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# Discrete Fixed Point Theorem Reconsidered*

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**Abstract**

The aim of this note is to indicate an example that demonstrates the incorrectness of Iimura's discrete fixed point theorem [*J. Mathematical Economics* 39 (2003) 725] and to present a corrected statement using the concept of integrally convex sets.

## 1 Introduction

Iimura (2003) gave a discrete fixed point theorem for correspondences (set-valued mappings) on discrete sets. The theorem claims that a discretely convex-valued direction-preserving correspondence defined on a contiguously convex set to itself has a fixed point (see Section 2 for the precise statement). In this note we indicate an example that demonstrates the incorrectness of this statement, and rectify the statement using the concept of integrally convex sets introduced by Favati and Tardella (1990).

## 2 A counterexample for contiguously convex sets

Let $\mathbf{R}$ and $\mathbf{Z}$ denote the sets of all reals and all integers, respectively. Given a positive integer $n$, we denote by $\mathbf{Z}^n$ the set of all integer vectors $z = (z_i \in \mathbf{Z} : i = 1, \ldots, n)$. A finite set $X \subseteq \mathbf{Z}^n$ is called *discretely convex* (or *hole free*, Murota, 2003) if

$$X = \overline{X} \cap \mathbf{Z}^n, \tag{1}$$

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where $\overline{X}$ denotes the convex hull of $X$ in $\mathbf{R}^n$. A finite set $X \subseteq \mathbf{Z}^n$ is said to be *contiguously convex* (Iimura, 2003) if

$$\forall y \in \overline{X},\ \exists x \in X : ||x - y||_\infty < 1, \tag{2}$$

where $||x-y||_\infty = \max\{|x_i - y_i| \mid i = 1, \ldots, n\}$. By the definitions, a contiguously convex set is discretely convex. For two integer vectors $z$ and $z'$, we define a relation $z \simeq z'$ as

$$z \simeq z' \Leftrightarrow ||z - z'||_\infty \leq 1. \tag{3}$$

Let $X$ be a nonempty finite subset of $\mathbf{Z}^n$ and $\Gamma : X \rightarrow\rightarrow X$ be a nonempty-valued correspondence. A point $x \in X$ is said to be a *fixed point* if $x \in \Gamma(x)$. For each $x \in X$, let $\pi_\Gamma(x)$ denote the projection of $x$ onto $\overline{\Gamma(x)}$, i.e.,

$$||\pi_\Gamma(x) - x||_2 = \min_{y \in \overline{\Gamma(x)}} ||y - x||_2, \tag{4}$$

where $||y - x||_2 = \left(\sum_{i=1}^n (y_i - x_i)^2\right)^{1/2}$. We denote $\pi_\Gamma(x) - x$ by $\tau(x)$, and define

$$\sigma(x) = (\mathrm{sign}(\tau_i(x)) \in \{+1, 0, -1\} : i = 1, \ldots, n), \tag{5}$$

where $\tau_i(x)$ denotes the $i$th component of $\tau(x)$.

According to Iimura (2003), a correspondence $\Gamma : X \rightarrow\rightarrow X$ is said to be *direction preserving*[1] if for all $x, x' \in X$ with $x \simeq x'$,

$$\sigma_i(x) > 0 \Longrightarrow \sigma_i(x') \geq 0 \qquad (i = 1, \ldots, n), \tag{6}$$

where $\sigma_i(x)$ denotes the $i$th component of $\sigma(x)$. The condition is equivalent to

$$\sigma_i(x) < 0 \Longrightarrow \sigma_i(x') \leq 0 \qquad (i = 1, \ldots, n). \tag{7}$$

Iimura (2003) made the following statement.

**Statement:** *Let $X \subset \mathbf{Z}^n$ be a nonempty finite contiguously convex set. If $\Gamma : X \rightarrow\rightarrow X$ is a nonempty- and discretely convex-valued direction preserving correspondence, then $\Gamma$ has a fixed point.*

A counterexample exists to the above statement. Consider the finite set $X \subseteq \mathbf{Z}^3$ defined as

$$X = \{a = (0,1,0), b = (1,0,0), c = (2,0,0), d = (3,0,0), e = (4,0,1)\} \tag{8}$$

and the correspondence $\Gamma : X \rightarrow\rightarrow X$ defined as

$$\Gamma(a) = \Gamma(b) = \{e\},\ \Gamma(c) = \{a, e\},\ \Gamma(d) = \Gamma(e) = \{a\}. \tag{9}$$

[1] We note that "direction preserving" can also be defined in terms of $\tau$ as: if for all $x, x' \in X$ with $x \simeq x'$, $\tau_i(x) > 0 \Longrightarrow \tau_i(x') \geq 0$ for all $i = 1, \ldots, n$.

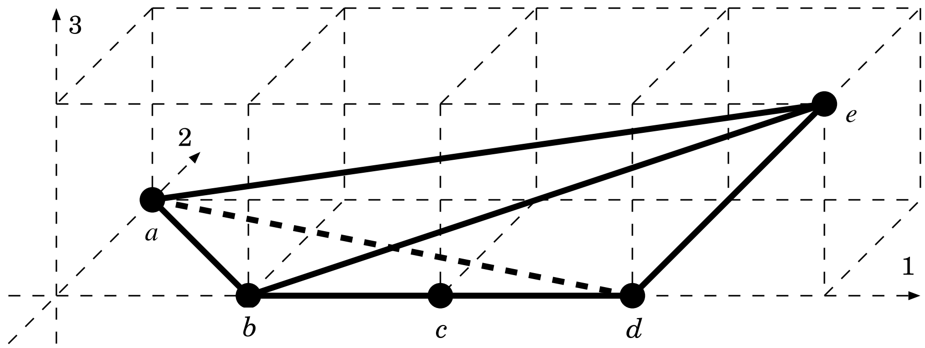


Figure 1: A contiguously convex set $X = \{a = (0,1,0), b = (1,0,0), c = (2,0,0), d = (3,0,0), e = (4,0,1)\}$ for the counterexample with $\Gamma(a) = \Gamma(b) = \{e\}$, $\Gamma(c) = \{a,e\}$, $\Gamma(d) = \Gamma(e) = \{a\}$.

Figure 1 shows that $X$ is a contiguously convex set and $\Gamma$ is a nonempty- and discretely convex-valued correspondence. Furthermore, $\Gamma$ is direction preserving, because

$$a \simeq b,\ b \simeq c,\ c \simeq d,\ d \simeq e, \tag{10}$$

the other pairs of distinct points are not in the relation $\simeq$, and $\tau$ and $\sigma$ are calculated as

$$\left.\begin{array}{lcr} \tau(a) & = & (\ \ 4,\ \ -1,\ \ \ 1) \\ \tau(b) & = & (\ \ 3,\ \ \ 0,\ \ \ 1) \\ \tau(c) & = & (\ \ 0,\ \ 1/2,\ \ 1/2) \\ \tau(d) & = & (-3,\ \ \ 1,\ \ \ 0) \\ \tau(e) & = & (-4,\ \ \ 1,\ \ -1) \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcr} \sigma(a) & = & (+1,\ \ -1,\ \ +1) \\ \sigma(b) & = & (+1,\ \ \ 0,\ \ +1) \\ \sigma(c) & = & (\ \ 0,\ \ +1,\ \ +1) \\ \sigma(d) & = & (-1,\ \ +1,\ \ \ 0) \\ \sigma(e) & = & (-1,\ \ +1,\ \ -1). \end{array}\right. \tag{11}$$

Obviously, $\Gamma$ has no fixed point.

## 3 Theorem for integrally convex sets

In this section, we give a discrete fixed point theorem for integrally convex sets.

For $y \in \mathbf{R}^n$, we define a neighborhood $N(y)$ of $y$ by

$$N(y) = \{z \in \mathbf{Z}^n \mid ||z - y||_\infty < 1\}. \tag{12}$$

A finite set of integer points $X \subseteq \mathbf{Z}^n$ is said to be *integrally convex* if it satisfies

$$y \in \overline{X} \Longrightarrow y \in \overline{X \cap N(y)} \qquad (\forall y \in \mathbf{R}^n), \tag{13}$$

i.e., any point $y \in \overline{X}$ can be represented as a convex combination of integer points in $N(y)$ (Favati and Tardella, 1990, see also Murota, 2003, Section 3.4). By the definitions, an integrally convex set is contiguously convex.

An integrally convex set admits a nice simplicial decomposition, which is the key property for establishing the discrete fixed point theorem for integrally convex sets.

**Lemma 1.** *For any finite integrally convex set $X \subset \mathbf{Z}^n$, there exists a simplicial decomposition*[2] *$\mathcal{S}$ of $\overline{X}$ such that for each $y \in \overline{X}$, all the vertices of the smallest simplex $S(y) \in \mathcal{S}$ containing $y$ belong to $N(y)$. For such $\mathcal{S}$ we have $y \in \overline{S(y) \cap N(y)}$ for all $y \in \overline{X}$ and $\{x\} \in \mathcal{S}$ for all $x \in X$.*

The proof is given at the end of this section.

**Theorem 2.** *Let $X \subset \mathbf{Z}^n$ be a nonempty finite integrally convex set. If $\Gamma : X \to\to X$ is a nonempty- and discretely convex-valued direction preserving correspondence, then $\Gamma$ has a fixed point.*

*Proof.* We make use of Brouwer's fixed point theorem, which says that every continuous mapping from a compact convex set of $\mathbf{R}^n$ to itself has a fixed point. We define a continuous mapping $\gamma$ from $\overline{X}$ to $\overline{X}$. For each point $x \in X$, we define $\gamma(x) = \pi_\Gamma(x)$. Since $X$ is a finite integrally convex set, there exists a simplicial decomposition $\mathcal{S}$ of $\overline{X}$ satisfying the conditions in Lemma 1. Let $y$ be an arbitrary point in $\overline{X}$. By Lemma 1, we have $y \in \overline{S(y) \cap N(y)}$. Let

$$y = \sum_{x \in S(y) \cap N(y)} \lambda_x x, \quad \sum \lambda_x = 1, \quad \lambda_x \geq 0, \tag{14}$$

be the uniquely determined convex combination. That is, $(\lambda_x : x \in S(y) \cap N(y))$ is the barycentric coordinate of $y$ in $\overline{S(y) \cap N(y)}$. Then, we define $\gamma(y)$ by

$$\gamma(y) = \sum_{x \in S(y) \cap N(y)} \lambda_x \pi_\Gamma(x). \tag{15}$$

Since $\pi_\Gamma(x) \in \overline{X}$ for all $x \in X$, we have $\gamma(y) \in \overline{X}$ for all $y \in \overline{X}$. Moreover, since $\mathcal{S}$ is a simplicial decomposition satisfying the conditions in Lemma 1, $\gamma$ is continuous. By Brouwer's fixed point theorem, $\gamma$ has a fixed point, say, $y \in \overline{X}$.

We next show that $\gamma$ has an integral fixed point. We have

$$\sum_{x \in S(y) \cap N(y)} \lambda_x x = y = \gamma(y) = \sum_{x \in S(y) \cap N(y)} \lambda_x \pi_\Gamma(x). \tag{16}$$

This says that

$$\sum_{x \in S(y) \cap N(y)} \lambda_x (\pi_\Gamma(x) - x) = \sum_{x \in S(y) \cap N(y)} \lambda_x \tau(x) = \mathbf{0}, \tag{17}$$

where $\mathbf{0}$ denotes the zero vector in $\mathbf{R}^n$. Since $\Gamma$ is direction preserving, we have $\tau(x) = \mathbf{0}$ if $\lambda_x > 0$. Therefore, there exists at least one $x \in S(y) \cap N(y)$ with $\tau(x) = \mathbf{0}$. Such $x$ is a fixed point of $\Gamma$, since $\tau(x) = \mathbf{0}$ implies $x \in \overline{\Gamma(x)} \cap \mathbf{Z}^n = \Gamma(x)$. □

[2] A simplicial decomposition $\mathcal{S}$ is a collection of simplices satisfying (a) $\overline{X} = \bigcup_{S \in \mathcal{S}} S$, (b) $S \in \mathcal{S}$, $S'$: a face of $S \Longrightarrow S' \in \mathcal{S}$, and (c) $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 \neq \emptyset \Longrightarrow S_1 \cap S_2$: a face of $S_1$ and $S_2$.

We finally show Lemma 1. For $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$, its *convex closure* is defined as the piecewise linear function $\overline{f} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ such that the epigraph of $\overline{f}$ coincides with the convex hull of $\{(z, \alpha) \in \mathbf{Z}^n \times \mathbf{R} \mid \alpha \geq f(z), \forall z \in \mathbf{Z}^n\}$. For $p \in \mathbf{R}^n$, we denote the inner product of $p$ and $y$ by $\langle p, y \rangle$, define a function $\overline{f}[-p]$ by

$$\overline{f}[-p](y) = \overline{f}(y) - \langle p, y \rangle \quad (\forall y \in \mathbf{R}^n), \tag{18}$$

and denote the set of minimizers of $\overline{f}[-p]$ by $\arg\min \overline{f}[-p]$, i.e.,

$$\arg\min \overline{f}[-p] \quad = \quad \{y \in \mathbf{R}^n \mid \overline{f}[-p](y) \leq \overline{f}[-p](y'), \forall y' \in \mathbf{R}^n\}. \tag{19}$$

The indicator function $\delta_X : \mathbf{Z}^n \to \{0, +\infty\}$ is defined as $\delta_X(x) = 0$ if $x \in X$; otherwise $\delta_X(x) = +\infty$.

**Proof of Lemma 1.** We construct a simplicial decomposition of $\overline{X}$ through the projection of faces of a piecewise linear function.

Let $\delta_X$ be the indicator function of $X$ and $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be defined as

$$g(x) = \delta_X(x) + \sum_{i=1}^{n} {x_i}^2 \quad (\forall x \in \mathbf{Z}^n). \tag{20}$$

Its convex closure $\overline{g}$ is a piecewise linear function. Since the second term of $g$ is a separable quadratic function, the projection of each face of $\overline{g}$, which can be represented as $\arg\min \overline{g}[-p]$ for some $p \in \mathbf{R}^n$, is included in the intersection of $\overline{X}$ with a hypercube $\{y \in \mathbf{R}^n \mid z \leq y \leq z + \mathbf{1}\}$ for some $z \in \mathbf{Z}^n$, where $\mathbf{1}$ denotes the vector of all ones. Note that $\arg\min \overline{g}[-p]$ generated by $p \in \mathbf{R}^n$ are not necessarily simplices, so let us slightly perturb $g$ as follows.

Let $d$ be an integer vector such that if $x \neq x'$ then $\langle d, x \rangle \neq \langle d, x' \rangle$ for all $x, x' \in X$. Since $X$ is a finite set, we can always choose such $d$, e.g., by letting $d = (1, L, L^2, \ldots, L^{n-1})$ with a sufficiently large positive integer $L$ (this gives a kind of lexicographic valuation in terms of $\langle d, x \rangle$ for all $x$ in $X$). Let $\varepsilon > 0$ and $h_\varepsilon : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be defined as

$$h_\varepsilon(x) = g(x) + \varepsilon \exp\langle d, x \rangle \qquad (\forall x \in \mathbf{Z}^n). \tag{21}$$

Its convex closure $\overline{h_\varepsilon}$ is a piecewise linear function, and, since $X$ is finite and $\arg\min \overline{g}[-p]$ is included in a hypercube for each $p \in \mathbf{R}^n$, there exists a sufficiently small $\varepsilon > 0$ such that for all $p \in \mathbf{R}^n$, $\arg\min \overline{h_\varepsilon}[-p]$ are also included in hypercubes. The vertices of $\arg\min \overline{h_\varepsilon}[-p]$ belong to $X$. Moreover, for each $p \in \mathbf{R}^n$, $S = \arg\min \overline{h_\varepsilon}[-p]$ is a simplex. To see this, suppose, by way of contradiction, that $S$ is not a simplex. Then, the vertices of $S$ are affinely dependent, and hence, there exist two disjoint families $\{x^i \in S \cap \mathbf{Z}^n \mid i \in I\}$ and $\{x^j \in S \cap \mathbf{Z}^n \mid j \in J\}$ of integer vectors and two families $\{\lambda_i \mid i \in I\}$ and $\{\lambda_j \mid j \in J\}$ of positive rational numbers, representing a rational point $y \in S$ as two different convex combinations, such that

$$\sum_{i \in I} \lambda_i = 1 = \sum_{j \in J} \lambda_j, \tag{22}$$

$$\sum_{i \in I} \lambda_i x^i = y = \sum_{j \in J} \lambda_j x^j, \tag{23}$$

$$\sum_{i \in I} \lambda_i h_\varepsilon(x^i) = \overline{h_\varepsilon}(y) = \sum_{j \in J} \lambda_j h_\varepsilon(x^j). \tag{24}$$

Since $\overline{g}$ is linear on $S$, we must have $\sum_{i \in I} \lambda_i g(x^i) = \sum_{j \in J} \lambda_j g(x^j)$ and

$$\sum_{i \in I} \lambda_i \exp\langle d, x^i \rangle = \sum_{j \in J} \lambda_j \exp\langle d, x^j \rangle. \tag{25}$$

This means that the base of the natural logarithm $e$ satisfies $F(e) = 0$ for

$$F(t) = \sum_{i \in I} \lambda_i t^{\langle d, x^i \rangle} - \sum_{j \in J} \lambda_j t^{\langle d, x^j \rangle}, \tag{26}$$

which is a nonvanishing polynomial in $t$ over the field of rational numbers. This contradicts the transcendentality of $e$. Thus $S$ must be a simplex.

The desired simplicial decomposition is obtained by

$$\mathcal{S} = \{\arg\min \overline{h_\varepsilon}[-p] \mid p \in \mathbf{R}^n\}. \tag{27}$$

It is easy to verify (a) $\overline{X} = \bigcup_{S \in \mathcal{S}} S$; (b) if $S \in \mathcal{S}$ and $S'$ is a face of $S$ then $S' \in \mathcal{S}$; and (c) if $S_1, S_2 \in \mathcal{S}$ and $S_1 \cap S_2 \neq \emptyset$ then $S_1 \cap S_2 \in \mathcal{S}$. It remains to show that, for each $y \in \overline{X}$, the vertices of the smallest simplex $S(y) \in \mathcal{S}$ containing $y$ (the existence of which is secured by (c)) all belong to $N(y)$. Suppose not. Then there exists a vertex $x$ of $S(y)$ with $x \notin N(y)$. Since $S(y)$ is included in a hypercube and $x$ is an integer vector, $S(y) \cap \overline{N(y)}$ is a proper face of $S(y)$ containing $y$. However, this contradicts the minimality of $S(y)$. Hence, all the vertices of $S(y)$ belong to $N(y)$. This together with the definition of $N(y)$ implies that $y \in \overline{S(y) \cap N(y)}$ for all $y \in \overline{X}$ and $\{x\} \in \mathcal{S}$ for all $x \in X$. □

## Concluding remark

We emphasize that Theorem 2 is valid for two applications in Iimura (2003), namely, existence of Walrasian equilibrium with indivisible commodities, and discrete non-cooperative games, because they are actually dealing with a direction preserving correspondence on a rectangular set, which is integrally convex.

After the completion of the manuscript, the authors learned that a Russian group of Danilov and Koshevoy (2004) also noticed the incorrectness of the proof in Iimura (2003) and gave a theorem similar to ours.

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