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Abstract

We consider two periodically weighted model-matching problems with regard to a discrete-time LTI plant by means of discrete-time linear periodically time-varying (LPTV) control. These two problems are formulated by projection operators and shown to be determined uniquely by corresponding LTI model-matching problems in which new free parameters arise from the periodically time-varying property of the controller. As a result, we can solve these problems approximately by using the usual LTI synthesis tools. The causality of the resulting LPTV controllers is guaranteed by a representation of a causal N-periodic system which we introduce as *dual lifted forms*. We also show an illustrative numerical example in which our proposing LPTV controller achieves superior performance to that of LTI controllers.

1 Introduction

Periodically time-varying digital control is one of the natural extensions to time-invariant digital control theory, and we can easily implement it on modern high-performance microcomputers. To clarify the advantages of periodic controllers compared with linear time-invariant (LTI) controllers and to find tractable design methods are important from a practical point of view.

There are many contributions in periodically time-varying control in the last few decades (See [Davis, 1972, Khargonekar et al., 1985, Francis and Georgiou, 1988, Goodwin and Feuer, 1992, Chen and Qiu, 1997, Mehr and Chen, 2002, Colaneri and Kučera, 1997, Bittanti and Colaneri, 2000] and references therein). And several researchers claim that

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periodic controllers have more desirable properties than LTI controllers. For example, Francis and Georgiou [1988] showed that zero placement of closed-loop systems can be achieved by periodic compensation. Khargonekar et al. [1985] used a *raised form* directly to design periodic controllers with arbitrary gain margins. However, these design methods are restricted to special qualitative objectives compared with LTI control theories. This motivates us to find a design method for linear periodically time-varying (LPTV) controllers that can deal with various control objectives.

In spite of the above advantages of periodic control, it has also been shown that any time-varying controller cannot be superior to the optimal LTI controller for $\mathcal{H}^\infty$ and p-norm control performance with respect to an LTI plant [Poolla and Ting, 1987, Zhang and Zhang, 1996]. This suggests that the usual $\mathcal{H}^\infty$ and p-norm control performance criteria ignore the potential of periodic controllers at least for LTI plants. Therefore, it is necessary to develop a suitable criterion by which we can see the difference between the potential of LPTV controllers and that of LTI controllers for LTI plants.

Our objective is to clarify the advantages of periodic controllers in optimization problems and to provide a design method for them. In this paper, we introduce both a performance criterion and a procedure for constructing a desirable LPTV controller. The former is formulated as two forms of periodic model-matching problem, which utilizes periodic filtering on input and output signal. The latter is given via a representation called *dual lifted forms* and relaxed LTI problems to which we can apply various useful design methods from LTI control theories.

A model-matching problem is known as a generalization of many control problems [Youla et al., 1976, Zhou et al., 1996] and also appropriate to formulate various design objectives for LPTV controllers. Colaneri and Kučera [1997] solved a periodic model matching problem for an LPTV plant with LPTV state feedback control so that the closed-loop transfer function is identical to a given LTI transfer function. Their design procedure can be executed automatically, however, they did not refer to the problem when the optimal controller is periodic and the mechanism of its superiority to LTI control is unclear. Chapellat et al. [1993] formulated and analyzed a periodic $\mathcal{H}^\infty$ problem for an LPTV plant and showed that time-varying controllers have not any advantage over periodic controllers. However, they did not provide any controller design method. Tanaka et al. [2002] dealt with the same problem in [Chapellat et al., 1993] and concluded that the cost of $\mathcal{H}^\infty$ control for LTI plant cannot be improved by any LPTV controller. This conclusion is identical to the performance limitation of time-varying control [Poolla and Ting, 1987, Zhang and Zhang, 1996].

We are interested in the difference between achievable performance of LPTV control and that of LTI control even for the case with an LTI plant. In this paper, we show that the interpolation conditions of a model matching problem can be related under some conditions and that this is the reason why an LPTV controller achieves better performances than LTI controllers. In addition, our considering two types of model matching problem result in periodic closed-loop and we deal with output feedback control for an LTI plant. To formulate the problems, we introduce a periodic filtering in the time domain on the disturbance input signal, or the regulated output signal, of the closed-loop, and show that periodic controllers have the advantage in periodically weighted signal space.

In one of our proposing problems, the disturbance input of a generalized plant is projected onto a signal space where the signal value can be nonzero only at a specified phase of each period in the time domain. This problem is equivalent to regulating an output signal caused by a periodically impulsive disturbance input of a stable closed-loop system. In the other problem, the regulated output of a generalized plant is projected onto a signal space where the signal value can be also nonzero only at a specified phase of each period in the time domain. This case is equivalent to regulating a sampled output signal caused by an arbitrary disturbance input. In other words, controlled outputs sampled at a specified phase of each period are required to be regulated more precisely than at the other phases. The periodic projection operators are represented by the lift operator.

The lift operator was used to transform LPTV systems into rational transfer matrices in Davis [1972], Khargonekar et al. [1985], Chen and Qiu [1997], Mehr and Chen [2002] and so on. Davis [1972] introduced a decimation operator relevant to the lifted forms and analyzed periodic systems using that operator. Goodwin and Feuer [1992] introduced a *modulated system*, which is represented with a frequency domain shift operator, and proposed a memory-less periodic control for a pole placement problem. However, their controllers are restricted to memory-less or a very conservative class [Khargonekar et al., 1985] because of causality problem. We reform the lift representations into *dual lifted forms* that are appropriate to solve model matching problems. The dual lifted forms guarantee causality in smart way that is equivalent to block lower triangular structure in the lifted signal space and simplify the multiplication of two causal LPTV systems.

Because the relationships from input to output of our proposing problems are described as LPTV operators, we cannot apply LTI design approaches directly to our periodically weighted model matching problems. We derive two relaxed LTI model-matching problems which give upper bounds and lower bounds of the performance of the original problems.

This paper is organized as follows. Section 2 is devoted to the problem formulation, where we introduce two types of periodic model-matching problem that clarify the advantages of LPTV controllers. In Section 3, we show several mathematical preliminaries, including the definition of a periodic system and the lift operator. Section 4 introduces *dual lifted forms* by which causal LPTV systems can be represented as LTI transfer matrices in the lifted signal space. In Section 5, we discuss the well-posedness of the LPTV controllers. Our main results are in Section 6, and we provide an approach for the problems of Section 2 using the tools introduced in Section 4. In Section 7, we give a criterion for block number reduction which is tightly connected to the order of given plant and the period of designed LPTV controller. An illustrative example is provided in Section 8 to show the effectiveness of our design method. Section 9 contains conclusions in this paper and summarizes the main results.

We use the following notation:

$\mathbb{Z}$	:	the set of integers
$\mathbb{C}$	:	the set of complex numbers
$\mathbb{R}$	:	the set of real numbers
q	:	a time forward shift operator
l^n	:	the set of discrete-time sequences of n-th dimension
$\mathbf{L}^{n\times m}$	:	the set of linear causal maps from l^m to l^n
$\mathbf{RL}^{n\times m}$	:	the subset of $\mathbf{L}^{n\times m}$ represented by matrices whose elements are rational transfer functions of z
$\mathbf{RH}_\infty$	:	the subset of $\mathbf{RL}$, whose poles are in the open unit disk on $\mathbb{C}$.

$$\begin{aligned} e_i &\triangleq \overbrace{(\ 0\cdots 0 \quad 1 \quad 0\cdots 0\ )}^{1\cdots \quad i \quad \cdots N}{}^T \in \mathbb{R}^N \\ E_i &\triangleq e_i \otimes I_r, \end{aligned}$$

where $\otimes$ represents the matrix Kronecker product. Given two matrices $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{p\times q}$, the matrix Kronecker product $A \otimes B$ is defined as follows (See [Zhou et al., 1996] for example.):

$$A \otimes B \triangleq \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}. \tag{1}$$

The z-transformation of a discrete-time r-dimension time sequence $\{\alpha_i\}_{i=0,1,\cdots}$, where i is the time index of l^r, is given by $\alpha(z) = \sum_{i=0}^{\infty} \alpha_i z^{-i}$. When given a rational transfer matrix $f(z)$, $f(z)$ is called inner if $f(z)$ satisfies $f(e^{-j\theta})^T f(e^{j\theta}) = I \ \ \forall\, \theta \in [0, 2\pi]$.

The definition of l_2 norm and l_∞ norm of signal $u \in l^r$ are $\|u\|_{l_2} = \left(\sum_{i=0}^{\infty} u_i^T u_i\right)^{\frac{1}{2}}$ and $\|u\|_{l_\infty} = \max_j \sup_{0\le k} |(u_j)_k|$, respectively. Signal spaces l_2^r and l_∞^r denote subsets of l^r, whose l_2 and l_∞ norm are finite. We may omit the dimension of a signal or a system when it is obvious from context.

2 Periodic Model-Matching Problems

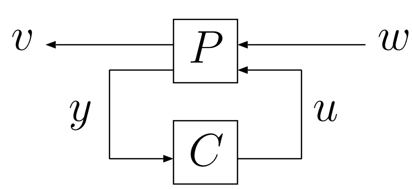

Fig. 1: A generalized plant with controller

We formulate two control problems for an LTI plant by LPTV control in this section. There are several contributions that deal with similar case, however, they are restricted with specific performance criteria [Khargonekar et al., 1985, Goodwin and Feuer, 1992] or state feedback compensation and LPTV plant [Colaneri and Kučera, 1997] or the normal $\mathcal{H}^\infty$ problem [Chapellat et al., 1993, Tanaka et al., 2002]. On the other hand, Poolla and Ting [1987], Zhang and Zhang [1996] showed that the optimal controller for an LTI plant with respect to the normal $\mathcal{H}^\infty$ control performance is LTI. One of our main objectives is to clarify the mechanism when an LPTV controller is superior to LTI controllers for an LTI plant and this problem does not appear in [Colaneri and Kučera, 1997, Tanaka et al., 2002]. Hence, we deal with more general case of output feedback performance criteria by formulating two model-matching problems with respect to an LTI generalized plant. We use the Youla parameterization of all stabilizing controllers, where the generalized plant $P(z) \in \mathbf{RL}$ and the stabilizing controller $C \in \mathbf{RL}$ (see Fig. 1) are given by

$$P(z) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right), \tag{2}$$

$$C = \mathrm{LFT}(\Pi, Q), \quad Q : \text{stable}, \tag{3}$$

$$\Pi \triangleq \left(\begin{array}{c|cc} A - B_2K + HC' & H & B_2 + HD_{22} \\ \hline K & 0 & -I \\ C' & I & D_{22} \end{array} \right), \tag{4}$$

$$C' \triangleq C_2 - D_{22}K, \tag{5}$$

where K and H are constant matrices such that $A - B_2K$ and $A + HC_2$ are stable and Q is a free parameter of the controller [Youla et al., 1976, Zhou et al., 1996, Francis and Doyle, 1987]. Then, the closed-loop transfer function from w to v is given by

$$T = T_1(z) - T_2(z)QT_3(z) \tag{6}$$

where $T_1, T_2, T_3 \in \mathbf{RH}_\infty$ is given by [Youla et al., 1976, Zhou et al., 1996]

$$T_1(z) \triangleq \left(\begin{array}{cc|c} A - B_2K & B_2K & B_1 \\ 0 & A + HC_2 & B_1 + HD_{21} \\ \hline C_1 - D_{12}K & D_{12}K & D_{11} \end{array} \right), \tag{7}$$

$$T_2(z) \triangleq \left(\begin{array}{c|c} A - B_2K & B_2 \\ \hline C_1 - D_{12}K & D_{12} \end{array} \right), \; T_3(z) \triangleq \left(\begin{array}{c|c} A + HC_2 & B_1 + HD_{21} \\ \hline C_2 & D_{21} \end{array} \right), \tag{8}$$

and the original model-matching problem (see Fig. 2) is:

P0. Find Q^* such that

$$Q^* = \arg \inf_{Q:\text{causal, stable}} \|T\|. \tag{9}$$

With respect to the usual model-matching problem P0, it has been shown that there is no LPTV controller that is superior to the optimal LTI controller under l_p induced norm evaluation, i.e.,

$$\inf_{Q:\text{causal, stable}} \|T\|_{l_p} = \inf_{Q \in \mathrm{RH}_\infty} \|T\|_{l_p} \tag{10}$$

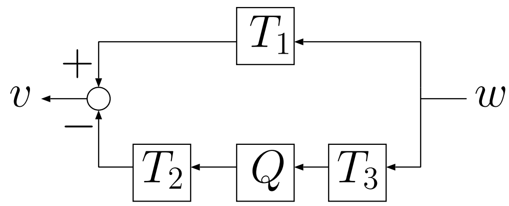

Fig. 2: A model-matching problem

is satisfied [Poolla and Ting, 1987, Zhang and Zhang, 1996]. In other words, this performance criterion cannot evaluate the advantages of causal LPTV controllers. This is because the performance criterion is time-invariant although the controller is time-varying. With this in mind, we can find other types of model matching problem that can clarify the performance difference between LPTV controllers and LTI controllers. The key idea is that LPTV controllers have greater potential than LTI controllers in periodically weighted signal space. The definitions of periodic model-matching problems are:

P1. Find an LPTV controller that optimally reduces the output signal caused by a periodic impulsive disturbance input of finite norm.

P2. Find an LPTV controller that optimally reduces the periodically sampled output signal caused by any disturbance input of finite norm.

These two problems can be seen as model matching problems weighted periodically in time domain.

We can find applicable areas of these problems such as hard disk drive, reciprocal engine, and so on. The common feature of these devices is the periodical dynamics that we are willing to deal with. A possible control subject is to reduce the effect by periodical impulsive disturbances such as gas explosions in a cylinder or to move a hard disk head to desirable positions more precisely at every clock time at which data arrives. Such control subjects can be formulated by the above problems, P1 and P2.

The mathematical definitions of P1 and P2 are given in Section 6. In the following three sections, we introduce useful mathematical tools to formulate these problems.

3 Lift Operators and Periodic Systems

In this section, we introduce a lift operator, LPTV systems and properties of them for the followings of this paper. Firstly, we introduce a lift operator W defined in Khargonekar et al. [1985], which is an isomorphism on time series in l^r:

$$W : l^r \to l^{rN} : \alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \alpha_0 & \alpha_N & \cdots \end{pmatrix} \\ \begin{pmatrix} \alpha_1 & \alpha_{N+1} & \cdots \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \alpha_{N-1} & \alpha_{2N-1} & \cdots \end{pmatrix} \end{pmatrix}, \tag{11}$$

and for $\alpha^{(i)}(z) \triangleq \sum_{j=0}^{\infty} \alpha_{Nj+i} z^{-j}$, where $i = 0, \cdots, N-1$, an inverse operator of W denoted by W^{-1} is given by

$$W^{-1} : l^{rN} \to l^{r} : \begin{pmatrix} \alpha^{(0)}(z) \\ \vdots \\ \alpha^{(N-1)}(z) \end{pmatrix} \to \alpha(z) = \sum_{i=0}^{N-1} \alpha^{(i)}(z^N) z^{-i}. \tag{12}$$

Note that we define the operator W so that the first row of $W\alpha$ corresponds to $\alpha_0, \alpha_N, \cdots$, so we assume that the phase of l^r with respect to each N-period is determined a priori. In the following argument, we oftenly state that $\alpha \in l^r$ is in the original signal space and $W\alpha \in l^{rN}$ is in the lifted signal space.

Similarly, we can decompose an LTI transfer function $G(z) \in \mathbf{RL}^{n\times m}$ into LTI subsystems uniquely [Davis, 1972, Khargonekar et al., 1985] as

$$G(z) = \sum_{i=0}^{N-1} z^{-i} G^{(i)}(z^N), \tag{13}$$

where $G^{(i)}(z) \in \mathbf{RL}^{n\times m}$ for $i = 0, 1, \cdots, N-1$. By letting (A, B, C, D) be a state space realization of $G(z)$, each $G^{(i)}(z)$ is represented by

$$G^{(i)}(z) = \left\{ \begin{array}{cc} D + C(zI - A^N)^{-1} A^{N-1} B & (i = 0) \\ CA^{i-1}B + CA^{i}(zI - A^N)^{-1} A^{N-1} B & (1 \le i \le N-1) \end{array} \right. . \tag{14}$$

Obviously, when $G(z)$ is stable, $G^{(i)}(z)$ is also stable for $i = 0, \cdots, N-1$. It is simple to check that the definition of $G^{(i)}(z)$ is independent of any similar transformation of (A, B, C, D) :

$$(A, B, C, D) \to \left(V^{-1}AV, V^{-1}B, CV, D\right).$$

Define two operators U_i and D_i as

$$U_i \triangleq W^{-1} E_{i+1}, \quad D_i \triangleq E_{i+1}^T W. \tag{15}$$

The meanings of U_i and D_i are up-sampling and down-sampling of discrete time sequences, respectively. The equation $D_i \cdot U_i = I_r$ is always satisfied for any index i by definition. Therefore, let an operator Pr^j denote

$$\mathrm{Pr}^j \triangleq U_j D_j, \tag{16}$$

then Pr^j is a projection and has the eigen signal space $\mathcal{W}_j$:

$$\mathcal{W}_j^r \ \triangleq\ \left\{ w \in l^r \mid \mathrm{Pr}^j w = w \right\}. \tag{17}$$

Property 1 *The projection operator* Pr^i *has two following properties:*

1. $\forall\ i, j \in \mathbb{Z}[0, N]$ $\ \mathrm{Pr}^i \mathrm{Pr}^j = \delta_{ij} \mathrm{Pr}^i$ *where* δ_{ij} *is the Kronecker delta.*

2. $\mathrm{Pr}^0 + \mathrm{Pr}^1 + \cdots + \mathrm{Pr}^{N-1} = I$.

The orthogonal property of Pr^i is useful to formulate the problems P1 and P2. And the second property is relating to the performance limitations of LPTV controllers. We will give a detail of this topic in Section 6.

Assume that an m-input n-output linear discrete-time system $G \in \mathbf{L}^{n\times m}$ has the following relationship from input to output in the original signal space:

$$y = Gu, \tag{18}$$

where $y \in l^n$ and $u \in l^m$. Then, we can give the definition of an LPTV system and the representation of transfer functions in lifted signal space.

Definition 1 *An linear discrete-time system $G \in \mathbf{L}^{n\times m}$ is called τ-periodic if G satisfies*

$$G = q^{\tau} G q^{-\tau}, \tag{19}$$

where τ can take any non-negative integer value.

From this definition, an α-periodic system where $\alpha \times k = \beta$ for some positive integer k is also β-periodic. Therefore, a set of all LTI systems is included in any class of N-periodic systems where $N \geq 1$ because an LTI system is 1-periodic.

By lifting the input signal and the output signal, we can represent (18) in the lifted signal space ($l^{mN} \to l^{nN}$):

$$Wy = WGW^{-1}Wu = \tilde{G}Wu. \tag{20}$$

The symbol $\tilde{G}$ denote $WGW^{-1} \in \mathbf{L}^{nN\times mN}$ in the following argument. Lifting procedure has convenient properties for further analysis.

Property 2 *With respect to the lift operator W and $\forall\ A, B \in \mathbf{L}^{n\times m}$, the following three equations and one statement are satisfied:*

1. $W(A+B)W^{-1} = WAW^{-1} + WBW^{-1}$
2. $WABW^{-1} = WAW^{-1} \cdot WBW^{-1}$
3. $(WAW^{-1})^{-1} = WA^{-1}W^{-1}$
4. *For $u \in l^m$, $\|Wu\| = \|u\|$ is satisfied under l_2 and l_∞ norms.*

From the fourth property, we can discuss induced norms of the transfer matrices represented in the lifted signal space instead of the original signal space. In other words, the induced norm in lifted signal space is equivalent to that of in the original signal space: $\|G\| = \left\|\tilde{G}\right\|$ is satisfied for $G \in \mathbf{L}^{n\times m}$.

Multiplying (6) by W and W^{-1} from both sides, we can see from Property 2 that

$$\tilde{T}(z) = \tilde{T}_1(z) - \tilde{T}_2(z)\tilde{Q}(z)\tilde{T}_3(z) \tag{21}$$

is satisfied when Q is linear N-periodic because N-periodic Q can be represented as $Q = W^{-1}\tilde{Q}(z)W$ with an LTI transfer function $\tilde{Q}(z)$ (see Section 4). Therefore, when Q is an N-periodic stable transfer matrix, the closed-loop transfer function T is also N-periodic and stable. The above procedure can be also seen as a normal LTI Youla parameterization with an LTI free parameter $\tilde{Q}(z)$ in the lifted signal space because if $P(z)$ has a double coprime factorization $P(z) = M_l^{-1}N_l = N_rM_r^{-1}$ [Zhou et al., 1996] where

$$\begin{bmatrix} X_r & Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & -Y_l \\ N_r & X_l \end{bmatrix} = I, \quad M_l, N_l, M_r, N_r, X_l, Y_l, X_r, Y_r \in \mathbf{RH}_\infty, \tag{22}$$

then $\tilde{P}(z) = WPW^{-1}$ also has a coprime factorization $\tilde{P}(z) = \tilde{M_l}^{-1}\tilde{N_l} = \tilde{N_r}\tilde{M_r}^{-1}$ with

$$\begin{bmatrix} \tilde{X_r} & \tilde{Y_r} \\ -\tilde{N_l} & \tilde{M_l} \end{bmatrix} \begin{bmatrix} \tilde{M_r} & -\tilde{Y_l} \\ \tilde{N_r} & \tilde{X_l} \end{bmatrix} = I, \quad \tilde{M_l}, \tilde{N_l}, \tilde{M_r}, \tilde{N_r}, \tilde{X_l}, \tilde{Y_l}, \tilde{X_r}, \tilde{Y_r} \in \mathbf{RH}_\infty \tag{23}$$

where $\tilde{M_l} = WM_lW^{-1}$, $\tilde{N_l} = WN_lW^{-1}$, and so on. Youla parameterization in the lifted signal space is summarized as follows.

Lemma 1 *[Freudenberg and Grizzle, 1989] A coprime factorization in the lifted signal space and causal LTI free parameters $\tilde{Q}(z) \in \mathbf{RH}_\infty$ whose constant part is block lower triangular gives all stabilizing causal LPTV controllers in the original signal space.*

We formulate an N-periodic system that is causal in the original signal space and is represented by rational transfer function in the lifted signal space in the following section. We also discuss the well-posedness of the LPTV controller (3) with an N-periodic free parameter Q in Section 5.

4 Dual Lifted Forms of N-Periodic Systems

We show that a causal LPTV system has a pair of dual representations in the lifted signal space. The dual lifted forms enable us to represent the multiplication of two causal LPTV systems very simply. It has already been shown that for $G \in \mathbf{RL}$, $\tilde{G} \triangleq WGW^{-1}$ has the block Toeplitz structure [Davis, 1972, Khargonekar et al., 1985, Francis and Georgiou, 1988, Goodwin and Feuer, 1992]:

$$\tilde{G}(z) = \begin{pmatrix} G^{(0)}(z) & z^{-1}G^{(N-1)}(z) & \cdots & z^{-1}G^{(1)}(z) \\ G^{(1)}(z) & G^{(0)}(z) & \cdots & z^{-1}G^{(2)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ G^{(N-1)}(z) & G^{(N-2)}(z) & \cdots & G^{(0)}(z) \end{pmatrix}, \tag{24}$$

where $G^{(i)}(z) \quad i = 0, \cdots, N-1$ are defined in (13), (14). With this structure in mind, we introduce a pair of row and column operations for $\tilde{G}(z)$. Let cs denote an operator on $\mathbf{RL}^{n\times m}$ as

$$\mathrm{cs} : \mathbf{RL}^{n\times m} \to \mathbf{RL}^{nN\times m} : \mathrm{cs}\ G(z) \triangleq \begin{pmatrix} G^{(0)}(z) \\ \vdots \\ G^{(N-1)}(z) \end{pmatrix}, \tag{25}$$

and rs denote an operator on $\mathbf{RL}^{n\times m}$ as

$$\text{rs} : \mathbf{RL}^{n\times m} \rightarrow \mathbf{RL}^{n\times mN} : \text{rs}\, G(z) \triangleq \begin{pmatrix} G^{(N-1)}(z) & \cdots & G^{(0)}(z) \end{pmatrix}. \tag{26}$$

Note that $G(z)$ is a matrix function, such that cs $G(z)$ and rs $G(z)$ consist of matrix blocks and the operator cs and rs are one to one maps due to the uniqueness of $G^{(i)}(z)$ defined in (13) and (14). In addition, the operators cs and rs can be represented by sampling operators:

$$\begin{aligned} \text{cs}\, G(z) &= W G(z) U_0, & (27)\\ \text{rs}\, G(z) &= D_{N-1} G(z) W^{-1}. & (28) \end{aligned}$$

The next lemma shows lifted forms of time shift operators which is previously introduced by Bittanti and Colaneri [2000] and it is used throughout the paper.

Lemma 2 *[Bittanti and Colaneri, 2000] Suppose W is a lift operator from l^r to l^{rN}. Let $\tilde{q}^\tau \triangleq W q^\tau W^{-1}$ and $\tilde{q}^{-\tau} \triangleq W q^{-\tau} W^{-1}$ for $\tau \in \mathbb{Z}[0, N]$. Then $\tilde{q}^\tau$ and $\tilde{q}^{-\tau}$ can be represented as LTI matrices:*

$$\tilde{q}^\tau = \left(\begin{array}{c|c} 0 & I_{N-\tau} \\ \hline zI_\tau & 0 \end{array}\right) \otimes I_r, \tag{29}$$

$$\tilde{q}^{-\tau} = \left(\begin{array}{c|c} 0 & z^{-1}I_\tau \\ \hline I_{N-\tau} & 0 \end{array}\right) \otimes I_r. \tag{30}$$

$$\begin{array}{ccc} G & \xrightarrow{W, W^{-1}} & \tilde{G} \\ {\scriptstyle q^\tau}\downarrow & & \downarrow{\scriptstyle \tilde{q}^\tau} \\ q^\tau G & \xrightarrow{W, W^{-1}} & \tilde{q}^\tau \tilde{G} \end{array}$$

Fig. 3: Relationship of $\tilde{G}$ and G

The relationship between G and $\tilde{G}$ can be illustrated as Fig. 3 by using the operators q^τ and $\tilde{q}^\tau$.

Property 3 *The operators $\tilde{q}^\tau$ and $\tilde{q}^{-\tau}$ have the following properties.*

1. $\forall \tau_1, \tau_2 \quad \tilde{q}^{\tau_1} \cdot \tilde{q}^{\tau_2} = \tilde{q}^{\tau_2} \cdot \tilde{q}^{\tau_1} = W q^{\tau_1+\tau_2} W^{-1}$,
2. $\forall \tau \quad \tilde{q}^\tau \cdot \tilde{q}^{-\tau} = I$,
3. $\tilde{q}^0 = I, \quad \tilde{q}^N = zI, \ \tilde{q}^{-N} = z^{-1}I$,
4. $\forall \tau \quad z^{-1}\tilde{q}^\tau = \tilde{q}^{\tau-N}, \quad z\tilde{q}^{-\tau} = \tilde{q}^{N-\tau}$.

From definitions of cs and rs and $\tilde{q}$, we can transform (24) into the following forms:

$$\tilde{G}(z) = \begin{bmatrix} \tilde{q}^0 \text{cs } G(z) & \cdots & \tilde{q}^{-N+1} \text{cs } G(z) \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} \text{rs } G(z) \cdot \tilde{q}^{-N+1} \\ \vdots \\ \text{rs } G(z) \cdot \tilde{q}^0 \end{bmatrix}. \tag{32}$$

The next theorem gives a similar transformation when G is linear and periodic.

Theorem 1 *For any linear causal N-periodic discrete-time system $G \in \mathbf{L}^{n\times m}$, $\tilde{G} = WGW^{-1}$ is an element of $\mathbf{RL}^{nN\times mN}$ and can be represented by the following pair of dual lifted forms:*

F1. $\tilde{G}(z) = \begin{bmatrix} \tilde{q}^0 \text{cs } G_0(z) & \tilde{q}^{-1} \text{cs } G_1(z) & \cdots & \tilde{q}^{-N+1} \text{cs } G_{N-1}(z) \end{bmatrix}$,
where $G_i(z) \in \mathbf{RL}^{n\times m}$ for $i = 0, \cdots, N-1$.

F2. $\tilde{G}(z) = \begin{bmatrix} \text{rs } G_{N-1}(z) \cdot \tilde{q}^{-N+1} \\ \text{rs } G_{N-2}(z) \cdot \tilde{q}^{-N+2} \\ \vdots \\ \text{rs } G_0(z) \cdot \tilde{q}^0 \end{bmatrix}$,
where $G_i(z) \in \mathbf{RL}^{n\times m}$ for $i = 0, \cdots, N-1$.

Proof. Because G is linear and N-periodic, G satisfies

$$G = q^N G q^{-N}. \tag{33}$$

From this and Property 3, $\tilde{G} \stackrel{\triangle}{=} WGW^{-1}$ satisfies

$$\tilde{G} = \tilde{q}^N \tilde{G} \tilde{q}^{-N} = q \tilde{G} q^{-1}. \tag{34}$$

This means that $\tilde{G}$ is an element of $\mathbf{RL}^{nN\times mN}$ and we can write $\tilde{G}(z)$ as

$$\tilde{G}(z) = \begin{pmatrix} G_{11}(z) & \cdots & G_{1N}(z) \\ \vdots & \ddots & \vdots \\ G_{N1}(z) & \cdots & G_{NN}(z) \end{pmatrix}. \tag{35}$$

Furthermore, $\lim_{z\to\infty} \tilde{G}(z)$ must be block lower triangular because of causality [Khargonekar et al., 1985, Freudenberg and Grizzle, 1989], i.e.,

$$\lim_{z\to\infty} \tilde{G}(z) = \begin{pmatrix} G_{11}(\infty) & 0 & 0 \\ \vdots & \ddots & 0 \\ G_{N1}(\infty) & \cdots & G_{NN}(\infty) \end{pmatrix}. \tag{36}$$

Therefore, the upper right blocks of $\tilde{G}$ can be represented as $z^{-1}\hat{G}_{ij}(z)$, where $\hat{G}_{ij}(z) \in \mathbf{RL}^{n\times m}$.[1] In addition, because of the uniqueness of the maps cs and rs, there always exists

[1]For example, $\frac{1}{z-1} = z^{-1} \cdot \frac{z}{z-1}$.

$G_i(z) \in \mathbf{RL}^{n\times m}$ such that $\tilde{q}^{-i}\mathrm{cs}\ G_i(z)$ equals the $i+1$-th column block WGW^{-1} in the F1 case and rs $G_{N-1-i}(z)\cdot\tilde{q}^{-N+1+i}$ is equal to the $i+1$-th row block of WGW^{-1} in the F2 case. This completes the proof of the theorem. □

As is shown in Section 6, the two forms F1 and F2 enable us to easily calculate the multiplication of two LPTV systems and this is a key idea to solve our model matching problems P1 and P2. The relationship between an LPTV system represented in dual lifted forms and its implicit period is given as follows.

Corollary 1 *Given a $\tilde{G}$ in F1 or F2 form, then $G = W^{-1}\tilde{G}W$ is τ-periodic (i.e. $\tilde{G}(z) = \tilde{q}^{\tau}\tilde{G}(z)\tilde{q}^{-\tau}$) iff the following two conditions are satisfied.*

1. *The equation $G_i(z) = G_{\tau+i}(z)$ is satisfied for $i = 0, \cdots, N-\tau-1$.*
2. *The equation $G_i(z) = G_{N-\tau+i}(z)$ is satisfied for $i = 0, \cdots, \tau-1$.*

In particular, $G(z)$ is LTI (i.e. 1-periodic) iff $G_0(z) = G_1(z) = \cdots = G_{N-1}(z)$.

See Appendix A for the proof.

Note that the two conditions in Corollary 1 always hold when $\tau = N$. Therefore, when given a system $\tilde{G} \in \mathbf{RL}^{nN\times mN}$ in F1 or F2 form, there always exists a linear causal N-periodic system $G \in \mathbf{L}^{n\times m}$ whose lifted form is $\tilde{G}$.

5 Well-Posedness of Periodic Controller

The well-posedness condition of the LPTV controller expressed by (3) is given in this section. Note that we can convert a parametrized controller $C = \mathrm{LFT}(\Pi, Q) \in \mathbf{L}^{n\times m}$ into

$$C = C_1(z) - C_2(z)Q(C_3(z) + C_4(z)Q)^{-1}, \tag{37}$$

where $C_1(z) \in \mathbf{RL}^{n\times m}$, $C_2(z) \in \mathbf{RL}^{n\times n}$, $C_4(z) \in \mathbf{RL}^{m\times n}$, and $C_3(z) \in \mathbf{RL}^{m\times m}$. Therefore, the well-posedness of C is equivalent to the existence of a causal inverse map of $C_3 + C_4Q$ in $\mathbf{L}^{m\times m}$. Consequently, the following theorem states the well-posedness condition of the parametrized controller C.

Theorem 2 *For causal N-periodic Q, $C_3 + C_4Q$ has a causal inverse in $\mathbf{L}^{m\times m}$ iff*

$$\lim_{z\to\infty} \det(I_m - D_{22}Q_i^{(0)}(z)) \neq 0 \tag{38}$$

is satisfied for $i = 0, \cdots, N-1$.

From this theorem, we know that only the constant part of Q is relevant to the well-posedness of the LPTV controller. Therefore, any LPTV controller parameterized as (3) can always be made well-posed by adding appropriate nonzero constant ϵ to the free parameter Q to avoid singular points of $C_3 + C_4Q$.

Proof. By lifting (37) from both sides we get

$$\tilde{C} = \tilde{C}_1 - \tilde{C}_2\tilde{Q}(\tilde{C}_3 + \tilde{C}_4\tilde{Q})^{-1}. \tag{39}$$

Because Q is causal N-periodic, we can represent $\tilde{Q}$ in F1 form:

$$\tilde{Q} = \left[\begin{array}{ccc} \tilde{q}^0 \text{cs } Q_0(z) & \cdots & \tilde{q}^{-N+1} \text{cs } Q_{N-1}(z) \end{array} \right]. \tag{40}$$

Therefore, $\tilde{C}_3 + \tilde{C}_4 \tilde{Q}$ is a rational matrix function of z in the lifted signal space and has an inverse iff its constant term is non-singular. Define D_∞ as

$$\begin{aligned} D_\infty &\triangleq \lim_{z\to\infty} (\tilde{C}_3 + \tilde{C}_4 \tilde{Q}) && (41)\\ &= I_N \otimes I_m - I_N \otimes D_{22} \cdot \lim_{z\to\infty} \tilde{Q}(z). && (42) \end{aligned}$$

Note that D_∞ is block lower triangular because C_3, C_4, Q are causal transfer functions. Hence, D_∞ is non-singular iff its diagonal blocks are non-singular. Moreover, D_∞^{-1} is also block lower triangular if it exists. Consequently, the condition:

$$\lim_{z\to\infty} \det(C_3^{(0)}(z) + C_4^{(0)}(z) Q_i^{(0)}(z)) \neq 0 \quad \forall\, i \in \mathbb{Z}[0, N-1] \tag{43}$$

is necessary and sufficient for the existence of a causal N-periodic inverse of $C_3 + C_4 Q$. □

6 Projection Approach for Problems P1 and P2

Now we are ready to state mathematical definitions of two periodic model-matching problems in Section 2.

P1. Find Q^* such that

$$Q^* = \arg \inf_{Q:\text{causal},\ WQW^{-1} \in \text{RH}_\infty} \| \mathcal{I}[T_1(z) - T_2(z) Q T_3(z)] \|, \tag{44}$$

P2. Find Q^* such that

$$Q^* = \arg \inf_{Q:\text{causal},\ WQW^{-1} \in \text{RH}_\infty} \| \mathcal{O}[T_1(z) - T_2(z) Q T_3(z)] \|, \tag{45}$$

where $\mathcal{I}[T]$ and $\mathcal{O}[T]$ are defined by

$$\begin{aligned} \mathcal{I}[T] &\triangleq T \cdot \text{Pr}^0, && (46)\\ \mathcal{O}[T] &\triangleq \text{Pr}^{N-1} \cdot T && (47) \end{aligned}$$

with $T = T_1(z) - T_2(z) Q T_3(z) \in \mathbf{L}^{n\times m}$, and U_i and D_i are defined in (15). The functionals $\mathcal{I}[T]$ and $\mathcal{O}[T]$ imply periodic sampling of the input signal of T and the output signal of T, respectively.

In this section, we discuss the problems P1 and P2 with causal LPTV free parameter Q. The key idea is to simplify the multiplication of two causal LPTV systems by using the dual lifted forms F1 and F2. Consequently, we can relax the problems P1 and P2 into LTI model matching forms. Firstly, we introduce two classes of causal and stable LPTV systems by using dual lifted forms. Assume that Q is causal N-periodic in the original signal space, then from the arguments in Section 4, $\tilde{Q} = WQW^{-1}$ always has the F1 form representation:

$$\tilde{Q}(z) = \left[\ \tilde{q}^0 \mathrm{cs}\ Q_0^{\mathrm{c}}(z) \quad \tilde{q}^{-1} \mathrm{cs}\ Q_1^{\mathrm{c}}(z) \quad \cdots\ \tilde{q}^{-N+1} \mathrm{cs}\ Q_{N-1}^{\mathrm{c}}(z) \right], \tag{48}$$

or the F2 form representation:

$$\tilde{Q}(z) = \begin{bmatrix} \mathrm{rs}\ Q_{N-1}^{\mathrm{r}}(z) \cdot \tilde{q}^{-N+1} \\ \vdots \\ \mathrm{rs}\ Q_0^{\mathrm{r}}(z) \cdot \tilde{q}^0 \end{bmatrix}. \tag{49}$$

Observe that $\tilde{Q}(z)$ is stable whenever $Q_i^{\mathrm{c}}(z)$ or $Q_i^{\mathrm{r}}(z)$ is an element in $\mathbf{RH}_\infty$ for $i = 0, \cdots, N-1$. Let define two classes of causal and stable LPTV Q parameters as

$$\mathcal{Q}^{\mathrm{c}} \triangleq \{Q \mid WQW^{-1} \text{ is given by (48) where } \ Q_i^{\mathrm{c}}(z) \in \mathbf{RH}_\infty \ \text{ for } i = 0, \cdots, N-1\}, \tag{50}$$

$$\mathcal{Q}^{\mathrm{r}} \triangleq \{Q \mid WQW^{-1} \text{ is given by (49) where } \ Q_i^{\mathrm{r}}(z) \in \mathbf{RH}_\infty \ \text{ for } i = 0, \cdots, N-1\}. \tag{51}$$

The two classes $\mathcal{Q}^{\mathrm{c}}$ and $\mathcal{Q}^{\mathrm{r}}$ are equivalent although their representation are different. We introduce two operators on $\mathbf{RL}$ next. Let ccs $G(z)$ denote

$$\begin{aligned} \mathrm{ccs}\ G(z) &\triangleq \left[\ G^{(0)}(z^N) \quad z^{-1}G^{(1)}(z^N) \quad \cdots \quad z^{-N+1}G^{(N-1)}(z^N)\ \right] \\ &= \left[\ D \quad 0 \quad \cdots \quad 0\ \right] \\ &\quad + C(z^N I - A^N)^{-1} \left[\ A^{N-1}B \quad z^{N-1}B \quad z^{N-2}AB \quad \cdots \quad zA^{N-2}B\ \right], \end{aligned} \tag{52}$$

and rrs $G(z)$ denote

$$\mathrm{rrs}\ G(z) \triangleq \begin{bmatrix} G^{(0)}(z^N) \\ \vdots \\ z^{-N+1}G^{(N-1)}(z^N) \end{bmatrix} = \begin{bmatrix} D \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} CA^{N-1} \\ z^{N-1}C \\ \vdots \\ zCA^{N-2} \end{bmatrix} (z^N I - A^N)^{-1}B. \tag{53}$$

where $G^{(i)}(z^N)$ is given by substituting z^N into $G^{(i)}(z)$ defined by (14).

Then, we are ready to state the main result in this paper. The following theorem shows that the problems P1 and P2 can be expressed by corresponding LTI transfer functions.

Theorem 3 *1. Let $Q \in \mathcal{Q}^{\mathrm{c}}$, then*

$$\mathcal{I}[T] = T^{\mathrm{I}}(z)\mathrm{Pr}^0, \tag{54}$$

where $T^{\mathrm{I}}(z)$ is defined as

$$T^{\mathrm{I}}(z) \triangleq T_1(z) - T_2(z)Q^{\mathrm{I}}(z) \cdot \mathrm{rrs}\ T_3(z) \tag{55}$$

with

$$Q^{\mathrm{I}}(z) = \left[\ Q_0^{\mathrm{c}}(z) \quad \cdots \quad Q_{N-1}^{\mathrm{c}}(z)\ \right]. \tag{56}$$

2. Let $Q \in \mathcal{Q}^{\mathrm{r}}$, then

$$\mathcal{O}[T] = \mathrm{Pr}^{N-1} T^{\mathrm{O}}(z), \tag{57}$$

where $T^{\mathrm{O}}(z)$ is defined as

$$T^{\mathrm{O}}(z) \triangleq T_1(z) - \mathrm{ccs}\ T_2(z) \cdot Q^{\mathrm{O}}(z) T_3(z) \tag{58}$$

with

$$Q^{\mathrm{O}}(z) \triangleq \begin{bmatrix} Q_0^{\mathrm{r}}(z) \\ \vdots \\ Q_{N-1}^{\mathrm{r}}(z) \end{bmatrix}. \tag{59}$$

Note that $T^{\mathrm{I}}(z)$ and $T^{\mathrm{O}}(z)$ contain all free parameters of $\mathcal{Q}^{\mathrm{c}}$ and $\mathcal{Q}^{\mathrm{r}}$. Hence, the statements of the theorem is not conservative at all. The proof of Theorem 3 is based on dual lifted forms F1 and F2. We utilize the fact that the multiplication of an LPTV system in F1 form and an LPTV system in F2 form results in very simple form.

Proof. We start with the proof of the first part of the theorem.

$$\begin{aligned} \mathcal{I}[T] &= T \mathrm{Pr}^0 \\ &= T_1 \mathrm{Pr}^0 - T_2 Q T_3 U_0 D_0 \\ &= T_1 \mathrm{Pr}^0 - T_2 W^{-1} \tilde{Q} \tilde{T}_3 E_1 D_0. \end{aligned} \tag{60}$$

Because $T_3(z)$ is LTI, $\tilde{T}_3(z)$ can be represented in F2 form as

$$\tilde{T}_3(z) = \begin{bmatrix} \mathrm{rs}\ T_3(z) \cdot \tilde{q}^{-N+1} \\ \vdots \\ \mathrm{rs}\ T_3(z) \cdot \tilde{q}^{0} \end{bmatrix}, \tag{61}$$

therefore

$$\begin{aligned} W^{-1} \tilde{Q} \tilde{T}_3 E_1 &= W^{-1} \left(\sum_{i=0}^{N-1} \tilde{q}^{-i} \mathrm{cs}\ Q_i^{\mathrm{c}} \cdot \mathrm{rs}\ T_3 \cdot \tilde{q}^{i-N+1} \right) E_1 \\ &= W^{-1} \sum_{i=0}^{N-1} \tilde{Q}_i^{\mathrm{c}} E_{i+1} E_{i+1}^T \tilde{T}_3 E_1 \\ &= W^{-1} \sum_{i=0}^{N-1} \tilde{Q}_i^{\mathrm{c}} E_{i+1} T_3^{(i)} \\ &= W^{-1} \sum_{i=0}^{N-1} \tilde{Q}_i^{\mathrm{c}}\ \mathrm{cs}\ V_i^{\mathrm{c}} \\ &= \left(\sum_{i=0}^{N-1} Q_i^{\mathrm{c}} V_i^{\mathrm{c}} \right) W^{-1} E_1 \\ &= \left(\sum_{i=0}^{N-1} Q_i^{\mathrm{c}} V_i^{\mathrm{c}} \right) U_0 \end{aligned} \tag{62}$$

where $V_i^{\rm c}(z) \stackrel{\triangle}{=} z^{-i}T_3^{(i)}(z^N)$ $(i = 0, \cdots, N-1)$. Therefore,

$$\mathcal{I}[T] \;=\; \left\{ T_1 - T_2 \sum_{i=0}^{N-1} Q_i^{\rm c} V_i^{\rm c} \right\} {\rm Pr}^0. \tag{63}$$

The proof of the second part is similar to that of the first part.

$$\begin{aligned} \mathcal{O}[T] &= {\rm Pr}^{N-1} T \\ &= {\rm Pr}^{N-1} T_1 - U_{N-1} D_{N-1} T_2 Q T_3 \\ &= {\rm Pr}^{N-1} T_1 - U_{N-1} E_N^T \tilde{T}_2 \tilde{Q} W T_3. \end{aligned} \tag{64}$$

Because $T_2(z)$ is LTI, $\tilde{T}_2(z)$ can be represented in F1 as

$$\tilde{T}_2(z) = \left[\begin{array}{ccc} \tilde{q}^0 {\rm cs}\ T_2(z) & \cdots & \tilde{q}^{-N+1} {\rm cs}\ T_2(z) \end{array} \right], \tag{65}$$

therefore

$$\begin{aligned} E_N^T \tilde{T}_2 \tilde{Q} W &= E_N^T \left(\sum_{i=0}^{N-1} \tilde{q}^{-i} {\rm cs}\ T_2 \cdot {\rm rs}\ Q_{N-1-i}^{\rm r} \cdot \tilde{q}^{i-N+1} \right) W \\ &= \sum_{i=0}^{N-1} E_N^T \tilde{T}_2 E_{i+1} E_{i+1}^T \tilde{Q}_{N-1-i}^{\rm r} W \\ &= \sum_{i=0}^{N-1} T_2^{(N-1-i)} E_{i+1}^T \tilde{Q}_{N-1-i}^{\rm r} W \\ &= \sum_{i=0}^{N-1} {\rm rs}\ V_{N-1-i}^{\rm r} \cdot \tilde{Q}_{N-1-i}^{\rm r} W \\ &= E_N W \left(\sum_{i=0}^{N-1} V_{N-1-i}^{\rm r} Q_{N-1-i}^{\rm r} \right) \\ &= D_{N-1} \left(\sum_{i=0}^{N-1} V_{N-1-i}^{\rm r} Q_{N-1-i}^{\rm r} \right) \end{aligned} \tag{66}$$

where $V_i^{\rm r}(z) \stackrel{\triangle}{=} z^{-i}T_2^{(i)}(z^N)$ $(i = 0, \cdots, N-1)$. Therefore,

$$\mathcal{O}[T] \;=\; {\rm Pr}^{N-1} \left\{ T_1 - \sum_{i=0}^{N-1} V_i^{\rm r} Q_i^{\rm r} T_3 \right\}. \tag{67}$$

□

Upper bounds of the achievable induced norm performance by LPTV controllers are given by the following corollary in combined with Theorem 3.

Corollary 2 *The following two statements always hold.*

1. When $Q \in \mathcal{Q}_i^{\mathrm{c}}$,

$$\sup_{w \in l_2,\ w \neq 0} \frac{\|\mathcal{I}[T]w\|}{\|w\|} = \sup_{w \in \mathcal{W}_0,\ w \neq 0} \frac{\left\|T^{\mathrm{I}}(z)w\right\|}{\|w\|} \leq \sup_{w \in l_2,\ w \neq 0} \frac{\left\|T^{\mathrm{I}}(z)w\right\|}{\|w\|}. \tag{68}$$

2. When $Q \in \mathcal{Q}_i^{\mathrm{r}}$,

$$\sup_{w \in l_2,\ w \neq 0} \frac{\|\mathcal{O}[T]w\|}{\|w\|} \leq \sup_{w \in l_2,\ w \neq 0} \frac{\left\|T^{\mathrm{O}}(z)w\right\|}{\|w\|}. \tag{69}$$

Proof. $\mathrm{Pr}^j w \in \mathcal{W}_j$ is satisfied for any $w \in l_2$, because Pr^j is a projection operator onto $\mathcal{W}_j$ and $\left\|\mathrm{Pr}^j w\right\| \leq \|w\|$. Therefore, the first equality of (68) is satisfied. The inequalities of (68) and (69) are trivial from Theorem 3. □

From Theorem 3 and Corollary 2, we can formulate LTI model-matching problems corresponding to P1 and P2 as P1' and P2':

P1'. Find $Q^{\mathrm{I}*} \in \mathbf{RH}_\infty$ such that

$$Q^{\mathrm{I}*} = \arg \inf_{Q^{\mathrm{I}} \in \mathbf{RH}_\infty} \left\|T^{\mathrm{I}}(z)\right\|. \tag{70}$$

P2'. Find $Q^{\mathrm{O}*} \in \mathbf{RH}_\infty$ such that

$$Q^{\mathrm{O}*} = \arg \inf_{Q^{\mathrm{O}} \in \mathbf{RH}_\infty} \left\|T^{\mathrm{O}}(z)\right\|. \tag{71}$$

These are standard LTI model-matching problems and we can solve them by following the results of Doyle et al. [1989] and Dahleh and Pearson [1987] for the case of $\mathcal{H}^\infty$ norm and l_1 norm, respectively.

Remark 1: Because $\left\|T^{\mathrm{I}}(z)\right\| = \left\|\sum_{j=0}^{N-1} T^{\mathrm{I}}(z)\mathrm{Pr}^j\right\|$ and $\left\|T^{\mathrm{O}}(z)\right\| = \left\|\sum_{j=0}^{N-1} \mathrm{Pr}^j T^{\mathrm{O}}(z)\right\|$ are satisfied from Property 1, minimizing $\left\|T^{\mathrm{I}}(z)\right\|$ instead of $\left\|T^{\mathrm{I}}(z)\mathrm{Pr}^0\right\|$ and minimizing $\left\|T^{\mathrm{O}}(z)\right\|$ instead of $\left\|\mathrm{Pr}^{N-1}T^{\mathrm{O}}(z)\right\|$ are conservative procedure. However, the following proposition guarantees that upper and lower bounds for P1 and P2 are given by approximate problems P1'and P2'.

Proposition 1 *For any LTI transfer matrix $T(z) \in$ **RL**,*

$$\|T(z)\| \geq \left\|T(z)\mathrm{Pr}^0\right\| = \left\|T(z)\mathrm{Pr}^j\right\| \geq \frac{1}{N}\|T(z)\|, \tag{72}$$

$$\|T(z)\| \geq \left\|\mathrm{Pr}^{N-1}T(z)\right\| = \left\|\mathrm{Pr}^j T(z)\right\| \geq \frac{1}{N}\|T(z)\| \tag{73}$$

hold for $j = 0, \cdots, N-1$.

Proof. We start with (72). Representing $\tilde{T}(z)$ in F1 form,

$$\begin{aligned} T(z)\mathrm{Pr}^j &= W^{-1}WTW^{-1}E_{j+1}E_{j+1}^TW \\ &= W^{-1}\tilde{q}^j \mathrm{cs}\, T \cdot D_j \\ &= q^j(T(z)\mathrm{Pr}^0)U_0D_j \end{aligned} \quad (74)$$

hold for $j = 0, \cdots, N-1$, where the third equality is satisfied from (27) and the definition of Pr^j. Because q and U_j are norm preserving operators, $\left\|T(z)\mathrm{Pr}^j\right\| = \left\|T(z)\mathrm{Pr}^0U_0D_j\right\|$ and $\|U_jD_0w\| = \|D_0w\| = \left\|\mathrm{Pr}^0w\right\|$ for $w \in l_2$. In addition, from $U_0D_j \cdot U_jD_0 = \mathrm{Pr}^0$ and $\mathrm{Pr}^0 \cdot \mathrm{Pr}^0 = \mathrm{Pr}^0$,

$$\sup_{w\in l_2,\ w\neq 0} \frac{\left\|T(z)\mathrm{Pr}^0U_0D_jw\right\|}{\|w\|} = \sup_{w\in l_2,\ w\neq 0} \frac{\left\|T(z)\mathrm{Pr}^0U_0D_jU_jD_0w\right\|}{\|U_jD_0w\|} \quad (75)$$

$$= \sup_{w\in l_2,\ w\neq 0} \frac{\left\|T(z)\mathrm{Pr}^0w\right\|}{\left\|\mathrm{Pr}^0w\right\|}. \quad (76)$$

From this, $\left\|T(z)\mathrm{Pr}^j\right\| = \left\|T(z)\mathrm{Pr}^0\right\|$ for $j = 0, \cdots, N-1$.

We proceed to the proof of (73). $\left\|\mathrm{Pr}^jT\right\| = \|D_jT(z)\|$ because of the norm preserving property of U_j. In addition,

$$\begin{aligned} D_jT(z) &= E_{j+1}^TWTW^{-1}W \\ &= \mathrm{rs}\, T \cdot \tilde{q}^{j-N+1}W \\ &= \mathrm{rs}\, T \cdot Wq^{j-N+1}, \end{aligned} \quad (77)$$

because $\tilde{T}(z)$ can be represented in F2 form. From this and (28),

$$D_jT(z) = D_{N-1}T(z)q^{j-N+1}. \quad (78)$$

Therefore, $\left\|\mathrm{Pr}^jT(z)\right\| = \left\|\mathrm{Pr}^{N-1}T(z)\right\|$ for $j = 0, \cdots, N-1$.

The first and second inequalities of (72) and (73) readily follow from Corollary 2 and the triangular inequality. This completes the proof. □

Remark 2: For the case $Q_0^{\mathrm{c}} = \cdots = Q_{N-1}^{\mathrm{c}} = Q(z)$ and $Q_0^{\mathrm{r}} = \cdots = Q_{N-1}^{\mathrm{r}} = Q(z)$, T^{O} and T^{I} are

$$T^{\mathrm{O}}(z) = T^{\mathrm{I}}(z) = T_1(z) - T_2(z)Q(z)T_3(z), \quad (79)$$

and this means P1' and P2' in this case are equivalent to the original model-matching problem P0 in which $Q \in \mathrm{RH}_\infty$. Therefore, LTI controllers for problems P1' and P2' are no different than those for P0, although LPTV controllers for P1' and P2' can be different from those for P0 because $Q_0^{\mathrm{c}}(z), \cdots, Q_{N-1}^{\mathrm{c}}(z)$ and $Q_0^{\mathrm{r}}(z), \cdots, Q_{N-1}^{\mathrm{r}}(z)$ can be different from each other.

7 Block-Number Reduction by Extended Freedom

Problems P1' and P2' are dependent on block matrices rrs $T_3(z)$ and ccs $T_2(z)$. And we can simplify the problems by using the special structure of the operators rrs and ccs. In this section, we show that P1' and P2' are equivalent to simplified LTI model matching problems where $T_3(z)$ or $T_2(z)$ disappears under several conditions.

We introduce two functions $M_N^{\rm c}(A,B,\xi)$ and $M_N^{\rm o}(A,C,\xi)$ as

$$M_N^{\rm c}(A,B,\xi) \triangleq \left(A^{N-1}+\xi_1 I+\xi_2 A+\cdots+\xi_{N-1}A^{N-2}\right)B, \tag{80}$$

$$M_N^{\rm o}(A,C,\xi) \triangleq C\left(A^{N-1}+\xi_1 I+\xi_2 A+\cdots+\xi_{N-1}A^{N-2}\right), \tag{81}$$

$$\xi \triangleq \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_{N-1} \end{pmatrix}^T \in \mathbb{R}^{N-1} \tag{82}$$

Then, an existence condition of stable inverses of rrs $G(z)$ and ccs $G(z)$ can be derived.

Lemma 3 *Let an LTI system $G(z)=(A,B,C,D)$. If there exists $\xi\in\mathbb{R}^{N-1}$ such that $M_N^{\rm c}(A,B,\xi)=0$ or $M_N^{\rm o}(A,C,\xi)=0$, then the two following statements hold.*

1. *If there exists a constant matrix $D^\#$ such that $DD^\#=I$, then there exist* $(\text{ccs } G)^\#(z)\in \mathbf{RH}_\infty$ *such that* $\text{ccs } G(z)\cdot(\text{ccs } G)^\#(z)=I$

2. *If there exists a constant matrix $D^\#$ such that $D^\# D=I$, then there exist* $(\text{rrs } G)^\#(z)\in \mathbf{RH}_\infty$ *such that* $(\text{rrs } G)^\#(z)\cdot\text{rrs } G(z)=I$

Proof. By selecting $(\text{ccs } G)^\#(z)$ and $(\text{rrs } G)^\#(z)$ as

$$(\text{ccs } G)^\#(z) \triangleq \begin{bmatrix} D^\# \\ z^{-N+1}\xi_1 D^\# \\ \vdots \\ z^{-1}\xi_{N-1}D^\# \end{bmatrix}, \tag{83}$$

$$(\text{rrs } G)^\#(z) \triangleq \begin{bmatrix} D^\# & z^{-N+1}\xi_1 D^\# & \cdots & z^{-1}\xi_{N-1}D^\# \end{bmatrix}, \tag{84}$$

it is obvious that $(\text{ccs } G)^\#(z), (\text{rrs } G)^\#(z)\in\mathbf{RH}_\infty$ and the following equations complete the proof.

$$\begin{aligned} \text{ccs } G(z)\cdot(\text{ccs } G)^\#(z) &= DD^\# + C(z^N I - A^N)^{-1}M_N^{\rm c}(A,B,\xi)D^\# \\ &= DD^\# + M_N^{\rm o}(A,C,\xi)(z^N I - A^N)^{-1}BD^\#, \end{aligned} \tag{85}$$

$$\begin{aligned} (\text{rrs } G)^\#(z)\cdot\text{rrs } G(z) &= D^\# D + D^\# M_N^{\rm o}(A,C,\xi)(z^N I - A^N)^{-1}B \\ &= D^\# D + D^\# C(z^N - A^N)^{-1}M_N^{\rm c}(A,B,\xi). \end{aligned} \tag{86}$$

□

The conditions of Lemma 3 except for the existence of $D^\#$ are always satisfied in two cases: (1) when N is larger than the size of A plus one; (2) when N is equal to the size of A and there exists uncontrollable mode of the pair (A,B) or unobservable mode of the pair (A,C). In the first case, we can achieve both $M_N^{\rm c}(A,B,\xi)=0$ and $M_N^{\rm o}(A,C,\xi)=0$ by selecting $\xi_1,\cdots,\xi_{N-1}$ as coefficients of the characteristic polynomial of A from Cayley-Hamilton theorem (See [Zhou et al., 1996]). In the second case, there exists ξ such that $M_N^{\rm c}(A,B,\xi)=0$ or $M_N^{\rm c}(A,B,\xi)=0$ because the controllability matrix or the observability matrix is not rank full. Conversely, it is sufficient to take $N=\text{size}(A)+1$ in order to construct stable inverse of the operator rrs or ccs.

By using this lemma, the problems P1' and P2' can be considerably simplified.

Theorem 4 *1. If there exists $\xi \in \mathbb{R}^{N-1}$ such that $M_N^{\mathrm{c}}(A+HC_2, B_1+HD_{21}, \xi) = 0$ or $M_N^{\mathrm{o}}(A+HC_2, C_2, \xi) = 0$ is satisfied, in addition, there exists a constant matrix $D_{21}^{\#}$ such that $D_{21}^{\#} D_{21} = I$, then*

$$\inf_{Q^{\mathrm{I}}(z)\in\mathbf{RH}_\infty} \left\| T^{\mathrm{I}}(z) \right\| = \inf_{Q^{\mathrm{I}'}(z)\in\mathbf{RH}_\infty} \left\| T_1(z) - T_2(z) Q^{\mathrm{I}'}(z) \right\| \tag{87}$$

2. If there exists $\xi \in \mathbb{R}^{N-1}$ such that $M_N^{\mathrm{c}}(A-B_2K, B_2, \xi) = 0$ or $M_N^{\mathrm{o}}(A-B_2K, C_1 - D_{12}K, \xi) = 0$ is satisfied, in addition, there exists a constant matrix $D_{12}^{\#}$ such that $D_{12} D_{12}^{\#} = I$, then

$$\inf_{Q^{\mathrm{O}}(z)\in\mathbf{RH}_\infty} \left\| T^{\mathrm{O}}(z) \right\| = \inf_{Q^{\mathrm{O}'}(z)\in\mathbf{RH}_\infty} \left\| T_1(z) - Q^{\mathrm{O}'}(z) T_3(z) \right\| \tag{88}$$

Proof. Because $Q^{\mathrm{I}}(z) \cdot \mathrm{rrs}\ T_3(z),\ \mathrm{ccs}\ T_2(z) \cdot Q^{\mathrm{O}}(z) \in \mathbf{RH}_\infty$ is satisfied,

$$\inf_{Q^{\mathrm{I}}(z)\in\mathbf{RH}_\infty} \left\| T^{\mathrm{I}}(z) \right\| \geq \inf_{Q^{\mathrm{I}'}(z)\in\mathbf{RH}_\infty} \left\| T_1(z) - T_2(z) Q^{\mathrm{I}'}(z) \right\|,$$
$$\inf_{Q^{\mathrm{O}}(z)\in\mathbf{RH}_\infty} \left\| T^{\mathrm{O}}(z) \right\| \geq \inf_{Q^{\mathrm{O}'}(z)\in\mathbf{RH}_\infty} \left\| T_1(z) - Q^{\mathrm{O}'}(z) T_3(z) \right\|$$

hold. On the other hand, the lower bonds can be achieved by substituting $Q^{\mathrm{I}}(z) = Q^{\mathrm{I}'}(z)(\mathrm{rrs}\ T_3)^{\#}(z)$ into $T^{\mathrm{I}}(z)$ and substituting $Q^{\mathrm{O}}(z) = (\mathrm{ccs}\ T_2)^{\#}(z) Q^{\mathrm{O}'}(z)$ into $T^{\mathrm{O}}(z)$ from Lemma 3. □

The fact that ccs $T_2(z)$ or rrs $T_3(z)$ can be removed from P1' and P2' indicates that unstable zeros of $T_2(z)$ or $T_3(z)$ have no effect on resulting control performances. This is the main reason why LPTV controllers are superior to LTI controllers with regard to our model matching problems.

In general, the conditions of Theorem 4 except for the existence of $D_{12}^{\#}$ and $D_{21}^{\#}$ are more easily satisfied as N increases. However, when N is a fixed number less than the size of A matrix and $V_i^{\mathrm{r}}(z) = T_2(z)$ or $V_i^{\mathrm{c}}(z) = T_3(z)$ is satisfied for some i where $V_i^{\mathrm{r}}(z)$ and $V_i^{\mathrm{c}}(z)$ are defined in the proof of Theorem 3, minimizing $\left\| T^{\mathrm{I}}(z) \right\|$ subject to $Q_i^{\mathrm{c}}(z) \in \mathbf{RH}_\infty$ or minimizing $\left\| T^{\mathrm{O}}(z) \right\|$ subject to $Q_i^{\mathrm{r}}(z) \in \mathbf{RH}_\infty$ is equivalent to minimizing $\|T\|$ subject to $Q(z) \in \mathbf{RH}_\infty$. For example, when $T_2(z) = \frac{z}{z^2-2}$ and $N = 2$, the equation $V_1^{\mathrm{r}}(z) = T_2(z)$ is satisfied. Therefore, when the number N is fixed, our proposed LPTV controller is effective in the case that $V_i^{\mathrm{r}}(z) \neq T_2(z)$ or $V_i^{\mathrm{c}} \neq T_3(z)$ is satisfied for $i = 0, \cdots, N-1$.

8 Design Example

In the next section, we show an illustrative example in which the controller designed by the method based on P2' achieves arbitrary small induced norm performance. In the example, we consider 2-periodic LPTV controller for the P2 problem with regard to l_2 induced norm. We also evaluate the resulting control performance compared with LTI controllers.

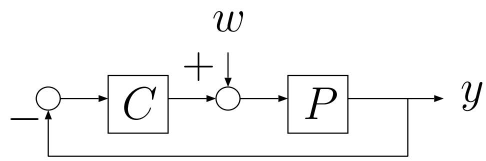

Fig. 4: A closed-loop system of the example problem

We consider the following plant $P(z)$:

$$P(z)=\frac{z-2}{z-3}. \tag{89}$$

Because $P(z)$ has an unstable pole at 3 and an unstable zero at 2, trivial controllers $C=0$ and $C=\infty$ do not stabilize the closed-loop. Selecting a feedback gain K and an observer gain H as $K=H=-2.5$ and executing the Youla parameterization gives the controller parameterized by Q:

$$C=C_1(z)-C_2(z)Q(C_3(z)+C_4(z)Q)^{-1}, \tag{90}$$

where

$$\begin{bmatrix} C_1(z) & C_2(z) \\ C_3(z) & C_4(z) \end{bmatrix}=\begin{bmatrix} \frac{6.25}{z-4.25} & \frac{z-0.5}{z-4.25} \\ \frac{z-4.25}{z-0.5} & \frac{z-2}{z-0.5} \end{bmatrix}. \tag{91}$$

We select Q from $\mathcal{Q}_1^{\mathrm{r}}$ which is a class of causal 2-periodic stable functions, then $\tilde{Q}=WQW^{-1}$ has the representation, in F2 form:

$$\tilde{Q} \triangleq \begin{bmatrix} \mathrm{rs}\ Q_1^{\mathrm{r}}\cdot\tilde{q}^{-1} \\ \mathrm{rs}\ Q_0^{\mathrm{r}}\cdot\tilde{q}^{0} \end{bmatrix}=\begin{bmatrix} Q_1^{\mathrm{r}(0)} & z^{-1}Q_1^{\mathrm{r}(1)} \\ Q_0^{\mathrm{r}(1)} & Q_0^{\mathrm{r}(0)} \end{bmatrix}. \tag{92}$$

Let T denote the closed-loop transfer function from w to y

$$T=T_1(z)+4T_2(z)QT_3(z), \tag{93}$$

where

$$T_1(z) = -2\frac{z-2}{1-2z}\left(1-\frac{3.75}{z-0.5}\right) \tag{94}$$

$$T_2(z) = T_3(z)=\frac{z-2}{1-2z}. \tag{95}$$

Then, there exists $T^{\mathrm{O}}(z)$ which satisfies $\mathcal{O}[T]=\mathrm{Pr}^1T^{\mathrm{O}}(z)$ from Theorem 3:

$$\begin{aligned} T^{\mathrm{O}}(z) &= T_1(z)+4\begin{bmatrix} T_2^{(0)}(z^2) & z^{-1}T_2^{(1)}(z^2) \end{bmatrix}\begin{bmatrix} Q_0^{\mathrm{r}} \\ Q_1^{\mathrm{r}} \end{bmatrix}T_3(z) \\ &= T_1(z)+4\ \mathrm{ccs}\ T_2(z)\cdot Q^{\mathrm{O}}(z)T_3(z). \end{aligned} \tag{96}$$

By following Theorem 4, we find $Q^{\mathrm{O}}(z)$ in the form:

$$Q^{\mathrm{O}}(z)=0.25\ (\mathrm{ccs}\ T_2(z))^{\#}(z)Q^{\mathrm{O}'}(z)=0.25\begin{bmatrix} -2 \\ z^{-1} \end{bmatrix}Q^{\mathrm{O}'}(z), \tag{97}$$

where $Q^{\mathrm{O}'}(z)$ is a new parameter in $\mathbf{RH}_\infty$. Substituting (97) into (96), we get

$$T^{\mathrm{O}}(z) = \frac{z-2}{1-2z}\left\{Q^{\mathrm{O}'}(z) - \left(2 - \frac{7.5}{z-0.5}\right)\right\}. \tag{98}$$

Therefore, we can design a suboptimal $Q^{\mathrm{O}'}(z)$ as

$$Q^{\mathrm{O}'}(z) \triangleq 2 + \epsilon - \frac{7.5}{z-0.5}, \tag{99}$$

where $\epsilon > 0$ is a constant that guarantees the well-posedness of the LPTV controller (see Section 5) and we set $\epsilon = 0.01$ as an example. Note that $\left\|T^{\mathrm{O}}(z)\right\|_\infty = \epsilon$ because $\frac{z-2}{1-2z}$ is an inner function. Substituting (99) into (97) yields $Q^{\mathrm{O}}(z)$, and using (92) we obtain $\tilde{Q}$:

$$\tilde{Q} = \left[\begin{array}{cc|cc} 0.25 & 0 & -1.875 & -3.75 \\ 0 & 0 & 0 & 4.253 \\ \hline 1 & 1 & 0 & 0 \\ -0.5 & 0 & 3.75 & -1.005 \end{array}\right]. \tag{100}$$

Substituting $\tilde{Q}$ into $\tilde{C}$ yields

$$\tilde{C} = \left[\begin{array}{cc|cc} 3.943 & -4.075 & -3.282 & -983.3 \\ -5.498 & 5.682 & 4.577 & -1312 \\ \hline -1.806 & 1.846 & 0 & 0 \\ -3.003 & 3.104 & 2.5 & -201 \end{array}\right]. \tag{101}$$

Then $\tilde{T} = (I + \tilde{P}\tilde{C})^{-1}\tilde{P}$ is given by

$$\tilde{T} = D + C(zI - A)^{-1}B \in \mathbf{RH}^{2\times 2}, \tag{102}$$

where

$$A = \begin{bmatrix} -1.832 & 6.345 & 12.4 \\ -0.762 & 2.084 & 4.344 \\ 1.632\times 10^{-2} & 0.1877 & 0.2477 \end{bmatrix}, \tag{103}$$

$$B = \begin{bmatrix} 0.4995 & 3.448 \\ -10.92 & 6.516 \\ 4.886 & -3.589 \end{bmatrix}, \tag{104}$$

$$C = \begin{bmatrix} -1.722 & 1.456 & 1.892 \\ 0.000632 & -0.001806 & -0.003717 \end{bmatrix}, \tag{105}$$

$$D = \begin{bmatrix} 1 & 0 \\ 7.5\times 10^{-3} & -5\times 10^{-3} \end{bmatrix}. \tag{106}$$

Note that the second rows of C and D are extremely small compared with the first rows. Actually, we readily calculate $\mathcal{H}^\infty$ norm $\left\|\mathrm{Pr}^0 T\right\|_\infty = 28.0$ and $\left\|\mathrm{Pr}^1 T\right\|_\infty = 0.01$. On the other hand, the optimal l_2 induced norm by LTI controllers is $\|T\|_\infty = 3.00$. Therefore, any LTI controllers cannot achieve $\left\|\mathrm{Pr}^1 T\right\|_\infty$ less than 1.50 from Proposition 1. This suggests that the resulting LPTV controller is superior to LTI controllers for our design objective P2 and contrasts with the fact (10). And we can see that the controller drastically refine $\left\|\mathrm{Pr}^1 T\right\|_\infty$ by ignoring $\left\|\mathrm{Pr}^0 T\right\|_\infty$.

9 Conclusions

We introduced two forms of periodic model-matching problem, P1 and P2, in which input signals or output signals of a closed-loop transfer function are projected onto the signal space $\mathcal{W}_0$ or $\mathcal{W}_{N-1}$ where any time sequence can take non-zero value at specified phase in each period. These problems are based on Youla parameterization of causal LPTV controllers in which only Q parameter is N-periodic.

To formulate causal LPTV discrete-time systems, we introduced the dual lifted forms F1 and F2. These two forms correspond to N LTI transfer matrices that have the same size as the original causal LPTV transfer matrix. Such N LTI transfer matrices are all identical when the original system is LTI.

We discussed the well-posedness of a causal LPTV controller whose Q parameter is formulated via dual lifted forms. The well-posedness is dependent only on the constant part of the Q parameter and we can always guarantee it by adding appropriate nonzero constant to the Q parameter.

Our main result states that the problems P1 and P2 with a class of LPTV controllers can be solved approximately by LTI model-matching problems P1' and P2' in which new stable LTI free parameters $Q^{\mathrm{I}}(z)$ and $Q^{\mathrm{O}}(z)$ are to be designed. These relaxed problems P1' and P2' give upper and lower bounds for the original problems P1 and P2. We showed a numerical design example such that the resulting LPTV controller achieves a suboptimal P2 performance that no LTI controller can attain. There was a tradeoff in the resulting performances and the LPTV controller consequently utilized the tradeoff to satisfy P2 control objective consequently.

The optimal performances achieved by our LPTV controllers are dependent on T_2 and T_3 and we gave a criterion for block number reduction which can be easily satisfied for sufficient large N. In particular, we can completely remove the effect by unstable zeros of $T_2(z)$ and $T_3(z)$ under some conditions and this enables us to design much effective causal LPTV controllers. However, the conditions are conservative and we will try to overcome this conservativeness. In addition, it is necessary to clarify the relationship between our periodic performance criteria and previously revealed advantages, such as arbitrary zero placement [Francis and Georgiou, 1988] or gain margin achievement [Khargonekar et al., 1985] etc.

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A Proof of Corollary 1

Proof. We prove this in the F1 form. Multiplying W and W^{-1} to $q^{\tau}Gq^{-\tau}$ from both sides

$$\begin{aligned}&Wq^{\tau}Gq^{-\tau}W^{-1} = \tilde{q}^{\tau}\tilde{G}(z)\tilde{q}^{-\tau}\\ &\quad = \begin{bmatrix} \tilde{q}^{0}\mathrm{cs}\ G_{\tau} & \cdots & \tilde{q}^{\tau-N+1}\mathrm{cs}\ G_{N-1}, & z^{-1}\tilde{q}^{\tau}\mathrm{cs}\ G_{0} & \cdots & z^{-1}\tilde{q}^{1}\mathrm{cs}\ G_{\tau-1} \end{bmatrix}\end{aligned}$$

$$= \left[\ \tilde{q}^0\text{cs } G_\tau \quad \cdots \quad \tilde{q}^{\tau-N+1}\text{cs } G_{N-1}, \quad \tilde{q}^{\tau-N}\text{cs } G_0 \quad \cdots \quad \tilde{q}^{-N+1}\text{cs } G_{\tau-1}\ \right] \tag{107}$$

Therefore,

$$\begin{aligned} \tilde{G} - \tilde{q}^\tau \tilde{G}(z)\tilde{q}^{-\tau} = \big[&\tilde{q}^0\text{cs } (G_0 - G_\tau) \ \cdots \ \tilde{q}^{\tau-N+1}\text{cs } (G_{N-\tau-1} - G_{N-1}), \\ &\tilde{q}^{\tau-N}\text{cs } (G_{N-\tau} - G_0) \ \ \cdots \ \ \tilde{q}^{-N+1}\text{cs } (G_{N-1} - G_{\tau-1})\big] . \end{aligned} \tag{108}$$

Because $\tilde{q}^\tau$ is an invertible operator, $\tilde{G} - \tilde{q}^\tau \tilde{G}\tilde{q}^{-\tau} = 0$ iff both conditions of Corollary 1 are satisfied. In particular, when $\tau = 1$, these conditions are

1. $G_{i+1}(z) = G_i(z) \quad \forall\ i \in \mathbb{Z}[0, N-2]$,
2. $G_{N-1}(z) = G_0(z)$.

This means $G_i(z) = G_j(z) \ \ \forall\ i, j$.

The case of F2 form is similar to that of F1 form, so we omit it here. □

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