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Strength of imperfect structures and materials: Probabilistic and group-theoretic approaches

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Abstract

The general theory of elastic stability invented by Koiter, 1945 [311] motivated the development of a series of asymptotic approaches to deal with initial postbuckling behavior of structures. These approaches, which played a pivotal role in the pre-computer age, are somewhat overshadowed by the progress of computational environment. Recently, the importance of the asymptotic approaches has been revived through the extension of their theoretical framework and the combination with group-theoretic bifurcation theory in nonlinear mathematics. The approaches serve as efficient and insightful strategy to tackle various aspects of buckling behaviors, including: probabilistic scatter of critical loads and experimentally observed bifurcation diagram. In particular, complex imperfect behaviors at double critical points of systems with dihedral-group symmetry, which labels the symmetry of cylindrical shells and reticulated polygonal domes, are investigated in a systematic manner. We review, through the perspective of theoretical engineers, the historical development and recent revival of the asymptotic approaches for imperfection-sensitive structures and materials.

1 Introduction

Koiter's asymptotic theory[1] of *general nonlinear theory of stability*[2] emerged as a pertinent tool to analyze the *initial postbuckling behavior* and *imperfection sensitivity* of structures. The asymptotic theory, however, seems somewhat overshadowed by numerical approaches in the computer age.

In nonlinear mathematics, catastrophe theory[3] and group-theoretic bifurcation theory[4] were developed to describe the qualitative characteristics of bifurcation behaviors, especially for coincidental and compound bifurcations. While catastrophe theory was quickly implemented into the framework of elastic stability theory,[5] the group-theoretic study of bifurcation was not necessarily common in structural engineering.

Recently, the importance of the asymptotic approaches has been revived through the extension of the framework of the Koiter theory and the combination with group-theoretic bifurcation theory. Theoretical procedures to determine the most influential (worst) initial imperfection[6] and to obtain the probability density function of critical loads for initial imperfections with known probabilistic characteristics[7] were developed. Weibull-like probability density function of critical loads was derived for initial imperfections subjected to a multivariate normal distribution.[8] These procedures were extended to experimentally observed bifurcation diagram and to a system with dihedral-group symmetry,[9] and, in turn, were applied to the description of the strength of structures and materials, such as shells, steels, soils, and concretes.[10] Bifurcation mechanism of deformation of materials was elucidated in the light of the asymptotic and group-theoretic approaches.[11]

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[1] The word asymptotic means that all results are local, valid only for sufficiently small values of initial imperfection parameters, and in a sufficiently close neighborhood of the critical point under consideration. Asymptotic theory appears in various fields of research (see, e.g., the reviews of Chernyschenko, 1998 [96]; Andrianov and Awrejcewicz, 2001 [16]; Andrianov, Awrejcewicz, and Barantsev, 2003 [17]).

[2] Koiter, 1945 [311]

[3] Chillingworth, 1975 [97]; Thom, 1975 [504]; Zeeman, 1976, 1976, 1977 [576, 577, 578]; Poston and Stewart, 1978 [429]; Saunders, 1980 [458]; Gilmore, 1981 [188]; Stewart, 1981 [490]; Callahan, 1982 [88]; Thompson, 1982 [513] Arnold, 1984 [35]; Arnold et al., 1994 [36]

[4] Sattinger, 1979 [456]; Golubitsky and Schaeffer, 1985 [195]; Golubitsky, Stewart, and Schaeffer, 1988 [198]

[5] Thompson, 1982, 1984 [513, 514]

[6] Ikeda and Murota, 1990, 1990 [262, 263]

[7] Ikeda and Murota, 1991 [264]

[8] Ikeda and Murota, 1993 [266]

[9] Murota and Ikeda, 1991, 1992 [367, 368]; Ikeda and Murota, 1999 [269]

[10] Ikeda, Murota, and Elishakoff, 1996 [271]; Ikeda, Chida, and Yanagisawa, 1997 [258]; Ikeda et al., 1997 [260]; Okazawa et al., 2002 [390]

[11] Ikeda, Murota, and Nakano, 1994 [273]; Ikeda et al., 1997, 2001, 2001 [274, 261, 278]; Murota, Ikeda, and Terada, 1999 [369]; Ikeda and Murota, 2002 [270]; Tanaka, Saiki, and Ikeda, 2002 [499]

In this paper, the development of asymptotic approaches to describe the probabilistic scatter of the strength of structures and their recent combination with group-theoretic bifurcation theory are reviewed as a summary of [270, 277, 390]. We shed light on a modern view of bifurcation theory in nonlinear mathematics by minimizing mathematical details so as to make it accessible for engineers. Emphasis is placed not on experimental and computational but on analytical approaches. We emphasize methodology, instead of studies on particular structures.[12] We deal mainly with imperfection-sensitive structures subjected to static bifurcation and buckling, and do not emphasize stochastic properties of structures or imperfection-insensitive structures with stockier proportions.[13] The research works in the Russian language are not contained due to our lack of command of the language.

The historical development of asymptotic and group-theoretic approaches is reviewed in Section 2. In Section 3, the general framework of the asymptotic approach is presented and is applied to simple critical points to arrive at bifurcation equation, imperfection sensitivity laws, probabilistic variation of strength, and experimentally observed bifurcation diagram. Such application is extended to hilltop branching points in Section 4. With the help of group-theoretic method presented in Section 5, it is extended to double critical points of structures with dihedral-group symmetry in Section 6.

2 HISTORICAL DEVELOPMENT

The experimental buckling loads of long thin-walled cylinders were found to fall markedly below the stability limit computed by the classical linearized theory of bifurcations.[14] In order to resolve such inadequacy, *nonlinearity*[15] and *initial imperfections* were implemented into the theory of elastic stability of shells.[16] Combined experimental, analytical and computational studies of imperfection-sensitive structures were motivated in this manner.

2.1 General theory of elastic stability

The mechanism of *postbuckling behavior* was fully understood by *general nonlinear theory of stability* in the masterful thesis of Koiter.[17] Nonlinear

[12] For shells, for example, see the reviews of Singer, 1980, 1999 [478, 479]; Simitses, 1986 [475]; Soldatos, 1994 [483]; Teng, 1996 [500].

[13] Imperfection-insensitive structures, which have stockier proportions and undergo elasto-plastic behavior, are studied intensively in ship and civil engineering.

[14] Euler, 1744 [164]

[15] The importance of nonlinearity in physical behavior was acknowledged at the beginning of the 20th century through the development of classical nonlinear mathematics (Poincaré, 1892–1899 [425]; Liapunov, 1906 [334]; Schmidt, 1910 [461]).

[16] Flügge, 1932 [171]; Donnel, 1934 [132]; von Kármán and Tsien, 1939 [557]; Cox, 1940 [116]; von Kármán, Dunn, and Tsien, 1940 [556]; Donnel and Wan, 1950 [133]

[17] Koiter, 1945 [311]

governing equations of structures subjected to buckling in general involve a large number of independent variables and nonlinear terms and, hence, are highly complex. In the *general nonlinear theory of stability*, to obtain asymptotic general forms of imperfect systems in the neighborhood of critical points, the nonlinear governing equation was simplified twofold:

- The governing equation is reduced[18] to the bifurcation equation with only a few active independent variables.
- The higher order terms of the bifurcation equation are truncated by the asymptotic assumption.

The general theory thus obtained was a pertinent and strong *analytical* tool to tackle nonlinear behavior of imperfect structural systems. Imperfection sensitivity laws[19] were derived to explain the mechanism of the erosion of the strength due to initial imperfections. The importance of this theory, however, was not recognized until the early 1960's when research on initial postbuckling behaviors sprung up worldwide.

It was noted that even small imperfections can sharply erode the strength of structural systems undergoing bifurcation: including, shells, beams, trusses, arches, frames, and so on.[20] In the United States, Koiter's original work was reconstructed in a form suitable for elastic continua and was applied to imperfection-sensitive structures.[21] Excellent reviews[22] provide an overview of the early development of *postbuckling theory* led by Koiter.[23]

In Europe, mainly in England, initial postbuckling behavior and related theories started to draw attention in the 1960's.[24] The direct experimental validation of imperfection sensitivity of structures was conducted.[25] The

[18] This reduction procedure is called the *Liapunov–Schmidt reduction* (Sattinger, 1979 [456]; Chow and Hale, 1982 [102]; Golubitsky and Schaeffer, 1985 [195]), the *Liapunov–Schmidt–Koiter reduction* (Peek and Kheyrkhahan, 1993 [415]), or the *elimination of passive coordinates* (Thompson and Hunt, 1973 [518]).

[19] Imperfection sensitivity laws mean interrelationships between the buckling load and imperfection magnitude.

[20] Koiter, 1963 [314]; Horton and Durham, 1965 [229]

[21] Budiansky and Hutchinson, 1964, 1966 [74, 75]; Budiansky, 1965, 1969 [70, 71]; Hutchinson, 1967 [250]; Hutchinson and Amazigo, 1967 [254]; Budiansky and Amazigo, 1968 [73]; Cohen, 1968 [112]; Fitch, 1968 [170]; Hutchinson and Koiter, 1970 [255]

[22] Hutchinson and Koiter, 1970 [255]; Budiansky, 1974 [72]; Koiter, 1976 [317]; Pignataro, Rizzi, and Luongo, 1991 [422]

[23] Koiter, 1956, 1963, 1963, 1966, 1967 [312, 313, 314, 315, 316]

[24] Britvec, 1963 [64]; Britvec and Chilver, 1963 [67]; Roorda, 1965, 1965, 1965, 1968 [440, 441, 442, 443]; Sewell, 1965, 1966, 1968 [466, 467, 468]; Thompson, 1965, 1968, 1969, 1969 [507, 508, 510, 511]; Chilver, 1967 [98]; Supple, 1967, 1968, 1969 [494, 495, 496]; Thompson and Walker, 1968 [528]; Pope, 1968 [428]; Thompson and Walker, 1968 [528]; Thompson and Hunt, 1969, 1969 [516, 517]; Huseyin, 1970 [246]; Roorda and Chilver, 1970 [447]

[25] Roorda, 1965, 1965 [440, 442]; Thompson, Tulk, and Walker, 1976 [527]

perturbation technique[26] was applied to the total potential energy function of a finite-dimensional system to derive asymptotic information on bifurcation buckling. A large number of imperfection modes were taken into account in the formulation by CIM (critical-imperfection-magnitude) method.[27]

The simultaneous buckling was studied in association with optimization. The principle of *simultaneous mode design* states, "A given form will be optimum if all failure modes which can possibly intersect occur simultaneously."[28] The danger of naive optimization without due regard to imperfection sensitivity and the erosion of optimization by compound branching were pointed out.[29] Various kinds of structures were found highly imperfection-sensitive when two or more bifurcation points are nearly or strictly coincident, and are subjected to interaction of buckling modes, such as local and global modes.[30] The study of coincidental buckling and the interaction between local and overall initial imperfections was thus motivated.[31]

Critical points were classified by investigating the linear, quadratic, cubic, quartic, ... terms of the potential function.[32] In particular, the normal forms of coincidental critical points were determined to show the diver-

[26] Bellman, 1964 [47]; Keller, 1968 [298]; Nayfeh, 1973 [374]; Simmonds and Mann, 1986 [477]; Bush, 1994 [79]; Kevorkian and Cole, 1996 [300]

[27] Palassopoulos, 1973, 1993, 1997 [398, 404, 406]; Palassopoulos and Shinozuka, 1973 [407]

[28] Spunt, 1971 [485]

[29] Koiter and Skaloud, 1963 [322]; Thompson and Supple, 1973 [526]; Thompson and Hunt, 1974 [519]

[30] Koiter and Skaloud, 1963 [322]; Hutchinson and Amazigo, 1967 [254]; van der Neut, 1968 [544]; Koiter and Kuiken, 1971 [319]; Thompson and Lewis, 1972 [524]; Tvergaard, 1973, 1973 [537, 538]; Byskov and Hutchinson, 1977 [86]; Gáspár, 1985 [179]; Sridharan and Ali, 1985 [487]; Kołakowski, 1987 [323]; Pignataro and Luongo, 1987 [420]; Rasmussen and Hancock, 1988 [435]; Menken, Groot, and Stallenberg, 1991 [359]; Batista and Batista, 1994 [44]; Pignataro, Pasca, and Franchin, 2000 [421]

[31] Chilver, 1967 [98]; Supple, 1967, 1968 [494, 495]; Sewell, 1970 [469]; Johns and Chilver, 1971 [288]; Ho, 1972, 1974 [226, 227]; Thompson and Hunt, 1973, 1975, 1984 [518, 520, 523]; Gilbert and Calladine, 1974 [187]; Johns, 1974 [286]; Keener, 1974, 1979 [295, 296]; Bauer, Keller, and Reiss, 1975 [45]; Huseyin, 1975 [247]; Svensson and Croll, 1975 [497]; Thompson and Schorrock, 1975 [525]; Koiter and Pignataro, 1976, 1976 [320, 321]; Maquoi and Massonnet, 1976 [352]; Tvergaard and Needleman, 1976 [541]; van der Neut, 1976 [546]; Byskov and Hutchinson, 1977 [86]; Hunt, 1977, 1979, 1981, 1986 [240, 241, 242, 243]; Huseyin and Mandadi, 1977 [249]; Maewal and Nachbar, 1977 [349]; Olhoff and Rasmussen, 1977 [393]; Byskov, 1979 [81]; Budiansky and Hutchinson, 1979 [76]; Reis and Roorda, 1979 [436]; Byskov and Hansen, 1980 [85]; Samuels, 1980 [454]; Foster, 1981 [172]; Hui, Tennyson, and Hansen, 1981 [239]; Haug and Choi, 1982 [217]; Thompson, 1982 [513]; Usami and Fukumoto, 1982 [543]; Sridharan, 1983 [486]; Golubitsky, Marsden, and Shaeffer, 1984 [192]; Hui, 1984 [231]; Masur, 1984 [355]; Benito and Sridharan, 1984-85 [52]; Cowell, 1986 [114]; Hunt, Williams, and Cowell, 1986 [245]; Byskov, Damkilde, and Jensen, 1988-89 [84]; Bousfield and Samuels, 1989 [61]; Abdelmoula, Damil, and Potier-Ferry, 1992 [1]; Ikeda, Providéncia, and Hunt, 1993 [279]; Ohsaki, 2000, 2003 [383, 387]

[32] Thompson, 1963, 1965, 1969 [505, 507, 510]; Sewell, 1966, 1968 [467, 468]; Thompson and Hunt, 1973 [518]

sity and complexity of these points, thereby overshadowing the systematics of static perturbation method. The emergence of catastrophe theory[33] was quite timely and was quickly introduced into elastic stability theory to generalize the classification of critical points.[34] The seven elementary catastrophes were correlated with structural problems. The fold catastrophe corresponds to a limit point or an asymmetric point of bifurcation, and the cusp catastrophe corresponds to a symmetric point of bifurcation.[35] Structural examples of the swallowtail and butterfly cuspoids,[36] hyperbolic umbilic,[37] hyperbolic umbilic and elliptic umbilic,[38] and parabolic umbilic catastrophes[39] were found and investigated in detail. The set of umbilic catastrophes were classified[40] with reference to Zeeman's umbilic bracelet.[41] Higher-order singularities, such as double-cusp catastrophe, falling beyond the seven elementary catastrophes, were found for structures.[42]

The general theory of elastic stability, developed in this manner, played a pivotal role in dealing with initial postbuckling behavior, especially from an analytical standpoint, and led to excellent textbooks.[43] The concept of imperfection sensitivity was quite pertinent in gathering knowledge as to the behavior of imperfect structures. Thereafter research of initial postbuckling behavior and imperfection sensitivity of structures mushroomed

[33] Chillingworth, 1975 [97]; Thom, 1975 [504]; Wassermann, 1976 [560]; Zeeman, 1976, 1976, 1977 [576, 577, 578]; Poston and Stewart, 1978 [429]; Saunders, 1980 [458]; Gilmore, 1981 [188]; Stewart, 1981 [490]; Callahan, 1982 [88]; Thompson, 1982 [513]; Arnold, 1984 [35]; Arnold et al., 1994 [36]

[34] Thompson and Hunt, 1975 [520]

[35] Thompson, 1975 [512]

[36] Hui and Hansen, 1980 [236]

[37] Thompson and Hunt, 1975, 1984 [520, 523]; Thompson and Schorrock, 1975 [525]; Thompson, 1982 [513]

[38] Thompson and Hunt, 1975 [520]; Hansen, 1977 [215]; Hunt, 1977, 1979, 1981 [240, 241, 242]; Huseyin and Mandadi, 1977 [249]

[39] Hui and Hansen, 1980, 1981 [237, 238]

[40] Thompson and Gaspar, 1977 [515]; Samuels, 1979 [453]

[41] Zeeman, 1976 [576]

[42] Thompson and Hunt, 1977 [521]; Poston and Stewart, 1978 [429]; Hui, 1986 [233]

[43] Britvec, 1973 [65]; Thompson and Hunt, 1973, 1984 [518, 523]; Budiansky, 1974 [72]; Huseyin, 1975, 1986 [247, 248]; Thompson, 1982 [513]; Potier-Ferry, 1987 [430]; El Naschie, 1990 [137]; Pignataro, Rizzi, and Luongo, 1991 [422]; Godoy, 2000 [189]

worldwide,[44] and was extended to plastic bifurcation of structures.[45]

2.2 Search for prototype initial imperfections

The studies of imperfect structures presented above dealt with pre-specified imperfection modes. There arose a question, "What is the imperfection to be employed?" The geometrical imperfection in the shape of the relevant buckling mode was used initially.[46] For example, classical axisymmetric buckling modes were used as initial shape imperfections for cylindrical shells,[47] and checkerboard imperfections were assumed for spherical shells.[48] The underlying belief that such initial imperfections realize the worst-case-scenario was addressed later.[49] In addition, dimple imperfections were employed as realistic imperfections.[50]

In this connection, study on the worst imperfection drew considerable attention.[51] The worst imperfection vector of an imperfect cubic potential system was proved to be in the direction of the perfect bifurcated path of the largest slope.[52] The worst imperfection shape of structures was studied

[44] Khot, 1970, 1970 [301, 302]; Amazigo and Fraser, 1971 [15]; Stephens, 1971 [489]; Masur, 1973 [354]; van der Neut, 1973 [545]; Bauld, 1974 [46]; Budiansky, 1974 [72]; Tvergaard, 1976 [539]; Budiansky and Hutchinson, 1979 [76]; Elishakoff, 1980 [140]; Almroth and Rankin, 1983 [7]; Loughlan, 1983 [346]; Thompson and Hunt, 1983 [522]; Yamaki, 1984 [573]; Simitses, Shaw, and Sheinman, 1985 [476]; Hui, 1986, 1986 [232, 233]; Hui and Chen, 1987 [234]; Hui and Du, 1987 [235]; Cohen and Haftka, 1989 [113]; Fan, 1989 [166]; Oyesanya, 1990 [396]; Padney and Sherbourne, 1991 [397]; Librescu and Chang, 1992 [335]; Kardomateas, 1993, 1997 [293, 294]; Ohsaki and Nakamura, 1994 [388]; Palassopoulos, 1994 [405]; Britvec, 1995 [66]; Calladine, 1995 [87]; Fu and Waas, 1995 [175]; Sorić, 1995 [484]; Triantafyllidis and Peek, 1995 [534]; Elishakoff, Li, and Starnes, 1996 [157]; Godoy and Mook, 1996 [191]; Lewis, 1997 [329]; Tabiei and Simitses, 1997 [498]; Tomblin, Barbero, and Godoy, 1997 [531]; Wu, 1999 [568]; Wu and Zhong, 1999 [569]; Lanzo, 2000 [327]; Li et al., 2000 [331]; Godoy and Banchio, 2001 [190]; Ohsaki, 2002 [386]; Jamal et al., 2003 [284]

[45] Hutchinson, 1972, 1973 [252, 253]; Needleman, 1975 [376]; Needleman and Tvergaard, 1976 [378]; Byskov, 1982, 1982-83 [82, 83]; Wunderlich, Rensch, and Obrecht, 1982 [571]; Ming and Wenda, 1990, 1991 [362, 363]; Su and Lu, 1991 [493]; Feldman and Aboudi, 1993 [168]; Lu, Obrecht, and Wunderlich, 1995 [347]; Cheng and Lu, 1997 [95]

[46] Koiter, 1963 [314]; Hutchinson and Amzigo, 1967 [254]; Budiansky and Amazigo, 1968 [73]; Hutchinson, 1968 [251]; Budiansky, 1969 [71]; Hansen and Roorda, 1974 [216]

[47] Koiter, 1963 [314]

[48] Hutchinson, 1967 [250]

[49] Tvergaard and Needleman, 1982 [542]; Ikeda and Murota, 2002 [270]

[50] Amazigo, Budiansky, and Carrier, 1970 [14]; Amazigo and Fraser, 1971 [15]; Amazigo, 1974 [11]

[51] Arbocz, 1974 [19]; Ho, 1974 [227]; Sadovsky, 1978 [451]; Samuels, 1980 [454]; Nishino and Hartono, 1989 [380]; Ikeda and Murota, 1990, 1990 [262, 263]; Murota and Ikeda, 1991 [367]; Bielski, 1992 [56]; Peek and Triantafyllidis, 1992 [416]; Triantafyllidis and Peek, 1992 [533]; Peek, 1993 [414]; Deng, 1994 [126]; Wunderlich and Albertin, 2000 [570]; Ikeda, Oide, and Terada, 2002 [277]

[52] Ho, 1974 [227]

in a more general setting,[53] extended to imperfections other than structural shapes,[54] and implemented into the framework of finite-element analysis.[55] Yet the application of the worst imperfection to realistic structures is not necessarily widespread, as was represented by Bernard Budiansky's question, "One other related thought, that is only vaguely in my mind, is this: Is it possible that a more predominant role should be given to *worst-case* imperfections?"[56]

In association with the development of techniques for measuring initial imperfections, detailed knowledge of geometric imperfections of shells was gathered.[57] Compilation and extensive analysis of the international initial data banks were pursued actively at the Delft University of Technology[58] and at the Israel Institute of Technology.[59] These data banks are useful in deriving characteristic initial imperfection distributions that a given fabrication process is likely to produce. At Imperial College, information on measured imperfections was gathered to produce characteristic imperfection shapes.[60]

2.3 Probabilistic scatter of critical loads

Through the search for the prototype imperfections, it came to be acknowledged that initial imperfections are subjected to probabilistic scatter and that the study of initial imperfections need to be combined with probabilistic treatment to make them practical. As first postulated by Bolotin, 1958 [57], the critical load of a structure can be expressed as a function of a number of random variables representing initial imperfections. The straightforward evaluation of the probability density function of the critical load is divided into the following two stages:

1. Obtain the probability densities of the initial imperfections.

2. Compute the set of critical loads for a given set of initial imperfections

[53] Peek and Triantafyllidis, 1992 [416]; Triantafyllidis and Peek, 1992 [533]; Peek, 1993 [414]

[54] Ikeda and Murota, 1990, 2002 [262, 270]; Murota and Ikeda, 1991 [367]

[55] Ikeda and Murota, 1990 [263]

[56] Arbocz and Singer, 2000 [31]

[57] Arbocz and Babcock, 1968, 1969 [24, 25]; Arbocz and Williams, 1977 [32]; Verduyn and Elishakoff, 1978 [553]; Guiggiani, 1989 [203]; Megson and Hallak, 1992 [358]; Chryssanthopoulos, Giavotto, and Poggi, 1995 [108]; Park and Kyriakides, 1996 [413]; Bernard, Coleman, and Bridge, 1999 [53]; Pircher and Wheeler, 2003 [424]

[58] Arbocz and Abramovich, 1979 [23]; Arbocz, 1981, 1982, 1982 [20, 21, 22]; Klompé, 1986, 1988, 1989 [307, 308, 309]; Dancy and Jacobs, 1988 [121]; Klompé and den Reyer, 1989 [310]; de Vries, 2001 [125]

[59] Singer, Abramovich, and Yaffe, 1978, 1981 [480, 481]; Abramovich, Singer, and Yaffe, 1981 [2]

[60] Scott, Harding, and Dowling, 1987 [463]; Chryssanthopoulos, Baker, and Dowling, 1991, 1991 [106, 107]; Chryssanthopoulos and Poggi, 1995 [109]

and, in turn, to obtain the probability density function of the critical load.

The first and the most difficult stage was tackled through a series of attempts. For columns and bars, random imperfections were employed.[61] For shells, random axisymmetric imperfections[62] and general (non-symmetric) random initial imperfections[63] were employed. Initial imperfections were often assumed to be Gaussian random variables.[64] In addition to the initial imperfection of structural shapes, various kinds of imperfections, such as loadings,[65] material properties (elastic moduli),[66] and thickness variation[67] were considered.

To tackle the second stage, a series of studies were conducted, as introduced below. The imperfection sensitivity law was used as a transfer function from an initial imperfection to the deterministic critical load and, in turn, to obtain the probabilistic variation of critical load for an imperfection with a known probabilistic property.[68] Such use of imperfection sensitivity, however, was limited to a certain prototype imperfection. As a remedy of this, an initial imperfection was represented as a random process and the method of stochastic differential equations was used to obtain an asymptotic relationship between the critical load and initial imperfection.[69] A branch of application of imperfection sensitivity was found in the optimiza-

[61] Boyce, 1961 [62]; Fraser, 1965 [173]; Fraser and Budiansky, 1969 [174]; Amazigo, Budiansky, and Carrier, 1970 [14]; Amazigo, 1971, 1976 [10, 12]; Bernard and Bogdanoff, 1971 [54]; Videc and Sanders, 1976 [555]; Elishakoff, 1979, 1980 [139, 141]; Miller and Hedgepeth, 1979 [361]; Day, Karwowski, and Papanicolaou, 1989 [124]; Palassopoulos, 1989 [400]

[62] Amazigo, 1969 [9]; Tennyson, Muggeridge, and Caswell, 1971 [502]; Amazigo and Budiansky, 1972 [13]; Roorda and Hansen, 1972 [448]; Slooten and Soong, 1972 [547]; Elishakoff and Arbocz, 1982 [151]

[63] Arbocz and Babcock, 1968, 1969 [24, 25]; Makarov, 1971 [350]; Slooten and Soong, 1972 [547]; Amazigo, 1974 [11]; Fersht, 1974 [169]; Hansen, 1975, 1977 [213, 214]; Palassopoulos, 1980 [399]; Elishakoff, 1985 [145]; Elishakoff and Arbocz, 1985 [152]; Arbocz and Hol, 1991 [30]; Bielewicz et al., 1994 [55]; Schenk, Schuëller, and Arbocz, 2000 [460]

[64] Amazigo, 1969 [9]; Hansen and Roorda, 1974 [216]; Hansen, 1975, 1977 [213, 214]; Elishakoff, 1978, 1979 [138, 139]

[65] Roorda, 1980 [446]; Elishakoff, 1983 [144]; Ikeda and Murota, 1990 [263]; Li, 1990 [330]; Li et al., 1995 [333]; Cederbaum and Arbocz, 1996, 1997 [90, 91]; Królak, Kołakowski, and Kotełko, 2001 [324]

[66] Shinozuka, 1987 [473]; Bucher and Shinozuka, 1988 [69]; Kardara, Bucher, and Shinozuka, 1989 [292]; Ikeda and Murota, 1990 [263]; Elishakoff, Li, and Starnes, 1994, 1996 [156, 157]

[67] Tvergaard, 1976 [540]; Bielski, 1992 [56]; Koiter et al., 1994 [318]; Li et al., 1995 [332]; Gusic, Combescure, and Jullien, 2000 [206]

[68] Bolotin, 1958 [57]; Thompson, 1967 [509]; Roorda, 1969 [444]; Roorda and Hansen, 1972 [448]

[69] Amazigo, 1969, 1971, 1974 [9, 10, 11]; Fraser and Budiansky, 1969 [174]; Amazigo, Budiansky, and Carrier, 1970 [14]; Amazigo and Budiansky, 1972 [13]; Slooten and Soong, 1972 [547]

tion of imperfection-sensitive structures.[70] Sensitivity coefficients—*design sensitivity coefficients*—of linear buckling load factor with respect to design variables, such as stiffness and nodal locations, were employed for design.

The probability of failure was employed to express the influence of the randomness of experimentally measured imperfections[71] and paved the way for the introduction of the results of statistical methods.[72] A series of studies based on measured data were conducted,[73] often with resort to the international initial imperfection data banks, from which stochastic imperfections with known average and autocorrelation can be produced. The Monte Carlo simulation came to be conducted to compute numerically the reliability of the buckling strength for measured or random initial imperfections.[74] An asymptotic approach was combined with statistical analysis.[75] Koiter's special theory[76] for axisymmetric imperfections was combined with the Monte Carlo method[77] and, in turn, to introduce imperfection-sensitivity concept into design procedure.[78] The Monte Carlo method was replaced by the first-order second-moment method to reduce computational costs considerably.[79]

In order to overcome possible limitations of probabilistic methods,[80] a few attempts to arrive at a lower bound of strength were conducted:

- *Knockdown factor* based on the so-called *lower bound design philosophy* is the most primitive but most robust way.[81] Engineers are reluctant to use the concept of imperfection sensitivity and prefer to rely on the *knockdown factor*, as was pointed out repeatedly.[82]

- The *reduced stiffness* method finds a lower bound of design strength of

[70] Haug, Choi, and Komkov, 1986 [218]; Palassopoulos, 1989, 1991 [400, 401]; Ohsaki and Nakamura, 1994 [388]; Reitinger and Ramm, 1995 [438]; Ohsaki, 2001 [385]

[71] Hansen and Roorda, 1974 [216]; Augusti and Barratta, 1976 [39]; Johns, 1976 [287]; Elishakoff, 1978, 1979 [138, 139]

[72] Bolotin, 1969, 1984 [58, 59]; Ang and Tang, 1975 [18]; Thoft-Christensen and Baker, 1982 [503]; Elishakoff, 1983 [143]; Augusti, Barratta, and Casciati, 1984 [40]; Ben-Haim, 1996 [49]; Haldar and Mahadevan, 2000 [208]

[73] Arbocz and Babcock, 1976, 1978 [26, 27]; Tennyson, 1976 [501]; Elishakoff and Arbocz, 1982 [151]; Elishakoff, 1982, 1988 [142, 146]; Arbocz and Hol, 1990, 1991 [29, 30]; Turčić, 1991 [536]; Ikeda, Murota, and Elishakoff, 1996 [271]; Doup, 1997 [134]; Pircher et al., 2001 [423]; Lin and Teng, 2003 [336]

[74] Edlund and Leopoldson, 1975 [136]; Hansen, 1977 [214]; Elishakoff, 1978, 1979, 1980 [138, 139, 141]; Elishakoff and Arbocz, 1982 [151]; Wang, 1990 [559]; Palassopoulos, 1992 [403]

[75] Palassopoulos, 1991, 1992 [402, 403]; Trendafilova and Ivanova, 1995 [532]

[76] Koiter, 1963 [314]

[77] Elishakoff and Arbocz, 1982 [151]

[78] Elishakoff, 1983, 1998 [144, 149]

[79] Karadeniz, van Manen, and Vrouwenvelder, 1982 [289]; Elishakoff, van Manen, Vermeulen, and Arbocz, 1987 [162]; Arbocz and Hol, 1991 [30]

[80] Elishakoff, 2000 [150]

[81] Weingarten, Morgan, and Seide, 1965 [562]; NASA, 1968 [373]

[82] Arbocz and Hol, 1990 [29]; Elishakoff, Li, and Starnes, 2001 [158]

a shell through the identification of the components of the membrane energy that are eroded by imperfections and mode interactions.[83]

- Convex modeling of uncertainty, robust reliability, and anti-optimization approach were developed to estimate the lower bounds based on the worst-case-scenario of problems where scarce knowledge is present and the use of probabilistic method cannot be justified.[84] The anti-optimization approach was employed also for other purposes, such as optimization.[85]

Finite element methods, such as STAGS,[86] were employed to deal with realistic imperfections of structures and, in turn, to investigate the stochastic properties and reliability of their strength. Koiter's asymptotic approach was combined with the finite element method to be consistent with computer aided engineering environments.[87]

Stochastic finite element methods (SFEM)[88] or finite element method for stochastic problems (FEMSP)[89] were employed to numerically tackle the probabilistic properties of structures. The perturbation method was employed for most cases to deal with random quantities involved, and the second-moment analysis was often employed to compute the mean and the variance of the displacement or stress. The elastic modulus was often modeled as a random field in SFEM. See the exhaustive review by Schuëller, 1997 [462] for more account of SFEM.

The response surface approach was used to evaluate the reliability of structures;[90] nonlinear finite element analyses, for example, were conducted

[83] Croll, 1981 [118]; Croll and Batista, 1981 [119]

[84] Ben-Haim and Elishakoff, 1990 [50]; Elishakoff, 1990, 1991 [147, 148]; Lindberg, 1992 [338]; Ben-Haim, 1993, 1996 [48, 49]; Elishakoff and Colombi, 1993 [154]; Elishakoff, Cai, and Starnes, 1994 [153]; Elishakoff, Haftka, and Fang, 1994 [155]; Elishakoff, Li, and Starnes, 1994, 2001 [156, 158]; Lombardi and Haftka, 1998 [345]; Zingales and Elishakoff, 2000 [583]; Papadimitriou, Beck, and Katafygiotis, 2001 [408]

[85] Adali et al., 1994, 1997 [3, 4]; Lee et al., 1994 [328]

[86] Almroth et al., 1973 [6]; Arbocz and Babcock, 1978 [27]

[87] Haftka, Mallet, and Nachbar, 1971 [207]; Arbocz and Hol, 1989, 1990 [28, 29]; Casciaro, Salerno, and Lanzo, 1992 [89]

[88] Astill, Nosseir, and Shinozuka, 1972 [37]; Nakagiri and Hisada, 1980, 1985 [371, 372]; Handa and Andersson, 1981 [212]; Hisada and Nakagiri, 1981 [224]; Vanmarcke and Grigoriu, 1983 [549]; der Kiureghian, 1985 [128]; Liu, Belytschko, and Mani, 1986, 1986 [341, 342]; Liu, Besterfield, and Mani, 1986 [343]; Vanmarcke et al., 1986 [550]; Benaroya and Rehak, 1988 [51]; der Kiureghian and Ke, 1988 [129]; Shinozuka and Yamazaki, 1988 [474]; Hisada and Noguchi, 1989 [225]; Ghanem and Spanos, 1991 [184]; Kleiber and Hien, 1992 [306]; Ramu and Ganesan, 1993 [434]; Brenner and Bucher, 1995 [63]; Hoshiya and Yoshida, 1995 [230]; Zhang and Ellingwood, 1995 [580]; Zhang et al., 1996 [579]; Papadopoulos and Papadrakakis, 1998 [409]; der Kiureghian and Zhang, 1999 [130]; Elishakoff and Ren, 1999 [160]; Matthies and Bucher, 1999 [357]; Haldar and Mahadevan, 2000 [209]

[89] Elishakoff and Ren, 2003 [161]

[90] Faravelli, 1989 [167]; Bucher and Bourgund, 1990 [68]; Rajashekhar and Ellingwood, 1993 [433]; Liu and Moses, 1994 [344]; Myers and Montgomery, 1995 [370]; Kim and

for the parameter sweep of a few initial imperfections and/or design parameters to evaluate the reliability.

2.4 Bifurcation mechanism for strength of materials

The study of the scatter of strength of materials followed a completely different course of development than that of structures. Weibull, 1939 [561] derived a famous distribution of the strength of materials due to fracture on the basis of the weakest link theory.

Bifurcation mechanism was taken into consideration in the description of strength of materials, as introduced below:

- In mechanical instability of stressed atomic crystal lattices[91] and in numerical simulation of a long tensile steel specimen undergoing plastic instability,[92] a nearly coincidental pair of critical points of a bifurcation point and a limit point of loading parameter was found. Such a pair of points was approximated by a *hilltop branching (bifurcation) point*, at which the pair of points coincide strictly. This hilltop point was shown to enjoy locally piecewise linear imperfection sensitivity,[93] which is less severe than the two-thirds power-law for a simple pitchfork bifurcation point.

- A structural form comprising of n unlinked separate identical cells underwent explosive bifurcation at a hilltop bifurcation point with n-fold criticality and as many as 2^n bifurcated paths.[94]

- Strictly coincidental hilltop branching points were found through optimization of structural systems.[95]

- Plastic bifurcation, through the formation of shear band, is acknowledged to govern the strength of materials, such as metals[96] and geomaterials.[97] The horizon of the study of (elastic) bifurcation is extending towards the failure and deformation of materials. Elastic, diffuse mode

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Na, 1997 [303]; Chryssanthopoulos, 1998 [105]; Venter, Haftka, and Starnes, 1998 [552]; Das and Zheng, 2000 [122]; Gayton, Bourinet, and Lemaire, 2003 [183]; Falsone and Impollonia, 2004 [165]; Gomes and Awruch, 2004 [199]; Gupta and Manohar, 2004 [205]; Romero, Swiler, and Giunta, 2004 [439]

[91] Thompson and Schorrock, 1975 [525]; Thompson, 1982 [513]

[92] Needleman, 1972 [375]; Hutchinson and Miles, 1974 [256]; Burke and Nix, 1979 [78]; Okazawa et al., 2002 [390]

[93] Thompson and Schorrock, 1975 [525]; Thompson, 1982 [513]; Ikeda, Oide, and Terada, 2002 [277]; Okazawa et al., 2002 [390]

[94] Ikeda, Providéncia, and Hunt, 1993 [279]

[95] Ohsaki, 2000, 2003 [383, 387]

[96] Hill and Hutchinson, 1975 [222]; Young, 1976 [575]; Needleman, 1979 [377]

[97] Vermeer, 1982 [554]; Prevost, 1984 [431]; Vardoulakis, 1986 [551]; Chau and Rudnicki, 1990 [93]

bifurcation was shown to occur in soil specimens through the combination of asymptotic method, group-theoretic bifurcation theory, and numerical procedure.[98] This combined procedure was successfully applied to the stochastic description of the strength of steels and low strength concretes.[99]

2.5 Symmetry and bifurcation phenomena

The underlying role of symmetry of structures undergoing bifurcation was gradually understood through the application of the asymptotic approach to individual structural systems. The symmetry systematically annihilates certain terms of the potential function to create coincidental and compound bifurcations, at which complex but interesting phenomena emerge. Major developments are:

- The reflection symmetry with respect to two independent variables was imposed on the Taylor expansion of the potential function, for a twofold coincidental bifurcation point.[100]

- For double-cusp catastrophe, all cubic terms of the potential are annihilated by the symmetry in combination of the two modes competing at the coincidental point.[101]

- Complex secondary bifurcations, often forming loops, were found in association with compound branching[102] and the relation with symmetries was studied under the names of semi-symmetry[103] and hidden symmetry.[104]

- The Augusti model[105] was studied by the parametric sweep of a design parameter to generate secondary branching at near-coincidence and direct branching at strict-coincidence.[106] At the coincidence (optimization), the model is endowed with three-axes symmetry, which is higher than two-axes symmetry at near-coincidence.

[98] Ikeda and Goto, 1993 [259]; Ikeda and Murota, 1996, 1997, 1999, 2002 [267, 268, 269, 270]; Ikeda, Chida, and Yanagisawa, 1997 [258]; Ikeda et al., 1997 [274]; Ikeda, Yamakawa, and Tsutsumi, 2003 [280]

[99] Ikeda et al., 1997, 2001 [260, 278]; Okazawa et al., 2002 [390]

[100] Chilver, 1967 [98]; Supple, 1967 [494]; Mandadi and Huseyin, 1978 [351]

[101] Thompson, 1984 [514]

[102] Chilver, 1967 [98]; Supple, 1967, 1968 [494, 495]; Thompson and Supple, 1973 [526]; Keener, 1974, 1979 [295, 296]; Bauer, Keller, and Reiss, 1975 [45]; Cowell, 1986 [114]; Bousfield and Samuels, 1989 [61]

[103] Thompson and Hunt, 1975 [520]; Hunt, 1977, 1979, 1981 [240, 241, 242]

[104] Golubitsky, Marsden, and Shaeffer, 1984 [192]; Hunt, 1986 [243]; Hunt, Williams, and Cowell, 1986 [245]

[105] Augusti, 1964 [38]

[106] Thompson and Hunt, 1973, 1984 [518, 523]

- Higher symmetry leads to complex bifurcation behavior. Complete spherical shells displayed rotationally-symmetric branching.[107] Diamond and other characteristic patterns were observed in cylindrical shells.[108] Axisymmetric and regular-polygonal shell and domes have double bifurcation points, at which complex but interesting phenomena take place, as we will see in Section 6.[109]

- Patterns appear ubiquitously for materials. The echelon mode can, for example, be found in various materials: soils,[110] rocks,[111] and metals.[112] Periodic shear bands of materials forming an echelon mode were simulated.[113] The cross-checker pattern was found in metals[114] and in the zebra patterns on the ocean floors.[115] A self-similar pattern model has been introduced.[116] The instability of a shell surface was employed to produce intriguing patterns of plants, such as sunflower and snapdragon.[117]

- Honeycomb structures, which have regular-hexagonal and in-plane translational symmetries, subjected to compression exhibited a series of characteristic deformation patterns during experiments.[118] A flower mode was observed experimentally[119] and simulated successfully by finite-element buckling/bifurcation analyses.[120] A gigantic honeycomb-like pattern[121] was found as a consequence of a secondary branching of an elasto-plastic honeycomb structure.[122]

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[107] Thompson, 1964 [506]; Lange and Kriegsmann, 1981 [326]

[108] Yoshimura, 1951 [574]; Esslinger and Geier, 1976 [163]; Yamaki, 1976, 1984 [572, 573]

[109] Ikeda, Murota, and Fujii, 1991 [272]; Wohlever and Healey, 1995 [566]; Fujii et al., 2001 [176]; Ikeda and Murota, 2002 [270]

[110] Ikeda, Murota, and Nakano, 1994 [273]; Ikeda and Murota, 1997 [268]

[111] Pollard, Segal, and Delaney, 1982 [427]; Davis, 1984 [123]; Petit, 1988 [417]; Smith, 1995 [482]

[112] Poirier, 1985 [426]; Bai and Dodd, 1992 [42]; Duszek–Perzyna and Perzyna, 1993 [135]

[113] Petryk and Thermann, 2002 [418]

[114] Voskamp and Hollox, 1998 [558]

[115] Nicolas, 1995 [379]

[116] Archambault et al., 1993 [33]

[117] Karam and Gibson, 1995, 1995 [290, 291]; Green, Steele, and Rennich, 1996 [202]; Green, 1999 [201]; Steele, 2000 [488]

[118] Gibson et al., 1989 [186]; Papka and Kyriakides, 1994 [410]; Gibson and Ashby, 1997 [185]; Triantafyllidis and Schraad, 1998 [535]; Zhu and Mills, 2000 [582]

[119] Papka and Kyriakides, 1999 [411]

[120] Chung and Waas, 1999, 2001 [110, 111]; Guo and Gibson, 1999 [204]; Papka and Kyriakides, 1999 [412]; Ohno, Okumura, and Noguchi, 2002 [381]; Okumura, Ohno, and Noguchi, 2002 [391]

[121] Large circles are formed as an assemblage of a number of deformed hexagonal cells and such circles are regularly arranged in space to form this pattern.

[122] Okumura, Ohno, and Noguchi, 2004 [392]

2.6 Mathematical treatment of symmetry in bifurcation

Mathematical treatment of symmetry in science can be found in introductory books.[123] Group is an established means to describe geometrical symmetry and its theoretical backgrounds are readily available.[124] The symmetry of molecules and crystals was studied in chemical crystallography to invent the point groups, which describe the spatial symmetry around a point.[125] Group-theoretic method, such as group representation theory, for crystallographic symmetries was employed to generate constitutive relations of materials.[126]

Among these groups, we are particularly interested in the dihedral groups D_n (for some integer n) and D_∞, which respectively represent regular-polygonal and circular symmetries. (The definition of these groups will be given in Section 6.1.) In fact, the symmetry of the structures introduced above can be labeled by groups as follows:

- Reflection symmetry with respect to an independent variable: D_1-symmetry.
- The Augusti model: D_1-symmetry at near-coincidence and D_3-symmetry at strict-coincidence.
- Shells and domes of revolution and perfectly spherical shells: D_n- (n large) or D_∞-symmetry in the circumferential direction.

We now shift an eye on bifurcation theory in nonlinear mathematics, which took a somewhat separate course of development, but formed the foundation of elastic stability theory and turned out to be vital in the classification and investigation of multiple bifurcation points of structures. Early development of branching theory can be traced to the use of nonlinear perturbation theory.[127] Perfect and imperfect bifurcation behavior drew keen mathematical interest,[128] and blossomed into universal unfolding[129] to describe general forms of imperfect systems at the presence of general initial imperfections.

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[123] Weyl, 1952 [565]; Stewart and Golubitsky, 1992 [491]; Baggot, 1994 [41]; Icke, 1995 [257]; Rosen, 1995 [449]

[124] Curtis and Reiner, 1962 [120]; Hamermesh, 1962 [211]; Hall, 1967 [210]; Miller, 1972 [360]; Hill, 1975 [223]; Serre, 1977 [465]; Rotman, 1984 [450]; Armstrong, 1988 [34]; James and Liebeck, 1993 [285]; Ludwig and Falter, 1996 [348]; Kim, 1999 [304]

[125] Ladd, 1989 [325]; Senechal, 1990 [464]; Prince, 1994 [432]; Kettle, 1995 [299]; Ludwig and Falter, 1996 [348]; Kim, 1999 [304]

[126] Kiral and Smith, 1974 [305]; Choudhury and Glockner, 1979 [101]; Zhong and del Piero, 1991 [581]; Cowin and Mehrabadi, 1995 [115]; Hong and Liu, 1999 [228]; Liu, 2003 [340]

[127] Poincaré, 1892–1899 [425]

[128] Keener and Keller, 1973 [297]; Hansen and Roorda, 1974 [216]; Keener, 1974 [295]; Chow, Hale, and Mallet-Panet, 1975, 1976 [103, 104]; Matkowsky and Reiss, 1977 [356]; Reiss, 1977 [437]; Golubitsky and Schaeffer, 1979, 1979 [193, 194]

[129] Golubitsky and Schaeffer, 1979, 1985 [193, 195]

Group-theoretic bifurcation theory made great progress in the 1980's to describe the mechanism of instability and chaos,[130] and of pattern formation of flows.[131] This theory was applied to the numerical computation of the global bifurcation problems with symmetry.[132] The loss of symmetry at the onset of bifurcation can be investigated theoretically by local bifurcation analysis using the Liapunov–Schmidt reduction and the exploitation of the symmetry of the bifurcation equation.[133] The rule of such bifurcation can be determined by the symmetry of the system under consideration, and possible critical points and bifurcated solutions can be classified systematically.

A number of attempts have been conducted to introduce the methodology of group-theoretic bifurcation theory to the study of structures. The direct, secondary, tertially, ... bifurcations of structures with axisymmetric and regular-polygonal symmetries, such as reticulated domes and cylindrical shells, were studied as a bifurcation problem of dihedral-group symmetry.[134] The emergence of the secondary branching, as was observed for the Augusti model, can be ascribed with the presence of an initial imperfection with partial symmetry (cf., Section 6.3).[135] Block-diagonalization method was employed to exploit symmetry and to reduce the computational cost involved.[136] Related mathematical results for a bifurcation problem of a system with dihedral-group symmetry[137] and block diagonalization[138] are available.

Hidden periodic symmetry was investigated by employing periodic bound-

[130] Sattinger, 1980 [457]; Golubitsky and Schaeffer, 1985 [195]; Golubitsky, Stewart, and Schaeffer, 1988 [198]; Mitropolsky and Lopatin, 1988 [364]; Allgower, Böhmer, and Golubitsky, 1992 [5]; Marsden and Ratiu, 1994 [353]; Olver, 1995 [395]; Golubitsky and Stewart, 2002 [197]; Ikeda and Murota, 2002 [270]

[131] Schaeffer, 1980 [459]; Iooss, 1986 [281]; Iooss and Joseph, 1990 [283]; Bakker, 1991 [43]; Crawford and Knobloch, 1991 [117]; Iooss and Adelmeyer, 1992 [282]; Chossat, 1994 [99]; Chossat and Iooss, 1994 [100]; Marsden and Ratiu, 1994 [353]; Seydel, 1994 [472]; Buescu et al., 2000 [77]; Moehlis and Knobloch, 2000 [365]; Okamoto and Shoji, 2001 [389]

[132] Werner and Spence, 1984 [564]; Weinitschke, 1985 [563]; Healey, 1988 [220]; Dellnitz and Werner, 1989 [127]; Gatermann and Hohmann, 1991 [181]; Stork and Werner, 1991 [492]

[133] Sattinger, 1978, 1979 [455, 456]; Vanderbauwhede, 1982 [548]; Golubitsky and Schaeffer, 1985 [195]; Golubitsky, Stewart, and Schaeffer, 1988 [198]; Allgower, Böhmer, and Golubitsky, 1992 [5]; Ikeda and Murota, 2002 [270]

[134] Healey, 1988 [219]; Ikeda, Murota, and Fujii, 1991 [272]; Gatermann, 1993 [180]; Gatermann and Werner, 1994 [182]; Wohlever and Healey, 1995 [566]; Ikeda and Murota, 2002 [270]

[135] Ikeda and Murota, 2002 [270]

[136] Healey, 1988 [219]; Zloković, 1989 [584]; Dinkevich, 1991 [131]; Healey and Treacy, 1991 [221]; Ikeda and Murota, 1991 [265]; Murota and Ikeda, 1991 [366]

[137] Sattinger, 1979 [456]; Fujii, Mimura, and Nishiura, 1982 [177]; Buzano, Geymonat, and Poston, 1985 [80]; Golubitsky and Stewart, 1986 [196]; Golubitsky, Stewart, and Schaeffer, 1988 [198]; Dellnitz and Werner, 1989 [127]

[138] Bossavit, 1986 [60]; Chen and Sameh, 1989 [94]; Gatermann and Hohmann, 1991 [181]; Stork and Werner, 1991 [492]; Gatermann and Werner, 1994 [182]; Govaerts, 2000 [200]

aries. The bifurcation hierarchy of a rectangular plate was investigated.[139] Bifurcation mechanism underlying echelon mode formation was made clear.[140] As a model for geometrical patterns of joints and folds, bifurcation mechanism for pattern formation in three-dimensional uniform materials was studied.[141] Group-theoretic study of honeycomb patterns is underway.[142]

Recently, the importance of the asymptotic approaches has been revived through the combination with group-theoretic bifurcation theory. As will be reviewed in Section 6, the explicit formulas for bifurcation equation, imperfection sensitivity laws, probabilistic variation of strength, and experimentally observed bifurcation diagram were obtained for the double bifurcation point of a system with dihedral-group symmetry.[143]

3 SIMPLE CRITICAL POINTS

The general framework of the asymptotic approach is presented and is applied to simple critical points. The explicit forms of bifurcation equation, imperfection sensitivity laws, probabilistic variation of strength, and experimentally observed bifurcation diagram are presented.

3.1 General framework

The general framework to describe the asymptotic behavior in the neighborhood of a critical point is presented. Among a few alternatives for deriving the bifurcation equation, we employ here the method[144] employing the Taylor expansion of the bifurcation equation in favor of its simplicity so as to make the discussion accessible for engineers.[145] This method can implement a large number of initial imperfections and, hence, is suited for the description of realistic structures.

We consider a system of nonlinear governing or equilibrium equations

$$\mathbf{F}(\mathbf{u}, f, \mathbf{v}) = \mathbf{0}, \tag{1}$$

where $\mathbf{u} \in \mathbf{R}^N$ indicates an N-dimensional independent variable vector ($\mathbf{R}$ is the set of real numbers); $f \in \mathbf{R}$ denotes a bifurcation parameter; and $\mathbf{v} \in \mathbf{R}^p$ denotes a p-dimensional imperfection parameter vector. We assume

[139] Ikeda and Nakazawa, 1998 [275]

[140] Ikeda, Murota, and Nakano, 1994 [273]; Murota, Ikeda, and Terada, 1999 [369]; Ikeda et al., 2001 [261]

[141] Tanaka, Saiki, and Ikeda, 2002 [499]

[142] Saiki, Ikeda, and Murota, 2004 [452]

[143] Murota and Ikeda, 1991, 1992 [367, 368]; Ikeda and Murota, 1993, 1999, 2002 [266, 269, 270]

[144] Iooss and Joseph, 1990 [283]; Ikeda and Murota, 2002 [270]

[145] The expansion of the potential was employed, for example, by Thompson, 1963 [505]; Supple, 1967 [494]; Chilver, 1967 [98].

that $\mathbf{F} : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^p \to \mathbf{R}^N$ is a sufficiently smooth nonlinear function in $\mathbf{u}$, f, and $\mathbf{v}$. Note that we deal with a finite-dimensional problem, mainly for mathematical simplicity.[146]

For a fixed $\mathbf{v}$, solutions $(\mathbf{u}, f) = (\mathbf{u}(\mathbf{v}), f(\mathbf{v}))$ of the above system of equations (1) make up solution curves. The solution points are divided into two types, *ordinary* or *critical* points, according to whether the *Jacobian matrix* (or the *tangent stiffness matrix* in structural mechanics)

$$J(\mathbf{u}, f, \mathbf{v}) = (J_{ij}) = \left(\frac{\partial F_i}{\partial u_j} \right) \tag{2}$$

is nonsingular or singular. That is,

$$\det(J(\mathbf{u}, f, \mathbf{v})) = \begin{cases} \text{nonzero} & \text{at an ordinary point,} \\ 0 & \text{at a critical (singular) point,} \end{cases} \tag{3}$$

where $\det(\cdot)$ denotes the determinant of the matrix therein.

In a sufficiently small neighborhood of an ordinary point, the implicit function theorem applies, and for each f there exists a unique $\mathbf{u} = \mathbf{u}(f)$ such that $(\mathbf{u}(f), f)$ is a solution to (1). Here the imperfection parameter $\mathbf{v}$ is kept fixed and, therefore, suppressed in the notation $\mathbf{u} = \mathbf{u}(f)$.

In the neighborhood of a critical point, say $(\mathbf{u}_\mathrm{c}, f_\mathrm{c}) = (\mathbf{u}_\mathrm{c}(\mathbf{v}), f_\mathrm{c}(\mathbf{v}))$, an interesting phenomenon can possibly occur, where $(\cdot)_\mathrm{c}$ denotes a variable related to the critical point. The Jacobian matrix $J_\mathrm{c} = J(\mathbf{u}_\mathrm{c}, f_\mathrm{c}, \mathbf{v})$ at $(\mathbf{u}_\mathrm{c}, f_\mathrm{c}, \mathbf{v})$ is singular by the definition of a critical point, i.e.,

$$\det[J(\mathbf{u}_\mathrm{c}, f_\mathrm{c}, \mathbf{v})] = 0, \tag{4}$$

and the behavior of $\mathbf{u} = \mathbf{u}(f)$ around $(\mathbf{u}_\mathrm{c}, f_\mathrm{c})$ is not governed by the implicit function theorem. This admits the possibility of *bifurcation*, the emergence of multiple solution paths.

The *multiplicity* M of a critical point $(\mathbf{u}_\mathrm{c}, f_\mathrm{c})$ is defined as the rank deficiency of the Jacobian matrix, that is,

$$M = \dim[\ker(J_\mathrm{c})] = N - \mathrm{rank}(J_\mathrm{c}),$$

where $\ker(J_\mathrm{c})$ denotes the *kernel space* of J_c defined as $\ker(J_\mathrm{c}) = \{\mathbf{u} \mid J_\mathrm{c}\mathbf{u} = \mathbf{0}\}$. The critical point $(\mathbf{u}_\mathrm{c}, f_\mathrm{c})$ is a simple point or a multiple point according to whether $M = 1$ or $M \geq 2$.

Let $\{\boldsymbol{\xi}_i \mid i = 1, \ldots, M\}$ and $\{\boldsymbol{\eta}_i \mid i = 1, \ldots, M\}$ be two families of the space[147] of independent vectors of $\mathbf{R}^N$ such that

$$\boldsymbol{\xi}_i^{\mathrm{T}} J_\mathrm{c} = \mathbf{0}^{\mathrm{T}}, \qquad J_\mathrm{c}\boldsymbol{\eta}_i = \mathbf{0}, \qquad i = 1, \ldots, M. \tag{5}$$

[146] We aim at presenting the main ideas for engineers without sacrificing the mathematical rigor. To this end we restrict ourselves to finite-dimensional equations. For a thorough treatment, the reader is referred to Sattinger, 1979 [456]; Chow and Hale, 1982 [102]; Golubitsky and Schaeffer, 1985 [195].

[147] We intend $\{\boldsymbol{\eta}_j\}$ to be a basis of the space of the vectors $\mathbf{u}$, and $\{\boldsymbol{\xi}_j\}$ a basis (of the values) of $\mathbf{F}$. The bases need not be orthogonal, although orthogonal bases are a natural choice for a potential system.

Such vectors $\boldsymbol{\xi}_i$ $(i = 1, \ldots, M)$ are called the *left critical eigenvectors*, and $\boldsymbol{\eta}_i$ $(i = 1, \ldots, M)$ the *right critical (eigen)vectors*. Note that $\{\boldsymbol{\xi}_i \mid i = 1, \ldots, M\}$ and $\{\boldsymbol{\eta}_i \mid i = 1, \ldots, M\}$ span the kernel of $(J_{\mathrm{c}})^{\mathrm{T}}$ and J_{c}, respectively. Also note that orthogonality is not imposed in general, although it is a natural and convenient requirement in some cases. Critical eigenvectors play a crucial role in deriving a reduced system of equations, the bifurcation equation, in Section 3.2.

For the imperfection parameter vector $\mathbf{v}$, we often express it as

$$\mathbf{v} = \mathbf{v}^0 + \epsilon \mathbf{d}, \tag{6}$$

where $\mathbf{v}^0$ denotes the value of the imperfection parameter vector $\mathbf{v}$ for the perfect system, $\mathbf{d}$ is called the *imperfection pattern vector* (normalized appropriately), and ϵ denotes the magnitude of initial imperfection that represents the amount of deviation from the perfect case (ϵ can be negative). We define the *imperfection sensitivity matrix*, $B(\mathbf{u}, f, \mathbf{v})$, as the $N \times p$ matrix consisting of the partial derivatives of F_i with respect to v_j, i.e.,

$$B(\mathbf{u}, f, \mathbf{v}) = \frac{\partial \mathbf{F}}{\partial \mathbf{v}}(\mathbf{u}, f, \mathbf{v}) = \left(\frac{\partial F_i}{\partial v_j}\right). \tag{7}$$

This matrix plays the major role in the analysis of the effect of imperfections.

In what follows we use $(\cdot)^0$ to denote variables associated with the perfect system. For example, we express by $(\mathbf{u}_{\mathrm{c}}^0, f_{\mathrm{c}}^0)$ a critical point for the perfect system described by $\mathbf{v} = \mathbf{v}^0$, and put $J_{\mathrm{c}}^0 = J(\mathbf{u}_{\mathrm{c}}^0, f_{\mathrm{c}}^0, \mathbf{v}^0)$ and $B_{\mathrm{c}}^0 = B(\mathbf{u}_{\mathrm{c}}^0, f_{\mathrm{c}}^0, \mathbf{v}^0)$.

3.2 Liapunov–Schmidt reduction

In this section we explain a standard procedure, the *Liapunov–Schmidt reduction*, that reduces the whole system of equations to M equations locally in a neighborhood of a critical point of multiplicity M. The Liapunov–Schmidt reduction can be done in a fairly general setting of functional equations, but we explain the main ideas for a simple critical point $(M = 1)$ of the finite-dimensional equation (1).

Let $(\mathbf{u}_{\mathrm{c}}^0, f_{\mathrm{c}}^0)$ be a simple critical point of the perfect system with $\mathbf{v} = \mathbf{v}_0$, and we set $M = 1$ in the formulation of Section 3.1. We express the state variable $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{u}_{\mathrm{c}}^0 + \sum_{j=1}^{N} w_j \boldsymbol{\eta}_j \tag{8}$$

in terms of incremental variables $(w_j \mid j = 1, \ldots, N)$, and express the bifurcation parameter f as

$$f = f_{\mathrm{c}}^0 + \widetilde{f}, \tag{9}$$

where $\widetilde{f}$ means the increment of f.

With the use of these incremental variables, the original system (1) of equations is rewritten as

$$\boldsymbol{\xi}_1^{\mathrm{T}} \mathbf{F}\Big(\mathbf{u}_{\mathrm{c}}^0 + \sum_{j=1}^{N} w_j \boldsymbol{\eta}_j, f_{\mathrm{c}}^0 + \widetilde{f}, \mathbf{v}\Big) = 0, \tag{10}$$

$$\boldsymbol{\xi}_i^{\mathrm{T}} \mathbf{F}\Big(\mathbf{u}_{\mathrm{c}}^0 + \sum_{j=1}^{N} w_j \boldsymbol{\eta}_j, f_{\mathrm{c}}^0 + \widetilde{f}, \mathbf{v}\Big) = 0, \qquad i = 2, \ldots, N. \tag{11}$$

Equations (10) and (11) give a decomposition of the original system (1) into two parts; the latter (11) is for the range space of J_{c}^0 and the former (10) is for its complement.

The Jacobian matrix on the left-hand side of (11) with respect to w_j $(j = 2, \ldots, N)$, evaluated at $w_j = 0$ $(j = 1, 2, \ldots, N)$, $\widetilde{f} = 0$, and $\mathbf{v} = \mathbf{v}^0$, is nonsingular. Therefore, by the implicit function theorem, (11) can be solved locally for w_j $(j = 2, \ldots, N)$ as

$$w_j = \varphi_j(w, \widetilde{f}, \mathbf{v}), \qquad j = 2, \ldots, N, \tag{12}$$

where $w \equiv w_1$ and

$$\varphi_j(0, 0, \mathbf{v}^0) = 0, \qquad j = 2, \ldots, N. \tag{13}$$

On substituting this into (10) we obtain a reduced equation

$$\widetilde{F}(w, \widetilde{f}, \mathbf{v}) = 0 \tag{14}$$

in w, where

$$\widetilde{F}(w, \widetilde{f}, \mathbf{v}) = \boldsymbol{\xi}_1^{\mathrm{T}} \mathbf{F}\Big(\mathbf{u}_{\mathrm{c}}^0 + w\boldsymbol{\eta}_1 + \sum_{j=2}^{N} \varphi_j(w, \widetilde{f}, \mathbf{v})\boldsymbol{\eta}_j, f_{\mathrm{c}}^0 + \widetilde{f}, \mathbf{v}\Big). \tag{15}$$

This reduced equation (14) is called the *bifurcation equation*. It is emphasized that the reduction to (14) is valid locally in a neighborhood of $(\mathbf{u}_{\mathrm{c}}^0, f_{\mathrm{c}}^0, \mathbf{v}^0)$.

The key to the reduction to the single equation (14) is the elimination of w_j $(j = 2, \ldots, N)$ on the basis of the nonsingularity of the Jacobian matrix of (11). Hence the equation (14) is also valid for an imperfect system with $\mathbf{v}$ approximately equal to $\mathbf{v}^0$.

The solutions $(w, \widetilde{f}, \mathbf{v})$ to the bifurcation equation (14) are in one-to-one correspondence through (12) with the solutions $(\mathbf{u}, f, \mathbf{v})$ of the original system (1), i.e.,

$$\mathbf{u} = \mathbf{u}(w, \widetilde{f}, \mathbf{v}) = \mathbf{u}_{\mathrm{c}}^0 + w\boldsymbol{\eta}_1 + \sum_{j=2}^{N} \varphi_j(w, \widetilde{f}, \mathbf{v})\boldsymbol{\eta}_j, \tag{16}$$

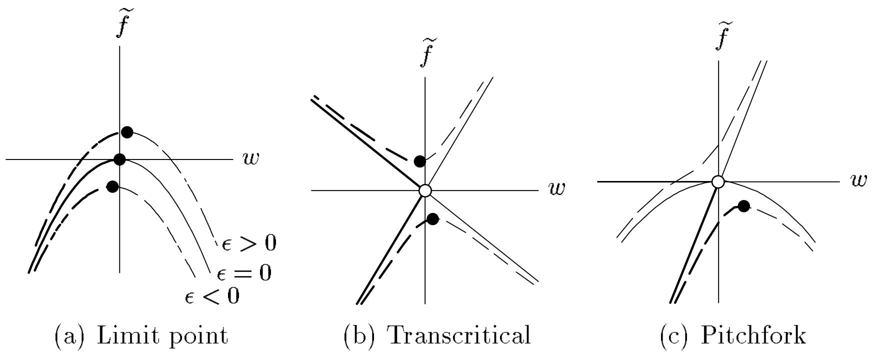


Figure 1: Solution curves in the neighborhood of simple critical points expressed by the leading terms of the bifurcation equation. ———: path for the perfect system; — — —: path for an imperfect system; thick line: stable; thin line: unstable; ∘: bifurcation point; •: limit point.

in the neighborhood of $(\mathbf{u}_c^0, f_c^0, \mathbf{v}^0)$. Hence the qualitative picture of the solution set of the original system (1) is isomorphic to that of the bifurcation equation (14). In this connection it is noted that the criticality condition (4) for the original system (1) is now translated into the criticality condition

$$\frac{\partial \widetilde{F}}{\partial w} = 0 \tag{17}$$

for the bifurcation equation. This means, in particular, that the critical load f_c of an imperfect system can be determined from (14) and (17).

3.3 Bifurcation equation and imperfection sensitivity laws

Although the direct use of the bifurcation equation (14) in the investigation of the bifurcation behavior is difficult in general, it is much simpler and pertinent to investigate its asymptotic behavior by expanding the bifurcation equation into power series and examining the leading terms.

Referring to $\mathbf{v} = \mathbf{v}^0 + \epsilon\mathbf{d}$ in (6), we regard ϵ as an independent variable for imperfection, whereas $\mathbf{d}$ is assumed to be constant. With the notation $\widehat{F}(w, \widetilde{f}, \epsilon) = \widetilde{F}(w, \widetilde{f}, \mathbf{v}^0 + \epsilon\mathbf{d})$, the bifurcation equation (14) can be expressed, alternatively, as

$$\widehat{F}(w, \widetilde{f}, \epsilon) = 0. \tag{18}$$

We can grasp the essential nature of (18) by expanding $\widehat{F}$ into a power series involving an appropriate number of terms

$$\widehat{F}(w, \widetilde{f}, \epsilon) \sim \sum_{i=0} \sum_{j=0} \sum_{k=0} A_{ijk} w^i \widetilde{f}^j \epsilon^k. \tag{19}$$

The coefficient A_{001}, which represents the influence of the imperfection on the bifurcation equation, is given by

$$A_{001} = \boldsymbol{\xi}_1^{\mathrm{T}} B_{\mathrm{c}}^0 \mathbf{d} \tag{20}$$

with B defined in (7). We assume $A_{001} \neq 0$ in the sequel, while imperfection patterns with $A_{001} = 0$ are called the *second order initial imperfections.*[148]

We can classify equilibrium points satisfying the bifurcation equation (18), based on the vanishing or nonvanishing of the coefficients A_{ijk} in the expansion (19). The bifurcation equation (18) for simple critical points reads:

$$\widehat{F}(w, \widetilde{f}, \epsilon) = \begin{cases} A_{200}w^2 + A_{010}\widetilde{f} + A_{001}\epsilon + \text{h.o.t.} = 0 & \text{at the limit point,} \\ A_{200}w^2 + A_{110}w\widetilde{f} + A_{020}\widetilde{f}^2 + A_{001}\epsilon + \text{h.o.t.} = 0 & \\ \qquad\qquad \text{at the transcritical bifurcation point,} & \\ A_{300}w^3 + A_{110}w\widetilde{f} + A_{020}\widetilde{f}^2 + A_{001}\epsilon + \text{h.o.t.} = 0 & \\ \qquad\qquad \text{at the pitchfork bifurcation point,} & \end{cases} \tag{21}$$

where h.o.t. denotes the higher-order terms. The transcritical and pitchfork points are often called asymmetric and symmetric (cusp) points of bifurcation, respectively.[149] Solution curves in the neighborhood of simple critical points expressed by the leading terms of the bifurcation equation are shown in Fig. 1.

The imperfection sensitivity laws can be derived by solving simultaneously the bifurcation equation (21) and the criticality condition $\partial \widehat{F} / \partial w = 0$ explained in (17). The explicit forms of these laws are[150]

$$\widetilde{f}_{\mathrm{c}} \sim \begin{cases} C(\mathbf{d})\epsilon & \text{at the limit point,} \\ C(\mathbf{d})|\epsilon|^{1/2} & \text{at the transcritical bifurcation point,} \\ C(\mathbf{d})\epsilon^{2/3} & \text{at the pitchfork bifurcation point.} \end{cases} \tag{22}$$

The coefficient $C(\mathbf{d})$ is given by

$$C(\mathbf{d}) = \begin{cases} -\dfrac{A_{001}}{A_{010}} & \text{at the limit point,} \\ -\left(\dfrac{4|A_{200}A_{001}|}{{A_{110}}^2 - 4A_{200}A_{020}}\right)^{1/2} & \begin{array}{l}\text{at the transcritical point} \\ \text{(exists for } A_{200}A_{001}\epsilon > 0\text{),}\end{array} \\ -\dfrac{3{A_{300}}^{1/3}}{A_{110}}\left(\dfrac{A_{001}}{2}\right)^{2/3} & \text{at the pitchfork bifurcation point.} \end{cases} \tag{23}$$

[148] Roorda, 1965, 1968 [442, 443]; Ohsaki, 2001 [384]

[149] Thompson and Hunt, 1973 [518]

[150] Koiter, 1945 [311]; Thompson, 1965 [507]; Ikeda and Murota, 1990, 1990, 2002 [262, 263, 270]

This way to derive the imperfection sensitivity laws is sometimes called the Liapunov–Schmidt–Koiter approach.[151] In contrast, there is a direct method to compute the location of a bifurcation point by solving numerically the extended system of a nonlinear governing equation.[152]

3.4 Probabilistic scatter of critical loads

The probabilistic properties of critical loads can be formulated systematically in an asymptotic sense (when initial imperfections are small). Given the joint probability density function of $\mathbf{d} = (d_1, \ldots, d_p)^{\mathrm{T}}$, the probability density function of $A_{001} = \boldsymbol{\xi}_1^{\mathrm{T}} B_{\mathrm{c}}^0 \mathbf{d}$ can be calculated. Then a simple transformation from A_{001} to the critical load f_{c} via (22) with (23) yields the probability density function of f_{c}.

We shall investigate the behavior of f_{c} when the initial imperfection $\mathbf{v} - \mathbf{v}^0 = \epsilon \mathbf{d}$ is subject to the normal distribution $\mathrm{N}(\mathbf{0}, \epsilon^2 W)$ with mean $\mathbf{0}$ and variance–covariance matrix $\epsilon^2 W$, where W is a positive-definite symmetric matrix. The probability density function of critical load is dependent on the types of critical points as follows:

$$\phi(\zeta) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \exp(-\zeta^2/2), & -\infty < \zeta < \infty \\ & \text{at the limit point,} \\ \dfrac{4|\zeta|}{\sqrt{2\pi}} \exp(-\zeta^4/2), & -\infty < \zeta < 0 \\ & \text{at the transcritical bifurcation point,} \\ \dfrac{3|\zeta|^{1/2}}{\sqrt{2\pi}} \exp(-|\zeta|^3/2), & -\infty < \zeta < 0, \\ & \text{at the unstable pitchfork bifurcation point,} \end{cases} \tag{24}$$

where $\zeta = (f_{\mathrm{c}} - f_{\mathrm{c}}^0)/\widehat{C}$ denotes the normalized critical load increment with a proper scaling $\widehat{C}$.

As a numerical example, we employ a finite beam on a nonlinear foundation with an initial shape imperfection. The normalized initial imperfection $\widetilde{u}(\eta)$ is assumed to be a Gaussian random function of the position η with given mean function and autocorrelation function. The differential equation of the beam was discretized and was numerically solved to arrive at the nondimensional buckling load f_{c}, which was governed by an unstable pitchfork bifurcation point. Figure 2 shows the numerical histogram of f_{c} produced by the Monte Carlo method for an ensemble of 1000 beams with the Gaussian imperfections prescribed above.[153]

[151] Peek and Triantafyllidis, 1992 [416]
[152] Seydel, 1979, 1979 [470, 471]; Werner and Spence, 1984 [564]; Wriggers and Simo, 1990 [567]
[153] Elishakoff, 1979 [139]

The random variation of critical loads of the beam is described by the present theory.[154] The values of $f_{\rm c}^0$ and $\widehat{C}$ listed on the right of Fig. 2 were computed so that the numerical histogram of $f_{\rm c}$ and the probability density function in (24) have the same mean and variance. For the unstable pitchfork bifurcation point, which governs the critical load in this case, the value of $f_{\rm c}^0 = 0.99$ computed is close to its theoretical value, 1.00. We computed the curves of the probability density function for various types of critical points in Fig. 2. The theoretical curve applicable to this case cannot be identified through a mere comparison of the histogram with the theoretical curves. This emphasizes the importance of the knowledge on the type of bifurcation point, an unstable pitchfork for this case.

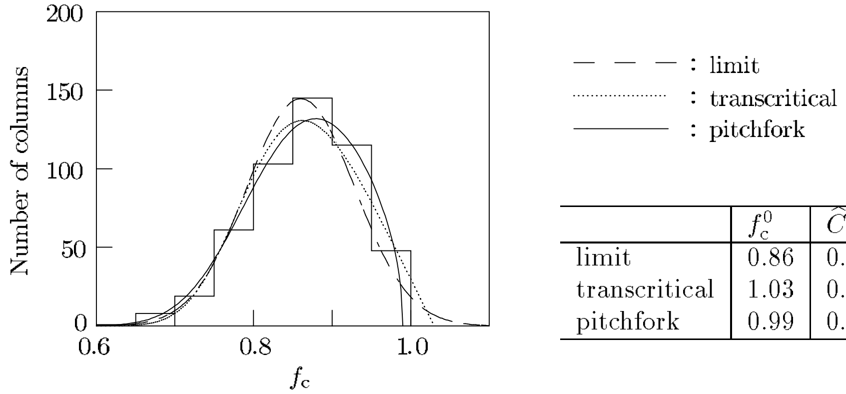
3.5 Experimentally observed bifurcation diagrams

There is a gap between bifurcation diagrams in mathematical theory and those in engineering practice in the *experiment* of materials undergoing bifurcation. In mathematical theory, a canonical coordinate for mathematical convenience is chosen to be the abscissa of a bifurcation diagram, whereas a physically meaningful variable is a natural choice of an abscissa in the bifurcation diagram obtained by an analysis or experiment in engineering. Bifurcation diagrams observed in engineering experiments may be qualitatively different from those in mathematics, as illustrated in Fig. 3 for a pitchfork bifurcation point (possible observed bifurcation diagrams in Fig. 3(a) and (b) in comparison with a mathematical bifurcation diagram in Fig. 3(c)).

The reason for such a qualitative difference may be explained as follows. A bifurcation diagram is obtained as the projection of the solution path in a higher-dimensional space to a two-dimensional plane. The resulting picture naturally depends on the chosen projection. A canonical choice of the projection yields the mathematical bifurcation diagram (see Fig. 3(c)), whereas an arbitrary choice would result in a diagram like the one in Fig. 3(b), which is qualitatively similar to the mathematical diagram. If the direction of the projection happens to be so special that it is perpendicular to the bifurcated path, the resulting diagram looks like the one in Fig. 3(a), which is qualitatively different from the mathematical diagram. Such an exceptional situation occurs quite often in engineering experiments as a natural consequence of geometrical symmetry. In order to fill the gap due to this difference, the theory on initial imperfections is tailored to be applicable to experimentally observed diagrams.[155]

The bifurcation equation (21) expresses an relationship between w and $\widetilde{f}$ with an imperfection parameter ϵ. We here transform the variable w into an observed variable u_* in Fig. 3(a) or u_1 in (b). (We hereafter call u_* the symmetric displacement and u_1 the nonsymmetric displacement.) Then

[154] Ikeda, Murota, and Elishakoff, 1996 [271]
[155] Ikeda and Murota, 1999, 2002 [269, 270]



	f_c^0	$\widehat{C}$
limit	0.86	0.069
transcritical	1.03	0.20
pitchfork	0.99	0.16

Figure 2: Comparison of a numerical histogram and semiempirical probability density functions of the critical load f_c for the finite beam on a nonlinear elastic foundation [271] (Nondimensional buckling load satisfies $f_c^0 = 1$).

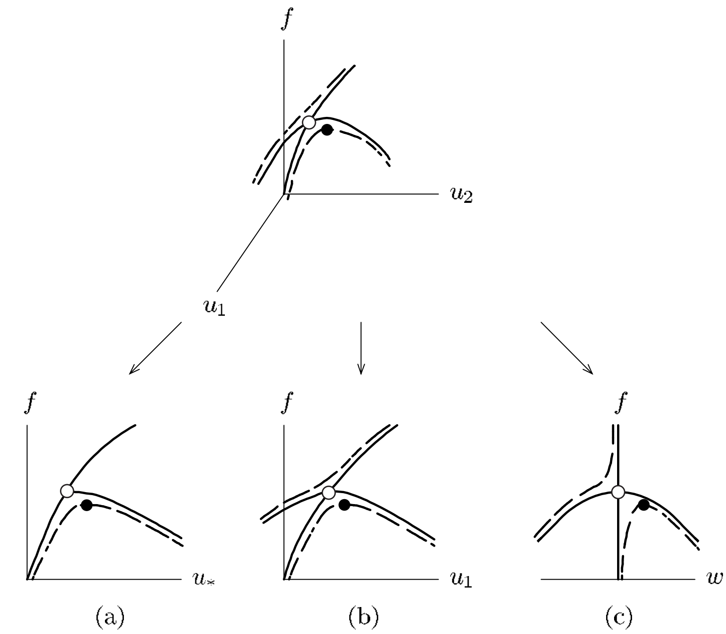


Figure 3: Choice of projections in drawing bifurcation diagrams. (a) and (b) experimentally observed bifurcation diagrams and (c) a mathematical diagram at an unstable pitchfork bifurcation point. ———: curve for the perfect system; — — —: curve for an imperfect system; ○: bifurcation point; ●: limit point.

from the bifurcation equation (21) for a pitchfork bifurcation point, we can arrive at the asymptotic expression for the curve of an imperfect system for an observed variable

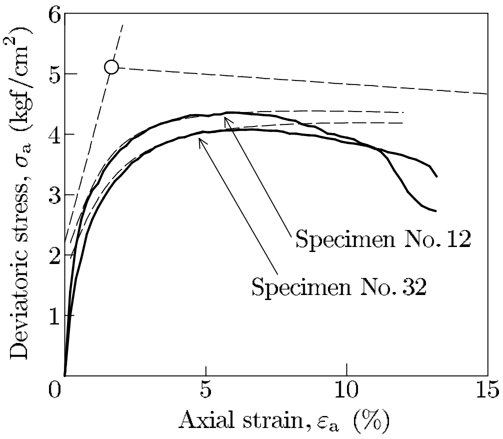
$$\left.\begin{array}{ll} \sqrt{\widetilde{u} - \widetilde{f}/E}\,[\,\widetilde{f} + p(\widetilde{u} - \widetilde{f}/E)] \pm q\epsilon + \text{h.o.t.} = 0, & \text{symmetric displacement} \\ (\widetilde{u} - \widetilde{f}/E)\widetilde{f} + p^*(\widetilde{u} - \widetilde{f}/E)^3 + q^*\epsilon + \text{h.o.t.} = 0, & \text{nonsymm. displacement} \end{array}\right\} \quad (25)$$

where $\widetilde{u}$ denotes the increment of u_* or u_1 from the value at the critical point of the perfect system; E, p, q, p^*, and q^* are constants.

As an example of symmetric displacement, we consider the experimental curves of the deviatoric stress $\sigma_{\rm a}$ versus the axial strain $\varepsilon_{\rm a}$ shown in Fig. 4. These curves are simulated by the first equation of (25) shown as the dashed lines to display fairly well correlation.

As an example of nonsymmetric displacement, we consider here an elastic four-sides-simply-supported rectangular plate in Fig. 5 subjected to in-plane pure bending. The von Kármán equation[156] with an initial deflection is considered. The perfect plate with initial displacement $u_0 = 0$ undergoes pitchfork bifurcation accompanied by the loss of upside-down symmetry. The bifurcation mode (critical eigenvector) is given by $\sin(\pi x/L_x)\sin(\pi y/L_y)$ $(0 \leq x \leq L_x,\ 0 \leq y \leq L_y)$, where L_x and L_y denote the width and depth of the plate, respectively. As the initial imperfection, we take the initial deflection $u_0(x, y) = \epsilon \cdot \sin(\pi x/L_x)\sin(\pi y/L_y)$. The solid lines on the left of Fig. 5 show a series of equilibrium paths for various values of ϵ. The asymptotic approximation shown by the dashed lines in Fig. 5, which is computed from the second equation of (25), is in good agreement with the computational curves.

[156] Timoshenko and Woinowsky–Krieger, 1959 [530]; Timoshenko and Gere, 1963 [529]



$(\varepsilon_a)_c^0$	1.65
$(\sigma_a)_c^0$	5.10
p	0.0319
E	1.75
$q\epsilon$	1.31 (No. 12)
	1.88 (No. 32)

Figure 4: Curves of the deviatoric stress σ_a versus the axial strain ε_a for the sand specimens and their simulation by the present method. ———: experimental (imperfect) curve; – – –: simulated curve; ○: bifurcation point; $1\,\text{kgf/cm}^2 = 98\,\text{kPa}$.

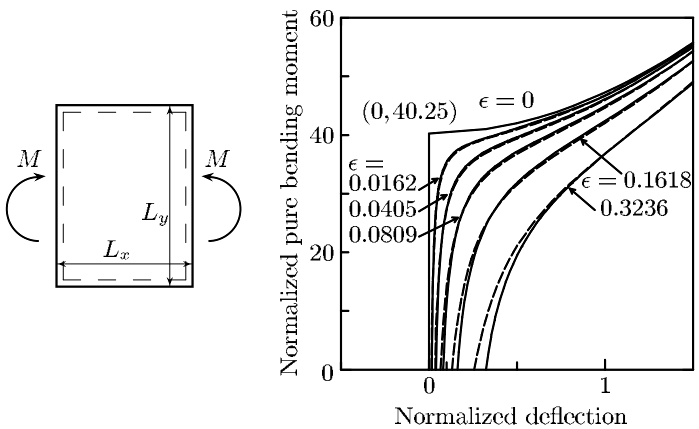


Figure 5: Rectangular plates and the equilibrium paths. Deflection u is measured at $(x, y) = (0.5L_x, 0.7L_y)$; aspect ratio $\alpha \equiv L_x/L_y = 0.8$; the depth–thickness ratio $\beta \equiv L_y/t = 200$; ———: exact numerical analysis; – – –: asymptotic simulation.

4 HILLTOP BRANCHING POINTS

In this section the asymptotic approach is applied to the hilltop branching point. The formulas for the imperfection sensitivity law and the stochastic scatter of critical loads are presented,[157] and are put to use in the description of the scatter of the strength of long steel members.[158]

4.1 Bifurcation equation and imperfection sensitivity laws

We derive here, for a potential system, the bifurcation equation at a hilltop bifurcation point, which is defined to be a double critical point ($M = 2$) occurring as a coincidence of a limit point and a pitchfork bifurcation point, as shown in Fig. 6.

For this double critical point, the system of bifurcation equations becomes

$$\widehat{F}_i(w_1, w_2, \widetilde{f}, \epsilon) = 0, \qquad i = 1, 2. \tag{26}$$

Note that $(w_1, w_2, \widetilde{f}, \epsilon) = (0, 0, 0, 0)$ corresponds to a hilltop bifurcation point for the perfect system. We expand (26) into power series

$$\widehat{F}_1(w_1, w_2, \widetilde{f}, \epsilon) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A_{ijkl} {w_1}^i {w_2}^j \widetilde{f}^k \epsilon^l, \tag{27}$$

$$\widehat{F}_2(w_1, w_2, \widetilde{f}, \epsilon) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} B_{ijkl} {w_1}^i {w_2}^j \widetilde{f}^k \epsilon^l. \tag{28}$$

We assume that the first equation (27) is associated with the pitchfork bifurcation point with a trivial solution $w_1 = 0$ and the second equation (28) is associated with the limit point. Since the first equation has trivial solution, it has a form of $\widehat{F}_1(w_1, w_2, \widetilde{f}, 0) = w_1 \widetilde{F}_1(w_1, w_2, \widetilde{f})$ for some function $\widetilde{F}_1(w_1, w_2, \widetilde{f})$.

[157] Ikeda, Oide, and Terada, 2002 [277]
[158] Okazawa et al., 2002 [390]

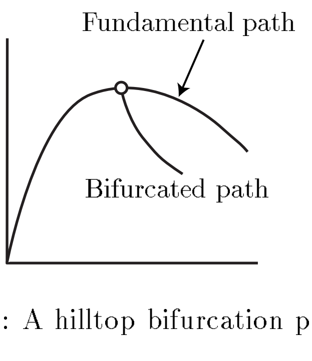


Figure 6: A hilltop bifurcation point: $\circ$.

By investigating vanishing of coefficients A_{ijkl} and B_{ijkl} $(i, j, k, l = 0, 1, \cdots)$ at the hilltop bifurcation point $(w_1, w_2, \widetilde{f}, \epsilon) = (0, 0, 0, 0)$, we obtain[159]

$$\widehat{F}_1(w_1, w_2, \widetilde{f}, \epsilon) = A_{3000}{w_1}^3 + 2B_{2000}w_1w_2 + A_{1010}w_1\widetilde{f} + A_{0001}\epsilon + \text{h.o.t.} = 0, \tag{29}$$

$$\widehat{F}_2(w_1, w_2, \widetilde{f}, \epsilon) = B_{2000}{w_1}^2 + B_{0200}{w_2}^2 + B_{0010}\widetilde{f} + B_{0001}\epsilon + \text{h.o.t.} = 0. \tag{30}$$

Imperfection sensitivity law of critical load is derived by determining the location by simultaneously solving (29), (30), and the criticality condition $\det \widehat{J} = 0$. Thus we obtain a piecewise linear law[160]

$$\widetilde{f}_{\rm c} \quad \sim \quad -\frac{\text{sign}(B_{2000})}{B_{0010}}\left(\frac{B_{0200}}{B_{2000}}\right)^{1/2} |A_{0001}\epsilon| - \frac{B_{0001}}{B_{0010}}\,\epsilon. \tag{31}$$

Remark 4.1 *The hilltop point occurring as a consequence of a limit point and a double bifurcation point of a system with dihedral-group symmetry also enjoys a piecewise linear law (Ikeda, Ohsaki, and Kanno, 2004 [276]).*

4.2 Probabilistic variation of critical loads

Probabilistic variation of $f_{\rm c}$ is investigated when initial imperfection pattern $\mathbf{v} - \mathbf{v}^0 = \epsilon\mathbf{d}$ is subject to normal distribution $\mathrm{N}(\mathbf{0}, \epsilon^2 W)$ with mean $\mathbf{0}$ and variance–covariance matrix $\epsilon^2 W$. The probability density function of the critical load is given by

$$\phi(\zeta) = \frac{2}{\sqrt{2\pi}}\exp(-\zeta^2/2)\Phi_{\rm N}(-r\zeta), \tag{32}$$

where $\zeta = (f_{\rm c} - f_{\rm c}^0)/\widehat{C}$ is the normalized critical load increment with a proper scaling $\widehat{C}$, r is a parameter, and $\Phi_{\rm N}(x)$ is the cumulative distribution function of the standard normal distribution. This function, the curve of which varies with the values of r, is shown in Fig. 7.

Finite-element, finite-strain, elastic-plastic analyses[161] are conducted on rectangular analysis domains for steel specimens subjected to uniform tension for a few aspect ratios of $L/H = 2 \sim 10$. As shown in the load–displacement curves in Fig. 8, limit point locations are identical for all aspect ratios and are denoted in this figure by $\bullet$. The first bifurcation points, denoted by $\circ$, approach the limit point as the specimen becomes slender. The load–displacement curve for $L/H = 10$, for which the critical load at the pitchfork bifurcation point is 0.2% smaller than the maximum load $f_{\rm c}^0 = 1086$ (kN) at the limit point, is used in the sequel to approximate the hilltop bifurcation point.

[159] Note that $A_{1100} = 2B_{2000}$ and $B_{1100} = A_{0200} = 0$ due to reciprocity by the existence of the potential.

[160] The piecewise linear law originally found in Thompson and Schorrock (1975) [525] corresponds to the case of $B_{0001}\epsilon = 0$ in (31).

[161] Details of the numerical analyses are given in Okazawa et al., 2002 [390].

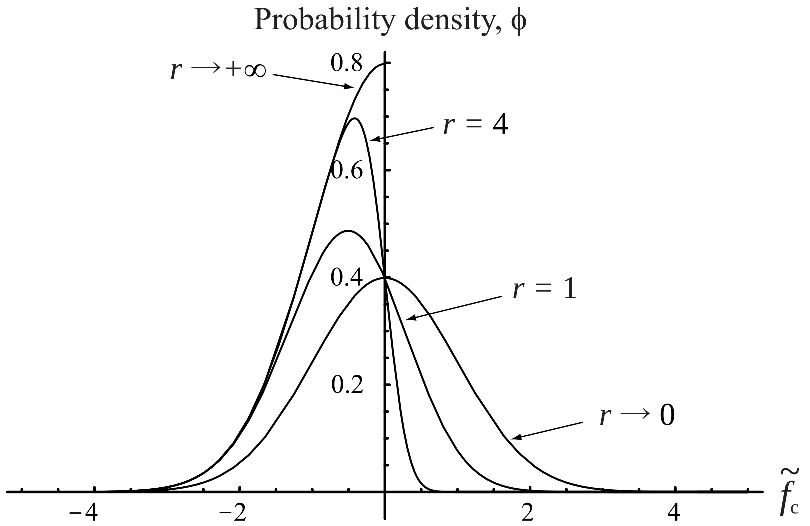


Figure 7: Probability density function of $\widetilde{f}_{\mathrm{c}}$ plotted for several values of r.

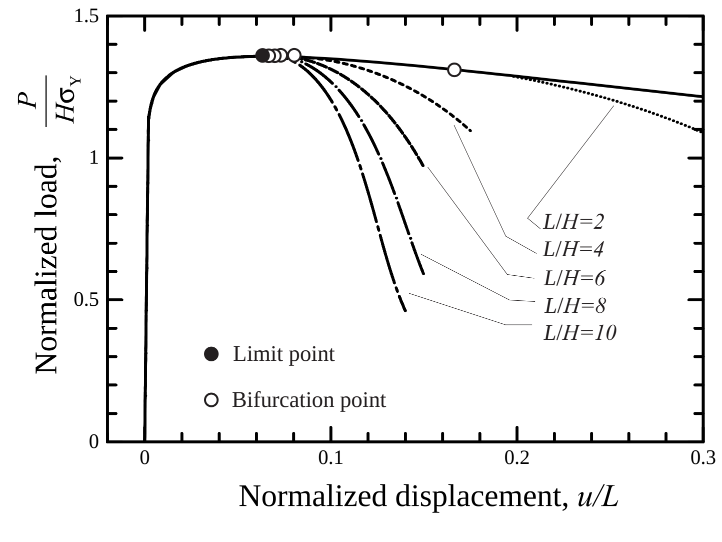


Figure 8: Normalized load–displacement curves of steel specimens with $L/H = 2, 4, 6, 8, 10$. P: applied load; u: axial displacement; L: member length; Young's modulus $E = 200$ GPa; Poisson's ratio $\nu = 0.333$; yield stress $\sigma_{\mathrm{Y}} = 400$ MPa; yield strain $e_{\mathrm{Y}} = \sigma_{\mathrm{Y}}/E = 1/500$.

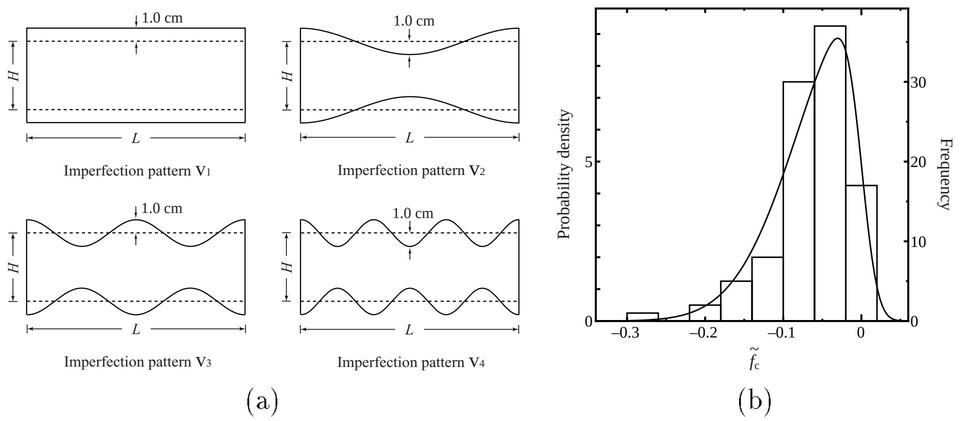


Figure 9: (a) Imperfection patterns imposed on members and (b) comparison of histograms and theoretical probability density functions.

We carry out the Monte Carlo simulation on imperfect steel members to arrive at the data bank of their strengths. We define the imperfection parameter vector as

$$\mathbf{v} = \epsilon(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + d_4\mathbf{v}_4), \tag{33}$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are harmonic modes shown in Fig. 9(a). We set $\mathbf{v}^0 = \mathbf{0}$ and $\mathbf{d} = (d_1, d_2, d_3, d_4)^{\mathrm{T}}$. We choose an ensemble of 100 imperfection patterns $\epsilon\mathbf{d}$ that is subject to a multivariate normal distribution $\epsilon\mathbf{d} \sim \mathrm{N}(\mathbf{0}, W)$ with $W = \mathrm{diag}(0.01^2, 0.1^2, 0.01^2, 0.01^2)$, where $\mathrm{diag}(\cdots)$ denotes a diagonal matrix with the diagonal components in the parentheses.

We have computed maximum loads for 100 imperfection patterns presented above. Figure 9(b) shows a histogram obtained in this manner and the curve for the theoretical probability density function (32). The Weibull-like histogram is represented well by the theoretical curve, which has passed the χ^2 test at a significance level of 0.05 or less.

5 GROUP-THEORETIC FRAMEWORK

In this section group-theoretic method for exploiting the symmetry of the governing equation, as well as that of the bifurcation equation, is presented briefly.[162] Systems that are endowed with an additional structure of symmetry undergo symmetry-breaking bifurcations and, in turn, produce patterns. This method is put to use in the mathematical analyses of double critical points of a system with dihedral group symmetry in Section 6.

5.1 Group representation

Some fundamental facts about linear representations of groups are introduced. For complete accounts, the reader is referred to textbooks, such as Miller, 1972 [360] and Serre, 1977 [465].

Let G be a *group*. This means that, for any pair of elements g and h of G, their product $g \cdot h$ is defined as an element of G and the following (i) through (iii) are satisfied: (i) the associative law holds: $(g \cdot h) \cdot k = g \cdot (h \cdot k)$, for any $g, h, k \in G$; (ii) there exists an element $e \in G$ (called the *identity element*) such that $e \cdot g = g \cdot e = g$ for any $g \in G$; (iii) for any $g \in G$ there exists $h \in G$ (called the *inverse* of g) such that $g \cdot h = h \cdot g = e$.

Example 1 The *dihedral group* of degree three is a group of six elements:

$$\mathrm{D}_3 = \{e, r, r^2, \sigma, \sigma r, \sigma r^2\}$$

with the relations $r^3 = \sigma^2 = (\sigma r)^2 = e$, where e is the identity element. The notation shows, e.g., that the product of σ and r equals the element denoted as σr. For the product of r and σr the assumed relations as well as the associative law yield $r \cdot \sigma r = \sigma^2 \cdot r \cdot \sigma r = \sigma \cdot (\sigma r)^2 = \sigma$.

A *representation* of G means a family of nonsingular matrices, say, $T(g)$ indexed by the elements g of G such that

$$T(gh) = T(g)T(h), \qquad g, h \in G. \tag{34}$$

The size of the matrices is called the degree of the representation. A representation T is said to be *unitary* if $T(g)$ is unitary for each $g \in G$.

Example 2 For the dihedral group of degree three, $\mathrm{D}_3 = \{e, r, r^2, \sigma, \sigma r, \sigma r^2\}$, a unitary representation is given by the family of matrices defined by

$$T(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T(r) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \qquad T(r^2) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

$$T(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T(\sigma r) = \begin{pmatrix} c & -s \\ -s & -c \end{pmatrix}, \qquad T(\sigma r^2) = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}$$

[162] We aim at presenting the main ideas for engineers without sacrificing the mathematical rigor. To this end we restrict ourselves to finite-dimensional equations and finite groups. For the full mathematical treatment, the reader is referred to Sattinger, 1979, 1980 [456, 457]; Golubitsky and Schaeffer, 1985 [195]; Golubitsky, Stewart, and Schaeffer, 1988 [198].

with $c = \cos(2\pi/3)$ and $s = \sin(2\pi/3)$.

A subspace W is said to be *invariant* if any vector in W remains in W under the transformation $T(g)$ for every $g \in G$. A representation T is said to be *irreducible* if there exists no invariant subspace distinct from $\{\mathbf{0}\}$ and the entire space.

Two representations, say, T_1 and T_2 are said to be *equivalent* if there exists a nonsingular matrix H such that

$$T_1(g) = H^{-1} T_2(g) H, \qquad g \in G.$$

Two representations are *inequivalent* if they are not equivalent. It is known that any representation is equivalent to a unitary representation.

There exist a finite number of inequivalent irreducible representations[163]of G. We denote by

$$\{T^\mu \mid \mu \in R(G)\}$$

a family of all inequivalent irreducible unitary representations of G, where $R(G)$ is the index set for the irreducible representations of G. For each μ we associate a subgroup defined by

$$G^\mu = \{g \in G \mid T^\mu(g) = I_{N^\mu}\}, \tag{35}$$

where N^μ denotes the dimension of the representation μ and I_{N^μ} is the identity matrix of order N^μ.

5.2 Symmetry of equations

We consider a system of nonlinear equilibrium or governing equations $\mathbf{F}(\mathbf{u}, f, \mathbf{v}) = \mathbf{0}$ in (1) that is endowed with an additional structure of symmetry.

According to the standard setting in group-theoretic bifurcation theory, the symmetry of the perfect structure (with $\mathbf{v} = \mathbf{v}^0$) is formulated as the identities

$$T(g)\mathbf{F}(\mathbf{u}, f, \mathbf{v}^0) = \mathbf{F}(T(g)\mathbf{u}, f, \mathbf{v}^0), \qquad g \in G, \tag{36}$$

in terms of a group G and a unitary representation T of G acting on the N-dimensional space of the independent variable vector $\mathbf{u}$. This relation (36) is called the *equivariance* of $\mathbf{F}$ to G.

To also express the symmetry in the imperfection parameter vector $\mathbf{v}$, we extend the equivariance (36) to the following form:

$$T(g)\mathbf{F}(\mathbf{u}, f, \mathbf{v}) = \mathbf{F}(T(g)\mathbf{u}, f, S(g)\mathbf{v}), \qquad g \in G, \tag{37}$$

using another unitary representation S of G on the p-dimensional space of the imperfection parameter vector $\mathbf{v}$. For the compatibility of (36) and

[163] To be precise, we fix an underlying field, which, in most applications, is the field of real numbers or the field of complex numbers.

(37) it is assumed that the imperfection vector $\mathbf{v}^0$ for the perfect system is *G-symmetric* in the sense that

$$\Sigma(\mathbf{v}^0; G, S) = G, \tag{38}$$

where

$$\Sigma(\mathbf{v}; G, S) = \{g \in G \mid S(g)\mathbf{v} = \mathbf{v}\} \tag{39}$$

denotes the subgroup of G that expresses the symmetry of $\mathbf{v}$.

Remark 5.2 *When the system of equilibrium equations $\mathbf{F}(\mathbf{u}, f, \mathbf{v})$ is derived from a potential function $U(\mathbf{u}, f, \mathbf{v})$, the equivariance of $\mathbf{F}$ to G is a consequence of the invariance of U to G. The invariance of U to G is formulated as*

$$U(T(g)\mathbf{u}, f, S(g)\mathbf{v}) = U(\mathbf{u}, f, \mathbf{v}), \qquad g \in G, \tag{40}$$

in terms of unitary representations T and S, and the differentiation of (40) shows that $\mathbf{F} = (\partial U/\partial \mathbf{u})^{\mathrm{T}}$ satisfies (37).

The equivariance (37) is inherited by the Jacobian matrix $J(\mathbf{u}, f, \mathbf{v})$ and the imperfection sensitivity matrix $B(\mathbf{u}, f, \mathbf{v})$. Differentiations of (37) with respect to $\mathbf{u}$ and $\mathbf{v}$, respectively, yield

$$T(g)J(\mathbf{u}, f, \mathbf{v}) = J(T(g)\mathbf{u}, f, S(g)\mathbf{v})T(g), \qquad g \in G, \tag{41}$$

$$T(g)B(\mathbf{u}, f, \mathbf{v}) = B(T(g)\mathbf{u}, f, S(g)\mathbf{v})S(g), \qquad g \in G. \tag{42}$$

This implies, in particular, that

$$T(g)J(\mathbf{u}, f, \mathbf{v}) = J(\mathbf{u}, f, \mathbf{v})T(g), \qquad g \in G, \tag{43}$$

$$T(g)B(\mathbf{u}, f, \mathbf{v}) = B(\mathbf{u}, f, \mathbf{v})S(g), \qquad g \in G \tag{44}$$

if

$$\Sigma(\mathbf{u}; G, T) = \Sigma(\mathbf{v}; G, S) = G. \tag{45}$$

Thus $J(\mathbf{u}, f, \mathbf{v})$ and $B(\mathbf{u}, f, \mathbf{v})$ both commute with the group actions for all $(\mathbf{u}, f, \mathbf{v})$ possessing the symmetry (45). A standard result of group representation theory then reveals that $J(\mathbf{u}, f, \mathbf{v})$ and $B(\mathbf{u}, f, \mathbf{v})$ can be transformed to block-diagonal forms through suitable basis changes independent of $(\mathbf{u}, f, \mathbf{v})$, and this fact often forms a technical pivot in deriving useful results. More issues on block diagonalization can be found in references.[164]

We consider a critical point $(\mathbf{u}_{\mathrm{c}}^0, f_{\mathrm{c}}^0)$ of the perfect system such that

$$\Sigma(\mathbf{u}_{\mathrm{c}}^0; G, T) = G. \tag{46}$$

[164] Bossavit, 1986 [60]; Chen and Sameh, 1989 [94]; Gatermann and Hohmann, 1991 [181]; Stork and Werner, 1991 [492]; Gatermann and Werner, 1994 [182]; Govaerts, 2000 [200]

It can easily be verified that the kernel of $J_{\rm c}^0 = J(\mathbf{u}_{\rm c}^0, f_{\rm c}^0, \mathbf{v}^0)$ is an invariant subspace. The critical point $(\mathbf{u}_{\rm c}^0, f_{\rm c}^0)$ is called *group-theoretic* if the kernel of $J_{\rm c}^0$ is irreducible and *parametric* otherwise. Group-theoretic multiple points are the ones which appear inherently in symmetric structures, whereas parametric multiple points are the ones which appear as a coincidence of a number of critical points. In what follows, we assume that $(\mathbf{u}_{\rm c}^0, f_{\rm c}^0)$ is a group-theoretic multiple point of multiplicity M.

Remark 5.3 *The coincidence of a pair of group-theoretic double bifurcation points was studied for ecological interacting and diffusing systems,*[165] *and for systems with various symmetries.*[166]

The Liapunov–Schmidt reduction described in Section 3.2 for a simple critical point can be carried out compatibly with symmetry for a multiple critical point. In a neighborhood of $(\mathbf{u}_{\rm c}^0, f_{\rm c}^0, \mathbf{v}^0)$, the full system of N equations (1) is reduced to M equations

$$\widetilde{\mathbf{F}}(\mathbf{w}, \widetilde{f}, \mathbf{v}) = \mathbf{0} \tag{47}$$

in $\mathbf{w} \in \mathbf{R}^M$, where $\widetilde{\mathbf{F}} : \mathbf{R}^M \times \mathbf{R} \times \mathbf{R}^p \to \mathbf{R}^M$ and $\widetilde{f} = f - f_{\rm c}^0$. The key ingredient for symmetry is the inheritance of the equivariance (37) of the full system to that of the reduced system

$$\widetilde{T}(g)\widetilde{\mathbf{F}}(\mathbf{w}, \widetilde{f}, \mathbf{v}) = \widetilde{\mathbf{F}}(\widetilde{T}(g)\mathbf{w}, \widetilde{f}, S(g)\mathbf{v}), \qquad g \in G, \tag{48}$$

where $\widetilde{T}$ is the representation induced from T on the M-dimensional kernel space of $J_{\rm c}^0$. It is this *inheritance of symmetry* that plays the key role in determining the symmetry of the bifurcating solutions.

5.3 Symmetry of solutions

In general, the symmetry of a solution $\mathbf{u}$ is lower than that of the equations $\mathbf{F}$, where the symmetry of a solution $\mathbf{u}$ is measured by a subset of G,

$$\Sigma(\mathbf{u}) = \Sigma(\mathbf{u}; G, T) = \{g \in G \mid T(g)\mathbf{u} = \mathbf{u}\}, \tag{49}$$

which forms a subgroup of G.

The most fundamental fact is that the symmetry of a solution remains invariant in the neighborhood of an ordinary point, at which the Jacobian matrix is nonsingular by definition. Interesting phenomena can happen in the neighborhood of a critical point.

Let μ denote the irreducible representation associated with the group-theoretic critical point $(\mathbf{u}_{\rm c}^0, f_{\rm c}^0)$. With the subgroup G^μ introduced in (35) we have an important relation

$$G^\mu \subseteq \Sigma(\mathbf{u}; G, T) \subseteq G. \tag{50}$$

[165] Fujii, Mimura, and Nishiura, 1982 [177]
[166] Golubitsky, Stewart, and Schaeffer, 1988 [198]

This says that the symmetry of a solution in the neighborhood of $(\mathbf{u}_c^0, f_c^0)$ can possibly be smaller than G, but is at least as large as G^μ. It is noted that G^μ denotes the symmetry common to all the critical eigenvectors. We also mention a crucial technical relation

$$\Sigma(\mathbf{w}; G, T) = \Sigma(\mathbf{u}; G, T), \tag{51}$$

which shows that the symmetry of the solution $\mathbf{u}$ to the full system of equations is determined by that of the solution $\mathbf{w}$ to the bifurcation equation. This fact allows us to focus on the solution $\mathbf{w}$ of the bifurcation equation in discussing the symmetry of a bifurcating solution $\mathbf{u}$.

In (50) the discrepancy between G^μ and $\Sigma(\mathbf{u}; G, T)$ is caused by the nonlinearity of the equation. Accordingly, the symmetry of the solution can be determined only through an analysis involving nonlinear terms. Equation (51) shows that an analysis of the bifurcation equation suffices for this.

In association with the repeated occurrence of bifurcation, we can find a hierarchy of subgroups

$$G \to G_1 \to G_2 \to \cdots \tag{52}$$

that characterizes the recursive change of symmetries. Actual forms of this hierarchy for particular groups are available.[167]

6 DOUBLE BIFURCATION POINTS: DIHEDRAL SYMMETRY

In this section we investigate systems with dihedral group symmetry. The variety of perfect and imperfect behaviors is enriched by the emergence of double critical points, at which two critical eigenvectors compete to generate a number of bifurcated solutions. These behaviors are analyzed by the group-theoretic method presented in Section 5.

6.1 Dihedral groups and their irreducible representations

The dihedral group of degree n is defined by

$$\mathrm{D}_n = \{c(2\pi i/n), \sigma c(2\pi i/n) \mid i = 0, 1, \ldots, n-1\}, \tag{53}$$

where $c(2\pi i/n)$ denotes a counterclockwise rotation about the origin of $2\pi i/n$ $(i = 0, 1, \ldots, n-1)$, σ is a reflection and $\sigma c(2\pi i/n)$ is the combined action of the rotation $c(2\pi i/n)$. This group, for example, describes the symmetry of a regular n-gon in the xy-plane.

[167] Ikeda and Murota, 2002 [270]; Tanaka, Saiki, and Ikeda, 2002 [499]

Subgroups of D_n consist of dihedral and cyclic groups whose degree m divides n; i.e., the family of the subgroups of D_n in the representation (53) is given by

$$\begin{aligned}&\{\mathrm{D}_m^{k,n} \mid k=1,\ldots,n/m;\ m \text{ divides } n \text{ and } 1 \leq m < n\},\\ &\{\mathrm{C}_m \mid m \text{ divides } n \text{ and } 1 \leq m \leq n\},\end{aligned} \tag{54}$$

where

$$\begin{aligned}\mathrm{D}_m^{k,n} &= \{c(2\pi i/m), \sigma c(2\pi[(k-1)/n + i/m]) \mid i = 0,1,\ldots,m-1\},\\ \mathrm{C}_m &= \{c(2\pi i/m) \mid i=0,1,\ldots,m-1\}.\end{aligned}$$

Note that $\mathrm{D}_m^{1,n} = \mathrm{D}_m$ and $\mathrm{C}_1 = \{e\}$. These subgroups express partial symmetries of regular n-gons. Cyclic groups C_m denote rotation-symmetric patterns; the group C_1 means a completely asymmetric pattern; and dihedral groups $\mathrm{D}_m^{k,n}$ indicate reflection symmetric patterns. The superscripts are introduced here to distinguish the difference in the direction of the reflection line.

The dihedral group D_n in (53) has one- and two-dimensional irreducible representations, which are respectively associated with a simple critical point and and a group-theoretic double bifurcation point.

The one-dimensional irreducible representations, which are labeled by $(+,+)_{\mathrm{D}_n}$, $(+,-)_{\mathrm{D}_n}$, $(-,+)_{\mathrm{D}_n}$, and $(-,-)_{\mathrm{D}_n}$, are defined by

$$\begin{aligned}&T^{(+,+)_{\mathrm{D}_n}}(c(2\pi/n)) = 1, && T^{(+,+)_{\mathrm{D}_n}}(\sigma) = 1,\\ &T^{(+,-)_{\mathrm{D}_n}}(c(2\pi/n)) = 1, && T^{(+,-)_{\mathrm{D}_n}}(\sigma) = -1,\\ &T^{(-,+)_{\mathrm{D}_n}}(c(2\pi/n)) = -1, && T^{(-,+)_{\mathrm{D}_n}}(\sigma) = 1,\\ &T^{(-,-)_{\mathrm{D}_n}}(c(2\pi/n)) = -1, && T^{(-,-)_{\mathrm{D}_n}}(\sigma) = -1,\end{aligned} \tag{55}$$

where $(-,+)_{\mathrm{D}_n}$ and $(-,-)_{\mathrm{D}_n}$ exist only for n even.

The two-dimensional representations, which are labeled by $(1)_{\mathrm{D}_n}$, …, $(N_n)_{\mathrm{D}_n}$, can be chosen to be unitary representations defined by

$$\left.\begin{aligned}&T^{(j)_{\mathrm{D}_n}}(c(2\pi/n)) = \begin{pmatrix}\cos(2\pi j/n) & -\sin(2\pi j/n)\\ \sin(2\pi j/n) & \cos(2\pi j/n)\end{pmatrix}, \quad j=1,\ldots,N_n\\ &T^{(j)_{\mathrm{D}_n}}(\sigma) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}\right\} \tag{56}$$

where $N_n = (n-2)/2$ for n even and $N_n = (n-1)/2$ for n odd.

The associated subgroups G^{μ} of (35) for those irreducible representations are given as follows:

$$\left.\begin{aligned}&G^{(+,+)_{\mathrm{D}_n}} = \mathrm{D}_n,\ G^{(+,-)_{\mathrm{D}_n}} = \mathrm{C}_n,\ G^{(-,+)_{\mathrm{D}_n}} = \mathrm{D}_{n/2},\ G^{(-,-)_{\mathrm{D}_n}} = \mathrm{D}_{n/2}^{2,n}\\ &G^{(j)_{\mathrm{D}_n}} = \mathrm{C}_{n/\widehat{n}},\end{aligned}\right\} \tag{57}$$

where

$$\widehat{n} = \frac{n}{\gcd(n, j)} \tag{58}$$

is an important parameter characterizing the double bifurcation point. Here $\gcd(n,j)$ denotes the greatest common divisor of n and j.

6.2 Perfect bifurcation behavior

With the general framework of Section 5, we can investigate the bifurcation behavior of a D_n-symmetric perfect system at a group-theoretic critical point.

For a simple critical point associated with the unit representation $\mu = (+,+)_{\mathrm{D}_n}$, the equivariance (48) to $G = \mathrm{D}_n$ of the bifurcation equation with $\mathbf{v} = \mathbf{v}^0$ plays no role, since $T^{(+,+)_{\mathrm{D}_n}}(g) = 1$ $(g \in \mathrm{D}_n)$ by (55) and $S(g)\mathbf{v}^0 = \mathbf{v}^0$ $(g \in \mathrm{D}_n)$ by (38). This means that this is a limit point.

For a simple critical point associated with $\mu = (+,-)_{\mathrm{D}_n}$, $(-,+)_{\mathrm{D}_n}$, or $(-,-)_{\mathrm{D}_n}$, the equivariance (48) to $G = \mathrm{D}_n$ of the bifurcation equation with $\mathbf{v} = \mathbf{v}^0$ is rewritten as

$$\widetilde{F}(-w, \widetilde{f}, \mathbf{v}^0) = -\widetilde{F}(w, \widetilde{f}, \mathbf{v}^0) \tag{59}$$

with the use of $S(g)\mathbf{v}^0 = \mathbf{v}^0$ for $g \in \mathrm{D}_n$ in (38). Therefore, the bifurcation equation for the perfect system is an odd function in w, as was the case for the exploitation of reflection symmetry conducted earlier.[168] This is a pitchfork bifurcation point. The symmetry of the bifurcating solution is represented by

$$\Sigma(\boldsymbol{\xi}_1) = \Sigma(\boldsymbol{\eta}_1) = G^{\mu} = \Sigma(\ker(J_{\mathrm{c}}^0)) = \Sigma(\mathbf{w}) = \Sigma(\mathbf{u}) = \mathrm{C}_n,\ \mathrm{D}_{n/2},\ \text{or } \mathrm{D}_{n/2}^{2,n} \tag{60}$$

given that the associated irreducible representation μ is $(+,-)_{\mathrm{D}_n}$, $(-,+)_{\mathrm{D}_n}$, or $(-,-)_{\mathrm{D}_n}$, respectively.

A double critical point on a D_n-symmetric path is associated with a two-dimensional irreducible representation $\mu = (j)_{\mathrm{D}_n}$ for some j. The positive integer $\widehat{n} = n/\gcd(n,j)$ in (58) characterizes this critical point. The superposition (linear combination) of a pair of critical eigenvectors

$$\boldsymbol{\xi}(\varphi) = \cos\varphi \cdot \boldsymbol{\xi}_1 + \sin\varphi \cdot \boldsymbol{\xi}_2, \tag{61}$$

$$\boldsymbol{\eta}(\theta) = \cos\theta \cdot \boldsymbol{\eta}_1 + \sin\theta \cdot \boldsymbol{\eta}_2 \tag{62}$$

serves also a critical eigenvector. The symmetry of $\boldsymbol{\xi}(\varphi)$ and $\boldsymbol{\eta}(\theta)$ is given by

$$\Sigma(\boldsymbol{\xi}(\varphi)) = \Sigma(\boldsymbol{\eta}(\theta)) = G^{\mu} = \mathrm{C}_{n/\widehat{n}} \tag{63}$$

(cf., (57)) for the general angles φ and θ $(0 \le \varphi < 2\pi,\ 0 \le \theta < 2\pi)$. Some critical eigenvectors have higher symmetry than $\mathrm{C}_{n/\widehat{n}}$. We can make

[168] Chilver, 1967 [98]; Supple, 1967 [494]

Table 1: Critical points on a D_n-symmetric path.

Multiplicity	Irreducible	Type of	Symmetry groups	
M	representation, μ	points	G^μ	Bifurcated paths
1	$(+,+)_{\mathrm{D}_n}$	Limit	D_n	No bifurcation
1	$(+,-)_{\mathrm{D}_n}$	Pitchfork	C_n	C_n
	$(-,+)_{\mathrm{D}_n}$ (n: even)		$\mathrm{D}_{n/2}$	$\mathrm{D}_{n/2}$
	$(-,-)_{\mathrm{D}_n}$ (n: even)		$\mathrm{D}^{2,n}_{n/2}$	$\mathrm{D}^{2,n}_{n/2}$
2	$(j)_{\mathrm{D}_n}$	Double	$\mathrm{C}_{n/\hat{n}}$	$\mathrm{D}^{i,n}_{n/\hat{n}}$
				$(i=1,\ldots,\hat{n})$

$\boldsymbol{\xi}_1 = \boldsymbol{\xi}(0)$ and $\boldsymbol{\eta}_1 = \boldsymbol{\eta}(0)$ invariant under reflection σ, i.e., $T(\sigma)\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1$ and $T(\sigma)\boldsymbol{\eta}_1 = \boldsymbol{\eta}_1$. Then there exist $2\hat{n}$ such eigenvectors, namely,

$$\Sigma(\boldsymbol{\xi}(\alpha_{i+\hat{n}j})) = \Sigma(\boldsymbol{\eta}(\alpha_{i+\hat{n}j})) = \mathrm{D}^{i,n}_{n/\hat{n}}, \qquad i=1,\ldots,\hat{n}, \quad j=0,1, \tag{64}$$

where

$$\alpha_k = -\pi\,\frac{k-1}{\hat{n}}, \qquad k=1,\ldots,2\hat{n}. \tag{65}$$

In obtaining the generic form of the system of bifurcation equations $\widetilde{\mathbf{F}}(\mathbf{w},\tilde{f},\mathbf{v}) = \mathbf{0}$ in (47) with $M=2$, we employ the D_n-equivariance (48) of the bifurcation equations formulated as

$$\widetilde{T}(g)\widetilde{\mathbf{F}}(\mathbf{w},\tilde{f},\mathbf{v}) = \widetilde{\mathbf{F}}(\widetilde{T}(g)\mathbf{w},\tilde{f},S(g)\mathbf{v}), \qquad g\in \mathrm{D}_n, \tag{66}$$

where $\widetilde{T}$ is the two-dimensional irreducible representation of D_n associated with the kernel of $J^0_{\mathrm{c}} = J(\mathbf{u}^0_{\mathrm{c}}, f^0_{\mathrm{c}}, \mathbf{v}^0)$. We may assume $\widetilde{T} = T^{(j)_{\mathrm{D}_n}}$ for some j in the notation of (56).

According to the analysis of the bifurcation equations, it was found that

(i) there exist $2\hat{n}$ half branches ($\hat{n}$ bifurcated paths) as shown in Fig. 10;

(ii) they bifurcate in the directions of $\boldsymbol{\eta}(\alpha_{i+\hat{n}j})$ ($i=1,\ldots,\hat{n},\ j=0,1$); and

(iii) the solutions $\mathbf{u}$ on the bifurcated path for $\theta = \alpha_{i+\hat{n}j}$ are $\mathrm{D}^{i,n}_{n/\hat{n}}$-symmetric.

It is emphasized that $\Sigma(\mathbf{u}) = \mathrm{D}^{i,n}_{n/\hat{n}}$ is strictly larger than $G^\mu = \mathrm{C}_{n/\hat{n}}$.

6.3 Local imperfect behavior at a double bifurcation point

We move here on to investigate the imperfect behavior at a double bifurcation point. In place of the variables (w_1, w_2) in the bifurcation equation (47), we employ the polar coordinates (r, θ) with $w_1 = r\cos\theta$ and $w_2 = r\sin\theta$. With (r, θ), we can rewrite the bifurcation equation (47) as

$$\widetilde{f} \sim -r^2 + \sum_{2\leq q\leq \widehat{n}/2-1} c_q r^{2q} + br^{\widehat{n}-2}\cos(\widehat{n}\theta) + \frac{|a\epsilon|}{r}\cos(\theta - \psi), \tag{67}$$

$$br^{\widehat{n}-1}\sin(\widehat{n}\theta) + |a\epsilon|\sin(\theta - \psi) \sim 0, \tag{68}$$

where a is a complex constant with $a\epsilon = |a\epsilon|\exp(\mathrm{i}\psi)$ and b and c_q are real constants. As we will see below, the solution paths that consist of $(r, \theta, \widetilde{f})$ satisfying these equations display complex but interesting behaviors dependent on the values of ψ and $\widehat{n}$.

Equation (68) does not contain $\widetilde{f}$ and (67) is solved for $\widetilde{f}$. This allows us to concentrate on (68). Two cases are distinguished:

(i) special case: $\psi = \pi(m-1)/\widehat{n}$ for some $m = 1, \ldots, 2\widehat{n}$; and

(ii) general case: $\psi \neq \pi(m-1)/\widehat{n}$ for any $m = 1, \ldots, 2\widehat{n}$.

The special case means the presence of the initial imperfection with partial symmetry, as was observed for the Augusti model.[169]

We first consider the special case, for which equation (68) admits a pair of rays

$$\theta = \psi,\ \ \psi + \pi \qquad (r\text{: arbitrary}) \tag{69}$$

as solutions, which satisfy

$$\sin(\widehat{n}\theta) = \sin(\theta - \psi) = 0. \tag{70}$$

These solutions, which are directed towards $\widetilde{f} \to \pm\infty$ as $r \to +0$, represent the fundamental and complementary paths. Equation (68) has another kind of solution represented as

$$r^{\widehat{n}-1} \sim \frac{-|a\epsilon|}{b}\,\frac{\sin(\theta - \psi)}{\sin(\widehat{n}\theta)}. \tag{71}$$

The curves of r and θ given in (69) and (71), which are dependent on the values of $\widehat{n}$ and ψ, are illustrated in Fig. 11(a)$\sim$(c).

As an example of $\widehat{n}$ odd, we consider $\widehat{n} = 3$ with two special cases of $\psi = 0$ and $\psi = \pi$. For $\psi = 0$ in Fig. 11(a), each of the fundamental and complementary paths on the ray $\theta = 0$ and π has an unstable pitchfork bifurcation point, at which branches a secondary bifurcated path, as was

[169] Augusti, 1964 [38]; Thompson and Hunt, 1973, 1984 [518, 523]

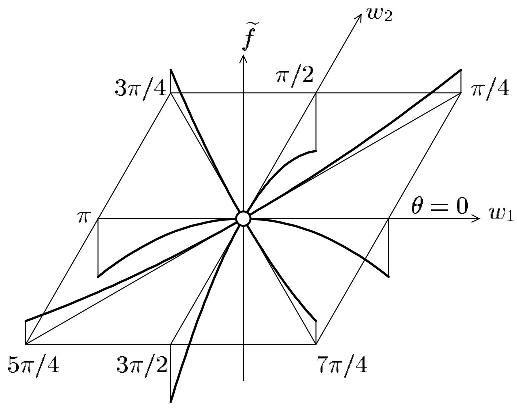


Figure 10: Spatial view of the perfect bifurcation behavior in the neighborhood of a double bifurcation point (○) with $\widehat{n}=4$.

observed for the Augusti model. For $\psi = \pi$ in Fig. 11(b), each of the fundamental and complementary paths on the ray $\theta = 0$ and π has a limit point.

As an example of $\widehat{n}$ even, we consider $\widehat{n} = 4$ with a special case of $\psi = 0$ shown in Fig. 11(c). The fundamental path on the ray $\theta = \pi$ has a limit point and the complementary path on the ray $\theta = 0$ has an unstable pitchfork bifurcation point.

We next consider the general case where $\psi \neq \pi(m-1)/\widehat{n}$ for any $m = 1, \ldots, 2\widehat{n}$. In this case, (70) has no solution, and (68) yields the solution (71) only, which yields $\widehat{n}+1$ solution paths. As shown in Fig. 11(d) and (e), unlike the special case, the directions of these two paths vary in the θ-direction in association with the change of r. The fundamental path has a limit point both for $\widehat{n} = 3$ and 6. Limit points are absent on aloof paths for $\widehat{n} = 3$, but are present for $\widehat{n} = 6$.

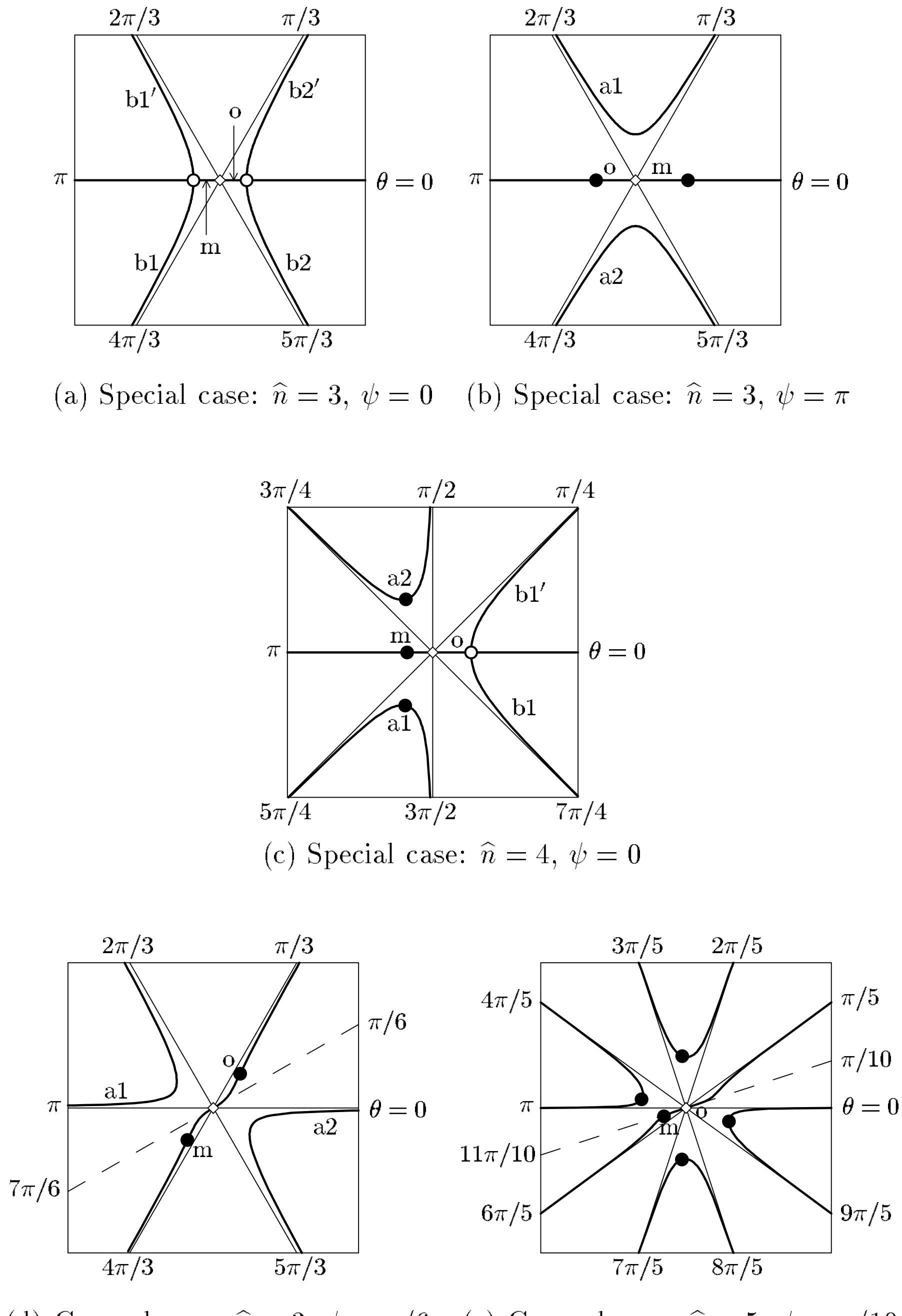
6.4 Imperfection sensitivity law

The imperfection sensitivity law, for the double bifurcation point on a D_n-symmetric path, was derived by solving simultaneously the bifurcation equations (47) and its criticality condition as

$$\widetilde{f}_{\mathrm{c}} \sim C(\mathbf{d})|\epsilon|^{\rho}, \tag{72}$$

where ρ and $C(\mathbf{d})$ are given, depending on the value of index $\widehat{n}$ in (58), as follows:

$$\left.\begin{array}{lll} \rho = 1/2, & C(\mathbf{d}) = -\tau(\psi)C_0 \cdot |a|^{1/2} & \text{if } \widehat{n} = 3 \\ \rho = 2/3, & C(\mathbf{d}) = -\hat{\tau}(\psi)C_0 \cdot |a|^{2/3} & \text{if } \widehat{n} = 4 \\ \rho = 2/3, & C(\mathbf{d}) = -C_0 \cdot |a|^{2/3} & \text{if } \widehat{n} \geq 5 \end{array}\right\} \tag{73}$$



(a) Special case: $\widehat{n} = 3$, $\psi = 0$ (b) Special case: $\widehat{n} = 3$, $\psi = \pi$

(c) Special case: $\widehat{n} = 4$, $\psi = 0$

(d) General case: $\widehat{n} = 3$, $\psi = \pi/6$ (e) General case: $\widehat{n} = 5$, $\psi = \pi/10$

Figure 11: Plane views of local imperfect behavior. ○: simple pitchfork bifurcation point; •: limit point; m: fundamental path; o: complementary path; ⋄ asymptote on the $\widetilde{f}$-axis; thin line: solution curve for the perfect system that serves as an asymptote; — — —: another asymptote.

in which a is a complex number with $\psi = \arg(a\epsilon)$, $\tau(\psi)$ is a nonlinear function in ψ and is positive, and $\hat{\tau}(\psi)$ is a nonlinear function in ψ that is dependent on individual systems.

6.5 Probabilistic scatter of critical loads

Following Section 3.4, we consider the random variation of the critical load $\widetilde{f}_{\rm c}$ when the imperfection pattern vector $\mathbf{d}$ is a random variable subject to a normal distribution $\mathrm{N}(\mathbf{0}, W)$ with a variance–covariance matrix W.

The probability density function of the critical load varies with the value of index $\widehat{n}$ and is more more complex than that for simple critical points, but can be simplified for $\widehat{n} \geq 5$ by exploiting the symmetry of an imperfect system. Namely, we have

$$\phi(\zeta) = \frac{3\zeta^2}{2} \exp(-|\zeta|^3/2), \quad -\infty < \zeta < 0, \tag{74}$$

where $\zeta = (f_{\rm c} - f_{\rm c}^0)/\widehat{C}$ is the normalized critical load increment for some scaling constant $\widehat{C}$. The cases for $\widehat{n} = 3$ and 4 can be found in [270].

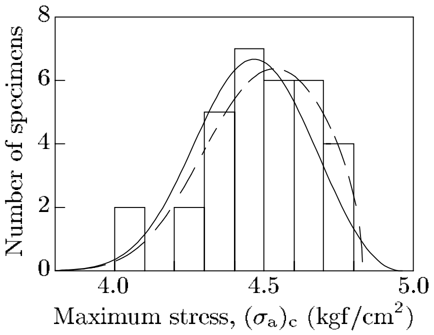
6.6 Imperfect behavior of materials

The probabilistic approach presented in Section 6.5 is applied to the description of the strength variation of cylindrical sand specimens. See Fig. 4 for a few examples of a set of 32 experimental curves of the deviatoric stress $\sigma_{\rm a}$ versus the axial strain $\varepsilon_{\rm a}$.

The initial states for the specimens are assumed to be D_n-symmetric with n large. Then the observed variable $\varepsilon_{\rm a}$ can be considered to be D_n-symmetric. We restrict ourselves to the possibility of a simple (pitchfork) bifurcation point and a double bifurcation point with $\widehat{n} \geq 5$, which are associated, respectively, with the symmetry-breaking processes:

$$\begin{cases} \mathrm{D}_n \to \mathrm{C}_n & \text{at the pitchfork bifurcation point,} \\ \mathrm{D}_n \to \mathrm{D}_{n/\widehat{n}} & \text{at the double bifurcation point with } \widehat{n} \geq 5. \end{cases}$$

The histogram of the maximum deviatoric stress for those specimens is compared in Fig. 12 with the probability density functions to show good correlation with the experimental histogram.



Type	Simple	Double
$E[(\sigma_a)_c]$ (kgf/cm^2)	4.49	
$Var[(\sigma_a)_c]$ ((kgf/cm^2)^2)	0.183^2	
$(\sigma_a)_c^0$ (kgf/cm^2)	4.83	4.96
$\widehat{C}$	0.424	0.448

Figure 12: Comparison of a histogram and the probability density functions of the maximum deviatoric stress $(\sigma_a)_c$ for 32 sand specimens. — — —: probability density function for a simple, unstable (pitchfork) bifurcation point; ———: probability density function for an unstable double bifurcation point.

7 CONCLUSIONS

The historical development and recent revival of the research on probabilistic strength scatter of imperfection-sensitive structures and materials have been reviewed. These approaches, which seem somewhat overshadowed by nonlinear finite element analysis in the computer age, can serve as efficient and insightful strategy to tackle various aspects of bifurcation behaviors with the help of group-theoretic bifurcation theory. It will be a natural trend in research in the future to transfer the results of asymptotic studies of structures to the study of materials.

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