

Now we present a theorem.

Theorem 1 *Let $\boldsymbol{z}$ and $\boldsymbol{z}'$ be indispensable moves satisfying the above conditions then $\boldsymbol{z}^* = \boldsymbol{z} + \boldsymbol{z}'$ is an indispensable move with its positive part*

$$\boldsymbol{z}^{*+} = \boldsymbol{z}^+ + \boldsymbol{z}'^+ - (\boldsymbol{\delta}_{i_0 j_0 k_1} + \boldsymbol{\delta}_{i_0 j_0 k_2})$$

and its negative part

$$\boldsymbol{z}^{*-} = \boldsymbol{z}^- + \boldsymbol{z}'^- - (\boldsymbol{\delta}_{i_0 j_0 k_1} + \boldsymbol{\delta}_{i_0 j_0 k_2}),$$

where $\boldsymbol{\delta}_{ijk}$ is a table with the element $+1$ only at the cell (i, j, k), and 0 otherwise.

For example, let $\boldsymbol{z}$ be a $2 \times 3 \times 3$ indispensable move of degree 6 with slice degree $\{3, 3\} \times \{2, 2, 2\} \times \{2, 2, 2\}$ and $\boldsymbol{z}'$ be a $3 \times 3 \times 4$ indispensable move of degree 8 with slice degree $\{2, 3, 3\} \times \{2, 3, 3\} \times \{2, 2, 2, 2\}$. Then the following $4 \times 5 \times 5$ move is made by the type-2 combination of $\boldsymbol{z}$ and $\boldsymbol{z}'$ satisfying the conditions and is an indispensable move. In this case, $S(\boldsymbol{z}) \cap S(\boldsymbol{z}') \subset \{(2,3,k) \mid k \in [K]\}$ and $k_1 = 2, k_2 = 3$.

$$
\begin{array}{|ccc|cc|}
\hline
+1 & -1 & 0 & 0 & 0 \\
-1 & 0 & +1 & 0 & 0 \\
0 & +1 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|cc|cc|}
\hline
-1 & +1 & 0 & 0 & 0 \\
+1 & 0 & -1 & 0 & 0 \\
\cline{2-3}
0 & 0 & 0 & 0 & 0 \\
0 & -1 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|cccc|}
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\cline{2-5}
0 & -1 & 0 & +1 & 0 \\
0 & +1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & +1 \\
\hline
\end{array}
\quad
\begin{array}{|c|cccc|}
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\cline{2-5}
0 & 0 & +1 & -1 & 0 \\
0 & 0 & -1 & 0 & +1 \\
0 & 0 & 0 & +1 & -1 \\
\hline
\end{array}
$$

Proof It is seen that $\boldsymbol{z}^{*+}$ and $\boldsymbol{z}^{*-}$ have the same marginal totals by definition. We write the reference set of the tables that have the same marginal totals to $\boldsymbol{z}^{*+}$ and $\boldsymbol{z}^{*-}$ as $\mathcal{F}^*$. We want to show that $\mathcal{F}^*$ is a two-element set, i.e., if $\boldsymbol{x} \in \mathcal{F}^*$ then $\boldsymbol{x} = \boldsymbol{z}^{*+}$ or $\boldsymbol{x} = \boldsymbol{z}^{*-}$. We consider the cells in the support of $\boldsymbol{x}$. Let $(i,j,k) \in \mathrm{supp}(\boldsymbol{x})$.

First we consider the case that $(i,j,k) \notin S(\boldsymbol{z}) \cup S(\boldsymbol{z}')$. In this case, (i,j,k) lies on at least one two-dimensional slice which is zero slice in $\boldsymbol{z}$ since $(i,j,k) \notin S(\boldsymbol{z})$. Similarly, (i,j,k) also lies on at least one two-dimensional slice which is zero slice in $\boldsymbol{z}'$ since $(i,j,k) \notin S(\boldsymbol{z}')$. These two slices have at least one line in common, and corresponding line sum of $\boldsymbol{x}$ must be zero, which contradicts that $(i,j,k) \in \mathrm{supp}(\boldsymbol{x})$. Next we consider the case that $(i,j,k) \in S(\boldsymbol{z}) \cap S(\boldsymbol{z}')$. In this case, (i,j,k) is in the line $\{(i_0,j_0,k) \mid k \in [K]\}$ by definition. However, $x_{i_0j_0\cdot}$ must be zero by definition and therefore this case is also contradiction. From these considerations, it is seen that each cell in the support of $\boldsymbol{x}$ belongs to exactly one of $S(\boldsymbol{z})$ or $S(\boldsymbol{z}')$. We write $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{y}'$ where

$$
y_{ijk} = \begin{cases} x_{ijk}, & \text{if } (i,j,k) \in S(\boldsymbol{z}), \\ 0, & \text{otherwise}, \end{cases}
$$

$$
y'_{ijk} = \begin{cases} x_{ijk}, & \text{if } (i,j,k) \in S(\boldsymbol{z}'), \\ 0, & \text{otherwise}. \end{cases}
$$

Let $n = \sum_{i,j,k} y_{ijk}$ and $n' = \sum_{i,j,k} y'_{ijk}$.

Here we consider the marginal totals of $\boldsymbol{y}$. By definition, only the five line sums described as (9) can differ between $\boldsymbol{y}$ and $\boldsymbol{z}^+, \boldsymbol{z}^-$. First we consider the line sums along the k-axis. It follows that

$$
\begin{aligned}
&y_{ij\cdot} = z^+_{ij\cdot} = z^-_{ij\cdot}, \quad \text{for } (i,j) \neq (i_0,j_0), \\
&y_{i_0j_0\cdot} = 0, \\
&z^+_{i_0j_0\cdot} = z^-_{i_0j_0\cdot} = 1.
\end{aligned}
$$

Therefore $n = \sum_{i,j} y_{ij\cdot} = \deg(\boldsymbol{z}) - 1$. Similarly $n' = \sum_{i,j} y'_{ij\cdot} = \deg(\boldsymbol{z}') - 1$ holds. Next we consider the line sums along the j-axis. It follows that

$$
y_{i\cdot k} = z^+_{i\cdot k} = z^-_{i\cdot k}, \quad \text{for } (i,k) \neq (i_0,k_1),(i_0,k_2).
$$

Since

$$
\begin{aligned}
z^+_{i_0\cdot k_1} = z^-_{i_0\cdot k_1} = z^+_{i_0\cdot k_2} = z^-_{i_0\cdot k_2} = 1, \\
z^{*+}_{i_0\cdot k_1} = z^{*-}_{i_0\cdot k_1} = z^{*+}_{i_0\cdot k_2} = z^{*-}_{i_0\cdot k_2} = 1
\end{aligned}
$$

and $n + 1 = \deg(\boldsymbol{z})$, it follows that

$$
(y_{i_0\cdot k_1}, y_{i_0\cdot k_2}) = (1,0) \text{ or } (0,1).
$$

Similarly, by considering the line sums along the i-axis, it follows that

$$
(y_{\cdot j_0k_1}, y_{\cdot j_0k_2}) = (1,0) \text{ or } (0,1).
$$

Moreover, since

$$\begin{aligned} z^+_{\cdot\cdot k_1} &= \sum_i z^+_{i\cdot k_1} = \sum_{i\neq i_0} z^+_{i\cdot k_1} + z^+_{i_0\cdot k_1} = \sum_{i\neq i_0} y_{i\cdot k_1} + 1 \\ &= \sum_j z^+_{\cdot j k_1} = \sum_{j\neq j_0} z^+_{\cdot j k_1} + z^+_{\cdot j_0 k_1} = \sum_{j\neq j_0} y_{\cdot j k_1} + 1 \end{aligned}$$

and

$$y_{\cdot\cdot k_1} = \sum_i y_{i\cdot k_1} = \sum_j y_{\cdot j k_1},$$

we have

$$0 = \left(\sum_i y_{i\cdot k_1} - \sum_j y_{\cdot j k_1}\right) - \left(\sum_{i\neq i_0} y_{i\cdot k_1} - \sum_{j\neq j_0} y_{\cdot j k_1}\right) = y_{i_0\cdot k_1} - y_{\cdot j_0 k_1}.$$

Similarly we have $y_{i_0\cdot k_2} = y_{\cdot j_0 k_2}$. From these considerations, only possible patterns are

$$(a) : (y_{i_0\cdot k_1}, y_{i_0\cdot k_2}, y_{\cdot j_0 k_1}, y_{\cdot j_0 k_2}) = (1, 0, 1, 0)$$

or

$$(b) : (y_{i_0\cdot k_1}, y_{i_0\cdot k_2}, y_{\cdot j_0 k_1}, y_{\cdot j_0 k_2}) = (0, 1, 0, 1).$$

In the case of (a), it follows that $\boldsymbol{y} = \boldsymbol{z}^+ - \boldsymbol{\delta}_{i_0 j_0 k_2}$ since $\boldsymbol{z}$ is an indispensable move. In this case, it also follows that $\boldsymbol{y}' = \boldsymbol{z}^{'+} - \boldsymbol{\delta}_{i_0 j_0 k_1}$ by definition, and therefore $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{y}' = \boldsymbol{z}^{*+}$. Similarly, in the case of (b), it is shown that $\boldsymbol{x}$ must be $\boldsymbol{z}^{*-}$ and Theorem 1 is proved. Q.E.D.

4 Separation and combination of two-dimensional slices

In this section we consider separation and combination of two-dimensional slices preserving the indispensability. In Section 4.1, we provide a sufficient condition that a move created by separation of a two-dimensional slice of an indispensable move is again an indispensable move. Conversely, we also consider combination of two-dimensional slices of an indispensable move in Section 4.2. Using these results, we can produce many larger indispensable moves from a set of indispensable moves that we have. In Appendix A, we give a list of indispensable moves for larger tables produced by the separations and combinations of two-dimensional slices of $3 \times 4 \times K$ and $4 \times 4 \times 4$ indispensable moves presented in Section 2.

4.1 Sufficient condition for indispensability in separations of a two-dimensional slice

First we consider separation of a two-dimensional slice of an indispensable move. To illustrate our result on separation, we consider two examples. First example is the following $3 \times 4 \times 4$ move of degree 10 with slice degree $\{3, 3, 4\} \times \{2, 2, 3, 3\} \times \{2, 2, 3, 3\}$:

$$\begin{array}{|cccc|} \hline 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ -1 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 \\ 0 & -1 & +1 & 0 \\ 0 & +1 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 \\ -1 & +1 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ \hline \end{array} \; .$$

The $i=3$-slice of this indispensable move seems to contain two *loops*, i.e., the following decomposition is observed.

$$\begin{array}{|cccc|}\hline 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 \\ -1 & +1 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ \hline \end{array} = \begin{array}{|cccc|}\hline 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array} + \begin{array}{|cccc|}\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ \hline \end{array}$$

In fact, the above separation of the $i=3$-slice creates another indispensable move ($4\times 4\times 4$ move(1) of degree 10 with slice degree$\{2,2,3,3\}\times\{2,2,3,3\}\times\{2,2,3,3\}$).

Next example is the following $3\times 3\times 5$ move of degree 10 with slice degree $\{3,3,4\}\times\{3,3,4\}\times\{2,2,2,2,2\}$

$$\begin{array}{|ccccc|}\hline -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ -1 & +1 & 0 & -1 & +1 \\ \hline \end{array}.$$

We consider the $i=3$-slice of this indispensable move. It is seen that this slice is again decomposed to two loops as follows:

$$\begin{array}{|ccccc|}\hline +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ -1 & +1 & 0 & -1 & +1 \\ \hline \end{array} = \begin{array}{|ccccc|}\hline +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 \\ \hline \end{array} + \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & -1 & +1 \\ \hline \end{array}.$$

In fact, the above separation of the $i=3$-slice creates a $4\times 3\times 5$ move. After permuting the axes and the indices, we see that the move is a $3\times 4\times 5$ indispensable move of degree 10 with slice degree $\{3,3,4\}\times\{2,2,3,3\}\times\{2,2,2,2,2\}$, which is included in our list. These two examples suggest the possibility that we can create a new indispensable move by separations of a two-dimensional slice of an already obtained indispensable move under some conditions.

Now we provide some definitions. Let $\boldsymbol{z}=\boldsymbol{z}^+-\boldsymbol{z}^-$ be a move of the size $I\times J\times K$ with the positive part $\boldsymbol{z}^+$ and the negative part $\boldsymbol{z}^-$. Without loss of generality, we consider the separation of $\boldsymbol{z}_{k=k_0}$, i.e., $k=k_0$-slice of $\boldsymbol{z}$. Note that $\boldsymbol{z}_{k=k_0}$ is an $I\times J$ two-dimensional integer array with zero row sums and zero column sums. In the following, we assume that the level indices $i_1,i_2,\ldots\in[I], j_1,j_2,\ldots\in[J]$ are all distinct, i.e.,

$$i_m\neq i_n \text{ and } j_m\neq j_n \text{ for all } m\neq n. \tag{10}$$

The following definition gives a fundamental tool.

Definition 3 *A loop of degree r is an $I\times J$ integer array $\boldsymbol{L}_r$, where $\boldsymbol{L}_r$ has the elements*

$$\begin{aligned} L_{i_1j_1}=L_{i_2j_2}=\cdots=L_{i_{r-1}j_{r-1}}=L_{i_rj_r}=1, \\ L_{i_1j_2}=L_{i_2j_3}=\cdots=L_{i_{r-1}j_r}=L_{i_rj_1}=-1, \end{aligned}$$

for some $i_1,\ldots,i_r\in[I], j_1,\ldots,j_r\in[J]$ and all the other elements in the two-way subarray $\{i_1,\ldots,i_r\}\times\{j_1,\ldots,j_r\}$ are zero.

Note that there is at most one $+1$ and -1 in each row and column of a loop. We call $\{i_1,\ldots,i_r\}\times\{j_1,\ldots,j_r\}$ the supporting rectangle of the loop $\boldsymbol{L}_r$. Now we clarify the separation of the two-dimensional slice of moves which we have seen in the above examples.

Lemma 3 *Let $\boldsymbol{z}_{k=k_0}$ be an $I \times J$ two-dimensional slice of a move $\boldsymbol{z}$. Then $\boldsymbol{z}_{k=k_0}$ can be expressed as a sum*

$$\boldsymbol{z} = a_1 \boldsymbol{L}_{r(1)} + \cdots + a_n \boldsymbol{L}_{r(n)}, \tag{11}$$

where $a_1, \ldots, a_n$ are positive integers, $r(1), \ldots, r(n) \leq \min(I, J)$, $\boldsymbol{L}_{r(1)}, \ldots, \boldsymbol{L}_{r(n)}$ are all distinct and there is no cancellation of signs in any cell.

Proof of this lemma is easy and omitted. See also Aoki and Takemura (2002) for detailed descriptions of the loops and the above lemma.

The separations of slices in the examples above correspond to the cases that the expression (11) is uniquely determined. Note that there are cases of unique separation of a slice even when the supporting rectangles of the loops in (11) have common cells. The following is an example of such a case:

$$\boxed{\begin{array}{ccc} +1 & -1 & 0 \\ +1 & +1 & -2 \\ -2 & 0 & +2 \end{array}} = \boxed{\begin{array}{ccc} +1 & -1 & 0 \\ 0 & +1 & -1 \\ -1 & 0 & +1 \end{array}} + \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ +1 & 0 & -1 \\ -1 & 0 & +1 \end{array}}.$$

Conversely, there are cases of non-unique separation of a slice even when the supporting rectangles of the loops in (11) are disjoint. The following is an example of such a case:

$$\begin{aligned} \boxed{\begin{array}{cccc} +1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 \end{array}} &= \boxed{\begin{array}{cccc} +1 & 0 & -1 & 0 \\ -1 & 0 & +1 & 0 \end{array}} + \boxed{\begin{array}{cccc} 0 & +1 & 0 & -1 \\ 0 & -1 & 0 & +1 \end{array}} \\ &= \boxed{\begin{array}{cccc} +1 & 0 & 0 & -1 \\ -1 & 0 & 0 & +1 \end{array}} + \boxed{\begin{array}{cccc} 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \end{array}}. \end{aligned}$$

Now we give the main theorem in this section.

Theorem 2 *Let $\boldsymbol{z}$ be an $I \times J \times K$ indispensable move. Suppose that a slice $\boldsymbol{z}_{k=k_0}$ is expressed uniquely as (11) and $a_1 + \cdots + a_n \geq 2$. Then*

(i) $I \times J \times (K + n - 1)$ move $\boldsymbol{z}^*$ that is created from $\boldsymbol{z}$ by the separation of $\boldsymbol{z}_{k=k_0}$ with respect to the loops of (11) and

(ii) $I \times J \times (K + a_1 + \cdots + a_n - 1)$ move $\boldsymbol{z}^{**}$ that is created from $\boldsymbol{z}^*$ such that the each k-slice that is created from the $\boldsymbol{z}_{k=k_0}$ is a single loop

are indispensable moves, respectively.

Proof. We only show a proof of (i) since (ii) is obvious when (i) is shown. The positive part and the negative part of $\boldsymbol{z}^*$ are in the same reference set of $I \times J \times (K + n - 1)$ contingency tables. We write this reference set as $\mathcal{F}^*$. Let $\widetilde{\boldsymbol{z}}^+$ and $\widetilde{\boldsymbol{z}}^-$ be the $I \times J \times (K + n - 1)$ contingency tables created from $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ by the separation (11), respectively. Since $\widetilde{\boldsymbol{z}}^+, \widetilde{\boldsymbol{z}}^- \in \mathcal{F}^*$, all we have to show is that $\boldsymbol{x} = \widetilde{\boldsymbol{z}}^+$ or $\boldsymbol{x} = \widetilde{\boldsymbol{z}}^-$ holds for any $\boldsymbol{x} \in \mathcal{F}^*$. Let κ denote the set of slice indices which are the separation of $\boldsymbol{z}_{k=k_0}$. We have $|\kappa| = n$ by definition. Let $\widehat{\boldsymbol{x}}$ be the $I \times J \times K$ contingency table which is made from $\boldsymbol{x}$ by the addition of the k-slices for $k \in \kappa$. Since $\widehat{\boldsymbol{x}}$ has the same marginal totals to $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$, $\widehat{\boldsymbol{x}} = \boldsymbol{z}^+$ or $\widehat{\boldsymbol{x}} = \boldsymbol{z}^-$ holds. We assume $\widehat{\boldsymbol{x}} = \boldsymbol{z}^+$ without loss of generality. Note that

$$x_{ijk} = \widetilde{z}^+_{ijk} \text{ for } i \in [I], j \in [J], k \notin \kappa. \tag{12}$$

Therefore we only have to show that $x_{ijk} = \widetilde{z}^+_{ijk}$ for $k \in \kappa$. Here consider $I \times J \times n$ subarrays of $\boldsymbol{x}$ and $\widetilde{\boldsymbol{z}}^+$ for $k \in \kappa$. From (12), we see that $\boldsymbol{x}$ and $\widetilde{\boldsymbol{z}}^+$ have the same two-dimensional marginals in these subarrays. Note that $\sum_{k \in \kappa} x_{ijk} = \sum_{k \in \kappa} \widetilde{z}^+_{ijk}$ in particular. From this, it is seen that

$$\{(i, j) \mid x_{ijk} > 0, \exists k \in \kappa\} = \{(i, j) \mid \widetilde{z}^+_{ijk} > 0, \exists k \in \kappa\}.$$

Moreover, since there is no cancellation of signs in (11), it is also seen that

$$\{(i,j) \mid \tilde{z}^{+}_{ijk} > 0, \exists k \in \kappa\} \cap \{(i,j) \mid \tilde{z}^{-}_{ijk} > 0, \exists k \in \kappa\} = \emptyset.$$

Therefore, if $\boldsymbol{x} \neq \tilde{\boldsymbol{z}}^{+}$, $\boldsymbol{x}$ and $\tilde{\boldsymbol{z}}^{-}$ have disjoint supports, then $\boldsymbol{x} - \tilde{\boldsymbol{z}}^{-} \neq \boldsymbol{z}^{*}$ is another separation of $\boldsymbol{z}$, which contradicts the assumption that the separation is unique. Q.E.D.

4.2 Combinations of two-dimensional slices

Next we consider combinations of two-dimensional slices of indispensable moves. It should be noted that the converse of the statement in Theorem 2 is not always true. To see this, consider again the following $3 \times 3 \times 5$ indispensable move of degree 10 with slice degree $\{3,3,4\} \times \{3,3,4\} \times \{2,2,2,2,2\}$

$$\boxed{\begin{array}{ccccc} -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & +1 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ -1 & +1 & 0 & -1 & +1 \end{array}}\,. \qquad (13)$$

Combining $k = 1$-slice and $k = 4$-slice of this move makes the following $3 \times 3 \times 4$ move.

$$\begin{array}{c} \begin{array}{cccc} 1+4 & 2 & 3 & 5 \end{array} \\ \boxed{\begin{array}{cccc} -1 & 0 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \end{array}} \end{array} \quad \begin{array}{c} \begin{array}{cccc} 1+4 & 2 & 3 & 5 \end{array} \\ \boxed{\begin{array}{cccc} 0 & +1 & -1 & 0 \\ -1 & 0 & +1 & 0 \\ +1 & -1 & 0 & 0 \end{array}} \end{array} \quad \begin{array}{c} \begin{array}{cccc} 1+4 & 2 & 3 & 5 \end{array} \\ \boxed{\begin{array}{cccc} +1 & -1 & 0 & 0 \\ +1 & 0 & 0 & -1 \\ -2 & +1 & 0 & +1 \end{array}} \end{array} \qquad (14)$$

It is seen that this is a dispensable move, although $[k = 1\text{-slice}]+[k = 4\text{-slice}]$ is a unique decomposition of the form (11). This example implies that we have to consider some additional conditions to assure that moves made by combining two-dimensional slices of indispensable moves are again indispensable moves.

Unfortunately, it seems difficult to derive a necessary and sufficient condition for this problem. We give a sufficient condition similar as Theorem 1 for this problem in Appendix B. However in practice it is easier to directly check whether a candidate move is indispensable or not than checking our sufficient condition in Appendix B. We remark this point at the end of Appendix B.

5 Discussion

In this paper, we provide an explicit form of the unique minimal Markov basis for $3 \times 4 \times K$ and $4 \times 4 \times 4$ contingency tables by considering all the sign patterns. Our approach is an elementary one and similar to that of Aoki and Takemura (2003a). Therefore this paper extends the results of Aoki and Takemura (2003a).

Our results in this paper enable us to construct a connected Markov chain over $3 \times 4 \times K$ and $4 \times 4 \times 4$ contingency tables. Adjusting this chain to have a given stationary distribution by the Metropolis procedure, we can perform various tests by the Monte Carlo method. It should be noted that, for some data sets, construction of a connected Markov chain over three-dimensional contingency tables with the given two-dimensional marginals is a difficult problem. Moreover, it is also difficult to determine whether a simple-minded Markov chain described in Section 1, i.e., a Markov chain constructed from the $2 \times 2 \times 2$ basic moves described in (3) alone, is connected or not for given marginal totals. Therefore our results are valuable since our definition of Markov basis takes into account *arbitrary* patterns of the marginal totals. In addition, our result of the unique minimal Markov basis for $3 \times 4 \times K$ contingency tables shows that it is sufficient to consider the moves with sizes up to $3 \times 4 \times 8$ to construct a connected Markov chain over $3 \times 4 \times K$ tables for any $K \geq 8$. This result is attractive since it cannot be derived by performing algebraic algorithms.

There are still many open problems on the Markov basis. One of the most interesting problems may be a problem concerning the existence of unique minimal Markov basis, which we have seen in Section 1. As is stated in Takemura and Aoki (2003) and Aoki and Takemura (2003b), a minimal Markov basis is not unique for many problems that we usually consider. For example, a minimal Markov basis for the model of complete independence in the log-linear model in the three-dimensional contingency tables, $p_{ijk} = \alpha_i \beta_j \gamma_k$, is not unique as shown in Takemura and Aoki (2003). It is also shown in Aoki and Takemura (2003b) that minimal Markov bases for many models of the hierarchical $2 \times 2 \times 2 \times 2$ log-linear models are not unique. Clearly the uniqueness of a minimal Markov basis depends on the models, i.e., the sufficient statistics that we fix. However it seems very difficult to determine whether the given model has unique minimal Markov basis or not.

The list of indispensable moves in Section 2 gives some further informations.

(i) An importance of the type-2 combination of indispensable moves are suggested in Section 3. However, $3 \times 4 \times 4$ move of degree 9 with slice degree $\{3,3,3\} \times \{2,2,2,3\} \times \{2,2,2,3\}$ suggests another possibility of forming larger indispensable moves. In Discussion of Aoki and Takemura (2003a), it is pointed out that this move of degree 9 has a structure that three $2 \times 2 \times 2$ basic moves combine *all at once* in such a way that any two of the three basic moves forms a type-1 combination, and this move suggests the difficulty in forming a conjecture on a minimal Markov basis for larger tables. In fact, our list in Appendix A contains many moves of odd degrees.

(ii) We found that some indispensable moves have entries ± 2, which leads to a next general problem.

> **Problem** What is the value $U_{I,J,K}$ depending on I, J and K, such that for each element $\boldsymbol{z}$ of a minimal Markov basis for the three-dimensional $I \times J \times K$ contingency tables with fixed two-dimensional marginals, $U_{I,J,K} = \max_{i,j,k} |z_{ijk}|$?

We see that $U_{3,4,K} = 1$ when $K \leq 5$, $U_{3,4,K} = 2$ when $K \geq 6$ and $U_{4,4,4} = 2$.

(iii) We found that some indispensable moves are asymmetric, where we define a symmetric move as a move $\boldsymbol{z}$ which can be transformed to $-\boldsymbol{z}$ by the permutations of indices for each axis of the move. See Aoki and Takemura (2003b) for detail. In our list of indispensable moves in Section 2,

$3 \times 4 \times 6$ move of degree 12 with slice degree $\{3,4,5\} \times \{2,3,3,4\} \times \{2,2,2,2,2,2\}$ and

$4 \times 4 \times 4$ move of degree 12 with slice degree $\{2,3,3,4\} \times \{2,3,3,4\} \times \{3,3,3,3\}$

are asymmetric, while all the other indispensable moves are symmetric. Furthermore, we found another indispensable move of size $3 \times 5 \times 6$ that has only one of $+2$ or -2:

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & -1 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 & +1 & -2 \\ \hline \end{array}\ .$$

This is an example of obvious asymmetric indispensable moves.

(iv) In Section 2, we label each indispensable move by three invariants for the permutations of indices for each axis and the permutations of axes of moves: size, degree and slice degree. However, we cannot completely distinguish all indispensable moves by these invariants. As is seen in Section 2, the following four pairs of indispensable moves have these three invariants in common:

$3 \times 4 \times 7$ moves of degree 14 with slice degree $\{4,4,6\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$,

$4 \times 4 \times 4$ moves of degree 10 with slice degree $\{2,2,3,3\} \times \{2,2,3,3\} \times \{2,2,3,3\}$,

$4 \times 4 \times 4$ moves of degree 12 with slice degree $\{3,3,3,3\} \times \{3,3,3,3\} \times \{3,3,3,3\}$,

$4 \times 4 \times 4$ moves of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{3,3,4,4\}$.

To distinguish these indispensable moves, other invariants can be considered. For example, consider a set of values $\{c_0, c_1, c_2, \ldots\}$ which we define as

$$c_n = \left| \left\{ (i,j,k) \;\middle|\; \sum_i z^+_{ijk} + \sum_j z^+_{ijk} + \sum_k z^+_{ijk} = n \right\} \right|.$$

It is seen that $\{c_0, c_1, c_2, \ldots\}$ is an invariant. We can distinguish two $4 \times 4 \times 4$ moves of degree 12 with slice degree $\{3,3,3,3\} \times \{3,3,3,3\} \times \{3,3,3,3\}$ by this invariant. In fact the values for the two indispensable moves are $\{0,12,24,28\}$ and $\{4,0,36,24\}$, respectively. We can also distinguish two $4 \times 4 \times 4$ moves of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{3,3,4,4\}$ by this invariant ($\{1,6,24,22,9,0,2\}$ and $\{2,4,24,24,8,0,2\}$, respectively). Unfortunately, however, we cannot distinguish two $3 \times 4 \times 7$ moves of degree 14 with slice degree $\{4,4,6\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$ and two $4 \times 4 \times 4$ moves of degree 10 with slice degree $\{2,2,3,3\} \times \{2,2,3,3\} \times \{2,2,3,3\}$ even when we consider the above invariant. To distinguish these indispensable moves, we have to consider other invariants. In this paper, however, we do not consider the identification of indispensable moves by invariants furthermore.

We found that substantial number of non-fundamental indispensable moves. We also found the following rare examples of non-circuit indispensable moves in Section 2.1 and Appendix A:

- $((3,4,8),(16),((4,6,6),(3,3,5,5),(2,2,2,2,2,2,2,2)),(FCs),\emptyset,$
  $((111,123,132,144,218,221,234,237,245,246,312,327,335,336,343,348),$
  $(112,121,134,143,211,227,235,236,244,248,318,323,332,337,345,346)))$
- $((3,5,7),(16),((4,6,6),(2,3,3,3,5),(2,2,2,2,2,2,4)),(FCs),\emptyset,$
  $((121,133,142,154,216,222,237,247,253,255,317,325,331,344,356,357),$
  $(122,131,144,153,217,225,233,242,256,257,316,321,337,347,354,355)))$

Appendix A: List of indispensable moves for larger tables produced by the separations and combinations of two-dimensional slices of $3 \times 4 \times K$ and $4 \times 4 \times 4$ indispensable moves

We list indispensable moves of larger sizes. All the indispensable moves listed below are produced by the separations and combinations of two-dimensional slices of $3 \times 4 \times K$ indispensable moves(Section 2.1), $4 \times 4 \times 4$ indispensable moves (Section 2.2) and themselves. We specify each indispensable move by its size, degree and slice degree. We also give a simplified information as the form (8). All the informations in the list are available from the author's web page.

A.1 Indispensable moves of degree 11

- $3 \times 5 \times 5$ move of degree 11 with slice degree $\{3,4,4\} \times \{2,2,2,2,3\} \times \{2,2,2,2,3\}$
  $((3,5,5),(11),((3,4,4),(2,2,2,2,3),(2,2,2,2,3)),(fcs),\emptyset,$
  $((111,125,152,215,234,243,251,322,335,344,353),(115,122,151,211,235,244,253,325,334,343,352)))$

$$
\begin{array}{|ccccc|}\hline
+1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
-1 & 0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & -1 \\
0 & 0 & +1 & -1 & 0 \\
+1 & 0 & -1 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & 0 & 0 \\
0 & +1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & +1 \\
0 & 0 & -1 & +1 & 0 \\
0 & -1 & +1 & 0 & 0 \\ \hline
\end{array}
$$

- $4 \times 4 \times 5$ move of degree 11 with slice degree $\{2,2,3,4\} \times \{2,3,3,3\} \times \{2,2,2,2,3\}$
  $((4,4,5),(11),((2,2,3,4),(2,3,3,3),(2,2,2,2,3)),(fcs),\emptyset,$
  $((121,135,225,242,314,333,345,413,422,431,444),(125,131,222,245,313,335,344,414,421,433,442)))$

$$
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & 0 & 0 \\
+1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 \\
0 & +1 & 0 & 0 & -1 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & -1 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & +1 & 0 & -1 \\
0 & 0 & 0 & -1 & +1 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & +1 & -1 & 0 \\
-1 & +1 & 0 & 0 & 0 \\
+1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & +1 & 0 \\ \hline
\end{array}
$$

- $4 \times 4 \times 5$ move of degree 11 with slice degree $\{2,3,3,3\} \times \{2,3,3,3\} \times \{2,2,2,2,3\}$
  $((4,4,5),(11),((2,3,3,3),(2,3,3,3),(2,2,2,2,3)),(fcs),\emptyset,$
  $((121,145,222,233,241,314,325,332,415,434,443),(125,141,221,232,243,315,322,334,414,433,445)))$

$$
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & 0 & 0 \\
+1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & +1 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 \\
0 & -1 & +1 & 0 & 0 \\
+1 & 0 & -1 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & +1 & -1 \\
0 & -1 & 0 & 0 & +1 \\
0 & +1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & -1 & +1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & +1 & 0 \\
0 & 0 & +1 & 0 & -1 \\ \hline
\end{array}
$$

A.2 Indispensable moves of degree 12

- $3 \times 5 \times 6$ move of degree 12 with slice degree $\{3,4,5\} \times \{2,2,2,3,3\} \times \{2,2,2,2,2,2\}$
  $((3,5,6),(12),((3,4,5),(2,2,2,3,3),(2,2,2,2,2,2)),(fcs),\emptyset,$
  $((121,143,152,214,235,246,253,315,322,336,341,354),(122,141,153,215,236,243,254,314,321,335,346,352)))$

$$
\begin{array}{|cccccc|}\hline
0 & 0 & 0 & 0 & 0 & 0 \\
+1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & +1 & 0 & 0 & 0 \\
0 & +1 & -1 & 0 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|cccccc|}\hline
0 & 0 & 0 & +1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & +1 & -1 \\
0 & 0 & -1 & 0 & 0 & +1 \\
0 & 0 & +1 & -1 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|cccccc|}\hline
0 & 0 & 0 & -1 & +1 & 0 \\
-1 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & +1 \\
+1 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & +1 & 0 & 0 \\ \hline
\end{array}
$$

- $3 \times 5 \times 6$ move of degree 12 with slice degree $\{4,4,4\} \times \{2,2,2,3,3\} \times \{2,2,2,2,2,2\}$
  $((3,5,6),(12),((4,4,4),(2,2,2,3,3),(2,2,2,2,2,2)),(fcs),\emptyset,$
  $((111,123,144,152,212,235,241,256,324,336,345,353),(112,124,141,153,211,236,245,252,323,335,344,356)))$

$$
\begin{array}{|cccccc|}\hline
+1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & +1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & +1 & 0 & 0 \\
0 & +1 & -1 & 0 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|cccccc|}\hline
-1 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & +1 & -1 \\
+1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & +1 \\ \hline
\end{array}
\quad
\begin{array}{|cccccc|}\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & +1 \\
0 & 0 & 0 & -1 & +1 & 0 \\
0 & 0 & +1 & 0 & 0 & -1 \\ \hline
\end{array}
$$

- $4 \times 4 \times 5$ move(1) of degree 12 with slice degree $\{2,2,4,4\} \times \{3,3,3,3\} \times \{2,2,2,2,4\}$
  $((4,4,5),(12),((2,2,4,4),(3,3,3,3),(2,2,2,2,4)),(fcs),\emptyset,$
  $((111,125,232,245,313,321,334,342,415,424,435,443),(115,121,235,242,311,324,332,343,413,425,434,445)))$

$$
\begin{array}{|ccccc|}\hline
+1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & +1 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & +1 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
-1 & 0 & +1 & 0 & 0 \\
+1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & +1 & 0 \\
0 & +1 & -1 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{|ccccc|}\hline
0 & 0 & -1 & 0 & +1 \\
0 & 0 & 0 & +1 & -1 \\
0 & 0 & 0 & -1 & +1 \\
0 & 0 & +1 & 0 & -1 \\ \hline
\end{array}
$$

- $4 \times 4 \times 5$ move(2) of degree 12 with slice degree $\{2,2,4,4\} \times \{3,3,3,3\} \times \{2,2,2,2,4\}$
(not fundamental, circuit)
$((4,4,5),(12),((2,2,4,4),(3,3,3,3),(2,2,2,2,4)),(Fcs),(315,325,435,445),$
$((111,125,232,245,313,321,335,344,415,424,433,442),(115,121,235,242,311,324,333,345,413,425,432,444)))$

$$\begin{array}{|ccccc|}\hline +1&0&0&0&-1\\ -1&0&0&0&+1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&+1&0&0&-1\\ 0&-1&0&0&+1\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline -1&0&+1&0&(0)\\ +1&0&0&-1&(0)\\ 0&0&-1&0&+1\\ 0&0&0&+1&-1\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&-1&0&+1\\ 0&0&0&+1&-1\\ 0&-1&+1&0&(0)\\ 0&+1&0&-1&(0)\\ \hline\end{array}$$

- $4 \times 4 \times 5$ move of degree 12 with slice degree $\{2,3,3,4\} \times \{2,3,3,4\} \times \{2,2,2,3,3\}$
(not fundamental, circuit)
$((4,4,5),(12),((2,3,3,4),(2,3,3,4),(2,2,2,3,3)),(Fcs),(244,445),$
$((135,144,221,234,242,313,324,345,415,423,432,441),(134,145,224,232,241,315,323,344,413,421,435,442)))$

$$\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-1&+1\\ 0&0&0&+1&-1\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ +1&0&0&-1&0\\ 0&-1&0&+1&0\\ -1&+1&0&(0)&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&+1&0&-1\\ 0&0&-1&+1&0\\ 0&0&0&0&0\\ 0&0&0&-1&+1\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&-1&0&+1\\ -1&0&+1&0&0\\ 0&+1&0&0&-1\\ +1&-1&0&0&(0)\\ \hline\end{array}$$

- $4 \times 4 \times 5$ move of degree 12 with slice degree $\{2,3,3,4\} \times \{3,3,3,3\} \times \{2,2,2,3,3\}$
$((4,4,5),(12),((2,3,3,4),(3,3,3,3),(2,2,2,3,3)),(fcs),(435),$
$((111,124,214,235,242,325,333,344,415,421,432,443),(114,121,215,232,244,324,335,343,411,425,433,442)))$

$$\begin{array}{|ccccc|}\hline +1&0&0&-1&0\\ -1&0&0&+1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&+1&-1\\ 0&0&0&0&0\\ 0&-1&0&0&+1\\ 0&+1&0&-1&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&-1&+1\\ 0&0&+1&0&-1\\ 0&0&-1&+1&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline -1&0&0&0&+1\\ +1&0&0&0&-1\\ 0&+1&-1&0&(0)\\ 0&-1&+1&0&0\\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 12 with slice degree $\{2,2,4,4\} \times \{2,3,3,4\} \times \{2,2,2,2,2,2\}$
$((4,4,6),(12),((2,2,4,4),(2,3,3,4),(2,2,2,2,2,2)),(fcs),\emptyset,$
$((122,141,233,244,315,321,336,343,416,425,434,442),(121,142,234,243,316,325,333,341,415,422,436,444)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ -1&+1&0&0&0&0\\ 0&0&0&0&0&0\\ +1&-1&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&+1&-1&0&0\\ 0&0&-1&+1&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&+1&-1\\ +1&0&0&0&-1&0\\ 0&0&-1&0&0&+1\\ -1&0&+1&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&-1&+1\\ 0&-1&0&0&+1&0\\ 0&0&0&+1&0&-1\\ 0&+1&0&-1&0&0\\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 12 with slice degree $\{2,2,4,4\} \times \{3,3,3,3\} \times \{2,2,2,2,2,2\}$
$((4,4,6),(12),((2,2,4,4),(3,3,3,3),(2,2,2,2,2,2)),(fcs),\emptyset,$
$((111,122,233,244,315,321,336,343,412,426,434,445),(112,121,234,243,311,326,333,345,415,422,436,444)))$

$$\begin{array}{|cccccc|}\hline +1&-1&0&0&0&0\\ -1&+1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&+1&-1&0&0\\ 0&0&-1&+1&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline -1&0&0&0&+1&0\\ +1&0&0&0&0&-1\\ 0&0&-1&0&0&+1\\ 0&0&+1&0&-1&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&+1&0&0&-1&0\\ 0&-1&0&0&0&+1\\ 0&0&0&+1&0&-1\\ 0&0&0&-1&+1&0\\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 12 with slice degree $\{2,3,3,4\} \times \{2,3,3,4\} \times \{2,2,2,2,2,2\}$
$((4,4,6),(12),((2,3,3,4),(2,3,3,4),(2,2,2,2,2,2)),(fcs),\emptyset,$
$((121,142,223,234,241,316,335,344,415,422,433,456),(122,141,221,233,244,315,334,346,416,423,435,442)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ +1&-1&0&0&0&0\\ 0&0&0&0&0&0\\ -1&+1&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ -1&0&+1&0&0&0\\ 0&0&-1&+1&0&0\\ +1&0&0&-1&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&-1&+1\\ 0&0&0&0&0&0\\ 0&0&0&-1&+1&0\\ 0&0&0&+1&0&-1\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&+1&-1\\ 0&+1&-1&0&0&0\\ 0&0&+1&0&-1&0\\ 0&-1&0&0&0&+1\\ \hline\end{array}$$

- $4 \times 5 \times 5$ move of degree 12 with slice degree $\{3,3,3,3\} \times \{2,2,2,2,4\} \times \{2,2,2,2,4\}$
$((4,5,5),(12),((3,3,3,3),(2,2,2,2,4),(2,2,2,2,4)),(fcs),\emptyset,$
$((111,125,152,215,233,251,322,345,354,435,444,453),(115,122,151,211,235,253,325,344,352,433,445,454)))$

$$\begin{array}{|ccccc|}\hline +1&0&0&0&-1\\ 0&-1&0&0&+1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1&+1&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline -1&0&0&0&+1\\ 0&0&0&0&0\\ 0&0&+1&0&-1\\ 0&0&0&0&0\\ +1&0&-1&0&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&+1&0&0&-1\\ 0&0&0&0&0\\ 0&0&0&-1&+1\\ 0&-1&0&+1&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&-1&0&+1\\ 0&0&0&+1&-1\\ 0&0&+1&-1&0\\ \hline\end{array}$$

- $4 \times 5 \times 5$ move of degree 12 with slice degree $\{3,3,3,3\} \times \{2,2,2,3,3\} \times \{2,2,2,3,3\}$
  $((4,5,5),(12),((3,3,3,3),(2,2,2,3,3),(2,2,2,3,3)),(fcs),\emptyset,$
  $((111,125,144,214,221,255,333,342,354,432,445,453),(114,121,145,211,225,254,332,344,353,433,442,455)))$

$$
\begin{array}{|ccccc|}\hline +1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline\end{array}
\quad
\begin{array}{|ccccc|}\hline -1 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ \hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ \hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & -1 \\ \hline\end{array}
$$

A.3 Indispensable moves of degree 13

- $3 \times 5 \times 6$ move of degree 13 with slice degree $\{4,4,5\} \times \{2,2,2,3,4\} \times \{2,2,2,2,2,3\}$
  $((3,5,6),(13),((4,4,5),(2,2,2,3,4),(2,2,2,2,2,3)),(fcs),\emptyset,((111,126,143,152,216,234,241,255,322,336,345,$
  $353,354),(116,122,141,153,211,236,245,254,326,334,343,352,355)))$

$$
\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & +1 & -1 & 0 \\ \hline\end{array}
$$

- $3 \times 5 \times 6$ move of degree 13 with slice degree $\{4,4,5\} \times \{2,2,3,3,3\} \times \{2,2,2,2,2,3\}$
  $((3,5,6),(13),((4,4,5),(2,2,3,3,3),(2,2,2,2,2,3)),(fcs),(356),((111,133,142,156,226,234,243,255,316,324,$
  $331,345,352),(116,131,143,152,224,233,245,256,311,326,334,342,355)))$

$$
\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & -1 & (0) \\ \hline\end{array}
$$

- $3 \times 6 \times 6$ move of degree 13 with slice degree $\{3,5,5\} \times \{2,2,2,2,2,3\} \times \{2,2,2,2,2,3\}$
  $((3,6,6),(13),((3,5,5),(2,2,2,2,2,3),(2,2,2,2,2,3)),(fcs),\emptyset,((111,126,162,216,234,243,255,261,322,336,$
  $345,354,363),(116,122,161,211,236,245,254,263,326,334,343,355,362)))$

$$
\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ +1 & 0 & -1 & 0 & 0 & 0 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ \hline\end{array}
$$

- $3 \times 6 \times 6$ move of degree 13 with slice degree $\{4,4,5\} \times \{2,2,2,2,2,3\} \times \{2,2,2,2,2,3\}$
  $((3,6,6),(13),((4,4,5),(2,2,2,2,2,3),(2,2,2,2,2,3)),(fcs),\emptyset,((111,126,133,162,216,245,254,261,323,332,$
  $346,355,364),(116,123,132,161,211,246,255,264,326,333,345,354,362)))$

$$
\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 \\ \hline\end{array}
\quad
\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 \\ \hline\end{array}
$$

- $4 \times 4 \times 5$ move(1) of degree 13 with slice degree $\{3,3,3,4\} \times \{3,3,3,4\} \times \{2,2,3,3,3\}$
  $((4,4,5),(13),((3,3,3,4),(3,3,3,4),(2,2,3,3,3)),(fcs),(443,444),((111,135,144,225,233,242,315,324,343,$
  $413,422,434,441),(115,134,141,222,235,243,313,325,344,411,424,433,442)))$

$$
\begin{array}{|ccccc|}\hline +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ -1 & 0 & 0 & +1 & 0 \\ \hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 \\ \hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ \hline\end{array}
\quad
\begin{array}{|ccccc|}\hline -1 & 0 & +1 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ +1 & -1 & (0) & (0) & 0 \\ \hline\end{array}
$$

- $4 \times 4 \times 5$ move(2) of degree 13 with slice degree $\{3,3,3,4\} \times \{3,3,3,4\} \times \{2,2,3,3,3\}$ (not fundamental, circuit)
  $((4,4,5),(13),((3,3,3,4),(3,3,3,4),(2,2,3,3,3)),(Fcs),(244,323,415,443),((111,135,144,223,234,242,315,$ $324,343,413,425,432,441),(115,134,141,224,232,243,313,325,344,411,423,435,442)))$

$$\begin{array}{|ccccc|}\hline +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ -1 & 0 & 0 & +1 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & +1 & -1 & (0) & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & (0) & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline -1 & 0 & +1 & 0 & (0) \\ 0 & 0 & -1 & 0 & +1 \\ 0 & +1 & 0 & 0 & -1 \\ +1 & -1 & (0) & 0 & 0 \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 13 with slice degree $\{2,3,3,5\} \times \{3,3,3,4\} \times \{2,2,2,2,2,3\}$
  $((4,4,6),(13),((2,3,3,5),(3,3,3,4),(2,2,2,2,2,3)),(fcs),\emptyset,((111,126,216,233,242,325,336,344,412,421,434,$ $443,445),(116,121,212,236,243,326,334,345,411,425,433,442,444)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & -1 & +1 & 0 \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move(1) of degree 13 with slice degree $\{2,3,4,4\} \times \{3,3,3,4\} \times \{2,2,2,2,2,3\}$
  $((4,4,6),(13),((2,3,4,4),(3,3,3,4),(2,2,2,2,2,3)),(fcs),(446),((111,146,223,236,242,314,325,333,341,416,$ $422,434,445),(116,141,222,233,246,311,323,334,345,414,425,436,442)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline -1 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & -1 & 0 & 0 & +1 & (0) \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move(2) of degree 13 with slice degree $\{2,3,4,4\} \times \{3,3,3,4\} \times \{2,2,2,2,2,3\}$
  $((4,4,6),(13),((2,3,4,4),(3,3,3,4),(2,2,2,2,2,3)),(fcs),\emptyset,((111,126,222,233,246,315,321,332,344,416,434,$ $443,445),(116,121,226,232,243,311,322,334,345,415,433,444,446)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & +1 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & +1 & -1 & +1 & -1 \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 13 with slice degree $\{3,3,3,4\} \times \{3,3,3,4\} \times \{2,2,2,2,2,3\}$
  $((4,4,6),(13),((3,3,3,4),(3,3,3,4),(2,2,2,2,2,3)),(fcs),(416),((111,126,142,222,234,243,316,333,345,415,$ $424,436,441),(116,122,141,224,233,242,315,336,343,411,426,434,445)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & +1 & (0) \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ \hline\end{array}$$

- $4 \times 5 \times 5$ move of degree 13 with slice degree $\{2,3,3,5\} \times \{2,2,2,3,4\} \times \{2,2,3,3,3\}$
  $((4,5,5),(13),((2,3,3,5),(2,2,2,3,4),(2,2,3,3,3)),(fcs),\emptyset,((113,144,221,245,254,335,343,352,414,425,432,$ $451,453),(114,143,225,244,251,332,345,353,413,421,435,452,454)))$

$$\begin{array}{|ccccc|}\hline 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ -1 & 0 & 0 & +1 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & +1 & 0 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & +1 & -1 & 0 \\ \hline\end{array}$$

- $4 \times 5 \times 5$ move of degree 13 with slice degree $\{2,3,4,4\} \times \{2,2,2,3,4\} \times \{2,2,3,3,3\}$
  $((4,5,5),(13),((2,3,4,4),(2,2,2,3,4),(2,2,3,3,3)),(fcs),\emptyset,((113,154,235,241,252,314,325,331,343,424,442,$ $453,455),(114,153,231,242,255,313,324,335,341,425,443,452,454)))$

$$\begin{array}{|ccccc|}\hline 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & +1 & -1 & +1 \\ \hline\end{array}$$

- $4 \times 5 \times 5$ move of degree 13 with slice degree $\{3,3,3,4\} \times \{2,2,2,3,4\} \times \{2,2,2,3,4\}$
$((4,5,5),(13),((3,3,3,4),(2,2,2,3,4),(2,2,2,3,4)),(fcs),\emptyset,((114,135,152,225,241,253,334,343,355,412,421,444,455),(112,134,155,221,243,255,335,344,353,414,425,441,452)))$

$$\begin{array}{|ccccc|}\hline 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ \hline\end{array}$$

- $4 \times 5 \times 5$ move of degree 13 with slice degree $\{3,3,3,4\} \times \{2,2,2,3,4\} \times \{2,2,3,3,3\}$
$((4,5,5),(13),((3,3,3,4),(2,2,2,3,4),(2,2,3,3,3)),(fcs),\emptyset,((113,125,134,214,243,251,323,342,355,435,441,452,454),(114,123,135,213,241,254,325,343,352,434,442,451,455)))$

$$\begin{array}{|ccccc|}\hline 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ +1 & -1 & 0 & 0 & 0 \\ -1 & +1 & 0 & +1 & -1 \\ \hline\end{array}$$

- $4 \times 5 \times 6$ move of degree 13 with slice degree $\{2,3,3,5\} \times \{2,2,3,3,3\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((2,3,3,5),(2,2,3,3,3),(2,2,2,2,2,3)),(fcs),\emptyset,((131,146,212,236,253,324,345,356,413,425,432,441,454),(136,141,213,232,256,325,346,354,412,424,431,445,453)))$

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ \hline\end{array}$$

- $4 \times 5 \times 6$ move(1) of degree 13 with slice degree $\{2,3,4,4\} \times \{2,2,2,3,4\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((2,3,4,4),(2,2,2,3,4),(2,2,2,2,2,3)),(fcs),\emptyset,((141,152,213,226,254,324,336,345,351,416,435,442,453),(142,151,216,224,253,326,335,341,354,413,436,445,452)))$

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ -1 & 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ \hline\end{array}$$

- $4 \times 5 \times 6$ move(2) of degree 13 with slice degree $\{2,3,4,4\} \times \{2,2,2,3,4\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((2,3,4,4),(2,2,2,3,4),(2,2,2,2,2,3)),(fcs),\emptyset,((141,152,213,246,251,325,336,342,354,416,424,435,453),(142,151,216,241,253,324,335,346,352,413,425,436,454)))$

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & -1 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & +1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ \hline\end{array}$$

- $4 \times 5 \times 6$ move(1) of degree 13 with slice degree $\{2,3,4,4\} \times \{2,2,3,3,3\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((2,3,4,4),(2,2,3,3,3),(2,2,2,2,2,3)),(fcs),\emptyset,((136,141,216,234,253,322,331,345,354,413,425,446,452),(131,146,213,236,254,325,334,341,352,416,422,445,453)))$

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ \hline\end{array}$$

- $4 \times 5 \times 6$ move(2) of degree 13 with slice degree $\{2,3,4,4\} \times \{2,2,3,3,3\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((2,3,4,4),(2,2,3,3,3),(2,2,2,2,2,3)),(fcs),\emptyset,((131,142,213,226,254,324,336,341,355,416,432,445,453),(132,141,216,224,253,326,331,345,354,413,436,442,455)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ +1&-1&0&0&0&0\\ -1&+1&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&+1&0&0&-1\\ 0&0&0&-1&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&+1&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&+1&0&-1\\ -1&0&0&0&0&+1\\ +1&0&0&0&-1&0\\ 0&0&0&-1&+1&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&-1&0&0&+1\\ 0&0&0&0&0&0\\ 0&+1&0&0&0&-1\\ 0&-1&0&0&+1&0\\ 0&0&+1&0&-1&0\\ \hline\end{array}$$

- $4 \times 5 \times 6$ move of degree 13 with slice degree $\{3,3,3,4\} \times \{2,2,2,3,4\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((3,3,3,4),(2,2,2,3,4),(2,2,2,2,2,3)),(fcs),\emptyset,((111,126,152,222,246,253,335,343,354,416,434,445,451),(116,122,151,226,243,252,334,345,353,411,435,446,454)))$

$$\begin{array}{|cccccc|}\hline +1&0&0&0&0&-1\\ 0&-1&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&+1&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&+1&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&+1\\ 0&-1&+1&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-1&+1&0\\ 0&0&+1&0&-1&0\\ 0&0&-1&+1&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline -1&0&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&+1&-1&0\\ 0&0&0&0&+1&-1\\ +1&0&0&-1&0&0\\ \hline\end{array}$$

- $4 \times 5 \times 6$ move(1) of degree 13 with slice degree $\{3,3,3,4\} \times \{2,2,3,3,3\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((3,3,3,4),(2,2,3,3,3),(2,2,2,2,2,3)),(fcs),\emptyset,((131,143,152,216,234,241,325,346,353,414,426,432,455),(132,141,153,214,231,246,326,343,355,416,425,434,452)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ +1&-1&0&0&0&0\\ -1&0&+1&0&0&0\\ 0&+1&-1&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&-1&0&+1\\ 0&0&0&0&0&0\\ -1&0&0&+1&0&0\\ +1&0&0&0&0&-1\\ 0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&+1&-1\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&+1\\ 0&0&+1&0&-1&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&+1&0&-1\\ 0&0&0&0&-1&+1\\ 0&+1&0&-1&0&0\\ 0&0&0&0&0&0\\ 0&-1&0&0&+1&0\\ \hline\end{array}$$

- $4 \times 5 \times 6$ move(2) of degree 13 with slice degree $\{3,3,3,4\} \times \{2,2,3,3,3\} \times \{2,2,2,2,2,3\}$
$((4,5,6),(13),((3,3,3,4),(2,2,3,3,3),(2,2,2,2,2,3)),(fcs),\emptyset,((111,136,152,223,244,256,326,335,343,412,431,445,454),(112,131,156,226,243,254,323,336,345,411,435,444,452)))$

$$\begin{array}{|cccccc|}\hline +1&-1&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&+1&0&0&0&-1\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&+1&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&-1&+1&0&0\\ 0&0&0&-1&0&+1\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&-1&0&0&+1\\ 0&0&0&0&+1&-1\\ 0&0&+1&0&-1&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline -1&+1&0&0&0&0\\ 0&0&0&0&0&0\\ +1&0&0&0&-1&0\\ 0&0&0&-1&+1&0\\ 0&-1&0&+1&0&0\\ \hline\end{array}$$

- $5 \times 5 \times 5$ move of degree 13 with slice degree $\{2,2,2,3,4\} \times \{2,2,2,3,4\} \times \{2,2,3,3,3\}$
$((5,5,5),(13),((2,2,2,3,4),(2,2,2,3,4),(2,2,3,3,3)),(fcs),\emptyset,((113,155,224,253,345,351,432,441,454,515,523,534,542),(115,153,223,254,341,355,434,442,451,513,524,532,545)))$

$$\begin{array}{|ccccc|}\hline 0&0&+1&0&-1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&-1&0&+1\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&-1&+1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&+1&-1&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1&0&0&0&+1\\ +1&0&0&0&-1\\ \hline\end{array}$$

$$\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&+1&0&-1&0\\ +1&-1&0&0&0\\ -1&0&0&+1&0\\ \hline\end{array}\quad\begin{array}{|ccccc|}\hline 0&0&-1&0&+1\\ 0&0&+1&-1&0\\ 0&-1&0&+1&0\\ 0&+1&0&0&-1\\ 0&0&0&0&0\\ \hline\end{array}$$

- $5 \times 5 \times 5$ move(1) of degree 13 with slice degree $\{2,2,2,3,4\} \times \{2,2,3,3,3\} \times \{2,2,3,3,3\}$
$((5,5,5),(13),((2,2,2,3,4),(2,2,3,3,3),(2,2,3,3,3)),(fcs),\emptyset,((135,153,231,245,313,354,424,442,455,514,$

522, 533, 541), (133, 155, 235, 241, 314, 353, 422, 445, 454, 513, 524, 531, 542)))

$$
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&-1&0&+1\\ 0&0&0&0&0\\ 0&0&+1&0&-1\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ +1&0&0&0&-1\\ -1&0&0&0&+1\\ 0&0&0&0&0\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&+1&-1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&-1&+1&0\\\hline\end{array}
$$

$$
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&-1&0&+1&0\\ 0&0&0&0&0\\ 0&+1&0&0&-1\\ 0&0&0&-1&+1\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&-1&+1&0\\ 0&+1&0&-1&0\\ -1&0&+1&0&0\\ +1&-1&0&0&0\\ 0&0&0&0&0\\\hline\end{array}
$$

- $5 \times 5 \times 5$ move(2) of degree 13 with slice degree $\{2,2,2,3,4\} \times \{2,2,3,3,3\} \times \{2,2,3,3,3\}$
  $((5,5,5),(13),((2,2,2,3,4),(2,2,3,3,3),(2,2,3,3,3)),(fcs),\emptyset,((133,144,245,254,315,343,421,434,452,513,$ $522,531,555),(134,143,244,255,313,345,422,431,454,515,521,533,552)))$

$$
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&+1&-1&0\\ 0&0&-1&+1&0\\ 0&0&0&0&0\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-1&+1\\ 0&0&0&+1&-1\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&-1&0&+1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&+1&0&-1\\ 0&0&0&0&0\\\hline\end{array}
$$

$$
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ +1&-1&0&0&0\\ -1&0&0&+1&0\\ 0&0&0&0&0\\ 0&+1&0&-1&0\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&+1&0&-1\\ -1&+1&0&0&0\\ +1&0&-1&0&0\\ 0&0&0&0&0\\ 0&-1&0&0&+1\\\hline\end{array}
$$

- $5 \times 5 \times 5$ move(3) of degree 13 with slice degree $\{2,2,2,3,4\} \times \{2,2,3,3,3\} \times \{2,2,3,3,3\}$
  $((5,5,5),(13),((2,2,2,3,4),(2,2,3,3,3),(2,2,3,3,3)),(fcs),\emptyset,((131,143,233,252,315,324,425,444,453,514,$ $532,541,555),(133,141,232,253,314,325,424,443,455,515,531,544,552)))$

$$
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ +1&0&-1&0&0\\ -1&0&+1&0&0\\ 0&0&0&0&0\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&-1&+1&0&0\\ 0&0&0&0&0\\ 0&+1&-1&0&0\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&0&-1&+1\\ 0&0&0&+1&-1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\\hline\end{array}
$$

$$
\begin{array}{|ccccc|}\hline 0&0&0&0&0\\ 0&0&0&-1&+1\\ 0&0&0&0&0\\ 0&0&-1&+1&0\\ 0&0&+1&0&-1\\\hline\end{array}
\quad
\begin{array}{|ccccc|}\hline 0&0&0&+1&-1\\ 0&0&0&0&0\\ -1&+1&0&0&0\\ +1&0&0&-1&0\\ 0&-1&0&0&+1\\\hline\end{array}
$$

A.4 Indispensable moves of degree 14

- $3 \times 5 \times 6$ move of degree 14 with slice degree $\{4,4,6\} \times \{2,2,3,3,4\} \times \{2,2,2,2,2,4\}$
  $((3,5,6),(14),((4,4,6),(2,2,3,3,4),(2,2,2,2,2,4)),(fcS),\emptyset,((111,132,143,156,224,235,242,256,316,326,$ $331,344,353,355),(116,131,142,153,226,232,244,255,311,324,335,343,356,356)))$

$$
\begin{array}{|cccccc|}\hline +1&0&0&0&0&-1\\ 0&0&0&0&0&0\\ -1&+1&0&0&0&0\\ 0&-1&+1&0&0&0\\ 0&0&-1&0&0&+1\\\hline\end{array}
\quad
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&+1&0&-1\\ 0&-1&0&0&+1&0\\ 0&+1&0&-1&0&0\\ 0&0&0&0&-1&+1\\\hline\end{array}
\quad
\begin{array}{|cccccc|}\hline -1&0&0&0&0&+1\\ 0&0&0&-1&0&+1\\ +1&0&0&0&-1&0\\ 0&0&-1&+1&0&0\\ 0&0&+1&0&+1&-2\\\hline\end{array}
$$

- $3 \times 5 \times 6$ move of degree 14 with slice degree $\{4,5,5\} \times \{2,3,3,3,3\} \times \{2,2,2,2,2,4\}$
  $((3,5,6),(14),((4,5,5),(2,3,3,3,3),(2,2,2,2,2,4)),(fcs),\emptyset,((121,132,143,154,215,226,233,246,251,316,$

322, 335, 344, 356), (122, 133, 144, 151, 216, 221, 235, 243, 256, 315, 326, 332, 346, 354)))

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ +1&-1&0&0&0&0\\ 0&+1&-1&0&0&0\\ 0&0&+1&-1&0&0\\ -1&0&0&+1&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&+1&-1\\ -1&0&0&0&0&+1\\ 0&0&+1&0&-1&0\\ 0&0&-1&0&0&+1\\ +1&0&0&0&0&-1\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&-1&+1\\ 0&+1&0&0&0&-1\\ 0&-1&0&0&+1&0\\ 0&0&0&+1&0&-1\\ 0&0&0&-1&0&+1\\ \hline\end{array}$$

- $3 \times 5 \times 7$ move(1) of degree 14 with slice degree $\{4, 4, 6\} \times \{2, 2, 3, 3, 4\} \times \{2, 2, 2, 2, 2, 2, 2\}$
  $((3, 5, 7), (14), ((4, 4, 6), (2, 2, 3, 3, 4), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((111, 132, 143, 154, 227, 235, 242, 256, 314, 326, 331, 347, 353, 355), (114, 131, 142, 153, 226, 232, 247, 255, 311, 327, 335, 343, 354, 356)))$

$$\begin{array}{|ccccccc|}\hline +1&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0\\ 0&-1&+1&0&0&0&0\\ 0&0&-1&+1&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&+1\\ 0&-1&0&0&+1&0&0\\ 0&+1&0&0&0&0&-1\\ 0&0&0&0&-1&+1&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline -1&0&0&+1&0&0&0\\ 0&0&0&0&0&+1&-1\\ +1&0&0&0&-1&0&0\\ 0&0&-1&0&0&0&+1\\ 0&0&+1&-1&+1&-1&0\\ \hline\end{array}$$

- $3 \times 5 \times 7$ move(2) of degree 14 with slice degree $\{4, 4, 6\} \times \{2, 2, 3, 3, 4\} \times \{2, 2, 2, 2, 2, 2, 2\}$
  $((3, 5, 7), (14), ((4, 4, 6), (2, 2, 3, 3, 4), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((111, 132, 143, 154, 226, 235, 242, 257, 314, 325, 331, 347, 353, 356), (114, 131, 142, 153, 225, 232, 247, 256, 311, 326, 335, 343, 354, 357)))$

$$\begin{array}{|ccccccc|}\hline +1&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0\\ 0&-1&+1&0&0&0&0\\ 0&0&-1&+1&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&-1&+1&0\\ 0&-1&0&0&+1&0&0\\ 0&+1&0&0&0&0&-1\\ 0&0&0&0&0&-1&+1\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline -1&0&0&+1&0&0&0\\ 0&0&0&0&+1&-1&0\\ +1&0&0&0&-1&0&0\\ 0&0&-1&0&0&0&+1\\ 0&0&+1&-1&0&+1&-1\\ \hline\end{array}$$

- $3 \times 5 \times 7$ move(1) of degree 14 with slice degree $\{4, 5, 5\} \times \{2, 2, 3, 3, 4\} \times \{2, 2, 2, 2, 2, 2, 2\}$
  $((3, 5, 7), (14), ((4, 5, 5), (2, 2, 3, 3, 4), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((111, 133, 142, 154, 212, 226, 231, 247, 255, 325, 336, 344, 353, 357), (112, 131, 144, 153, 211, 225, 236, 242, 257, 326, 333, 347, 354, 355)))$

$$\begin{array}{|ccccccc|}\hline +1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&+1&0&0&0&0\\ 0&+1&0&-1&0&0&0\\ 0&0&-1&+1&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline -1&+1&0&0&0&0&0\\ 0&0&0&0&-1&+1&0\\ +1&0&0&0&0&-1&0\\ 0&-1&0&0&0&0&+1\\ 0&0&0&0&+1&0&-1\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&+1&-1&0\\ 0&0&-1&0&0&+1&0\\ 0&0&0&+1&0&0&-1\\ 0&0&+1&-1&-1&0&+1\\ \hline\end{array}$$

- $3 \times 5 \times 7$ move (2) of degree 14 with slice degree $\{4, 5, 5\} \times \{2, 2, 3, 3, 4\} \times \{2, 2, 2, 2, 2, 2, 2\}$
  $((3, 5, 7), (14), ((4, 5, 5), (2, 2, 3, 3, 4), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((131, 144, 152, 153, 215, 226, 237, 243, 251, 316, 327, 332, 345, 354), (132, 143, 151, 154, 216, 227, 231, 245, 253, 315, 326, 337, 344, 352)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0\\ 0&0&-1&+1&0&0&0\\ -1&+1&+1&-1&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&+1&-1&0\\ 0&0&0&0&0&+1&-1\\ -1&0&0&0&0&0&+1\\ 0&0&+1&0&-1&0&0\\ +1&0&-1&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&-1&+1&0\\ 0&0&0&0&0&-1&+1\\ 0&+1&0&0&0&0&-1\\ 0&0&0&-1&+1&0&0\\ 0&-1&0&+1&0&0&0\\ \hline\end{array}$$

- $3 \times 5 \times 7$ move of degree 14 with slice degree $\{4, 5, 5\} \times \{2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 2, 2\}$
  $((3, 5, 7), (14), ((4, 5, 5), (2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((121, 133, 142, 154, 216, 222, 237, 245, 253, 315, 326, 331, 344, 357), (122, 131, 144, 153, 215, 226, 233, 242, 257, 316, 321, 337, 345, 354)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0\\ -1&0&+1&0&0&0&0\\ 0&+1&0&-1&0&0&0\\ 0&0&-1&+1&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&-1&+1&0\\ 0&+1&0&0&0&-1&0\\ 0&0&-1&0&0&0&+1\\ 0&-1&0&0&+1&0&0\\ 0&0&+1&0&0&0&-1\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&+1&-1&0\\ -1&0&0&0&0&+1&0\\ +1&0&0&0&0&0&-1\\ 0&0&0&+1&-1&0&0\\ 0&0&0&-1&0&0&+1\\ \hline\end{array}$$

- $3 \times 6 \times 6$ move(1) of degree 14 with slice degree $\{4, 4, 6\} \times \{2, 2, 2, 2, 3, 3\} \times \{2, 2, 2, 2, 2, 4\}$
  $((3, 6, 6), (14), ((4, 4, 6), (2, 2, 2, 2, 3, 3), (2, 2, 2, 2, 2, 4)), (fcs), \emptyset, ((111, 126, 152, 163, 234, 246, 253, 265, 316,$

322, 336, 345, 354, 361), (116, 122, 153, 161, 236, 245, 254, 263, 311, 326, 334, 346, 352, 365)))

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & -1 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ \hline \end{array}$$

- $3 \times 6 \times 6$ move(2) of degree 14 with slice degree $\{4,4,6\} \times \{2,2,2,2,3,3\} \times \{2,2,2,2,2,4\}$
  (not fundamental, circuit)
  ((3, 6, 6), (14), ((4, 4, 6), (2, 2, 2, 2, 3, 3), (2, 2, 2, 2, 2, 4)), (Fcs), (156), ((111, 126, 152, 163, 234, 246, 253, 265, 316, 322, 335, 344, 356, 361), (116, 122, 153, 161, 235, 244, 256, 263, 311, 326, 334, 346, 352, 365)))

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & (0) \\ -1 & 0 & +1 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ \hline \end{array}$$

- $3 \times 6 \times 7$ move of degree 14 with slice degree $\{4,4,6\} \times \{2,2,2,2,3,3\} \times \{2,2,2,2,2,2,2\}$
  ((3, 6, 7), (14), ((4, 4, 6), (2, 2, 2, 2, 3, 3), (2, 2, 2, 2, 2, 2, 2)), (fcs), $\emptyset$, ((111, 122, 153, 164, 236, 247, 254, 265, 312, 323, 335, 346, 357, 361), (112, 123, 154, 161, 235, 246, 257, 264, 311, 322, 336, 347, 353, 365)))

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline \end{array}$$

- $3 \times 6 \times 7$ move of degree 14 with slice degree $\{4,5,5\} \times \{2,2,2,2,3,3\} \times \{2,2,2,2,2,2,2\}$
  ((3, 6, 7), (14), ((4, 5, 5), (2, 2, 2, 2, 3, 3), (2, 2, 2, 2, 2, 2, 2)), (fcs), $\emptyset$, ((111, 123, 154, 162, 212, 235, 246, 251, 267, 324, 336, 347, 355, 363), (112, 124, 151, 163, 211, 236, 247, 255, 262, 323, 335, 346, 354, 367)))

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ \hline \end{array}$$

- $4 \times 4 \times 5$ move(1) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{2,2,3,3,4\}$
  (not fundamental, circuit)
  ((4, 4, 5), (14), ((3, 3, 3, 5), (3, 3, 3, 5), (2, 2, 3, 3, 4)), (FcS), (335, 444), ((111, 123, 144, 222, 233, 245, 313, 334, 345, 415, 424, 435, 441, 442), (113, 124, 141, 223, 235, 242, 315, 333, 344, 411, 422, 434, 445, 445)))

$$\begin{array}{|ccccc|}\hline +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ 0 & -1 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & (0) \\ 0 & 0 & 0 & -1 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline -1 & 0 & 0 & 0 & +1 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ +1 & +1 & 0 & (0) & -2 \\ \hline \end{array}$$

- $4 \times 4 \times 5$ move(2) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{2,2,3,3,4\}$
  (not fundamental, circuit)
  ((4, 4, 5), (14), ((3, 3, 3, 5), (3, 3, 3, 5), (2, 2, 3, 3, 4)), (FcS), (244, 413), ((111, 133, 144, 222, 234, 245, 313, 324, 345, 415, 423, 435, 441, 442), (113, 134, 141, 224, 235, 242, 315, 323, 344, 411, 422, 433, 445, 445)))

$$\begin{array}{|ccccc|}\hline -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & +1 & 0 & (0) & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline +1 & 0 & (0) & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ -1 & -1 & 0 & 0 & +2 \\ \hline \end{array}$$

- $4 \times 4 \times 5$ move(1) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,4,4\} \times \{2,3,3,3,3\}$
  (not fundamental, circuit)
  $((4,4,5),(14),((3,3,3,5),(3,3,4,4),(2,3,3,3,3)),(Fcs),(132,245,424),((112,124,133,213,225,244,$
  $315,332,341,422,431,434,443,445),(113,122,134,215,224,243,312,331,345,425,432,433,441,444)))$

$$\begin{array}{|ccccc|}\hline 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & (0) & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline\end{array}\;\begin{array}{|ccccc|}\hline 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & (0) \\ \hline\end{array}\;\begin{array}{|ccccc|}\hline 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ \hline\end{array}\;\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & (0) & -1 \\ +1 & -1 & -1 & +1 & 0 \\ -1 & 0 & +1 & -1 & +1 \\ \hline\end{array}$$

- $4 \times 4 \times 5$ move(2) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,4,4\} \times \{2,3,3,3,3\}$
  $((4,4,5),(14),((3,3,3,5),(3,3,4,4),(2,3,3,3,3)),(fcs),\emptyset,((112,124,133,213,225,244,315,331,342,$
  $422,434,435,441,443),(113,122,134,215,224,243,312,335,341,425,431,433,442,444)))$

$$\begin{array}{|ccccc|}\hline 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline\end{array}\;\begin{array}{|ccccc|}\hline 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ \hline\end{array}\;\begin{array}{|ccccc|}\hline 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ -1 & +1 & 0 & 0 & 0 \\ \hline\end{array}\;\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ -1 & 0 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & 0 \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 14 with slice degree $\{2,3,4,5\} \times \{3,3,3,5\} \times \{2,2,2,2,2,4\}$
  $((4,4,6),(14),((2,3,4,5),(3,3,3,5),(2,2,2,2,2,4)),(fcS),\emptyset,((111,146,225,233,246,312,323,334,341,$
  $416,422,436,444,445),(116,141,223,236,245,311,322,333,344,412,425,434,446,446)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 & +1 & -2 \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 14 with slice degree $\{2,4,4,4\} \times \{3,3,4,4\} \times \{2,2,2,2,2,4\}$
  $((4,4,6),(14),((2,4,4,4),(3,3,4,4),(2,2,2,2,2,4)),(fcs),\emptyset,((131,142,213,226,236,241,314,323,332,$
  $345,416,425,434,446),(132,141,216,223,231,246,313,325,334,342,414,426,436,445)))$

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move(1) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{2,2,2,2,3,3\}$
  $((4,4,6),(14),((3,3,3,5),(3,3,3,5),(2,2,2,2,3,3)),(fcs),(446),((111,125,142,215,233,246,326,335,$
  $344,416,422,434,441,443),(115,122,141,216,235,243,325,334,346,411,426,433,442,444)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ +1 & -1 & +1 & -1 & 0 & (0) \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move(2) of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,3,5\} \times \{2,2,2,2,3,3\}$
  (not fundamental, circuit)
  $((4,4,6),(14),((3,3,3,5),(3,3,3,5),(2,2,2,2,3,3)),(Fcs),(436),((111,125,142,215,236,243,326,334,$
  $345,416,422,433,441,444),(115,122,141,216,233,245,325,336,344,411,426,434,442,443)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & -1 & 0 & (0) \\ +1 & -1 & -1 & +1 & 0 & 0 \\ \hline\end{array}$$

- $4 \times 4 \times 6$ move of degree 14 with slice degree $\{3,3,4,4\} \times \{3,3,4,4\} \times \{2,2,2,2,2,4\}$
  $((4,4,6),(14),((3,3,4,4),(3,3,4,4),(2,2,2,2,2,4)),(fcs),\emptyset,((111,136,142,223,236,244,315,324,331,$
  $346,412,425,433,446),(112,131,146,224,233,246,311,325,336,344,415,423,436,442)))$

$$\begin{array}{|cccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ \hline\end{array}\;\begin{array}{|cccccc|}\hline 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ \hline\end{array}$$

- $4 \times 4 \times 7$ move(1) of degree 14 with slice degree $\{2,3,4,5\} \times \{2,3,4,5\} \times \{2,2,2,2,2,2,2\}$
$((4,4,7),(14),((2,3,4,5),(2,3,4,5),(2,2,2,2,2,2,2)),(fcs),\emptyset,((133,145,222,231,243,314,321,337,346,$
$416,424,435,442,447),(135,143,221,233,242,316,324,331,347,414,422,437,445,446)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	+1	0	−1	0	0
0	0	−1	0	+1	0	0

0	0	0	0	0	0	0
−1	+1	0	0	0	0	0
+1	0	−1	0	0	0	0
0	−1	+1	0	0	0	0

0	0	0	+1	0	−1	0
+1	0	0	−1	0	0	0
−1	0	0	0	0	0	+1
0	0	0	0	0	+1	−1

0	0	0	−1	0	+1	0
0	−1	0	+1	0	0	0
0	0	0	0	+1	0	−1
0	+1	0	0	−1	−1	+1

- $4 \times 4 \times 7$ move(2) of degree 14 with slice degree $\{2,3,4,5\} \times \{2,3,4,5\} \times \{2,2,2,2,2,2,2\}$
$((4,4,7),(14),((2,3,4,5),(2,3,4,5),(2,2,2,2,2,2,2)),(fcs),\emptyset,((131,142,223,234,245,316,324,332,347,$
$417,426,435,441,443),(132,141,224,235,243,317,326,334,342,416,423,431,445,447)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
+1	−1	0	0	0	0	0
−1	+1	0	0	0	0	0

0	0	0	0	0	0	0
0	0	+1	−1	0	0	0
0	0	0	+1	−1	0	0
0	0	−1	0	+1	0	0

0	0	0	0	0	+1	−1
0	0	0	+1	0	−1	0
0	+1	0	−1	0	0	0
0	−1	0	0	0	0	+1

0	0	0	0	0	−1	+1
0	0	−1	0	0	+1	0
−1	0	0	0	+1	0	0
+1	0	+1	0	−1	0	−1

- $4 \times 4 \times 7$ move of degree 14 with slice degree $\{2,3,4,5\} \times \{2,4,4,4\} \times \{2,2,2,2,2,2,2\}$
$((4,4,7),(14),((2,3,4,5),(2,4,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((121,132,223,231,244,315,322,337,346,$
$416,425,434,443,447),(122,131,221,234,243,316,325,332,347,415,423,437,444,446)))$

0	0	0	0	0	0	0
+1	−1	0	0	0	0	0
−1	+1	0	0	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
−1	0	+1	0	0	0	0
+1	0	0	−1	0	0	0
0	0	−1	+1	0	0	0

0	0	0	0	+1	−1	0
0	+1	0	0	−1	0	0
0	−1	0	0	0	0	+1
0	0	0	0	0	+1	−1

0	0	0	0	−1	+1	0
0	0	−1	0	+1	0	0
0	0	0	+1	0	0	−1
0	0	+1	−1	0	−1	+1

- $4 \times 4 \times 7$ move of degree 14 with slice degree $\{2,3,4,5\} \times \{3,3,3,5\} \times \{2,2,2,2,2,2,2\}$
$((4,4,7),(14),((2,3,4,5),(3,3,3,5),(2,2,2,2,2,2,2)),(fcs),\emptyset,((111,142,223,235,244,312,326,333,347,$
$416,424,437,441,445),(112,141,224,233,245,316,323,337,342,411,426,435,444,447)))$

+1	−1	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	+1	0	0	0	0	0

0	0	0	0	0	0	0
0	0	+1	−1	0	0	0
0	0	−1	0	+1	0	0
0	0	0	+1	−1	0	0

0	+1	0	0	0	−1	0
0	0	−1	0	0	+1	0
0	0	+1	0	0	0	−1
0	−1	0	0	0	0	+1

−1	0	0	0	0	+1	0
0	0	0	+1	0	−1	0
0	0	0	0	−1	0	+1
+1	0	0	−1	+1	0	−1

- $4 \times 4 \times 7$ move of degree 14 with slice degree $\{2,3,4,5\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
$((4,4,7),(14),((2,3,4,5),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((111,132,223,234,245,317,324,331,346,$

412, 426, 435, 443, 447), (112, 131, 224, 235, 243, 311, 326, 334, 347, 417, 423, 432, 445, 446)))

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & -1 & +1 \\ \hline \end{array}$$

- $4 \times 4 \times 7$ move(1) of degree 14 with slice degree $\{2,4,4,4\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
  $((4,4,7),(14),((2,4,4,4),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((111,132,213,224,231,245,312,323,337,$ $346,426,435,444,447),(112,131,211,223,235,244,313,326,332,347,424,437,445,446)))$

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 & -1 & -1 & +1 \\ \hline \end{array}$$

- $4 \times 4 \times 7$ move(2) of degree 14 with slice degree $\{2,4,4,4\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
  $((4,4,7),(14),((2,4,4,4),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((131,142,213,225,234,241,316,323,332,$ $347,414,427,436,445),(132,141,214,223,231,245,313,327,336,342,416,425,434,447)))$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ \hline \end{array}$$

- $4 \times 4 \times 7$ move of degree 14 with slice degree $\{3,3,3,5\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
  $((4,4,7),(14),((3,3,3,5),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((111,123,132,212,235,244,326,333,347,$ $414,421,437,445,446),(112,121,133,214,232,245,323,337,346,411,426,435,444,447)))$

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & -1 & +1 & +1 & -1 \\ \hline \end{array}$$

- $4 \times 4 \times 7$ move of degree 14 with slice degree $\{3,3,4,4\} \times \{3,3,4,4\} \times \{2,2,2,2,2,2,2\}$
  $((4,4,7),(14),((3,3,4,4),(3,3,4,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((111,133,142,224,236,245,317,325,331,$

343, 412, 427, 434, 446), (112, 131, 143, 225, 234, 246, 311, 327, 333, 345, 417, 424, 436, 442)))

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ \hline \end{array}$$

- $4 \times 5 \times 5$ move of degree 14 with slice degree $\{2,3,4,5\} \times \{2,2,2,4,4\} \times \{2,3,3,3,3\}$
  $((4,5,5),(14),((2,3,4,5),(2,2,2,4,4),(2,3,3,3,3)),(fcs),\emptyset,((142,153,214,245,251,322,335,343,344,$
  $411,424,433,452,455),(143,152,211,244,255,324,333,342,345,414,422,435,451,453)))$

$$\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & -1 & +1 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline +1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & -1 & 0 & +1 \\ \hline \end{array}$$

- $4 \times 5 \times 5$ move(1) of degree 14 with slice degree $\{3,3,3,5\} \times \{2,2,3,3,4\} \times \{2,3,3,3,3\}$
  (not fundamental, circuit)
  $((4,5,5),(14),((3,3,3,5),(2,2,3,3,4),(2,3,3,3,3)),(Fcs),(353,445),((111,132,153,225,234,242,333,$
  $345,354,412,424,443,451,455),(112,133,151,224,232,245,334,343,355,411,425,442,453,454)))$

$$\begin{array}{|ccccc|}\hline +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & (0) & +1 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & (0) \\ +1 & 0 & -1 & -1 & +1 \\ \hline \end{array}$$

- $4 \times 5 \times 5$ move(2) of degree 14 with slice degree $\{3,3,3,5\} \times \{2,2,3,3,4\} \times \{2,3,3,3,3\}$
  $((4,5,5),(14),((3,3,3,5),(2,2,3,3,4),(2,3,3,3,3)),(fcs),(445),((111,132,153,224,233,245,334,342,$
  $355,412,425,443,451,454),(112,133,151,225,234,243,332,345,354,411,424,442,453,455)))$

$$\begin{array}{|ccccc|}\hline +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & (0) \\ +1 & 0 & -1 & +1 & -1 \\ \hline \end{array}$$

- $4 \times 5 \times 5$ move(1) of degree 14 with slice degree $\{3,3,4,4\} \times \{2,3,3,3,3\} \times \{2,3,3,3,3\}$
  $((4,5,5),(14),((3,3,4,4),(2,3,3,3,3),(2,3,3,3,3)),(fcs),(322,455),((122,135,153,224,242,255,314,$
  $323,332,341,413,431,445,454),(123,132,155,222,245,254,313,324,331,342,414,435,441,453)))$

$$\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & +1 & 0 \\ 0 & (0) & +1 & -1 & 0 \\ -1 & +1 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 & (0) \\ \hline \end{array}$$

- $4 \times 5 \times 5$ move(2) of degree 14 with slice degree $\{3,3,4,4\} \times \{2,3,3,3,3\} \times \{2,3,3,3,3\}$
  $((4,5,5),(14),((3,3,4,4),(2,3,3,3,3),(2,3,3,3,3)),(fcs),\emptyset,((122,135,153,224,242,255,314,323,331,$
  $345,413,432,441,454),(123,132,155,222,245,254,313,324,335,341,414,431,442,453)))$

$$\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ \hline \end{array}$$

- $4 \times 5 \times 6$ move(1) of degree 14 with slice degree $\{2,3,4,5\} \times \{2,3,3,3,3\} \times \{2,2,2,2,2,4\}$
$((4,5,6),(14),((2,3,4,5),(2,3,3,3,3),(2,2,2,2,2,4)),(fcs),\emptyset,((121,136,216,242,253,324,331,343,355,$
$412,426,435,444,456),(126,131,212,243,256,321,335,344,353,416,424,436,442,455)))$

$$
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ +1&0&0&0&0&-1\\ -1&0&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&-1&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&+1&-1&0&0&0\\ 0&0&+1&0&0&-1\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ -1&0&0&+1&0&0\\ +1&0&0&0&-1&0\\ 0&0&+1&-1&0&0\\ 0&0&-1&0&+1&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&+1&0&0&0&-1\\ 0&0&0&-1&0&+1\\ 0&0&0&0&+1&-1\\ 0&-1&0&+1&0&0\\ 0&0&0&0&-1&+1\\ \hline\end{array}
$$

- $4 \times 5 \times 6$ move(2) of degree 14 with slice degree $\{2,3,4,5\} \times \{2,3,3,3,3\} \times \{2,2,2,2,2,4\}$
$((4,5,6),(14),((2,3,4,5),(2,3,3,3,3),(2,2,2,2,2,4)),(fcs),\emptyset,((121,136,212,246,253,326,335,344,356,$
$413,424,431,442,455),(126,131,213,242,256,324,336,346,355,412,421,435,444,453)))$

$$
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ +1&0&0&0&0&-1\\ -1&0&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&+1&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&+1\\ 0&0&+1&0&0&-1\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&-1&0&+1\\ 0&0&0&0&+1&-1\\ 0&0&0&+1&0&-1\\ 0&0&0&0&-1&+1\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&-1&+1&0&0&0\\ -1&0&0&+1&0&0\\ +1&0&0&0&-1&0\\ 0&+1&0&-1&0&0\\ 0&0&-1&0&+1&0\\ \hline\end{array}
$$

- $4 \times 5 \times 6$ move(3) of degree 14 with slice degree $\{2,3,4,5\} \times \{2,3,3,3,3\} \times \{2,2,2,2,2,4\}$
(not fundamental, circuit)
$((4,5,6),(14),((2,3,4,5),(2,3,3,3,3),(2,2,2,2,2,4)),(Fcs),(326,336,446,456),((121,136,212,246,253,$
$324,331,345,356,413,426,435,442,454),(126,131,213,242,256,321,335,346,354,412,424,436,445,453)))$

$$
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ +1&0&0&0&0&-1\\ -1&0&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&+1&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&+1\\ 0&0&+1&0&0&-1\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ -1&0&0&+1&0&(0)\\ +1&0&0&0&-1&(0)\\ 0&0&0&0&+1&-1\\ 0&0&0&-1&0&+1\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&-1&+1&0&0&0\\ 0&0&0&-1&0&+1\\ 0&0&0&0&+1&-1\\ 0&+1&0&0&-1&(0)\\ 0&0&-1&+1&0&(0)\\ \hline\end{array}
$$

- $4 \times 5 \times 6$ move of degree 14 with slice degree $\{3,3,3,5\} \times \{2,2,3,3,4\} \times \{2,2,2,2,2,4\}$
$((4,5,6),(14),((3,3,3,5),(2,2,3,3,4),(2,2,2,2,2,4)),(fcs),\emptyset,((111,136,152,232,243,254,326,344,355,$
$416,425,433,446,451),(116,132,151,233,244,252,325,346,354,411,426,436,443,455)))$

$$
\begin{array}{|cccccc|}\hline +1&0&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&+1\\ 0&0&0&0&0&0\\ -1&+1&0&0&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&+1&-1&0&0&0\\ 0&0&+1&-1&0&0\\ 0&-1&0&+1&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&-1&+1\\ 0&0&0&0&0&0\\ 0&0&0&+1&0&-1\\ 0&0&0&-1&+1&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline -1&0&0&0&0&+1\\ 0&0&0&0&+1&-1\\ 0&0&+1&0&0&-1\\ 0&0&-1&0&0&+1\\ +1&0&0&0&-1&0\\ \hline\end{array}
$$

- $4 \times 5 \times 6$ move of degree 14 with slice degree $\{3,3,3,5\} \times \{2,3,3,3,3\} \times \{2,2,2,2,3,3\}$
$((4,5,6),(14),((3,3,3,5),(2,3,3,3,3),(2,2,2,2,3,3)),(fcs),(456),((111,125,132,226,245,253,335,344,$
$356,412,421,436,443,454),(112,121,135,225,243,256,336,345,354,411,426,432,444,453)))$

$$
\begin{array}{|cccccc|}\hline +1&-1&0&0&0&0\\ -1&0&0&0&+1&0\\ 0&+1&0&0&-1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&-1&+1\\ 0&0&0&0&0&0\\ 0&0&-1&0&+1&0\\ 0&0&+1&0&0&-1\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&+1&-1\\ 0&0&0&+1&-1&0\\ 0&0&0&-1&0&+1\\ \hline\end{array}
\begin{array}{|cccccc|}\hline -1&+1&0&0&0&0\\ +1&0&0&0&0&-1\\ 0&-1&0&0&0&+1\\ 0&0&+1&-1&0&0\\ 0&0&-1&+1&0&(0)\\ \hline\end{array}
$$

- $4 \times 5 \times 6$ move of degree 14 with slice degree $\{3,3,4,4\} \times \{2,2,2,4,4\} \times \{2,2,2,2,2,4\}$
$((4,5,6),(14),((3,3,4,4),(2,2,2,4,4),(2,2,2,2,2,4)),(fcs),\emptyset,((111,146,153,226,243,252,316,335,344,$
$351,422,434,446,455),(116,143,151,222,246,253,311,334,346,355,426,435,444,452)))$

$$
\begin{array}{|cccccc|}\hline +1&0&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&+1\\ -1&0&+1&0&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&-1&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&+1&0&0&-1\\ 0&+1&-1&0&0&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline -1&0&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&-1&+1&0\\ 0&0&0&+1&0&-1\\ +1&0&0&0&-1&0\\ \hline\end{array}
\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&+1&0&0&0&-1\\ 0&0&0&+1&-1&0\\ 0&0&0&-1&0&+1\\ 0&-1&0&0&+1&0\\ \hline\end{array}
$$

- $4\times5\times6$ move of degree 14 with slice degree $\{3,3,4,4\}\times\{2,2,2,4,4\}\times\{2,2,2,2,3,3\}$
$((4,5,6),(14),((3,3,4,4),(2,2,2,4,4),(2,2,2,2,3,3)),(fcs),\emptyset,((111,142,155,226,243,254,315,323,336,341,435,444,452,456),(115,141,152,223,244,256,311,326,335,343,436,442,454,455)))$

$$\begin{array}{|cccccc|}\hline +1&0&0&0&-1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-1&+1&0&0&0&0\\0&-1&0&0&+1&0\\\hline\end{array}\;\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\0&0&-1&0&0&+1\\0&0&0&0&0&0\\0&0&+1&-1&0&0\\0&0&0&+1&0&-1\\\hline\end{array}\;\begin{array}{|cccccc|}\hline -1&0&0&0&+1&0\\0&0&+1&0&0&-1\\0&0&0&0&-1&+1\\+1&0&-1&0&0&0\\0&0&0&0&0&0\\\hline\end{array}\;\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&+1&-1\\0&-1&0&+1&0&0\\0&+1&0&-1&-1&+1\\\hline\end{array}$$

- $4\times5\times6$ move of degree 14 with slice degree $\{3,3,4,4\}\times\{2,2,3,3,4\}\times\{2,2,2,2,2,4\}$
$((4,5,6),(14),((3,3,4,4),(2,2,3,3,4),(2,2,2,2,2,4)),(fcs),\emptyset,((111,132,156,223,244,256,316,331,345,354,426,435,443,452),(116,131,152,226,243,254,311,335,344,356,423,432,445,456)))$

$$\begin{array}{|cccccc|}\hline -1&0&0&0&0&+1\\0&0&0&0&0&0\\+1&-1&0&0&0&0\\0&0&0&0&0&0\\0&+1&0&0&0&-1\\\hline\end{array}\;\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\0&0&-1&0&0&+1\\0&0&0&0&0&0\\0&0&+1&-1&0&0\\0&0&0&+1&0&-1\\\hline\end{array}\;\begin{array}{|cccccc|}\hline +1&0&0&0&0&-1\\0&0&0&0&0&0\\-1&0&0&0&+1&0\\0&0&0&+1&-1&0\\0&0&0&-1&0&+1\\\hline\end{array}\;\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\0&0&+1&0&0&-1\\0&+1&0&0&-1&0\\0&0&-1&0&+1&0\\0&-1&0&0&0&+1\\\hline\end{array}$$

- $4\times5\times7$ move(1) of degree 14 with slice degree $\{2,3,4,5\}\times\{2,2,3,3,4\}\times\{2,2,2,2,2,2,2\}$
$((4,5,7),(14),((2,3,4,5),(2,2,3,3,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((131,152,213,244,255,327,336,345,351,414,426,432,447,453),(132,151,214,245,253,326,331,347,355,413,427,436,444,452)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\+1&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&+1&0&0&0&0&0\\\hline\end{array}\;\begin{array}{|ccccccc|}\hline 0&0&+1&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&+1&-1&0&0\\0&0&-1&0&+1&0&0\\\hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\0&0&0&0&0&-1&+1\\-1&0&0&0&0&+1&0\\0&0&0&0&+1&0&-1\\+1&0&0&0&-1&0&0\\\hline\end{array}\;\begin{array}{|ccccccc|}\hline 0&0&-1&+1&0&0&0\\0&0&0&0&0&+1&-1\\0&+1&0&0&0&-1&0\\0&0&0&-1&0&0&+1\\0&-1&+1&0&0&0&0\\\hline\end{array}$$

- $4\times5\times7$ move(2) of degree 14 with slice degree $\{2,3,4,5\}\times\{2,2,3,3,4\}\times\{2,2,2,2,2,2,2\}$
$((4,5,7),(14),((2,3,4,5),(2,2,3,3,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((132,151,231,243,254,316,327,344,355,415,426,433,447,452),(131,152,233,244,251,315,326,347,354,416,427,432,443,455)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&+1&0&0&0&0&0\\0&0&0&0&0&0&0\\+1&-1&0&0&0&0&0\\\hline\end{array}\;\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\+1&0&-1&0&0&0&0\\0&0&+1&-1&0&0&0\\-1&0&0&+1&0&0&0\\\hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&-1&+1&0\\0&0&0&0&0&-1&+1\\0&0&0&0&0&0&0\\0&0&0&+1&0&0&-1\\0&0&0&-1&+1&0&0\\\hline\end{array}\;\begin{array}{|ccccccc|}\hline 0&0&0&0&+1&-1&0\\0&0&0&0&0&+1&-1\\0&-1&+1&0&0&0&0\\0&0&-1&0&0&0&+1\\0&+1&0&0&-1&0&0\\\hline\end{array}$$

- $4\times5\times7$ move of degree 14 with slice degree $\{2,3,4,5\}\times\{2,3,3,3,3\}\times\{2,2,2,2,2,2,2\}$
$((4,5,7),(14),((2,3,4,5),(2,3,3,3,3),(2,2,2,2,2,2,2)),(fcs),\emptyset,((121,132,214,243,255,326,331,345,357,$

413, 422, 437, 446, 454), (122, 131, 213, 245, 254, 321, 337, 346, 355, 414, 426, 432, 443, 457)))

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ \hline\end{array}$$

- $4 \times 5 \times 7$ move of degree 14 with slice degree $\{2, 4, 4, 4\} \times \{2, 2, 3, 3, 4\} \times \{2, 2, 2, 2, 2, 2, 2\}$
$((4, 5, 7), (14), ((2, 4, 4, 4), (2, 2, 3, 3, 4), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((131, 152, 213, 235, 244, 251, 326, 332, 345, 357, 414, 427, 446, 453), (132, 151, 214, 231, 245, 253, 327, 335, 346, 352, 413, 426, 444, 457)))$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ \hline\end{array}$$

- $4 \times 5 \times 7$ move of degree 14 with slice degree $\{3, 3, 3, 5\} \times \{2, 2, 3, 3, 4\} \times \{2, 2, 2, 2, 2, 2, 2\}$
$((4, 5, 7), (14), ((3, 3, 3, 5), (2, 2, 3, 3, 4), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((111, 133, 152, 232, 245, 254, 327, 344, 356, 413, 426, 435, 447, 451), (113, 132, 151, 235, 244, 252, 326, 347, 354, 411, 427, 433, 445, 456)))$

$$\begin{array}{|ccccccc|}\hline +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline\end{array}$$

- $4 \times 5 \times 7$ move of degree 14 with slice degree $\{3, 3, 4, 4\} \times \{2, 2, 2, 4, 4\} \times \{2, 2, 2, 2, 2, 2, 2\}$
$((4, 5, 7), (14), ((3, 3, 4, 4), (2, 2, 2, 4, 4), (2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((111, 143, 152, 225, 244, 253, 312, 336, 341, 357, 424, 437, 446, 455), (112, 141, 153, 224, 243, 255, 311, 337, 346, 352, 425, 436, 444, 457)))$

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline\end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ \hline\end{array}$$

- $4 \times 5 \times 7$ move of degree 14 with slice degree $\{3,3,4,4\} \times \{2,2,3,3,4\} \times \{2,2,2,2,2,2,2\}$
$((4,5,7),(14),((3,3,4,4),(2,2,3,3,4),(2,2,2,2,2,2,2)),(fcs),\emptyset,((111,132,153,224,245,256,312,337,346,351,425,433,447,454),(112,133,151,225,246,254,311,332,347,356,424,437,445,453)))$

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ \hline \end{array}$$

- $4 \times 6 \times 6$ move of degree 14 with slice degree $\{3,3,4,4\} \times \{2,2,2,2,3,3\} \times \{2,2,2,2,2,4\}$
$((4,6,6),(14),((3,3,4,4),(2,2,2,2,3,3),(2,2,2,2,2,4)),(fcs),\emptyset,((111,126,152,233,246,264,322,336,355,363,416,444,451,465),(116,122,151,236,244,263,326,333,352,365,411,446,455,464)))$

$$\begin{array}{|cccccc|}\hline +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ \hline \end{array} \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ \hline \end{array} \begin{array}{|cccccc|}\hline -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ \hline \end{array}$$

- $5 \times 5 \times 5$ move of degree 14 with slice degree $\{2,2,2,4,4\} \times \{2,2,3,3,4\} \times \{2,3,3,3,3\}$
$((5,5,5),(14),((2,2,2,4,4),(2,2,3,3,4),(2,3,3,3,3)),(fcs),\emptyset,((131,145,242,255,313,354,424,435,452,453,514,522,533,541),(135,141,245,252,314,353,422,433,454,455,513,524,531,542)))$

$$\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & +1 & -1 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & -1 & +1 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

- $5 \times 5 \times 5$ move(1) of degree 14 with slice degree $\{2,2,3,3,4\} \times \{2,3,3,3,3\} \times \{2,3,3,3,3\}$
$((5,5,5),(14),((2,2,3,3,4),(2,3,3,3,3),(2,3,3,3,3)),(fcs),(555),((112,123,231,244,322,334,355,424,445,453,513,535,541,552),(113,122,234,241,324,335,352,423,444,455,512,531,545,553)))$

$$\begin{array}{|ccccc|}\hline 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ \hline \end{array}$$

$$\begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccc|}\hline 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & (0) \\ \hline \end{array}$$

- $5 \times 5 \times 5$ move(2) of degree 14 with slice degree $\{2,2,3,3,4\} \times \{2,3,3,3,3\} \times \{2,3,3,3,3\}$
$((5,5,5),(14),((2,2,3,3,4),(2,3,3,3,3),(2,3,3,3,3)),(fcs),\emptyset,((112,123,231,244,322,345,354,425,434,$

453, 513, 535, 541, 552), (113, 122, 234, 241, 325, 344, 352, 423, 435, 454, 512, 531, 545, 553)))

$$\boxed{\begin{array}{ccccc} 0 & +1 & -1 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & -1 & 0 & +1 & 0 \end{array}}$$

$$\boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 \end{array}}$$

- $5 \times 5 \times 5$ move(3) of degree 14 with slice degree $\{2, 2, 3, 3, 4\} \times \{2, 3, 3, 3, 3\} \times \{2, 3, 3, 3, 3\}$
  ((5, 5, 5), (14), ((2, 2, 3, 3, 4), (2, 3, 3, 3, 3), (2, 3, 3, 3, 3)), (fcs), $\emptyset$, ((121, 135, 213, 252, 334, 345, 353, 425, 442, 454, 512, 524, 531, 543), (125, 131, 212, 253, 335, 343, 354, 424, 445, 452, 513, 521, 534, 542)))

$$\boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & +1 & -1 & 0 \end{array}}$$

$$\boxed{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & +1 & 0 \end{array}} \quad \boxed{\begin{array}{ccccc} 0 & +1 & -1 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}}$$

- $5 \times 5 \times 6$ move of degree 14 with slice degree $\{2, 2, 3, 3, 4\} \times \{2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 4\}$
  ((5, 5, 6), (14), ((2, 2, 3, 3, 4), (2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 4)), (fcs), $\emptyset$, ((121, 136, 232, 246, 316, 344, 353, 413, 426, 455, 525, 531, 542, 554), (126, 131, 236, 242, 313, 346, 354, 416, 425, 453, 521, 532, 544, 555)))

$$\boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 & 0 & 0 \end{array}}$$

$$\boxed{\begin{array}{cccccc} 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \end{array}}$$

- $5 \times 5 \times 6$ move of degree 14 with slice degree $\{2, 3, 3, 3, 3\} \times \{2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 3, 3\}$
  ((5, 5, 6), (14), ((2, 3, 3, 3, 3), (2, 3, 3, 3, 3), (2, 2, 2, 2, 3, 3)), (fcs), $\emptyset$, ((121, 132, 213, 245, 254, 314, 343, 356, 425, 431, 446, 522, 536, 555), (122, 131, 214, 243, 255, 313, 346, 354, 421, 436, 445, 525, 532, 556)))

$$\boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & +1 \end{array}}$$

$$\boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{array}}$$

A.5 Indispensable moves of degree 15

- $3 \times 5 \times 7$ move of degree 15 with slice degree $\{4,5,6\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  (not fundamental, circuit)
  $((3,5,7),(15),((4,5,6),(2,3,3,3,4),(2,2,2,2,2,2,3)),(Fcs),(257,337),((121,137,142,153,217,222,236,$
  $244,255,314,325,331,343,356,357),(122,131,143,157,214,225,237,242,256,317,321,336,344,353,355)))$

0	0	0	0	0	0	0
+1	−1	0	0	0	0	0
−1	0	0	0	0	0	+1
0	+1	−1	0	0	0	0
0	0	+1	0	0	0	−1

0	0	0	−1	0	0	+1
0	+1	0	0	−1	0	0
0	0	0	0	0	+1	−1
0	−1	0	+1	0	0	0
0	0	0	0	+1	−1	(0)

0	0	0	+1	0	0	−1
−1	0	0	0	+1	0	0
+1	0	0	0	0	−1	(0)
0	0	+1	−1	0	0	0
0	0	−1	0	−1	+1	+1

- $4 \times 4 \times 7$ move of degree 15 with slice degree $\{2,4,4,5\} \times \{3,3,4,5\} \times \{2,2,2,2,2,2,3\}$
  (not fundamental, circuit)
  $((4,4,7),(15),((2,4,4,5),(3,3,4,5),(2,2,2,2,2,2,3)),(Fcs),(347,437),((131,147,213,222,234,241,312,$
  $326,337,345,417,424,435,443,446),(137,141,212,224,231,243,317,322,335,346,413,426,434,445,447)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1
+1	0	0	0	0	0	−1

0	+1	−1	0	0	0	0
0	−1	0	+1	0	0	0
+1	0	0	−1	0	0	0
−1	0	+1	0	0	0	0

0	−1	0	0	0	0	+1
0	+1	0	0	0	−1	0
0	0	0	0	+1	0	−1
0	0	0	0	−1	+1	(0)

0	0	+1	0	0	0	−1
0	0	0	−1	0	+1	0
0	0	0	+1	−1	0	(0)
0	0	−1	0	+1	−1	+1

- $4 \times 4 \times 7$ move(1) of degree 15 with slice degree $\{3,3,4,5\} \times \{3,3,4,5\} \times \{2,2,2,2,2,2,3\}$
  (not fundamental, circuit)
  $((4,4,7),(15),((3,3,4,5),(3,3,4,5),(2,2,2,2,2,2,3)),(Fcs),(147,437),((111,137,142,223,234,247,315,$
  $324,336,341,417,425,432,443,446),(117,132,141,224,237,243,311,325,334,346,415,423,436,442,447)))$

+1	0	0	0	0	0	−1
0	0	0	0	0	0	0
0	−1	0	0	0	0	+1
−1	+1	0	0	0	0	(0)

0	0	0	0	0	0	0
0	0	+1	−1	0	0	0
0	0	0	+1	0	0	−1
0	0	−1	0	0	0	+1

−1	0	0	0	+1	0	0
0	0	0	+1	−1	0	0
0	0	0	−1	0	+1	0
+1	0	0	0	0	−1	0

0	0	0	0	−1	0	+1
0	0	−1	0	+1	0	0
0	+1	0	0	0	−1	(0)
0	−1	+1	0	0	+1	−1

- $4 \times 4 \times 7$ move(2) of degree 15 with slice degree $\{3,3,4,5\} \times \{3,3,4,5\} \times \{2,2,2,2,2,2,3\}$
  (not fundamental, circuit)
  $((4,4,7),(15),((3,3,4,5),(3,3,4,5),(2,2,2,2,2,2,3)),(Fcs),(247,437),((111,137,142,227,234,243,315,$
  $323,331,346,412,425,436,444,447),(112,131,147,223,237,244,311,325,336,343,415,427,434,442,446)))$

+1	−1	0	0	0	0	0
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1
0	+1	0	0	0	0	−1

0	0	0	0	0	0	0
0	0	−1	0	0	0	+1
0	0	0	+1	0	0	−1
0	0	+1	−1	0	0	(0)

−1	0	0	0	+1	0	0
0	0	+1	0	−1	0	0
+1	0	0	0	0	−1	0
0	0	−1	0	0	+1	0

0	+1	0	0	−1	0	0
0	0	0	0	+1	0	−1
0	0	0	−1	0	+1	(0)
0	−1	0	+1	0	−1	+1

- $4 \times 5 \times 6$ move of degree 15 with slice degree $\{3,3,4,5\} \times \{2,2,3,3,5\} \times \{2,2,2,2,3,4\}$
  $((4,5,6),(15),((3,3,4,5),(2,2,3,3,5),(2,2,2,2,3,4)),(fcS),\emptyset,((111,125,156,234,243,256,315,333,342,$

351, 426, 435, 446, 452, 454), (115, 126, 151, 233, 246, 254, 311, 335, 343, 352, 425, 434, 442, 456, 456)))

+1	0	0	0	−1	0
0	0	0	0	+1	−1
0	0	0	0	0	0
0	0	0	0	0	0
−1	0	0	0	0	+1

0	0	0	0	0	0
0	0	0	0	0	0
0	0	−1	+1	0	0
0	0	+1	0	0	−1
0	0	0	−1	0	+1

−1	0	0	0	+1	0
0	0	0	0	0	0
0	0	+1	0	−1	0
0	+1	−1	0	0	0
+1	−1	0	0	0	0

0	0	0	0	0	0
0	0	0	0	−1	+1
0	0	0	−1	+1	0
0	−1	0	0	0	+1
0	+1	0	+1	0	−2

- $4 \times 5 \times 7$ move(1) of degree 15 with slice degree $\{2,4,4,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  $((4,5,7),(15),((2,4,4,5),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),(457),((121,157,222,234,243,251,316,337,$ $344,355,415,427,432,446,453),(127,151,221,232,244,253,315,334,346,357,416,422,437,443,455)))$

0	0	0	0	0	0	0
+1	0	0	0	0	0	−1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1

0	0	0	0	0	0	0
−1	+1	0	0	0	0	0
0	−1	0	+1	0	0	0
0	0	+1	−1	0	0	0
+1	0	−1	0	0	0	0

0	0	0	0	−1	+1	0
0	0	0	0	0	0	0
0	0	0	−1	0	0	+1
0	0	0	+1	0	−1	0
0	0	0	0	+1	0	−1

0	0	0	0	+1	−1	0
0	−1	0	0	0	0	+1
0	+1	0	0	0	0	−1
0	0	−1	0	0	+1	0
0	0	+1	0	−1	0	(0)

- $4 \times 5 \times 7$ move(2) of degree 15 with slice degree $\{2,4,4,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  $((4,5,7),(15),((2,4,4,5),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),(457),((121,157,222,234,243,251,317,335,$ $344,356,415,427,432,446,453),(127,151,221,232,244,253,315,334,346,357,417,422,435,443,456)))$

0	0	0	0	0	0	0
+1	0	0	0	0	0	−1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1

0	0	0	0	0	0	0
−1	+1	0	0	0	0	0
0	−1	0	+1	0	0	0
0	0	+1	−1	0	0	0
+1	0	−1	0	0	0	0

0	0	0	0	−1	0	+1
0	0	0	0	0	0	0
0	0	0	−1	+1	0	0
0	0	0	+1	0	−1	0
0	0	0	0	0	+1	−1

0	0	0	0	+1	0	−1
0	−1	0	0	0	0	+1
0	+1	0	0	−1	0	0
0	0	−1	0	0	+1	0
0	0	+1	0	0	−1	(0)

- $4 \times 5 \times 7$ move of degree 15 with slice degree $\{3,3,4,5\} \times \{2,2,3,3,5\} \times \{2,2,2,2,2,2,3\}$
  $((4,5,7),(15),((3,3,4,5),(2,2,3,3,5),(2,2,2,2,2,2,3)),(fcs),\emptyset,((111,127,152,233,245,254,317,336,343,$ $351,422,434,447,455,456),(117,122,151,234,243,255,311,333,347,356,427,436,445,452,454)))$

+1	0	0	0	0	0	−1
0	−1	0	0	0	0	+1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	+1	0	0	0	0	0

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	+1	−1	0	0	0
0	0	−1	0	+1	0	0
0	0	0	+1	−1	0	0

−1	0	0	0	0	0	+1
0	0	0	0	0	0	0
0	0	−1	0	0	+1	0
0	0	+1	0	0	0	−1
+1	0	0	0	0	−1	0

0	0	0	0	0	0	0
0	+1	0	0	0	0	−1
0	0	0	+1	0	−1	0
0	0	0	0	−1	0	+1
0	−1	0	−1	+1	+1	0

- $4 \times 5 \times 7$ move(1) of degree 15 with slice degree $\{3,3,4,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  $((4,5,7),(15),((3,3,4,5),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),(457),((111,122,157,233,247,254,326,335,$

343, 352, 417, 421, 434, 456, 455), (117, 121, 152, 234, 243, 257, 322, 333, 346, 355, 411, 426, 435, 447, 454)))

$$\begin{array}{|ccccccc|}\hline +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & -1 & +1 & 0 & (0) \\ \hline \end{array}$$

- $4 \times 5 \times 7$ move(2) of degree 15 with slice degree $\{3,3,4,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  $((4,5,7),(15),((3,3,4,5),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),(457),((111,122,157,233,245,254,326,337,$
  $343,352,417,421,434,446,455),(117,121,152,234,243,255,322,333,346,357,411,426,437,445,454)))$

$$\begin{array}{|ccccccc|}\hline +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & (0) \\ \hline \end{array}$$

- $4 \times 5 \times 7$ move of degree 15 with slice degree $\{3,3,4,5\} \times \{3,3,3,3,3\} \times \{2,2,2,2,2,2,3\}$
  $((4,5,7),(15),((3,3,4,5),(3,3,3,3,3),(2,2,2,2,2,2,3)),(fcs),(437),((111,127,132,237,243,254,315,321,$
  $346,353,412,426,434,447,455),(112,121,137,234,247,253,311,326,343,355,415,427,432,446,454)))$

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & -1 & 0 & +1 & 0 & 0 & (0) \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ \hline \end{array}$$

- $4 \times 5 \times 7$ move of degree 15 with slice degree $\{3,4,4,4\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  $((4,5,7),(15),((3,4,4,4),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),(457),((122,131,157,221,234,243,255,317,$
  $323,346,352,416,437,445,454),(121,137,152,223,231,245,254,316,322,343,357,417,434,446,455)))$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & (0) \\ \hline \end{array}$$

- $4 \times 6 \times 6$ move of degree 15 with slice degree $\{2,4,4,5\} \times \{2,2,2,3,3,3\} \times \{2,2,2,2,3,4\}$
$((4,6,6),(15),((2,4,4,5),(2,2,2,3,3,3),(2,2,2,2,3,4)),(fcs),\emptyset,((141,156,215,243,251,262,312,325,336,$ $364,426,434,446,455,463),(146,151,212,241,255,263,315,326,334,362,425,436,443,456,464)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ +1&0&0&0&0&-1\\ -1&0&0&0&0&+1\\ 0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&-1&0&0&+1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&+1&0&0&0\\ +1&0&0&0&-1&0\\ 0&+1&-1&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&+1&0&0&-1&0\\ 0&0&0&0&+1&-1\\ 0&0&0&-1&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&0&+1&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&-1&+1\\ 0&0&0&+1&0&-1\\ 0&0&-1&0&0&+1\\ 0&0&0&0&+1&-1\\ 0&0&+1&-1&0&0\\ \hline\end{array}$$

- $4 \times 6 \times 6$ move of degree 15 with slice degree $\{3,3,3,6\} \times \{2,2,2,2,3,4\} \times \{2,2,2,3,3,3\}$
(not fundamental, circuit)
$((4,6,6),(15),((3,3,3,6),(2,2,2,2,3,4),(2,2,2,3,3,3)),(Fcs),(456),((111,125,164,236,253,262,324,345,$ $356,415,432,446,454,461,463),(115,124,161,232,256,263,325,346,354,411,436,445,453,462,464)))$

$$\begin{array}{|cccccc|}\hline +1&0&0&0&-1&0\\ 0&0&0&-1&+1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&0&+1&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&+1&0&0&-1\\ 0&+1&-1&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&+1&-1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&+1&-1\\ 0&0&0&-1&0&+1\\ 0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline -1&0&0&0&+1&0\\ 0&0&0&0&0&0\\ 0&+1&0&0&0&-1\\ 0&0&0&0&-1&+1\\ 0&0&-1&+1&0&(0)\\ +1&-1&+1&-1&0&0\\ \hline\end{array}$$

- $4 \times 6 \times 6$ move of degree 15 with slice degree $\{3,3,4,5\} \times \{2,2,2,2,3,4\} \times \{2,2,2,3,3,3\}$
$((4,6,6),(15),((3,3,4,5),(2,2,2,2,3,4),(2,2,2,3,3,3)),(fcs),\emptyset,((142,151,164,216,225,263,315,336,344,$ $352,423,434,455,461,466),(144,152,161,215,223,266,316,334,342,355,425,436,451,463,464)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&+1&0&-1&0&0\\ +1&-1&0&0&0&0\\ -1&0&0&+1&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&-1&+1\\ 0&0&-1&0&+1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&+1&0&0&-1\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&+1&-1\\ 0&0&0&0&0&0\\ 0&0&0&-1&0&+1\\ 0&-1&0&+1&0&0\\ 0&+1&0&0&-1&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&+1&0&-1&0\\ 0&0&0&+1&0&-1\\ 0&0&0&0&0&0\\ -1&0&0&0&+1&0\\ +1&0&-1&-1&0&+1\\ \hline\end{array}$$

- $4 \times 6 \times 7$ move of degree 15 with slice degree $\{2,4,4,5\} \times \{2,2,2,3,3,3\} \times \{2,2,2,2,2,2,3\}$
$((4,6,7),(15),((2,4,4,5),(2,2,2,3,3,3),(2,2,2,2,2,2,3)),(fcs),\emptyset,((151,162,213,224,237,245,335,346,357,$ $361,417,423,444,452,466),(152,161,217,223,235,244,337,345,351,366,413,424,446,457,462)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0\\ -1&+1&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&+1&0&0&0&-1\\ 0&0&-1&+1&0&0&0\\ 0&0&0&0&-1&0&+1\\ 0&0&0&-1&+1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&+1&0&-1\\ 0&0&0&0&-1&+1&0\\ -1&0&0&0&0&0&+1\\ +1&0&0&0&0&-1&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&-1&0&0&0&+1\\ 0&0&+1&-1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&+1&0&-1&0\\ 0&+1&0&0&0&0&-1\\ 0&-1&0&0&0&+1&0\\ \hline\end{array}$$

- $4 \times 6 \times 7$ move of degree 15 with slice degree $\{3,3,4,5\} \times \{2,2,2,3,3,3\} \times \{2,2,2,2,2,2,3\}$
$((4,6,7),(15),((3,3,4,5),(2,2,2,3,3,3),(2,2,2,2,2,2,3)),(fcs),\emptyset,((111,127,142,233,255,264,317,341,356,$

365, 422, 434, 446, 453, 467), (117, 122, 141, 234, 253, 265, 311, 346, 355, 367, 427, 433, 442, 456, 464)))

$$\begin{array}{|ccccccc|}\hline +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ \hline \end{array}$$

- $4 \times 6 \times 7$ move of degree 15 with slice degree $\{3,4,4,4\} \times \{2,2,2,3,3,3\} \times \{2,2,2,2,2,2,3\}$
  $((4,6,7),(15),((3,4,4,4),(2,2,2,3,3,3),(2,2,2,2,2,2,3)),(fcs),\emptyset,((111,147,152,217,223,251,264,336,342,$ $354,365,427,435,446,463),(117,142,151,211,227,254,263,335,346,352,364,423,436,447,465)))$

$$\begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ \hline \end{array}$$

- $5 \times 5 \times 6$ move(1) of degree 15 with slice degree $\{2,2,3,3,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,3,4\}$
  $((5,5,6),(15),((2,2,3,3,5),(2,3,3,3,4),(2,2,2,2,3,4)),(fcs),\emptyset,((121,155,225,246,316,333,352,434,445,$ $453,512,526,536,544,551),(125,151,226,245,312,336,353,433,444,455,516,521,534,546,552)))$

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccc|}\hline 0 & +1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

- $5 \times 5 \times 6$ move(2) of degree 15 with slice degree $\{2,2,3,3,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,3,4\}$
  $((5,5,6),(15),((2,2,3,3,5),(2,3,3,3,4),(2,2,2,2,3,4)),(fcs),\emptyset,((121,156,232,255,323,345,351,416,435,$

444, 514, 526, 536, 543, 552), (126, 151, 235, 252, 321, 343, 355, 414, 436, 445, 516, 523, 532, 544, 556)))

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ +1&0&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&0&0&0&+1\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&+1&0&0&-1&0\\ 0&0&0&0&0&0\\ 0&-1&0&0&+1&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ -1&0&+1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&+1&0\\ +1&0&0&0&-1&0\\ \hline\end{array}$$

$$\begin{array}{|cccccc|}\hline 0&0&0&-1&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&+1&-1\\ 0&0&0&+1&-1&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&+1&0&-1\\ 0&0&-1&0&0&+1\\ 0&-1&0&0&0&+1\\ 0&0&+1&-1&0&0\\ 0&+1&0&0&0&-1\\ \hline\end{array}$$

- $5 \times 5 \times 6$ move of degree 15 with slice degree $\{2,3,3,3,4\} \times \{2,3,3,3,4\} \times \{2,2,2,2,3,4\}$
  $((5,5,6),(15),((2,3,3,3,4),(2,3,3,3,4),(2,2,2,2,3,4)),(fcs),\emptyset,((141,156,222,245,251,324,333,352,416,$ $425,434,515,536,546,553),(146,151,225,241,252,322,334,353,415,424,436,516,533,545,556)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ +1&0&0&0&0&-1\\ -1&0&0&0&0&+1\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&+1&0&0&-1&0\\ 0&0&0&0&0&0\\ -1&0&0&0&+1&0\\ +1&-1&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&-1&0&+1&0&0\\ 0&0&+1&-1&0&0\\ 0&0&0&0&0&0\\ 0&+1&-1&0&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&-1&+1\\ 0&0&0&-1&+1&0\\ 0&0&0&+1&0&-1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|cccccc|}\hline 0&0&0&0&+1&-1\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&+1\\ 0&0&0&0&-1&+1\\ 0&0&+1&0&0&-1\\ \hline\end{array}$$

- $5 \times 5 \times 7$ move(1) of degree 15 with slice degree $\{2,2,3,3,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  $((5,5,7),(15),((2,2,3,3,5),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),\emptyset,((127,131,222,257,337,343,354,416,444,$ $455,515,521,533,546,552),(121,137,227,252,333,344,357,415,446,454,516,522,531,543,555)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&+1\\ +1&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&+1&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&+1\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&+1\\ 0&0&+1&-1&0&0&0\\ 0&0&0&+1&0&0&-1\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&-1&+1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&+1&0&-1&0\\ 0&0&0&-1&+1&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&+1&-1&0\\ +1&-1&0&0&0&0&0\\ -1&0&+1&0&0&0&0\\ 0&0&-1&0&0&+1&0\\ 0&+1&0&0&-1&0&0\\ \hline\end{array}$$

- $5 \times 5 \times 7$ move(2) of degree 15 with slice degree $\{2,2,3,3,5\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
  $((5,5,7),(15),((2,2,3,3,5),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),\emptyset,((121,152,233,257,324,347,351,415,437,$ $446,516,522,535,544,553),(122,151,237,253,321,344,357,416,435,447,515,524,533,546,552)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&+1&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&+1\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ -1&0&0&+1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&+1\\ +1&0&0&0&0&0&-1\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&+1&-1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&+1\\ 0&0&0&0&0&+1&-1\\ 0&0&0&0&0&0&0\\ \hline\end{array}\quad\begin{array}{|ccccccc|}\hline 0&0&0&0&-1&+1&0\\ 0&+1&0&-1&0&0&0\\ 0&0&-1&0&+1&0&0\\ 0&0&0&+1&0&-1&0\\ 0&-1&+1&0&0&0&0\\ \hline\end{array}$$

- $5 \times 5 \times 7$ move(1) of degree 15 with slice degree $\{2,2,3,4,4\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
$((5,5,7),(15),((2,2,3,4,4),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),\emptyset,((141,157,232,247,316,325,354,424,433,442,451,515,523,537,556),(147,151,237,242,315,324,356,423,432,441,454,516,525,533,557)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&+1\\ +1&0&0&0&0&0&-1\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&+1\\ 0&+1&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&+1&-1&0\\ 0&0&0&+1&-1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-1&0&+1&0\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&+1&-1&0&0&0\\ 0&+1&-1&0&0&0&0\\ +1&-1&0&0&0&0&0\\ -1&0&0&+1&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&-1&+1&0\\ 0&0&-1&0&+1&0&0\\ 0&0&+1&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&+1\\ \hline\end{array}$$

- $5 \times 5 \times 7$ move(2) of degree 15 with slice degree $\{2,2,3,4,4\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
$((5,5,7),(15),((2,2,3,4,4),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),\emptyset,((121,157,232,253,314,327,345,425,433,446,451,517,536,544,552),(127,151,233,252,317,325,344,421,436,445,453,514,532,546,557)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&+1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ +1&0&0&0&0&0&-1\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&+1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&+1&-1&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&-1&0&0&+1\\ 0&0&0&0&+1&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&+1&-1&0&0\\ 0&0&0&0&0&0&0\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ +1&0&0&0&-1&0&0\\ 0&0&-1&0&0&+1&0\\ 0&0&0&0&+1&-1&0\\ -1&0&+1&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&+1&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&+1&0&0&0&-1&0\\ 0&0&0&-1&0&+1&0\\ 0&-1&0&0&0&0&+1\\ \hline\end{array}$$

- $5 \times 5 \times 7$ move(1) of degree 15 with slice degree $\{2,3,3,3,4\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
$((5,5,7),(15),((2,3,3,3,4),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),\emptyset,((121,152,223,237,251,316,345,357,417,434,446,522,533,544,555),(122,151,221,233,257,317,346,355,416,437,444,523,534,545,552)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ -1&0&+1&0&0&0&0\\ 0&0&-1&0&0&0&+1\\ 0&0&0&0&0&0&0\\ +1&0&0&0&0&0&-1\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&+1&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&+1&-1&0\\ 0&0&0&0&-1&0&+1\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&-1&+1\\ 0&0&0&0&0&0&0\\ 0&0&0&+1&0&0&-1\\ 0&0&0&-1&0&+1&0\\ 0&0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&+1&-1&0&0&0&0\\ 0&0&+1&-1&0&0&0\\ 0&0&0&+1&-1&0&0\\ 0&-1&0&0&+1&0&0\\ \hline\end{array}$$

- $5 \times 5 \times 7$ move(2) of degree 15 with slice degree $\{2,3,3,3,4\} \times \{2,3,3,3,4\} \times \{2,2,2,2,2,2,3\}$
$((5,5,7),(15),((2,3,3,3,4),(2,3,3,3,4),(2,2,2,2,2,2,3)),(fcs),\emptyset,((121,152,234,245,253,327,343,351,416,435,447,517,522,536,554),(122,151,235,243,254,321,347,353,417,436,445,516,527,534,552)))$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&+1&-1&0&0\\ 0&0&-1&0&+1&0&0\\ 0&0&+1&-1&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&+1\\ 0&0&0&0&0&0&0\\ 0&0&+1&0&0&0&-1\\ +1&0&-1&0&0&0&0\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&+1&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&+1&-1&0\\ 0&0&0&0&-1&0&+1\\ 0&0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&-1&+1\\ 0&+1&0&0&0&0&-1\\ 0&0&0&-1&0&+1&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&+1&0&0&0\\ \hline\end{array}$$

- $5 \times 5 \times 7$ move of degree 15 with slice degree $\{2,3,3,3,4\} \times \{3,3,3,3,3\} \times \{2,2,2,2,2,2,3\}$
  $((5,5,7),(15),((2,3,3,3,4),(3,3,3,3,3),(2,2,2,2,2,2,3)),(fcs),\emptyset,((111,127,217,233,242,326,337,355,435,443,454,512,521,544,556),(117,121,212,237,243,327,335,356,433,444,455,511,526,542,554)))$

$$\begin{array}{|ccccccc|}\hline +1&0&0&0&0&0&-1\\ -1&0&0&0&0&0&+1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&-1&0&0&0&0&+1\\ 0&0&0&0&0&0&0\\ 0&0&+1&0&0&0&-1\\ 0&+1&-1&0&0&0&0\\ 0&0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&+1&-1\\ 0&0&0&0&-1&0&+1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&+1&-1&0\\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-1&0&+1&0&0\\ 0&0&+1&-1&0&0&0\\ 0&0&0&+1&-1&0&0\\ \hline\end{array}\quad \begin{array}{|ccccccc|}\hline -1&+1&0&0&0&0&0\\ +1&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&+1&0&0&0\\ 0&0&0&-1&0&+1&0\\ \hline\end{array}$$

- $5 \times 6 \times 6$ move of degree 15 with slice degree $\{2,2,3,4,4\} \times \{2,2,2,2,3,4\} \times \{2,2,2,3,3,3\}$
  $((5,6,6),(15),((2,2,3,4,4),(2,2,2,2,3,4),(2,2,2,3,3,3)),(fcs),\emptyset,((151,164,214,265,326,345,362,415,433,446,454,522,536,553,561),(154,161,215,264,322,346,365,414,436,445,453,526,533,551,562)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ +1&0&0&-1&0&0\\ -1&0&0&+1&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&+1&-1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-1&+1&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&-1&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&+1&-1\\ 0&0&0&0&0&0\\ 0&+1&0&0&-1&0\\ \hline\end{array}$$

$$\begin{array}{|cccccc|}\hline 0&0&0&-1&+1&0\\ 0&0&0&0&0&0\\ 0&0&+1&0&0&-1\\ 0&0&0&0&-1&+1\\ 0&0&-1&+1&0&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&+1&0&0&0&-1\\ 0&0&-1&0&0&+1\\ 0&0&0&0&0&0\\ -1&0&+1&0&0&0\\ +1&-1&0&0&0&0\\ \hline\end{array}$$

- $5 \times 6 \times 6$ move(1) of degree 15 with slice degree $\{2,3,3,3,4\} \times \{2,2,2,2,3,4\} \times \{2,2,2,3,3,3\}$
  $((5,6,6),(15),((2,3,3,3,4),(2,2,2,2,3,4),(2,2,2,3,3,3)),(fcs),\emptyset,((151,164,212,255,261,326,344,363,435,446,454,515,523,536,562),(154,161,215,251,262,323,346,364,436,444,455,512,526,535,563)))$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ +1&0&0&-1&0&0\\ -1&0&0&+1&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&+1&0&0&-1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&0&0&+1&0\\ +1&-1&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&-1&0&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&+1&0&-1\\ 0&0&0&0&0&0\\ 0&0&+1&-1&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&+1&-1\\ 0&0&0&-1&0&+1\\ 0&0&0&+1&-1&0\\ 0&0&0&0&0&0\\ \hline\end{array}\quad \begin{array}{|cccccc|}\hline 0&-1&0&0&+1&0\\ 0&0&+1&0&0&-1\\ 0&0&0&0&-1&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&+1&-1&0&0&0\\ \hline\end{array}$$

- $5 \times 6 \times 6$ move(2) of degree 15 with slice degree $\{2,3,3,3,4\} \times \{2,2,2,2,3,4\} \times \{2,2,2,3,3,3\}$
  $((5,6,6),(15),((2,3,3,3,4),(2,2,2,2,3,4),(2,2,2,3,3,3)),(fcs),\emptyset,((114,165,221,252,264,335,346,363,443,$

$456, 462, 515, 524, 536, 551), (115, 164, 224, 251, 262, 336, 343, 365, 446, 452, 463, 514, 521, 535, 556)))$

0	0	0	+1	−1	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	−1	+1	0

0	0	0	0	0	0
+1	0	0	−1	0	0
0	0	0	0	0	0
0	0	0	0	0	0
−1	+1	0	0	0	0
0	−1	0	+1	0	0

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	+1	−1
0	0	−1	0	0	+1
0	0	0	0	0	0
0	0	+1	0	−1	0

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	+1	0	0	−1
0	−1	0	0	0	+1
0	+1	−1	0	0	0

0	0	0	−1	+1	0
−1	0	0	+1	0	0
0	0	0	0	−1	+1
0	0	0	0	0	0
+1	0	0	0	0	−1
0	0	0	0	0	0

- $5 \times 6 \times 6$ move(3) of degree 15 with slice degree $\{2, 3, 3, 3, 4\} \times \{2, 2, 2, 2, 3, 4\} \times \{2, 2, 2, 3, 3, 3\}$
  $((5, 6, 6), (15), ((2, 3, 3, 3, 4), (2, 2, 2, 2, 3, 4), (2, 2, 2, 3, 3, 3)), (fcs), \emptyset, ((114, 165, 222, 251, 264, 335, 346, 363, 415, 436, 454, 521, 543, 556, 562), (115, 164, 221, 254, 262, 336, 343, 365, 414, 435, 456, 522, 546, 551, 563)))$

0	0	0	+1	−1	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	−1	+1	0

0	0	0	0	0	0
−1	+1	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
+1	0	0	−1	0	0
0	−1	0	+1	0	0

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	+1	−1
0	0	−1	0	0	+1
0	0	0	0	0	0
0	0	+1	0	−1	0

0	0	0	−1	+1	0
0	0	0	0	0	0
0	0	0	0	−1	+1
0	0	0	0	0	0
0	0	0	+1	0	−1
0	0	0	0	0	0

0	0	0	0	0	0
+1	−1	0	0	0	0
0	0	0	0	0	0
0	0	+1	0	0	−1
−1	0	0	0	0	+1
0	+1	−1	0	0	0

A.6 Indispensable moves of degree 16

- $3 \times 5 \times 7$ move(1) of degree 16 with slice degree $\{4, 6, 6\} \times \{2, 3, 3, 3, 5\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  $((3, 5, 7), (16), ((4, 6, 6), (2, 3, 3, 3, 5), (2, 2, 2, 2, 2, 2, 4)), (fcS), \emptyset, ((121, 133, 142, 154, 217, 222, 237, 245, 253, 256, 315, 326, 331, 344, 357, 357), (122, 131, 144, 153, 215, 226, 233, 242, 257, 257, 317, 321, 337, 345, 354, 356)))$

0	0	0	0	0	0	0
+1	−1	0	0	0	0	0
−1	0	+1	0	0	0	0
0	+1	0	−1	0	0	0
0	0	−1	+1	0	0	0

0	0	0	0	−1	0	+1
0	+1	0	0	0	−1	0
0	0	−1	0	0	0	+1
0	−1	0	0	+1	0	0
0	0	+1	0	0	+1	−2

0	0	0	0	+1	0	−1
−1	0	0	0	0	+1	0
+1	0	0	0	0	0	−1
0	0	0	+1	−1	0	0
0	0	0	−1	0	−1	+2

- $3 \times 5 \times 7$ move(2) of degree 16 with slice degree $\{4, 6, 6\} \times \{2, 3, 3, 3, 5\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  (not fundamental, not circuit)
  $((3, 5, 7), (16), ((4, 6, 6), (2, 3, 3, 3, 5), (2, 2, 2, 2, 2, 2, 4)), (FCs), \emptyset, ((121, 133, 142, 154, 216, 222, 237, 247, 253, 255, 317, 325, 331, 344, 356, 357), (122, 131, 144, 153, 217, 225, 233, 242, 256, 257, 316, 321, 337, 347, 354, 355)))$

0	0	0	0	0	0	0
+1	−1	0	0	0	0	0
−1	0	+1	0	0	0	0
0	+1	0	−1	0	0	0
0	0	−1	+1	0	0	0

0	0	0	0	0	+1	−1
0	+1	0	0	−1	0	0
0	0	−1	0	0	0	+1
0	−1	0	0	0	0	+1
0	0	+1	0	+1	−1	−1

0	0	0	0	0	−1	+1
−1	0	0	0	+1	0	0
+1	0	0	0	0	0	−1
0	0	0	+1	0	0	−1
0	0	0	−1	−1	+1	+1

- $3 \times 5 \times 8$ move of degree 16 with slice degree $\{4, 6, 6\} \times \{2, 3, 3, 3, 5\} \times \{2, 2, 2, 2, 2, 2, 2, 2\}$
  $((3, 5, 8), (16), ((4, 6, 6), (2, 3, 3, 3, 5), (2, 2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((121, 133, 142, 154, 216, 222, 238, 245, 253,$

257, 315, 327, 331, 344, 356, 358), (122, 131, 144, 153, 215, 227, 233, 242, 256, 258, 316, 321, 338, 345, 354, 357)))

0	0	0	0	0	0	0	0
+1	−1	0	0	0	0	0	0
−1	0	+1	0	0	0	0	0
0	+1	0	−1	0	0	0	0
0	0	−1	+1	0	0	0	0

0	0	0	0	−1	+1	0	0
0	+1	0	0	0	0	−1	0
0	0	−1	0	0	0	0	+1
0	−1	0	0	+1	0	0	0
0	0	+1	0	0	−1	+1	−1

0	0	0	0	+1	−1	0	0
−1	0	0	0	0	0	+1	0
+1	0	0	0	0	0	0	−1
0	0	0	+1	−1	0	0	0
0	0	0	−1	0	+1	−1	+1

- $3 \times 6 \times 7$ move of degree 16 with slice degree $\{4,6,6\} \times \{2,2,3,3,3,3\} \times \{2,2,2,2,2,4\}$
  $((3,6,7),(16),((4,6,6),(2,2,3,3,3,3),(2,2,2,2,2,4)),(fcs),\emptyset,((131,143,152,164,217,226,232,245,257,$ $263,315,327,337,341,354,366),(132,141,154,163,215,227,237,243,252,266,317,326,331,345,357,364)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
+1	−1	0	0	0	0	0
−1	0	+1	0	0	0	0
0	+1	0	−1	0	0	0
0	0	−1	+1	0	0	0

0	0	0	0	−1	0	+1
0	0	0	0	0	+1	−1
0	+1	0	0	0	0	−1
0	0	−1	0	+1	0	0
0	−1	0	0	0	0	+1
0	0	+1	0	0	−1	0

0	0	0	0	+1	0	−1
0	0	0	0	0	−1	+1
−1	0	0	0	0	0	+1
+1	0	0	0	−1	0	0
0	0	0	+1	0	0	−1
0	0	0	−1	0	+1	0

- $4 \times 4 \times 7$ move of degree 16 with slice degree $\{2,4,4,6\} \times \{3,3,5,5\} \times \{2,2,2,2,2,2,4\}$
  $((4,4,7),(16),((2,4,4,6),(3,3,5,5),(2,2,2,2,2,2,4)),(fcS),\emptyset,((131,147,212,224,233,241,315,322,336,$ $347,413,426,437,437,444,445),(137,141,213,222,231,244,312,326,337,345,415,424,433,436,447,447)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
+1	0	0	0	0	0	−1
−1	0	0	0	0	0	+1

0	+1	−1	0	0	0	0
0	−1	0	+1	0	0	0
−1	0	+1	0	0	0	0
+1	0	0	−1	0	0	0

0	−1	0	0	+1	0	0
0	+1	0	0	0	−1	0
0	0	0	0	0	+1	−1
0	0	0	0	−1	0	+1

0	0	+1	0	−1	0	0
0	0	0	−1	0	+1	0
0	0	−1	0	0	−1	+2
0	0	0	+1	+1	0	−2

- $4 \times 4 \times 7$ move of degree 16 with slice degree $\{3,3,4,6\} \times \{3,3,5,5\} \times \{2,2,2,2,2,2,4\}$
  $((4,4,7),(16),((3,3,4,6),(3,3,5,5),(2,2,2,2,2,2,4)),(fcS),\emptyset,((111,137,142,223,237,244,315,324,331,$ $346,412,425,433,436,447,447),(112,131,147,224,233,247,311,325,336,344,415,423,437,437,442,446)))$

+1	−1	0	0	0	0	0
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1
0	+1	0	0	0	0	−1

0	0	0	0	0	0	0
0	0	+1	−1	0	0	0
0	0	−1	0	0	0	+1
0	0	0	+1	0	0	−1

−1	0	0	0	+1	0	0
0	0	0	+1	−1	0	0
+1	0	0	0	0	−1	0
0	0	0	−1	0	+1	0

0	+1	0	0	−1	0	0
0	0	−1	0	+1	0	0
0	0	+1	0	0	+1	−2
0	−1	0	0	0	−1	+2

- $4 \times 4 \times 8$ move of degree 16 with slice degree $\{2,4,4,6\} \times \{3,3,5,5\} \times \{2,2,2,2,2,2,2,2\}$
  $((4,4,8),(16),((2,4,4,6),(3,3,5,5),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((131,142,213,225,234,241,316,323,337,$ $348,414,427,432,438,445,446),(132,141,214,223,231,245,313,327,338,346,416,425,434,437,442,448)))$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
+1	−1	0	0	0	0	0	0
−1	+1	0	0	0	0	0	0

0	0	+1	−1	0	0	0	0
0	0	−1	0	+1	0	0	0
−1	0	0	+1	0	0	0	0
+1	0	0	0	−1	0	0	0

0	0	−1	0	0	+1	0	0
0	0	+1	0	0	0	−1	0
0	0	0	0	0	0	+1	−1
0	0	0	0	0	−1	0	+1

0	0	0	+1	0	−1	0	0
0	0	0	0	−1	0	+1	0
0	+1	0	−1	0	0	−1	+1
0	−1	0	0	+1	+1	0	−1

- $4 \times 4 \times 8$ move of degree 16 with slice degree $\{3,3,4,6\} \times \{3,3,5,5\} \times \{2,2,2,2,2,2,2,2\}$
$((4,4,8),(16),((3,3,4,6),(3,3,5,5),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((111,133,142,224,236,245,317,325,331,348,412,427,434,438,443,446),(112,131,143,225,234,246,311,327,338,345,417,424,433,436,442,448)))$

$$\begin{array}{|cccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|cccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ \hline \end{array} \qquad \begin{array}{|cccccccc|}\hline 0 & +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & -1 & +1 & 0 & -1 & 0 & +1 \\ 0 & -1 & +1 & 0 & 0 & +1 & 0 & -1 \\ \hline \end{array}$$

- $4 \times 5 \times 7$ move of degree 16 with slice degree $\{2,4,4,6\} \times \{2,3,3,3,5\} \times \{2,2,2,2,2,2,4\}$
$((4,5,7),(16),((2,4,4,6),(2,3,3,3,5),(2,2,2,2,2,2,4)),(fcS),\emptyset,((121,157,222,233,244,251,316,335,343,357,417,427,432,446,454,455),(127,151,221,232,243,254,317,333,346,355,416,422,435,444,457,457)))$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ \hline \end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & +1 & 0 & -2 \\ \hline \end{array}$$

- $4 \times 5 \times 7$ move of degree 16 with slice degree $\{3,3,4,6\} \times \{2,3,3,3,5\} \times \{2,2,2,2,2,2,4\}$
$((4,5,7),(16),((3,3,4,6),(2,3,3,3,5),(2,2,2,2,2,2,4)),(fcS),\emptyset,((111,122,157,233,244,257,325,336,343,352,417,421,437,445,454,456),(117,121,152,237,243,254,322,333,345,356,411,425,436,444,457,457)))$

$$\begin{array}{|ccccccc|}\hline +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 & 0 \\ \hline \end{array} \qquad \begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & +1 & -2 \\ \hline \end{array}$$

- $4 \times 5 \times 7$ move of degree 16 with slice degree $\{3,3,5,5\} \times \{2,3,3,4,4\} \times \{2,2,2,2,2,2,4\}$
$((4,5,7),(16),((3,3,5,5),(2,3,3,4,4),(2,2,2,2,2,2,4)),(fcs),\emptyset,((121,143,152,234,245,253,316,327,337,341,354,417,422,435,446,457),(122,141,153,235,243,254,317,321,334,346,357,416,427,437,445,452)))$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ \hline \end{array} \qquad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ \hline \end{array}$$

- $4 \times 5 \times 8$ move of degree 16 with slice degree $\{2,3,5,6\} \times \{2,3,3,4,4\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((2,3,5,6),(2,3,3,4,4),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((121,142,234,245,253,317,322,338,344,356,418,426,433,441,455,457),(122,141,233,244,255,318,326,334,342,357,417,421,438,445,453,456)))$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-1&+1&0&0&0&0\\ 0&0&0&-1&+1&0&0&0\\ 0&0&+1&0&-1&0&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&+1&-1\\ 0&+1&0&0&0&-1&0&0\\ 0&0&0&-1&0&0&0&+1\\ 0&-1&0&+1&0&0&0&0\\ 0&0&0&0&0&+1&-1&0\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&-1&+1\\ -1&0&0&0&0&+1&0&0\\ 0&0&+1&0&0&0&0&-1\\ +1&0&0&0&-1&0&0&0\\ 0&0&-1&0&+1&-1&+1&0\\ \hline\end{array}$$

- $4 \times 5 \times 8$ move(1) of degree 16 with slice degree $\{2,4,4,6\} \times \{2,3,3,3,5\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((2,4,4,6),(2,3,3,3,5),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((121,152,223,234,245,251,316,338,344,357,417,422,433,446,455,458),(122,151,221,233,244,255,317,334,346,358,416,423,438,445,452,457)))$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0&0\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ -1&0&+1&0&0&0&0&0\\ 0&0&-1&+1&0&0&0&0\\ 0&0&0&-1&+1&0&0&0\\ +1&0&0&0&-1&0&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&+1&-1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&+1\\ 0&0&0&+1&0&-1&0&0\\ 0&0&0&0&0&0&+1&-1\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&-1&+1&0\\ 0&+1&-1&0&0&0&0&0\\ 0&0&+1&0&0&0&0&-1\\ 0&0&0&0&-1&+1&0&0\\ 0&-1&0&0&+1&0&-1&+1\\ \hline\end{array}$$

- $4 \times 5 \times 8$ move(2) of degree 16 with slice degree $\{2,4,4,6\} \times \{2,3,3,3,5\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((2,4,4,6),(2,3,3,3,5),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((121,152,223,234,245,251,316,338,344,357,418,422,433,447,455,456),(122,151,221,233,244,255,318,334,347,356,416,423,438,445,452,457)))$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0&0\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ +1&0&-1&0&0&0&0&0\\ 0&0&+1&-1&0&0&0&0\\ 0&0&0&+1&-1&0&0&0\\ -1&0&0&0&+1&0&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&-1&0&+1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&+1&0&0&0&-1\\ 0&0&0&-1&0&0&+1&0\\ 0&0&0&0&0&+1&-1&0\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&+1&0&-1\\ 0&-1&+1&0&0&0&0&0\\ 0&0&-1&0&0&0&0&+1\\ 0&0&0&0&+1&0&-1&0\\ 0&+1&0&0&-1&-1&+1&0\\ \hline\end{array}$$

- $4 \times 5 \times 8$ move of degree 16 with slice degree $\{3,3,4,6\} \times \{2,3,3,3,5\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((3,3,4,6),(2,3,3,3,5),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((111,123,152,234,246,255,327,338,344,353,412,421,435,447,456,458),(112,121,153,235,244,256,323,334,347,358,411,427,438,446,452,455)))$

$$\begin{array}{|cccccccc|}\hline +1&-1&0&0&0&0&0&0\\ -1&0&+1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&+1&-1&0&0&0&0&0\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&+1&-1&0&0&0\\ 0&0&0&-1&0&+1&0&0\\ 0&0&0&0&+1&-1&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&+1&0\\ 0&0&0&-1&0&0&0&+1\\ 0&0&0&+1&0&0&-1&0\\ 0&0&+1&0&0&0&0&-1\\ \hline\end{array} \qquad \begin{array}{|cccccccc|}\hline -1&+1&0&0&0&0&0&0\\ +1&0&0&0&0&0&-1&0\\ 0&0&0&0&+1&0&0&-1\\ 0&0&0&0&0&-1&+1&0\\ 0&-1&0&0&-1&+1&0&+1\\ \hline\end{array}$$

- $4 \times 5 \times 8$ move of degree 16 with slice degree $\{3,3,4,6\} \times \{3,3,3,3,4\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((3,3,4,6),(3,3,3,3,4),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((111,123,152,234,246,255,313,327,336,$ $348,418,422,437,445,451,454),(113,122,151,236,245,254,318,323,337,346,411,427,434,448,452,455)))$

$$\begin{array}{|cccccccc|}\hline +1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|cccccccc|}\hline 0 & 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ +1 & -1 & 0 & +1 & -1 & 0 & 0 & 0 \\ \hline \end{array}$$

- $4 \times 5 \times 8$ move(1) of degree 16 with slice degree $\{3,3,5,5\} \times \{2,3,3,4,4\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((3,3,5,5),(2,3,3,4,4),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((111,133,152,224,246,255,312,327,331,$ $348,354,426,438,445,447,453),(112,131,153,226,245,254,311,324,338,347,352,427,433,446,448,455)))$

$$\begin{array}{|cccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|cccccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & -1 & +1 & -1 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 & 0 \\ \hline \end{array}$$

- $4 \times 5 \times 8$ move(2) of degree 16 with slice degree $\{3,3,5,5\} \times \{2,3,3,4,4\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((3,3,5,5),(2,3,3,4,4),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((121,143,152,234,245,253,317,326,338,$ $341,354,418,422,435,447,456),(122,141,153,235,243,254,318,321,334,347,356,417,426,438,445,452)))$

$$\begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ -1 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ \hline \end{array}$$

- $4 \times 5 \times 8$ move of degree 16 with slice degree $\{3,4,4,5\} \times \{2,3,3,3,5\} \times \{2,2,2,2,2,2,2,2\}$
$((4,5,8),(16),((3,4,4,5),(2,3,3,3,5),(2,2,2,2,2,2,2,2)),(fcs),\emptyset,((131,143,152,224,236,241,255,318,327,$ $344,353,417,425,432,456,458),(132,141,153,225,231,244,256,317,324,343,358,418,427,436,452,455)))$

$$\begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ \hline \end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & +1 & 0 & +1 \\ \hline \end{array}$$

- $4 \times 6 \times 6$ move of degree 16 with slice degree $\{2,4,4,6\} \times \{2,2,3,3,3,3\} \times \{2,2,2,2,4,4\}$
$((4,6,6),(16),((2,4,4,6),(2,2,3,3,3,3),(2,2,2,2,4,4)),(fcs),\emptyset,((135,146,231,245,255,262,313,326,354,$
$365,416,424,436,442,451,463),(136,145,235,242,251,265,316,324,355,363,413,426,431,446,454,462)))$

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	+1	−1
0	0	0	0	−1	+1
0	0	0	0	0	0
0	0	0	0	0	0

0	0	0	0	0	0
0	0	0	0	0	0
+1	0	0	0	−1	0
0	−1	0	0	+1	0
−1	0	0	0	+1	0
0	+1	0	0	−1	0

0	0	+1	0	0	−1
0	0	0	−1	0	+1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	+1	−1	0
0	0	−1	0	+1	0

0	0	−1	0	0	+1
0	0	0	+1	0	−1
−1	0	0	0	0	+1
0	+1	0	0	0	−1
+1	0	0	−1	0	0
0	−1	+1	0	0	0

- $4 \times 6 \times 6$ move(1) of degree 16 with slice degree $\{3,3,4,6\} \times \{2,2,3,3,3,3\} \times \{2,2,2,2,4,4\}$
$((4,6,6),(16),((3,3,4,6),(2,2,3,3,3,3),(2,2,2,2,4,4)),(fcs),\emptyset,((115,131,146,226,255,262,334,341,352,$
$363,416,425,435,443,454,466),(116,135,141,225,252,266,331,343,354,362,415,426,434,446,455,463)))$

0	0	0	0	+1	−1
0	0	0	0	0	0
+1	0	0	0	−1	0
−1	0	0	0	0	+1
0	0	0	0	0	0
0	0	0	0	0	0

0	0	0	0	0	0
0	0	0	0	−1	+1
0	0	0	0	0	0
0	0	0	0	0	0
0	−1	0	0	+1	0
0	+1	0	0	0	−1

0	0	0	0	0	0
0	0	0	0	0	0
−1	0	0	+1	0	0
+1	0	−1	0	0	0
0	+1	0	−1	0	0
0	−1	+1	0	0	0

0	0	0	0	−1	+1
0	0	0	0	+1	−1
0	0	0	−1	+1	0
0	0	+1	0	0	−1
0	0	0	+1	−1	0
0	0	−1	0	0	+1

- $4 \times 6 \times 6$ move(2) of degree 16 with slice degree $\{3,3,4,6\} \times \{2,2,3,3,3,3\} \times \{2,2,2,2,4,4\}$
$((4,6,6),(16),((3,3,4,6),(2,2,3,3,3,3),(2,2,2,2,4,4)),(fcs),\emptyset,((111,135,146,226,252,265,334,345,355,$
$363,416,422,431,443,454,466),(116,131,145,222,255,266,335,343,354,365,411,426,434,446,452,463)))$

+1	0	0	0	0	−1
0	0	0	0	0	0
−1	0	0	0	+1	0
0	0	0	0	−1	+1
0	0	0	0	0	0
0	0	0	0	0	0

0	0	0	0	0	0
0	−1	0	0	0	+1
0	0	0	0	0	0
0	0	0	0	0	0
0	+1	0	0	−1	0
0	0	0	0	+1	−1

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	+1	−1	0
0	0	−1	0	+1	0
0	0	0	−1	+1	0
0	0	+1	0	−1	0

−1	0	0	0	0	+1
0	+1	0	0	0	−1
+1	0	0	−1	0	0
0	0	+1	0	0	−1
0	−1	0	+1	0	0
0	0	−1	0	0	+1

- $4 \times 6 \times 7$ move(1) of degree 16 with slice degree $\{2,4,4,6\} \times \{2,2,3,3,3,3\} \times \{2,2,2,2,2,2,4\}$
$((4,6,7),(16),((2,4,4,6),(2,2,3,3,3,3),(2,2,2,2,2,2,4)),(fcs),\emptyset,((131,147,232,241,254,263,315,327,356,$
$364,417,426,437,443,452,465),(137,141,231,243,252,264,317,326,354,365,415,427,432,447,456,463)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
+1	0	0	0	0	0	−1
−1	0	0	0	0	0	+1
0	0	0	0	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	+1	0	0	0	0	0
+1	0	−1	0	0	0	0
0	−1	0	+1	0	0	0
0	0	+1	−1	0	0	0

0	0	0	0	+1	0	−1
0	0	0	0	0	−1	+1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	−1	0	+1	0
0	0	0	+1	−1	0	0

0	0	0	0	−1	0	+1
0	0	0	0	0	+1	−1
0	−1	0	0	0	0	+1
0	0	+1	0	0	0	−1
0	+1	0	0	0	−1	0
0	0	−1	0	+1	0	0

- $4 \times 6 \times 7$ move(2) of degree 16 with slice degree $\{2,4,4,6\} \times \{2,2,3,3,3,3\} \times \{2,2,2,2,2,2,4\}$
$((4,6,7),(16),((2,4,4,6),(2,2,3,3,3,3),(2,2,2,2,2,2,4)),(fcs),\emptyset,((131,147,237,243,252,267,316,325,357,$

$364, 415, 424, 432, 441, 456, 463), (137, 141, 232, 247, 257, 263, 315, 324, 356, 367, 416, 425, 431, 443, 452, 464)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1
+1	0	0	0	0	0	−1
0	0	0	0	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	+1	0	0	0	0	−1
0	0	−1	0	0	0	+1
0	−1	0	0	0	0	+1
0	0	+1	0	0	0	−1

0	0	0	0	+1	−1	0
0	0	0	+1	−1	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	+1	−1
0	0	0	−1	0	0	+1

0	0	0	0	−1	+1	0
0	0	0	−1	+1	0	0
+1	−1	0	0	0	0	0
−1	0	+1	0	0	0	0
0	+1	0	0	0	−1	0
0	0	−1	+1	0	0	0

- $4 \times 6 \times 7$ move(3) of degree 16 with slice degree $\{2, 4, 4, 6\} \times \{2, 2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  (not fundamental, circuit)
  $((4, 6, 7), (16), ((2, 4, 4, 6), (2, 2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 4)), (Fcs), (437, 447), ((131, 147, 237, 243, 252, 264, 316, 327, 354, 365, 417, 425, 432, 441, 456, 463), (137, 141, 232, 247, 254, 263, 317, 325, 356, 364, 416, 427, 431, 443, 452, 465)))$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1
+1	0	0	0	0	0	−1
0	0	0	0	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	+1	0	0	0	0	−1
0	0	−1	0	0	0	+1
0	−1	0	+1	0	0	0
0	0	+1	−1	0	0	0

0	0	0	0	0	−1	+1
0	0	0	0	+1	0	−1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	−1	0	+1	0
0	0	0	+1	−1	0	0

0	0	0	0	0	+1	−1
0	0	0	0	−1	0	+1
+1	−1	0	0	0	0	(0)
−1	0	+1	0	0	0	(0)
0	+1	0	0	0	−1	0
0	0	−1	0	+1	0	0

- $4 \times 6 \times 7$ move(1) of degree 16 with slice degree $\{3, 3, 4, 6\} \times \{2, 2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  $((4, 6, 7), (16), ((3, 3, 4, 6), (2, 2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 4)), (fcs), \emptyset, ((111, 132, 147, 227, 253, 264, 335, 342, 354, 366, 417, 423, 431, 446, 455, 467), (117, 131, 142, 223, 254, 267, 332, 346, 355, 364, 411, 427, 435, 447, 453, 466)))$

+1	0	0	0	0	0	−1
0	0	0	0	0	0	0
−1	+1	0	0	0	0	0
0	−1	0	0	0	0	+1
0	0	0	0	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	0	−1	0	0	0	+1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	+1	−1	0	0	0
0	0	0	+1	0	0	−1

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	−1	0	0	+1	0	0
0	+1	0	0	0	−1	0
0	0	0	+1	−1	0	0
0	0	0	−1	0	+1	0

−1	0	0	0	0	0	+1
0	0	+1	0	0	0	−1
+1	0	0	0	−1	0	0
0	0	0	0	0	+1	−1
0	0	−1	0	+1	0	0
0	0	0	0	0	−1	+1

- $4 \times 6 \times 7$ move(2) of degree 16 with slice degree $\{3, 3, 4, 6\} \times \{2, 2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  $((4, 6, 7), (16), ((3, 3, 4, 6), (2, 2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 4)), (fcs), \emptyset, ((111, 132, 147, 223, 254, 267, 335, 342, 356,$

$364, 417, 427, 431, 446, 453, 465), (117, 131, 142, 227, 253, 264, 332, 346, 354, 365, 411, 423, 435, 447, 456, 467)))$

$$\begin{array}{|ccccccc|}\hline -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccccc|}\hline +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ \hline\end{array}$$

- $4 \times 6 \times 7$ move(3) of degree 16 with slice degree $\{3, 3, 4, 6\} \times \{2, 2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  $((4, 6, 7), (16), ((3, 3, 4, 6), (2, 2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 4)), (fcs), \emptyset, ((111, 137, 142, 223, 257, 264, 336, 347, 355, 367, 412, 424, 431, 445, 453, 466), (112, 131, 147, 224, 253, 267, 337, 345, 357, 366, 411, 423, 436, 442, 455, 464)))$

$$\begin{array}{|ccccccc|}\hline +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 \\ \hline\end{array} \quad \begin{array}{|ccccccc|}\hline -1 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ \hline\end{array}$$

- $4 \times 6 \times 8$ move of degree 16 with slice degree $\{2, 4, 4, 6\} \times \{2, 2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 2, 2, 2\}$
  $((4, 6, 8), (16), ((2, 4, 4, 6), (2, 2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((131, 142, 233, 241, 255, 264, 316, 328, 357, 365, 417, 426, 432, 444, 453, 468), (132, 141, 231, 244, 253, 265, 317, 326, 355, 368, 416, 428, 433, 442, 457, 464)))$

$$\begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ \hline\end{array} \quad \begin{array}{|cccccccc|}\hline 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ \hline\end{array}$$

- $4 \times 6 \times 8$ move of degree 16 with slice degree $\{3, 3, 4, 6\} \times \{2, 2, 3, 3, 3, 3\} \times \{2, 2, 2, 2, 2, 2, 2, 2\}$
  $((4, 6, 8), (16), ((3, 3, 4, 6), (2, 2, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((111, 133, 142, 224, 256, 265, 337, 343, 358,$

$366, 412, 425, 431, 448, 454, 467), (112, 131, 143, 225, 254, 266, 333, 348, 356, 367, 411, 424, 437, 442, 458, 465)))$

+1	−1	0	0	0	0	0	0
0	0	0	0	0	0	0	0
−1	0	+1	0	0	0	0	0
0	+1	−1	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	+1	−1	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	−1	0	+1	0	0
0	0	0	0	+1	−1	0	0

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	−1	0	0	0	+1	0
0	0	+1	0	0	0	0	−1
0	0	0	0	0	−1	0	+1
0	0	0	0	0	+1	−1	0

−1	+1	0	0	0	0	0	0
0	0	0	−1	+1	0	0	0
+1	0	0	0	0	0	−1	0
0	−1	0	0	0	0	0	+1
0	0	0	+1	0	0	0	−1
0	0	0	0	−1	0	+1	0

- $5 \times 5 \times 7$ move(1) of degree 16 with slice degree $\{2, 3, 3, 3, 5\} \times \{2, 3, 3, 4, 4\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  $((5, 5, 7), (16), ((2, 3, 3, 3, 5), (2, 3, 3, 4, 4), (2, 2, 2, 2, 2, 2, 4)), (fcs), \emptyset, ((121, 147, 222, 241, 253, 334, 343, 355, 417, 435, 456, 516, 527, 537, 544, 552), (127, 141, 221, 243, 252, 335, 344, 353, 416, 437, 455, 517, 522, 534, 547, 556)))$

0	0	0	0	0	0	0
+1	0	0	0	0	0	−1
0	0	0	0	0	0	0
−1	0	0	0	0	0	+1
0	0	0	0	0	0	0

0	0	0	0	0	0	0
−1	+1	0	0	0	0	0
0	0	0	0	0	0	0
+1	0	−1	0	0	0	0
0	−1	+1	0	0	0	0

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	+1	−1	0	0
0	0	+1	−1	0	0	0
0	0	−1	0	+1	0	0

0	0	0	0	0	−1	+1
0	0	0	0	0	0	0
0	0	0	0	+1	0	−1
0	0	0	0	0	0	0
0	0	0	0	−1	+1	0

0	0	0	0	0	+1	−1
0	−1	0	0	0	0	+1
0	0	0	−1	0	0	+1
0	0	0	+1	0	0	−1
0	+1	0	0	0	−1	0

- $5 \times 5 \times 7$ move(2) of degree 16 with slice degree $\{2, 3, 3, 3, 5\} \times \{2, 3, 3, 4, 4\} \times \{2, 2, 2, 2, 2, 2, 4\}$
  (not fundamental, circuit)
  $((5, 5, 7), (16), ((2, 3, 3, 3, 5), (2, 3, 3, 4, 4), (2, 2, 2, 2, 2, 2, 4)), (Fcs), (527, 547), ((121, 147, 233, 244, 252, 327, 342, 355, 416, 437, 453, 517, 525, 534, 541, 556), (127, 141, 234, 242, 253, 325, 347, 352, 417, 433, 456, 516, 521, 537, 544, 555)))$

0	0	0	0	0	0	0
−1	0	0	0	0	0	+1
0	0	0	0	0	0	0
+1	0	0	0	0	0	−1
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	−1	+1	0	0	0
0	+1	0	−1	0	0	0
0	−1	+1	0	0	0	0

0	0	0	0	0	0	0
0	0	0	0	+1	0	−1
0	0	0	0	0	0	0
0	−1	0	0	0	0	+1
0	+1	0	0	−1	0	0

0	0	0	0	0	−1	+1
0	0	0	0	0	0	0
0	0	+1	0	0	0	−1
0	0	0	0	0	0	0
0	0	−1	0	0	+1	0

0	0	0	0	0	+1	−1
+1	0	0	0	−1	0	(0)
0	0	0	−1	0	0	+1
−1	0	0	+1	0	0	(0)
0	0	0	0	+1	−1	0

- $5 \times 5 \times 8$ move of degree 16 with slice degree $\{2, 3, 3, 3, 5\} \times \{2, 3, 3, 4, 4\} \times \{2, 2, 2, 2, 2, 2, 2, 2\}$
  $((5, 5, 8), (16), ((2, 3, 3, 3, 5), (2, 3, 3, 4, 4), (2, 2, 2, 2, 2, 2, 2, 2)), (fcs), \emptyset, ((121, 142, 223, 241, 254, 335, 344, 356, 418,$

436, 457, 517, 522, 538, 545, 553), (122, 141, 221, 244, 253, 336, 345, 354, 417, 438, 456, 518, 523, 535, 542, 557)))

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ +1&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ -1&+1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline\end{array} \quad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ -1&0&+1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ +1&0&0&-1&0&0&0&0\\ 0&0&-1&+1&0&0&0&0\\ \hline\end{array} \quad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&+1&-1&0&0\\ 0&0&0&+1&-1&0&0&0\\ 0&0&0&-1&0&+1&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&-1&+1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&+1&0&-1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&+1&0\\ \hline\end{array} \quad \begin{array}{|cccccccc|}\hline 0&0&0&0&0&0&+1&-1\\ 0&+1&-1&0&0&0&0&0\\ 0&0&0&0&-1&0&0&+1\\ 0&-1&0&0&+1&0&0&0\\ 0&0&+1&0&0&0&-1&0\\ \hline\end{array}$$

- $5 \times 6 \times 6$ move(1) of degree 16 with slice degree $\{2,3,3,4,4\} \times \{2,2,2,2,4,4\} \times \{2,2,3,3,3,3\}$
((5, 6, 6), (16), ((2, 3, 3, 4, 4), (2, 2, 2, 2, 4, 4), (2, 2, 3, 3, 3, 3)), (fcs), $\emptyset$, ((153, 164, 211, 225, 263, 334, 346, 362, 415, 442, 456, 461, 523, 536, 554, 555), (154, 163, 215, 223, 261, 336, 342, 364, 411, 446, 455, 462, 525, 534, 553, 556)))

$$\begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&+1&-1&0&0\\ 0&0&-1&+1&0&0\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline +1&0&0&0&-1&0\\ 0&0&-1&0&+1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&+1&0&0&0\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&+1&0&-1\\ 0&-1&0&0&0&+1\\ 0&0&0&0&0&0\\ 0&+1&0&-1&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccc|}\hline -1&0&0&0&+1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&+1&0&0&0&-1\\ 0&0&0&0&-1&+1\\ +1&-1&0&0&0&0\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&+1&0&-1&0\\ 0&0&0&-1&0&+1\\ 0&0&0&0&0&0\\ 0&0&-1&+1&+1&-1\\ 0&0&0&0&0&0\\ \hline\end{array}$$

- $5 \times 6 \times 6$ move(2) of degree 16 with slice degree $\{2,3,3,4,4\} \times \{2,2,2,2,4,4\} \times \{2,2,3,3,3,3\}$
((5, 6, 6), (16), ((2, 3, 3, 4, 4), (2, 2, 2, 2, 4, 4), (2, 2, 3, 3, 3, 3)), (fcs), $\emptyset$, ((113, 164, 225, 251, 263, 344, 356, 362, 414, 423, 435, 446, 536, 552, 555, 561), (114, 163, 223, 255, 261, 346, 352, 364, 413, 425, 436, 444, 535, 551, 556, 562)))

$$\begin{array}{|cccccc|}\hline 0&0&+1&-1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&+1&0&0\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&-1&0&+1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ +1&0&0&0&-1&0\\ -1&0&+1&0&0&0\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&+1&0&-1\\ 0&-1&0&0&0&+1\\ 0&+1&0&-1&0&0\\ \hline\end{array}$$

$$\begin{array}{|cccccc|}\hline 0&0&-1&+1&0&0\\ 0&0&+1&0&-1&0\\ 0&0&0&0&+1&-1\\ 0&0&0&-1&0&+1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&-1&+1\\ 0&0&0&0&0&0\\ -1&+1&0&0&+1&-1\\ +1&-1&0&0&0&0\\ \hline\end{array}$$

- $5 \times 6 \times 6$ move(3) of degree 16 with slice degree $\{2,3,3,4,4\} \times \{2,2,2,2,4,4\} \times \{2,2,3,3,3,3\}$
((5, 6, 6), (16), ((2, 3, 3, 4, 4), (2, 2, 2, 2, 4, 4), (2, 2, 3, 3, 3, 3)), (fcs), $\emptyset$, ((113, 164, 221, 255, 263, 346, 354, 362, 414,

435, 453, 456, 525, 536, 542, 561), (114, 163, 225, 253, 261, 342, 356, 364, 413, 436, 454, 455, 521, 535, 546, 562)))

$$\begin{array}{|cccccc|}\hline 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & +1 & 0 & -1 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|cccccc|}\hline 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \\ \hline\end{array}$$

- $5 \times 6 \times 7$ move of degree 16 with slice degree $\{2,3,3,4,4\} \times \{2,2,3,3,3,3\} \times \{2,2,2,2,2,2,4\}$
  $((5,6,7),(16),((2,3,3,4,4),(2,2,3,3,3,3),(2,2,2,2,2,2,4)),(fcs),\emptyset,((131,147,212,237,253,327,345,364,433,$
  $441,456,465,517,524,552,566),(137,141,217,233,252,324,347,365,431,445,453,466,512,527,556,564)))$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccccc|}\hline 0 & +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 & 0 \\ \hline\end{array} \quad \begin{array}{|ccccccc|}\hline 0 & -1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 \\ \hline\end{array}$$

Appendix B: Sufficient condition for indispensability in combinations of two-dimensional slices

Let $\boldsymbol{z} = \boldsymbol{z}^+ - \boldsymbol{z}^-$ be an indispensable move of the size $I \times J \times K$ with the positive part $\boldsymbol{z}^+$ and the negative part $\boldsymbol{z}^-$. Without loss of generality, we consider the combination of $k = K-1$-slice and $k = K$-slice of $\boldsymbol{z}$. Let $\boldsymbol{z}^* = \boldsymbol{z}^{*+} - \boldsymbol{z}^{*-}$ be a move of the size $I \times J \times (K-1)$ with the positive part $\boldsymbol{z}^{*+}$ and the negative part $\boldsymbol{z}^{*-}$, which is made by the combination of $\boldsymbol{z}_{k=K-1}$ and $\boldsymbol{z}_{k=K}$ from $\boldsymbol{z}$, i.e., the elements of $\boldsymbol{z}^*$ are given by

$$z^*_{ijk} = \begin{cases} z_{ijk}, & \text{for } k \in [K-2], \\ z_{ijK-1} + z_{ijK}, & \text{for } k = K-1, \end{cases} \tag{15}$$

for $i \in [I], j \in [J]$. We also assume that there is no cancellation of signs in any cell of $\boldsymbol{z}_{k=K-1}$ and $\boldsymbol{z}_{k=K}$. Hence we only consider the case of $\deg(\boldsymbol{z}) = \deg(\boldsymbol{z}^*)$. $\boldsymbol{z}^{*+}$ and $\boldsymbol{z}^{*-}$ are obtained by combining $k = K-1$- and $k = K$-slices of $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$, respectively, and have the same two-dimensional marginal totals, which are calculated as

$$\begin{aligned} z^{*+}_{ij\cdot} &= z^+_{ij\cdot}, \\ z^{*+}_{i\cdot k} &= \begin{cases} z^+_{i\cdot k}, & \text{for } k \in [K-2], \\ z^+_{i\cdot K-1} + z^+_{i\cdot K}, & \text{for } k = K-1, \end{cases} \\ z^{*+}_{\cdot jk} &= \begin{cases} z^+_{\cdot jk}, & \text{for } k \in [K-2], \\ z^+_{\cdot jK-1} + z^+_{\cdot jK}, & \text{for } k = K-1, \end{cases} \end{aligned}$$

for $i \in [I], j \in [J]$. Our aim is to derive a sufficient condition that $\boldsymbol{z}^* = \boldsymbol{z}^{*+} - \boldsymbol{z}^{*-}$ is an indispensable move, i.e.,

$$\mathcal{F}(\{z^{*+}_{ij\cdot}\}, \{z^{*+}_{i\cdot k}\}, \{z^{*+}_{\cdot jk}\}) = \{\boldsymbol{z}^{*+}, \boldsymbol{z}^{*-}\}.$$

Here we consider a separation of the one-dimensional marginal totals of the two-dimensional slice $\boldsymbol{z}^{*+}_{k=K-1}$, which is expressed as integer vectors $\boldsymbol{p}_1 = \{p_{1i}\}, \boldsymbol{p}_2 = \{p_{2i}\}, i \in [I]$, and $\boldsymbol{q}_1 = \{q_{1j}\}, \boldsymbol{q}_2 = \{q_{2j}\}, j \in [J]$, satisfying

$$\left\{ \begin{array}{l} z^{*+}_{i\cdot K-1} = p_{1i} + p_{2i}, \ \text{for } i \in [I], \\ z^{*+}_{\cdot jK-1} = q_{1j} + q_{2j}, \ \text{for } j \in [J], \\ \displaystyle\sum_{i=1}^{I} p_{1i} = \sum_{j=1}^{J} q_{1j} \geq 2, \ \sum_{i=1}^{I} p_{2i} = \sum_{j=1}^{J} q_{2j} \geq 2. \end{array} \right. \tag{16}$$

Then, for given $\{z^{*+}_{ij\cdot}\}, \{z^{*+}_{i\cdot k}\}, \{z^{*+}_{\cdot jk}\}, i \in [I], j \in [J], k \in [K-1]$ and $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2$, we consider the following simultaneous equation (for cell frequencies $\boldsymbol{y} = \{y_{ijk}\}$ of the size $I \times J \times K$).

$$\left\{ \begin{array}{l} y_{ij\cdot} = z^{*+}_{ij\cdot}, \\ y_{i\cdot k} = z^{*+}_{i\cdot k}, \ y_{i\cdot K-1} = p_{1i}, \ y_{i\cdot K} = p_{2i}, \\ y_{\cdot jk} = z^{*+}_{\cdot jk}, \ y_{\cdot jK-1} = q_{1i}, \ y_{\cdot jK} = q_{2i}, \\ \qquad i \in [I], \ j \in [J], \ k \in [K-2]. \end{array} \right. \tag{17}$$

By definition, Equation (17) has solutions $\boldsymbol{y} = \boldsymbol{z}^+$ and $\boldsymbol{y} = \boldsymbol{z}^-$ when

$$\left\{ \begin{array}{l} p_{1i} = z^+_{i\cdot K-1}, \ p_{2i} = z^+_{i\cdot K}, \ i \in [I], \\ q_{1j} = z^+_{\cdot jK-1}, \ q_{2j} = z^+_{\cdot jK}, \ j \in [J] \end{array} \right. \tag{18}$$

or

$$\left\{ \begin{array}{l} p_{1i} = z^+_{i\cdot K}, \ p_{2i} = z^+_{i\cdot K-1}, \ i \in [I], \\ q_{1j} = z^+_{\cdot jK}, \ q_{2j} = z^+_{\cdot jK-1}, \ j \in [J]. \end{array} \right. \tag{19}$$

Our sufficient concerns the situation that (17) has solutions only when the condition (18) or (19) holds.

Theorem 3 *Let $\boldsymbol{z}$ be an $I \times J \times K$ indispensable move and let $\boldsymbol{z}^*$ be an $I \times J \times (K-1)$ move satisfying* $\deg(\boldsymbol{z}) = \deg(\boldsymbol{z}^*)$*, which is made from $\boldsymbol{z}$ by combining $k = K-1$- and $k = K$-slices of $\boldsymbol{z}$ as (15). Then $\boldsymbol{z}^*$ is an indispensable move when the following two conditions are satisfied.*

(a) The simultaneous equations (17) has solutions only when the condition (18) or (19) holds.

(b) $\max(z^{*+}_{1\cdot K-1}, \ldots, z^{*+}_{I\cdot K-1}, z^{*+}_{\cdot 1K-1}, \ldots, z^{*+}_{\cdot JK-1}) \geq 2$.

Proof. We argue by contradiction. Suppose $\boldsymbol{z}^*$ is a dispensable move. Then there is some $\boldsymbol{x} \in \mathcal{F}(\{z^{*+}_{ij\cdot}\}, \{z^{*+}_{i\cdot k}\}, \{z^{*+}_{\cdot jk}\})$ where $\boldsymbol{x} \neq \boldsymbol{z}^{*+}$ and $\boldsymbol{x} \neq \boldsymbol{z}^{*-}$. If $\boldsymbol{x}_{k=K-1} = \boldsymbol{z}^{*+}_{k=K-1}$ or $\boldsymbol{x}_{k=K-1} = \boldsymbol{z}^{*-}_{k=K-1}$ holds, we can make an $I \times J \times K$ table $\widetilde{\boldsymbol{x}}$ by the separation of $\boldsymbol{x}_{k=K-1}$ satisfying $\widetilde{\boldsymbol{x}} \in \mathcal{F}(\{z^+_{ij\cdot}\}, \{z^+_{i\cdot k}\}, \{z^+_{\cdot jk}\})$, which contradicts the assumption $\boldsymbol{x} \neq \boldsymbol{z}^{*+}$ and $\boldsymbol{x} \neq \boldsymbol{z}^{*-}$ since $\boldsymbol{z}$ is an indispensable move. Hence we only have to consider the case that $\boldsymbol{x}_{k=K-1} \neq \boldsymbol{z}^{*+}_{k=K-1}$ and $\boldsymbol{x}_{k=K-1} \neq \boldsymbol{z}^{*-}_{k=K-1}$. Define an $I \times J \times (K-1)$ move $\boldsymbol{v}$ as $\boldsymbol{v} = \boldsymbol{z}^{*+} - \boldsymbol{x}$ and an $I \times J \times (K-1)$ table $\boldsymbol{u} = \{u_{ijk}\}$ as $u_{ijk} = \min(z^{*+}_{ijk}, \ x_{ijk})$ for $i \in [I], j \in [J], k \in [K-1]$.

Case 1. First we consider the case that there is some $i \in [I], j \in [J]$ such that $u_{ijK-1} > 0$. Note that $\boldsymbol{z}^{*+} = \boldsymbol{v}^+ + \boldsymbol{u}$ and $\boldsymbol{x} = \boldsymbol{v}^- + \boldsymbol{u}$ hold. Separation of the $k = K-1$-slice of $\boldsymbol{z}^{*+}$ to $\boldsymbol{v}^+_{k=K-1}$ and $\boldsymbol{u}_{k=K-1}$ makes a solution for (17) where

$$\begin{array}{l} p_{1i} = v^+_{i\cdot K-1},\ p_{2i} = u_{i\cdot K-1},\ i \in [I], \\ q_{1i} = v^+_{\cdot jK-1},\ q_{2i} = u_{\cdot jK-1},\ j \in [J]. \end{array} \tag{20}$$

Similarly, separation of the $k = K-1$-slice of $\boldsymbol{x}$ to $\boldsymbol{v}^-_{k=K-1}$ and $\boldsymbol{u}_{k=K-1}$ makes another solution for (17) where $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2$ are defined as (20). From the condition *(a)* and the assumption that $\boldsymbol{z}$ is an indispensable move, it follows that $\boldsymbol{x} = \boldsymbol{z}^{*+}$ or $\boldsymbol{x} = \boldsymbol{z}^{*-}$, which is a contradiction.

Case 2. Next we consider the case that $u_{ijK-1} = 0$ for all $i \in [I], j \in [J]$. In this case, $\boldsymbol{z}^{*+}$ and $\boldsymbol{x}$ do not have positive elements at common cells in the $k = K-1$-slice. In addition, an $I \times J$ two-dimensional $k = K-1$-slice of $\boldsymbol{z}^{*+} - \boldsymbol{x}$ has zero marginal totals. Hence $(\boldsymbol{z}^{*+} - \boldsymbol{x})_{k=K-1}$ can be expressed as a finite sum of loops with the form (11). Furthermore, $(\boldsymbol{z}^{*+} - \boldsymbol{x})_{k=K-1}$ is expressed as at least two loops, i.e., $a_1 + \cdots + a_n \geq 2$ in the expression (11), because otherwise it contradicts the condition *(b)*. According to this expression, we can write $(\boldsymbol{z}^{*+} - \boldsymbol{x})_{k=K-1} = \boldsymbol{L} + \widetilde{\boldsymbol{L}}$, where $\boldsymbol{L}$ and $\widetilde{\boldsymbol{L}}$ are (possibly sum of) $I \times J$ loops such that there is no cancellation of signs in any cell. Here, separation of the $k = K-1$-slice of $\boldsymbol{z}^{*+}$ to $\boldsymbol{L}^+$ and $\widetilde{\boldsymbol{L}}^+$ makes a solution for (17) where

$$\begin{array}{l} p_{1i} = L^+_{i\cdot},\ p_{2i} = \widetilde{L_{i\cdot}}^+,\ i \in [I], \\ q_{1i} = L^+_{\cdot j},\ q_{2i} = \widetilde{L_{\cdot j}}^+,\ j \in [J]. \end{array} \tag{21}$$

Similarly, separation of the $k = K-1$-slice of $\boldsymbol{x}$ to $\boldsymbol{L}^-$ and $\widetilde{\boldsymbol{L}}^-$ makes another solution for (17) where $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2$ are defined as (21). Then we see that Case 2 is also a contradiction for the same reason as Case 1. Q.E.D.

To see whether a move $\boldsymbol{z}^*$ is an indispensable move or not according to Theorem 2, we have to consider all the possible patterns of $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2$ satisfying the condition (16). As stated in Section 3.2, it is usually a much more complicated task than simply investigating $\boldsymbol{z}^*$ itself. For example, consider the $3 \times 3 \times 5$ indispensable move of degree 10 with slice degree $\{3,3,4\} \times \{3,3,4\} \times \{2,2,2,2,2\}$ displayed as (13) again. Considering the move (14) which is made by the combination of the $k = 1$- and $k = 4$-slices of (13), possible patterns of $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2$ include

$$\boldsymbol{p}_1 = (1,1,0),\ \boldsymbol{p}_2 = (0,0,2),\ \boldsymbol{q}_1 = (1,1,0),\ \boldsymbol{q}_2 = (0,0,2)$$

and

$$\boldsymbol{p}_1 = (1,0,1),\ \boldsymbol{p}_2 = (0,1,1),\ \boldsymbol{q}_1 = (0,1,1),\ \boldsymbol{q}_2 = (1,0,1).$$

Note that these two patterns permit solutions for (17), while the original indispensable move (13) is the difference of the two solutions for (17) when

$$\boldsymbol{p}_1 = (1,0,1),\ \boldsymbol{p}_2 = (0,1,1),\ \boldsymbol{q}_1 = (1,0,1),\ \boldsymbol{q}_2 = (0,1,1).$$

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