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Shigeru Kubota and Kazuyuki Aihara

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

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Institute of Industrial Science, University of Tokyo,
4-6-1 Komaba, Meguro-ku, Tokyo, 153-8505 Japan
kubota@sat.t.u-tokyo.ac.jp

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Abstract

We show that there is a Lyapunov functional for the neural field equation, a neural network model which represents highly dense cortical neurons as a spatially continuous field, and that the system necessarily converges to an equilibrium point as far as the length of the field is finite. We also show that the Lyapunov functional is a natural extension of the Lyapunov function of the Hopfield model. The results suggest that the two models have generally common global dynamics characterized by the intimately related Lyapunov functional/function.

1 Introduction

The neural field is a mathematical model of the cortex which describes the dynamics of the cortical neurons in the continuum limit, and a number of analytical studies have been performed [1–5]. Most of these studies have concerned the existence and stability of characteristic solutions, such as localized excitation [1, 2] or traveling front [3, 4], while global dynamics, behavior when the system starts under arbitrary initial conditions, has not been sufficiently analyzed.

On the other hand, it is well known that the Hopfield model [6, 7] with symmetric connections shows global convergence, which is assured by the existence of the Lyapunov function. The neural field equation and the analog Hopfield model [7] agree on the point that the time-averaged activity of neurons, i.e., the firing rate, is considered as neural outputs, whereas the spatial arrangement of neurons is treated differently as a continuum in the field equation, versus the discrete elements in the Hopfield model. It is important to understand the relation between these two neural systems to reconstruct our knowledge.

From these viewpoints, we set two main goals. First, we show the existence of a Lyapunov functional for the neural field equation with symmetric connections, which ensures global convergence of the system. Second, we show that this Lyapunov functional can be considered as an extension of the Lyapunov function of the analog Hopfield model to the continuous neural field.

2 Lyapunov Functional for Neural Field Equation

We consider one-dimensional neural field equation described as

$$\tau \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{x_{\min}}^{x_{\max}} w(x,y) f[u(y,t)] dy + S(x) - h, \tag{1}$$

where the finite interval $[x_{min}, x_{max}]$ denotes the domain of the field, $u(x,t)$ is the average membrane potential of neurons at position x at time t , $\tau(> 0)$ is the time constant, $w(x,y)$ is the connectivity function which represents the average intensity of connections from neurons at place y to neurons at place x, $f(u)$ is the output function which determines the firing rate of the neuron dependent on its membrane potential, $S(x)$ is the input stimulus externally applied to neurons at position x, and h is the threshold (or equivalent homogeneous input). Let us define

$$z(x,t) = f[u(x,t)] \tag{2}$$

to be the firing rate of neurons at place x at time t, and assume that $f(u)$ is a monotone increasing sigmoid function taking a value between 0 and $z_{max}(> 0)$, so that, for all u,

$$f'(u) > 0. \tag{3}$$

We also assume $| \int_0^z f^{-1}(z')dz'| < \infty$ for $0 \leq z \leq z_{\max}$. A typical example of $f(u)$ is $1/[1 + \exp(-u/\epsilon)]$, but we do not restrict the discussion to this case. The connection is assumed to be symmetric so that

$$w(x,y) = w(y,x). \tag{4}$$

We define $E(= E[z(x)])$ to be a functional of $z(x)$ described as

$$E = \int_{x_{min}}^{x_{max}} \Bigg\{ -\frac{1}{2} z(x) \int_{x_{min}}^{x_{max}} w(x,y) z(y) dy + \int_0^{z(x)} f^{-1}(z') dz' - [S(x) - h] z(x) \Bigg\} dx. \tag{5}$$

The differentiation yields

$$\frac{dE}{dt} = \int_{x_{min}}^{x_{max}} \frac{\partial z(x,t)}{\partial t} \Bigg\{ -\frac{1}{2} \int_{x_{min}}^{x_{max}} [w(x,y) + w(y,x)] z(y) dy \\ + f^{-1}[z(x)] - [S(x) - h] \Bigg\} dx. \tag{6}$$

From (2) and (4), we have

$$\frac{dE}{dt} = -\int_{x_{min}}^{x_{max}} \frac{\partial z(x,t)}{\partial t} \left\{ \int_{x_{min}}^{x_{max}} w(x,y) f[u(y)] dy - u(x) + S(x) - h \right\} dx. \tag{7}$$

Since the term in curly brackets is the same as the right-hand side of (1), we find

$$\begin{aligned} \frac{dE}{dt} &= -\tau \int_{x_{min}}^{x_{max}} \frac{\partial z(x,t)}{\partial t} \frac{\partial u(x,t)}{\partial t} dx \\ &= -\tau \int_{x_{min}}^{x_{max}} f'[u(x)] \left(\frac{\partial u(x,t)}{\partial t} \right)^2 dx. \end{aligned} \tag{8}$$

By considering (3), we obtain the following relationship:

$$\frac{dE}{dt} \le 0; \quad \frac{dE}{dt} = 0 \Leftrightarrow \frac{\partial u(x,t)}{\partial t} = 0 \quad \text{for all } x. \tag{9}$$

Therefore, E is a Lyapunov functional of the field equation (1). Since E is bounded, the state of the field necessarily converges to an equilibrium point.

We define the variation of $z(x)$ as $\delta z(x) = \epsilon \eta(x)$, and the corresponding first variation of $E[z(x)]$ as δE. By neglecting the terms of order ϵ^n with $n \ge 2$, we have

$$\begin{aligned} &E[z(x) + \epsilon\eta(x)] \\ &= E[z(x)] + \int_{x_{min}}^{x_{max}} \epsilon\eta(x) \Bigg\{ -\frac{1}{2} \int_{x_{min}}^{x_{max}} [w(x,y) + w(y,x)] z(y) dy \\ &+ f^{-1}[z(x)] - [S(x) - h] \Bigg\} dx. \end{aligned} \tag{10}$$

With (2) and (4), δE can be written as

$$\begin{aligned} \delta E &= E[z(x) + \epsilon\eta(x)] - E[z(x)] \\ &= -\int_{x_{min}}^{x_{max}} \epsilon\eta(x) \left\{ \int_{x_{min}}^{x_{max}} w(x,y) f[u(y)] dy - u(x) + S(x) - h \right\} dx. \end{aligned} \tag{11}$$

Again, the term in curly brackets is the same as the right-hand side of (1). Hence, at equilibrium, this term becomes 0 for all x, so that $\delta E = 0$ for any variation $\epsilon\eta(x)$. Furthermore, if $\delta E = 0$ for any variation $\epsilon\eta(x)$, the term in curly brackets must be 0 for all x, indicating the equilibrium state. Thus, the system is at equilibrium, if and only if the first variation of the Lyapunov functional δE becomes 0 for any variation $\delta z(x)$. In other words, $z(x)$ at an equilibrium state is equivalent to the stationary function of the functional E. Therefore, the value of the Lyapunov functional E keeps decreasing until the system reaches a point where any small changes cannot affect the value of the functional.

Since the only assumption necessary for connectivity function $w(x, y)$ is symmetry, all the discussion above can still hold independent of whether the domain of the field is periodic. Note that for the infinite domain, the system does not always converge to an equilibrium because it is possible for the activity of neurons to expand into the whole field without limitation [1].

3 Relation between Lyapunov Functional of Field Equation and Lyapunov function of Hopfield Model

Here, to explore the relation between the Lyapunov functional and the Lyapunov function of the analog Hopfield model [7], let us divide the domain of the field $[x_{min}, x_{max}]$ into N intervals $[x_{i-1}, x_i] (i = 1, ...N)$ with length Δx, where

$$x_i = x_{min} + i\Delta x \quad (i = 0, ..., N) \tag{12}$$

and

$$\Delta x = (x_{max} - x_{min})/N. \tag{13}$$

Let $u_i (i = 1, ..., N)$ be the average membrane potential of neurons in the interval $[x_{i-1}, x_i]$, $w_{ij} (i = 1, ..., N, j = 1, ..., N)$ be the average intensity of connections from neurons in the interval $[x_{j-1}, x_j]$ to neurons in the interval $[x_{i-1}, x_i]$, and $S_i (i = 1, ..., N)$ be the input stimulus externally applied to neurons in the interval $[x_{i-1}, x_i]$.

Let us do the following replacement in the field equation (1):

$$u(x) \to u_i, \quad u(y) \to u_j, \quad w(x, y) \to w_{ij}, \quad S(x) \to S_i, \tag{14}$$

and also replace the integral by a summation. Then, we obtain

$$\tau \frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} w_{ij} f(u_j) \Delta x + S_i - h. \tag{15}$$

Note that the condition of symmetric connection (4) is also replaced by $w_{ij} = w_{ji}$. Let us define $T_{ij}(i = 1, ..., N,\ j = 1, ..., N)$ and $z_i(i = 1, ..., N)$ as

$$T_{ij} = w_{ij}\Delta x \tag{16}$$

and

$$z_i = f(u_i), \tag{17}$$

respectively. Then, Eq. (15) can be rewritten as

$$\tau \frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} T_{ij} z_j + S_i - h. \tag{18}$$

This equation is exactly the same as the analog Hopfield model [7]. Since $T_{ij} = T_{ji}$, the Lyapunov function

$$I = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} T_{ij} z_i z_j + \sum_{i=1}^{N}\int_0^{z_i} f^{-1}(z')dz' - \sum_{i=1}^{N}(S_i - h)z_i \tag{19}$$

exists for (18), and satisfies $dI/dt \leq 0$ [7]. We multiply (19) by Δx and use (16) to obtain

$$\begin{aligned} I\Delta x = &-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} z_i z_j (\Delta x)^2 + \sum_{i=1}^{N}\left[\int_0^{z_i} f^{-1}(z')dz'\right]\Delta x \\ &- \sum_{i=1}^{N}(S_i - h) z_i \Delta x. \end{aligned} \tag{20}$$

In the limit of $N \to \infty$, if we use the inverse transformation of (14) (from (2) and (17), replacement of $u_i \to u(x)$ and $u_j \to u(y)$ leads to the replacement of $z_i \to z(x)$ and $z_j \to z(y)$, respectively) and also replace the summation by the integral, (20) becomes

$$\begin{aligned} I\Delta x \to &-\frac{1}{2}\int_{x_{\min}}^{x_{\max}}\int_{x_{\min}}^{x_{\max}} w(x,y)z(x)z(y)dxdy + \int_{x_{min}}^{x_{max}}\int_0^{z(x)} f^{-1}(z')dz'dx \\ &- \int_{x_{min}}^{x_{max}} [S(x) - h]z(x)dx = E. \end{aligned} \tag{21}$$

This equation means that $I\Delta x$ agrees with E in the limit of $N \to \infty$. Thus, we can understand that the Lyapunov functional of the field equation is an extension of the Lyapunov function of the Hopfield model to continuously distributed neurons.

4 Lyapunov Functional in the Field with Lateral-Inhibitory Connection

The existence of the Lyapunov functional not only ensures global convergence, but also provides a new analytical method for the theoretical study of pattern formation of the neural field, since we can find a stable (unstable) equilibrium solution by simply seeking out a local minimum (maximum) of the Lyapunov functional. As an example, here we apply the Lyapunov functional to the field with lateral-inhibitory connections. This type of field has been analyzed by Amari [1], and the existence of various types of pattern dynamics has been demonstrated. Now, we give a unified explanation for the pattern dynamics as the changing property of the Lyapunov functional.

Let us assume the connectivity function is homogeneous, i.e., $w(x, y) = w(x - y)$, as well as symmetric. Since the connection is lateral-inhibitory, $w(x)$ satisfies $w(x) > 0$ for $|x| < x_m$ and $w(x) < 0$ for $|x| > x_m$ with $x_m > 0$. We also assume $\lim_{x\to\infty} w(x) = 0$ and $W_\infty \equiv \int_0^\infty w(x)dx > 0$. An example of $w(x)$ is shown in Fig. 1(a). The output function $f(u)$ is assumed to be the step-function satisfying $f(u) = 0$ for $u \leq 0$ and $f(u) = 1$ for $u > 0$. We consider the localized excitation of the field, which is defined as the state where only the neurons in a finite interval (x_1, x_2) are active so that

$$\{x|u(x) > 0\} = (x_1, x_2). \tag{22}$$

We also consider the field without external input so that $S(x) = 0$.

Let $f_s(u)$ be a general sigmoid function taking a value between 0 and 1. Then, the following relationship holds with $f_\epsilon(u) \equiv f_s(u/\epsilon)$:

$$\int_0^z f_\epsilon^{-1}(z')dz' = \epsilon \int_0^z f_s^{-1}(z')dz'. \tag{23}$$

Since $f_\epsilon(u)$ agrees with the step-function $f(u)$ in the limit of $\epsilon \to 0$, the second term in curly brackets in (5) is 0. We also find the relations $z(x) = 1$ for $x \in (x_1, x_2)$ and $z(x) = 0$ for $x \notin (x_1, x_2)$ from (22). Hence, (5) can be written as

$$E = -\frac{1}{2}\int_{x_1}^{x_2}\int_{x_1}^{x_2} w(x - y)dxdy + h(x_2 - x_1). \tag{24}$$

When we define $W(x) \equiv \int_0^x w(x')dx'$, (24) can be reduced to

$$E = -\int_0^{x_2 - x_1} W(x)dx + h(x_2 - x_1). \tag{25}$$

Furthermore, if we define the length of the excited region $a \equiv x_2 - x_1$, we have

$$E = -\int_0^a W(x)dx + ha. \tag{26}$$

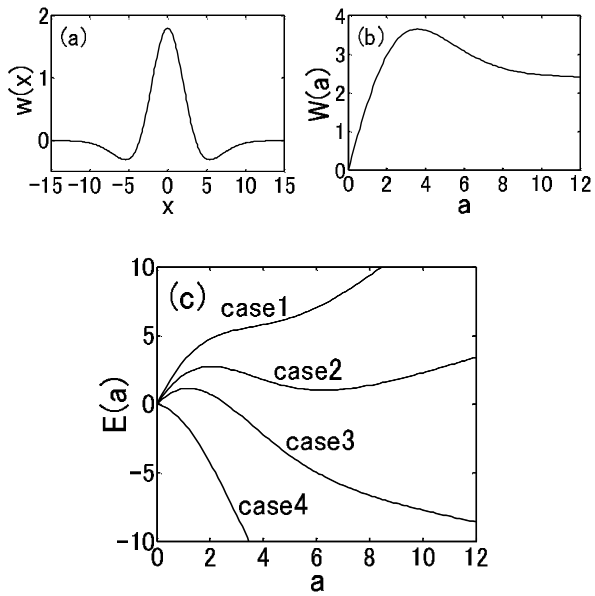

Figure 1: Example of the Lyapunov functional of the field equation with lateral-inhibitory connections, where $w(x) = 3\exp[-x^2/(2 \cdot 2.2^2)] - 1.2\exp[-x^2/(2 \cdot 3.9^2)]$. (a) Connectivity function $w(x)$. (b) Integral of connectivity function $W(a)$. (c) Relation between the Lyapunov functional E and the length of excited region a for four cases. The value of the threshold h for case 1-4 is 4, 3, 2, and -0.5, respectively.

E is only a function of a, so that we express this relation as $E(a)$. Differentiation of (26) with respect to a yields

$$\frac{dE}{da} = -W(a) + h. \tag{27}$$

Since $w(x)$ is a lateral-inhibitory function, $W(a)(a > 0)$ has one peak and converges to W_∞ in the limit of $a \to \infty$, as shown in Fig. 1(b). We define $W_m \equiv \max_{a>0} W(a)$ to be the maximum value of $W(a)$. Equation (27) indicates the relations $dE/da > 0$ for $W(a) < h$ and $dE/da < 0$ for $W(a) > h$. Thus, $E(a)$ has a shape shown in Fig. 1(c) for case 1 : $h > W_m$, case 2 : $W_\infty < h < W_m$, case 3 : $0 < h < W_\infty$, and case 4 : $h < 0$, respectively. In case 1, $E(a)$ is a monotone increasing function, so that the value of a decreases with time and converges to 0. Thus, neurons in the whole field finally become quiescent. In case 2, $E(a)$ has one local maximum and one local minimum, which correspond to the unstable and stable equilibrium solution of the localized excitation, respectively. Thus, the system converges either to this stable equilibrium solution or to the state of $a = 0$ dependent on the initial value of a . In case 3, $E(a)$ has one local maximum, and this state corresponds to the unstable equilibrium solution, so that the system converges either to the state of $a = 0$ or to the state where neurons in the whole field are active dependent on the initial value of a. In case 4, since $E(a)$ is a monotone decreasing function, the value of a increases with time and finally neurons in the whole field become active. Thus, the Lyapunov functional provides a unified and intuitive approach to understanding each type of pattern dynamics.

5 Discussion

We have demonstrated that there exists a Lyapunov functional for the field equation which is an extension of Lyapunov function of the Hopfield model. The existence of the Lyapunov functional tells us that the global dynamics of the field equation can be simply understood as the behavior that searches for the local minimum of the functional. Thus, it might be possible to use the Lyapunov functional for an optimization problem just like the Hopfield model [8]. Once the objective functional has been given in the form of the Lyapunov functional, we can at least find local minimum solutions by solving the corresponding field equation numerically.

The fact that the Lyapunov functional of the field equation is a natural extension of the Lyapunov function of the Hopfield model to the spatially continuous neural field is important, since this relation implies that both neural networks have generally common global dynamics characterized by the Lyapunov functional/function. Therefore, we expect that part of the analytical results regarding global stability of the Hopfield model with some types of asymmetric connections [9] or delays [10] can also be extended to the field equation with corresponding asymmetric connections or delays. This is a problem for the future work.

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