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COMBINATORIAL ANALYSIS OF GENERIC MATRIX PENCILS *

SATORU IWATA† AND RYO SHIMIZU†

Abstract. This paper investigates the Kronecker canonical form of matrix pencils under the genericity assumption that the set of nonzero entries is algebraically independent. We provide a combinatorial characterization of the sums of the row/column indices supported by efficient bipartite matching algorithms. We also give a simple alternative proof for a theorem of Poljak on the generic ranks of matrix powers.

Key words. Kronecker canonical form, matrix pencil, combinatorial matrix theory, bipartite matching, Dulmage-Mendelsohn decomposition

1. Introduction. A matrix pencil is a polynomial matrix whose nonzero entries are of degree at most one. Based on the theory of elementary divisors, Weierstrass established a criterion for strict equivalence, as well as a canonical form, of regular matrix pencils. Somewhat later, Kronecker investigated singular pencils to obtain a canonical form for matrix pencils in general under strict equivalence transformations, which is now called the Kronecker canonical form.

The Kronecker canonical form finds a variety of applications in control theory of linear dynamical systems [2, 22, 30, 31]. It is also in a close relation to the index of differential algebraic equations [13, 14, 28]. The literature in these application areas often refers to Gantmacher [12] for theoretical backgrounds on matrix pencils.

Numerically stable algorithms are already available for computing the Kronecker canonical form [1, 3, 8, 19, 21, 34, 35]. Nevertheless, these algorithms are not very accurate in the presence of round-off errors. The numerical difficulty is inherent in the problem as the Kronecker canonical form is highly sensitive to perturbation. This recently motivates extensive research on perturbation of matrix pencils [9, 10].

On the other side, matrix pencils arising in applications are often very sparse and their entries that represent physical characteristics are not precise in value because of noises. Hence one may assume that there is no algebraic dependency among the nonzero entries. It is then desirable to predict the structure of the Kronecker canonical form efficiently from the combinatorial information such as the zero/nonzero pattern without numerical computation. Such a structural approach has been conducted for regular matrix pencils by Duff and Gear [5] and Pantelides [28] in the context of differential algebraic equations. Murota [25] described a complete characterization of the Kronecker canonical form of regular matrix pencils in terms of the maximum degree of minors, which is tantamount to the maximum weight of bipartite matchings under the genericity assumption. An extension of this characterization to mixed matrix pencils is also presented in [26, 27]. A recent paper of van der Woude [33] provides another combinatorial characterization based on the Smith normal form [24].

In this paper, we extend the structural approach to the analysis of singular matrix pencils. The possible existence of the minimal row/column indices (rectangular blocks in the canonical form) makes this problem much more complicated than the regular case. In fact, it remains open to design an efficient algorithm for determining the row/column indices. The main result of this paper, however, provides a combinatorial

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characterization of the sums of the minimal row/column indices using the Dulmage-Mendelsohn decomposition and weighted bipartite matchings. Thus our combinatorial characterization is supported by efficient algorithms.

The structure of the Kronecker canonical form is closely related to the series of larger constant matrices, called the expanded matrices. We investigate those expanded matrices to show that their ranks are equal to the term-ranks in most cases. As a byproduct, we give a simple alternative proof for a theorem of Poljak [29] on the generic ranks of matrix powers.

The outline of this paper is as follows. Section 2 recapitulates the Kronecker canonical form. In Section 3, we introduce generic matrix pencils. In Section 4, we explain the Dulmage-Mendelsohn decomposition, which plays an essential rôle in our result on the minimal row/column indices presented in Section 5. Sections 6 and 7 are devoted to the analysis of expanded matrices. The simple proof for the theorem of Poljak is shown in Section 8. Sections 6 and 8 are independent of Sections 4 and 5.

2. The Kronecker canonical form of matrix pencils. Let $D(s) = sA + B$ be an $m \times n$ matrix pencil of rank r with the row set R and the column set C. We denote by $D(s)[X, Y]$ the submatrix of $D(s)$ determined by $X \subseteq R$ and $Y \subseteq C$. A matrix pencil $\bar{D}(s)$ is said to be strictly equivalent to $D(s)$ if there exists a pair of nonsingular constant matrices U and V such that $\bar{D}(s) = UD(s)V$. A matrix pencil $D(s) = sA + B$ is said to be regular if $\det D(s) \neq 0$ as a polynomial in s. It is strictly regular if both A and B are nonsingular matrices.

For a positive integer μ, we consider $\mu \times \mu$ matrix pencils N_μ and K_μ defined by

$$N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & s \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \qquad K_\mu = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & s & 1 \\ 0 & \cdots & \cdots & 0 & s \end{pmatrix}.$$

For a positive integer ε, we further denote by L_ε an $\varepsilon \times (\varepsilon + 1)$ matrix pencil

$$L_\varepsilon = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s & 1 \end{pmatrix}.$$

We also denote by $L_\eta^\top$ the transpose matrix of L_η.

The following theorem establishes the Kronecker canonical form of matrix pencils under strict equivalence transformations.

THEOREM 2.1 (Kronecker, Weierstrass). *For any matrix pencil $D(s)$, there exists a pair of nonsingular constant matrices U and V such that $\bar{D}(s) = UD(s)V$ is in a block-diagonal form*

$$\bar{D}(s) = \text{block-diag}(H_\nu, K_{\rho_1}, \cdots, K_{\rho_c}, N_{\mu_1}, \cdots, N_{\mu_d}, L_{\varepsilon_1}, \cdots, L_{\varepsilon_p}, L_{\eta_1}^\top, \cdots, L_{\eta_q}^\top, O),$$

where $\rho_1 \geq \cdots \geq \rho_c > 0$, $\mu_1 \geq \cdots \geq \mu_d > 0$, $\varepsilon_1 \geq \cdots \geq \varepsilon_p > 0$, $\eta_1 \geq \cdots \geq \eta_q > 0$, and H_ν is a strictly regular matrix pencil of size ν. The numbers c, d, p, q, ν, $\rho_1, \cdots, \rho_c$, $\mu_1, \ldots, \mu_d$, $\varepsilon_1, \cdots, \varepsilon_p$, $\eta_1, \cdots, \eta_q$ are uniquely determined. □

The block-diagonal matrix pencil $\bar{D}(s)$ in Theorem 2.1 is often referred to as the Kronecker canonical form of $D(s)$. The numbers $\mu_1, \ldots, \mu_d$ are called the indices of nilpotency. The numbers $\varepsilon_1, \cdots, \varepsilon_p$ and $\eta_1, \cdots, \eta_q$ are the minimal column and row indices, respectively. These numbers together with $\nu, \rho_1, \cdots, \rho_c$ are collectively called the structural indices of $D(s)$. Note that the matrix pencil $D(s)$ is regular if and only if $p = q = 0$.

For a polynomial $g(s)$ in s, let $\deg g(s)$ and $\operatorname{ord} g(s)$ denote the highest and lowest degrees of nonvanishing terms of $g(s)$, respectively. For each $k = 1, \ldots, r$, we denote

$$\delta_k(D) = \max\{\deg \det D(s)[X,Y] \mid |X| = |Y| = k, X \subseteq R, Y \subseteq C\},$$
$$\zeta_k(D) = \min\{\operatorname{ord} \det D(s)[X,Y] \mid |X| = |Y| = k, X \subseteq R, Y \subseteq C\}.$$

Note that $\delta_k(D)$ is concave in k and $\zeta_k(D)$ is convex in k. The following well-known lemma asserts that δ_k and ζ_k are invariant under strict equivalence transformations.

LEMMA 2.2. *If $\bar{D}(s)$ is strictly equivalent to $D(s)$, then $\delta_k(\bar{D}) = \delta_k(D)$ and $\zeta_k(\bar{D}) = \zeta_k(D)$ hold.* □

In the Kronecker canonical form of $D(s)$, we have $c = r - \max\{k \mid \zeta_k(D) = 0\}$ and $d = r - \max\{k \mid \delta_k(D) = k\}$. Moreover, we have $\rho_i = \zeta_{r-i+1}(D) - \zeta_{r-i}(D)$ for $i = 1, \ldots, c$ and $\mu_i = \delta_{r-i}(D) - \delta_{r-i+1}(D) + 1$ for $i = 1, \ldots, d$. These equalities imply

$$\sum_{i=1}^{d} \mu_i = r - \delta_r(D), \qquad \sum_{i=1}^{c} \rho_i = \zeta_r(D). \tag{2.1}$$

Since the sum of the structural indices is equal to the rank of $D(s)$, we have

$$\nu + \sum_{i=1}^{p} \varepsilon_i + \sum_{i=1}^{q} \eta_i = \delta_r(D) - \zeta_r(D). \tag{2.2}$$

For an $m \times n$ matrix pencil $D(s) = sA + B$, we construct a $(k+1)m \times kn$ matrix $\Psi_k(D)$ and a $km \times (k+1)n$ matrix $\Phi_k(D)$ defined by

$$\Psi_k(D) = \begin{pmatrix} A & O & \cdots & O \\ B & A & \ddots & \vdots \\ O & B & \ddots & O \\ \vdots & \ddots & \ddots & A \\ O & \cdots & O & B \end{pmatrix}, \qquad \Phi_k(D) = \begin{pmatrix} A & B & O & \cdots & O \\ O & A & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O \\ O & \cdots & O & A & B \end{pmatrix}.$$

We denote $\psi_k(D) = \operatorname{rank} \Psi_k(D)$ and $\varphi_k(D) = \operatorname{rank} \Phi_k(D)$. We also construct a pair of $km \times kn$ matrices $\Theta_k(D)$ and $\Omega_k(D)$ defined by

$$\Theta_k(D) = \begin{pmatrix} A & B & O & \cdots & O \\ O & A & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O \\ \vdots & & \ddots & A & B \\ O & \cdots & \cdots & O & A \end{pmatrix}, \qquad \Omega_k(D) = \begin{pmatrix} B & O & \cdots & \cdots & O \\ A & B & \ddots & & \vdots \\ O & A & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B & O \\ O & \cdots & O & A & B \end{pmatrix}.$$

We denote $\theta_k(D) = \operatorname{rank} \Theta_k(D)$ and $\omega_k(D) = \operatorname{rank} \Omega_k(D)$. Then it is easy to see that the ranks of these expanded matrices are expressed by the structural indices as follows.

THEOREM 2.3. *Let $D(s)$ be a matrix pencil of rank r with structural indices $(\nu, \rho_1, \cdots, \rho_c, \mu_1, \ldots, \mu_d, \varepsilon_1, \ldots, \varepsilon_p, \eta_1, \ldots, \eta_q)$. Then we have*

$$\psi_k(D) = rk + \sum_{i=1}^{p} \min\{k, \varepsilon_i\}, \qquad \varphi_k(D) = rk + \sum_{i=1}^{q} \min\{k, \eta_i\},$$

$$\theta_k(D) = rk - \sum_{i=1}^{d} \min\{k, \mu_i\}, \qquad \omega_k(D) = rk - \sum_{i=1}^{c} \min\{k, \rho_i\}.$$

□

COROLLARY 2.4. *If $D(s)$ is of column-full rank, so is $\Psi_k(D)$ for each k. If $D(s)$ is of row-full rank, so is $\Phi_k(D)$ for each k.*

Proof. If $D(s)$ is of column-full rank, the Kronecker canonical form has no minimal column indices. Hence Theorem 2.3 implies $\psi_k(D) = rk$. Similarly, if $D(s)$ is of row-full rank, the Kronecker canonical form has no minimal row indices, which together with Theorem 2.3 implies $\varphi_k(D) = rk$. □

Theorem 2.3 together with (2.1) implies the following corollary.

COROLLARY 2.5. *Let $D(s)$ be a matrix pencil of rank r. For $k \geq r$, we have $\theta_k(D) = r(k-1) + \delta_r(D)$ and $\omega_k(D) = rk - \zeta_r(D)$.*

Proof. For $k \geq r$, we have $\theta_k(D) = rk - \sum_{i=1}^{d} \mu_i$ and $\omega_k(D) = rk - \sum_{i=1}^{c} \rho_i$ by Theorem 2.3. Then it follows from (2.1) that $\theta_k(D) = r(k-1) + \delta_r(D)$ and $\omega_k(D) = rk - \zeta_k(D)$ hold for $k \geq r$. □

3. Generic matrix pencils. Given a matrix A with the row set R and the column set C, we construct a bipartite graph $G(A) = (R, C; E)$ with the vertex sets R and C and the edge set E that consists of nonzero entries of A. A subset $M \subseteq E$ is called a matching if no two edges in M share an end-vertex. The term-rank of A, denoted by $\text{t-rank}\, A$, is the maximum size of a matching in $G(A)$. The term-rank provides an upper bound on the rank of A. Under the genericity assumption that the set of nonzero entries are algebraically independent, this upper bound is tight. That is, $\text{rank}\, A = \text{t-rank}\, A$ holds for a generic matrix. A set function τ defined by $\tau(X) = \text{t-rank}\, A[X, C]$ for $X \subseteq R$ is submodular, i.e,

$$\tau(X) + \tau(Z) \geq \tau(X \cup Z) + \tau(X \cap Z)$$

holds for any $X, Z \subseteq R$. This submodularity will be used later in Section 7.

A matrix pencil $D(s) = sA + B$ is called a generic matrix pencil if the nonzero entries in A and B are indeterminates (independent parameters). To be more precise, suppose $D(s)$ is a matrix pencil over a field $\mathbf{F}$. That is, A and B are matrices over the field $\mathbf{F}$. Then $D(s)$ is a generic matrix pencil if the set $\mathcal{T}$ of nonzero entries in A and B is algebraically independent over the prime field $\mathbf{K}$ of $\mathbf{F}$. A typical setting in practice is $\mathbf{F} = \mathbf{R}$ and $\mathbf{K} = \mathbf{Q}$.

For a matrix pencil $D(s) = sA + B$ with the row set R and the column set C, let E and F be the sets of nonzero entries in A and in B, respectively. Then we construct a bipartite graph $G(D) = (R, C; E \cup F)$ with the vertex sets R and C and the edge set $E \cup F$. Note that $G(D)$ allows parallel edges. Each edge e has weight $w(e)$ defined by $w(e) = 1$ for $e \in E$ and $w(e) = 0$ for $e \in F$. A subset M of $E \cup F$ is called a matching if no two edges in M share an end-vertex. The maximum size of a matching in $G(D)$ is the term-rank, denoted by $\text{t-rank}\, D(s)$. The weight $w(M) = \sum_{e \in M} w(e)$ of a matching M is equal to the number of edges in $M \cap E$. We denote by $\widehat{\delta}_k(D)$

the maximum weight of a matching of size k. We also denote by $\widehat{\zeta}_k(D)$ the minimum weight of a matching of size k.

The following two lemmas demonstrate that some fundamental quantities of a generic matrix pencil coincide with their combinatorial counterparts. It should be emphasized here that these combinatorial counterparts are easy to compute with efficient combinatorial algorithms for bipartite matchings [11, 15, 16, 20, 23, 32].

LEMMA 3.1. *For a generic matrix pencil $D(s)$, we have* $\operatorname{rank} D(s) = \text{t-rank}\, D(s)$. □

LEMMA 3.2. *For a generic matrix pencil $D(s)$, we have $\delta_k(D) = \widehat{\delta}_k(D)$ and $\zeta_k(D) = \widehat{\zeta}_k(D)$.* □

4. Dulmage-Mendelsohn Decomposition. In this section, we recapitulate the Dulmage-Mendelsohn decomposition of bipartite graphs [6, 7, 8] following the exposition in [27, §2.2.3].

With a matrix pencil $D(s)$, we associate a submodular function f defined as follows. Let $\Gamma(Y) \subseteq R$ denote the set of vertices adjacent to $Y \subseteq C$ in $G(D)$. Then the function f defined by

$$f(Y) = |\Gamma(Y)| - |Y| \qquad (Y \subseteq C)$$

is submodular, i.e.,

$$f(Y) + f(Z) \geq f(Y \cup Z) + f(Y \cap Z)$$

holds for any $Y, Z \subseteq C$.

The term-rank of the matrix pencil $D(s)$ is characterized by the minimum value of this submodular function f, i.e.,

$$\text{t-rank}\, D(s) = \min\{f(Y) \mid Y \subseteq C\} + |C|,$$

which follows from the Hall-Ore theorem for bipartite graphs. The set of minimizers of a submodular function forms a distributive lattice.

Let Y_0 denote the unique minimal minimizer of f and Y_∞ denote the unique maximal minimizer of f. We put

$$\begin{aligned} C_0 &= Y_0, & R_0 &= \Gamma(Y_0), \\ C_* &= Y_\infty \setminus Y_0, & R_* &= \Gamma(Y_\infty) \setminus \Gamma(Y_0), \\ C_\infty &= C \setminus Y_\infty, & R_\infty &= R \setminus \Gamma(Y_\infty). \end{aligned}$$

Then $D(s)$ is in a block-triangular form with respect to the partitions $(R_0; R_*; R_\infty)$ and $(C_0; C_*; C_\infty)$. That is, $D(s)[R_*, C_0] = O$, $D(s)[R_*, C_0] = O$, $D(s)[R_\infty, C_*] = O$.

Furhtermore, if $D(s)$ is a generic matrix, we have

$$\begin{aligned} \operatorname{rank} D(s)[R_0, C_0] &= |R_0|, \\ \operatorname{rank} D(s)[R_*, C_*] &= |R_*| = |C_*|, \\ \operatorname{rank} D(s)[R_\infty, C_\infty] &= |C_\infty|. \end{aligned}$$

We call $D_0(s) = D(s)[R_0, C_0]$ the horizontal tail and denote its rank by $r_0 = |R_0|$. We also call $D_\infty(s) = D(s)[R_\infty, C_\infty]$ the vertical tail and denote its rank by $r_\infty = |C_\infty|$.

This block-triangularization is called the Dulmage-Mendelsohn decomposition (DM-decomposition) of $D(s)$. The DM-decomposition can be computed efficiently with the aid of the bipartite matching algorithms.

5. The Kronecker canonical form via DM-decomposition. In this section, we investigate the Kronecker canonical form of a generic matrix pencil $D(s)$ via the DM-decomposition.

LEMMA 5.1. *For the horizontal tail D_0 of a generic matrix pencil $D(s)$, we have $\psi_k(D) = \psi_k(D_0) + k(r - r_0)$ and $\widehat{\psi}_k(D) = \widehat{\psi}_k(D_0) + k(r - r_0)$.*

Proof. Recall $r = |R_0| + |C \setminus C_0|$ and $r_0 = |R_0|$. Since $D_*(s) = D(s)[R \setminus R_0, C \setminus C_0]$ is of column-full rank, so is $\Psi_k(D_*)$ by Corollary 2.4, namely, $\psi_k(D_*) = k|C \setminus C_0| = k(r - r_0)$. Since $\psi_k(D_*) \leq \widehat{\psi}_k(D_*) \leq k(r - r_0)$, we also have $\widehat{\psi}_k(D_*) = k(r - r_0)$. Then it follows from $D(s)[R \setminus R_0, C_0] = O$ that $\psi_k(D) = \psi_k(D_0) + \psi_k(D_*) = \psi_k(D_0) + k(r - r_0)$ and $\widehat{\psi}_k(D) = \widehat{\psi}_k(D_0) + \widehat{\psi}_k(D_*) = \widehat{\psi}_k(D_0) + k(r - r_0)$. □

Lemma 5.1 together with Theorem 2.3 implies that the minimal column indices of $D(s)$ coincide with those of $D_0(s)$. We now investigate the Kronecker canonical form of the horizontal tail D_0.

Let $g_0(s)$ be the monic determinantal divisor

$$g_0(s) = \gcd\{\det D(s)[R_0, Y] \mid |Y| = r_0, Y \subseteq C_0\},$$

where gcd designates the greatest common divisor whose leading coefficient is equal to one. The following lemma is a special case of a theorem of Murota [24] (see also [27, Theorem 6.3.8]). We describe its proof here for completeness.

LEMMA 5.2. *The monic determinantal divisor $g_0(s)$ is a monomial in s.*

Proof. We first claim that $g_0(s)$ belongs to $\mathbf{K}[s]$. For any column $j \in C_0$, there exists a column subset $Y \subseteq C_0 \setminus \{j\}$ such that $|Y| = r_0$ and $\det D(s)[R_0, Y] \neq 0$. Therefore, for any independent parameter $t \in \mathcal{T}$, the monic determinantal divisor $g_0(s)$ is free from t. Thus $g_0(s)$ is a polynomial over $\mathbf{K}$.

We now suppose that $g_0(s)$ is not a monomial in s. Let $\overline{\mathbf{K}}$ be the algebraic closure of $\mathbf{K}$. Then there exists a root $\xi \in \overline{\mathbf{K}} \setminus \{0\}$ that satisfies $g_0(\xi) = 0$. For a regular submatrix $D(s)[R_0, Y]$, we have $\det D(\xi)[R_0, Y] = 0$, which contradicts the assumption that $\mathcal{T}$ is algebraically independent over $\overline{\mathbf{K}}$. □

The determinantal divisor is invariant under strict equivalence transformations. Hence $g_0(s)$ is equal to the determinantal divisor of the Kronecker canonical form $\bar{D}_0$ of D_0. Then Lemma 5.2 implies that $\bar{D}_0$ does not contain a strictly regular block.

THEOREM 5.3. *The sum of the minimal column indices is obtained by*

$$\sum_{i=1}^{p} \varepsilon_i = \widehat{\delta}_{r_0}(D_0) - \widehat{\zeta}_{r_0}(D_0). \tag{5.1}$$

Proof. Since the Kronecker canonical form $\bar{D}_0$ of D_0 does not contain a strictly regular block or a rectangular block $L_\eta^\top$, it follows from (2.2) and Lemma 3.2 that $\sum_{i=1}^{p} \varepsilon_i = \delta_{r_0}(D_0) - \zeta_{r_0}(D_0) = \widehat{\delta}_{r_0}(D_0) - \widehat{\zeta}_{r_0}(D_0)$. □

A similar argument applied to the vertical tail D_∞ leads to the following results. Lemma 5.4 implies by Theorem 2.3 that the minimal row indices of $D(s)$ coincide with those of $D_\infty(s)$. Lemma 5.5 shows that the Kronecker canonical form $\bar{D}_\infty(s)$ of $D_\infty(s)$ does not contain the strictly regular block.

LEMMA 5.4. *For the vertical tail D_∞ of a generic matrix pencil $D(s)$, we have $\varphi_k(D) = \varphi_k(D_\infty) + k(r - r_\infty)$ and $\widehat{\varphi}_k(D) = \widehat{\varphi}_k(D_\infty) + k(r - r_\infty)$.* □

LEMMA 5.5. *The monic determinantal divisor*

$$g_\infty(s) = \gcd\{\det D(s)[X, C_\infty] \mid X \subseteq R_\infty, |X| = r_\infty\}$$

is a monomial in s. □

THEOREM 5.6. *The sum of the minimal row indices is obtained by*

$$\sum_{i=1}^{q} \eta_i = \widehat{\delta}_{r_\infty}(D_\infty) - \widehat{\zeta}_{r_\infty}(D_\infty). \tag{5.2}$$

□

As an immediate consequence of Theorems 5.3 and 5.6, we have the following theorem implied by (2.2).

THEOREM 5.7. *The size ν of the strictly regular block in the Kronecker canonical form $\bar{D}(s)$ of $D(s)$ is obtained by*

$$\nu = \widehat{\delta}_r(D) - \widehat{\zeta}_r(D) - \widehat{\delta}_{r_0}(D_0) + \widehat{\zeta}_{r_0}(D_0) - \widehat{\delta}_{r_\infty}(D_\infty) + \widehat{\zeta}_{r_\infty}(D_\infty). \tag{5.3}$$

□

Note that all these right-hand sides of (5.1), (5.2), and (5.3) can be computed efficiently by the DM-decomposition and weighted bipartite matching algorithms. We have thus obtained a useful combinatorial characterization of the sums of the minimal row/column indices as well as the size of the strictly regular block in the Kronecker canonical form.

Among the structural indices of a generic matrix pencil, $\mu_1, \cdots, \mu_d$ and $\rho_1, \cdots, \rho_c$ are known to be efficiently computable by weighted bipartite matching algorithms. The results in this section enables us to compute ν, $\sum_{i=1}^{p} \varepsilon_i$ and $\sum_{i=1}^{q} \eta_i$ as well. It still remains open to determine the values of the minimal row/column indices. The obtained partial results, however, provide sufficient information to discern if the Kronecker canonical form contains vertical/horizontal rectangular blocks.

6. Expanded matrices for indices of nilpotency. We now turn to the ranks of the expanded matrices, which are in a close relation to the structural indices as shown in Theorem 2.3, for a generic matrix pencil $D(s) = sA + B$. Even though the set of nonzero entries in A and B is algebraically independent, the expanded matrices are not generic matrices. It will be shown, however, that the ranks of the expanded matrices are equal to their term-ranks in most cases. In this section, we deal with the expanded matrices $\Theta_k(D)$ and $\Omega_k(D)$, which are particularly related to $\mu_1, \cdots, \mu_d$ and $\rho_1, \cdots, \rho_c$. The other expanded matrices $\Psi_k(D)$ and $\Phi_k(D)$ will be investigated in Section 7.

In order to examine the ranks of $\Theta_k(D)$ and $\Omega_k(D)$, we consider the bipartite graphs $G(\Theta_k(D))$ and $G(\Omega_k(D))$ associated with the expanded matrices. It will turn out that these bipartite graphs allow maximum matchings with periodic structures. As a consequence, the ranks of these expanded matrices are equal to their term-ranks denoted by $\widehat{\theta}_k(D)$ and $\widehat{\omega}_k(D)$.

We first investigate $\Theta_k(D)$. Let $\bar{R}$ and $\bar{C}$ be the row set and the column set of $\Theta_k(D)$. Then $\bar{R} = R^1 \cup \cdots \cup R^k$ and $\bar{C} = C^1 \cup \cdots \cup C^k$, where R_h and C_h are the copies of the row set R and the column set C of D for $h = 1, \ldots, k$. For vertices $u \in R$ and $v \in C$, we denote by u^h and v^h the corresponding vertices in R^h and C^h. The edge set of the bipartite graph $G(\Theta_k(D))$ consists of $\bar{E} = E^1 \cup \cdots \cup E^k$ and $\bar{F} = F^1 \cup \cdots \cup F^{k-1}$, where E^h and F^h are the copies of E and F. The edges in E^h connect R^h and C^h, whereas the edges in F^h connect R^h and C^{h+1}. In other words, $E^h = \{(u^h, v^h) \mid (u, v) \in E\}$ and $F^h = \{(u^h, v^{h+1}) \mid (u, v) \in F\}$.

For a matching M in $G(D)$, let M° be the set of edges (u^h, v^h) with $(u, v) \in E \cap M$ for $h = 1, \ldots, k$ and (u^h, v^{h+1}) with $(u, v) \in F \cap M$ for $h = 1, \ldots, k-1$. Then M°

forms a matching in $G(\Theta_k(D))$. A matching in $G(\Theta_k(D))$ is called a periodic matching if it can be represented as M° with a certain matching M in $G(D)$.

We now introduce the weight w_k on the edge set of $G(D)$ by $w_k(e) = k$ for $e \in E$ and $w_k(e) = k - 1$ for $e \in F$. For a matching M in $G(D)$, we consider the weight of M by $w_k(M) = \sum_{e \in M} w_k(e)$.

LEMMA 6.1. *Let M be a matching that maximizes $w_k(M)$ in $G(D)$. Then the corresponding periodic matching M° is a maximum matching in $G(\Theta_k(D))$.*

Proof. Suppose to the contrary that M° is not a maximum matching in $G(\Theta_k(D))$. Then there exists an augmenting path with respect to M° in $G(\Theta_k(D))$. Let P° be such an augmenting path with minimum number of arcs. The corresponding set of edges in $G(D)$ forms an augmenting path P with respect to M in $G(D)$. Then we analyze the weight of a matching $M' = M \triangle P$, which is the symmetric difference of M and P. Suppose that the end-vertices of P° are $u^h \in R^h$ and $v^l \in C^l$. If $h < l$, then the weight of M' satisfies $w_k(M') - w_k(M) = (l-h)(k-1) - k(l-h-1) = k-l+h$. On the other hand, if $h \geq l$, then we have $w_k(M') - w_k(M) = (h-l+1)k - (h-l)(k-1) = k+h-l$. Thus in either case, we have $w_k(M') = w_k(M) + k + h - k > w_k(M)$, which contradicts the maximality of $w_k(M)$. □

Let M° be a maximum periodic matching in $G(\Theta_k(D))$ corresponding to the maximum-weight matching M in $G(D)$. We denote by ∂M° the set of end-vertices of the edges in M°. Consider the submatrix $\Theta_k(D)[X, Y]$ determined by $X = \bar{R} \cap \partial M$ and $Y = \bar{C} \cap \partial M$. Then the expansion of $\det \Theta_k(D)[X, Y]$ contains a nonzero term

$$\prod_{(u,v) \in M \cap E} {A_{uv}}^k \prod_{(u,v) \in M \cap F} {B_{uv}}^{k-1},$$

where A_{uv} and B_{uv} denote the (u, v)-components of A and B. Each A_{uv} appears exactly k times in $\Theta_k(D)$ and B_{uv} appears exactly $k - 1$ times in $\Theta_k(D)$. Hence no other matching cancels this term in the expansion. Thus $\Theta_k(D)[X, Y]$ is a nonsingular submatrix of size $|M^\circ|$, which implies $\theta_k(D) = \widehat{\theta}_k(D)$ by Lemma 6.1.

A similar argument on $G(\Omega_k(D))$ leads to $\omega_k(D) = \widehat{\omega}_k(D)$. Thus we obtain the following theorem.

THEOREM 6.2. *For a generic matrix pencil $D(s)$, we have $\theta_k(D) = \widehat{\theta}_k(D)$ and $\omega_k(D) = \widehat{\omega}_k(D)$.*

7. Expanded matrices for column/row indices. This section is devoted to a combinatorial analysis of the ranks of the expanded matrices $\Psi_k(D)$ and $\Phi_k(D)$ for a generic matrix pencil $D(s)$. Let $\widehat{\psi}_k(D)$ and $\widehat{\varphi}_k(D)$ denote the term-ranks of these matrices. Since these expanded matrices admit the same indeterminates to appear in different places, it is not immediately clear that the ranks are equal to the term-ranks. In fact, there is an example of such a expanded matrix whose rank is less than its term-rank. Our analysis, however, makes it possible to assert that they are equal for sufficiently large k as well as $k = 1, 2$.

THEOREM 7.1. *For a generic matrix pencil $D(s)$ of rank r, let r_0 be the rank of the horizontal tail D_0. If $k \geq r_0$, we have*

$$\psi_k(D) = \widehat{\psi}_k(D) = kr + \widehat{\delta}_{r_0}(D_0) - \widehat{\zeta}_{r_0}(D_0).$$

Proof. Due to the submodularity of the term-rank, we have

$$\widehat{\psi}_k(D_0) + \widehat{\varphi}_{k-1}(D_0) \leq \widehat{\theta}_k(D_0) + \widehat{\omega}_k(D_0).$$

Since D_0 is of row-full rank, Corollary 2.4 implies $\widehat{\varphi}_{k-1}(D_0) = (k-1)r_0$. It follows from Corollary 2.5 and Theorem 6.2 that $\widehat{\theta}_k(D_0) = \theta_k(D_0) = r_0(k-1) - \delta_{r_0}(D_0)$ and $\widehat{\omega}_k(D_0) = \omega_k(D_0) = r_0 k - \zeta_{r_0}(D_0)$ hold for $k \geq r_0$ Thus, we obtain

$$\widehat{\psi}_k(D_0) \leq kr_0 + \delta_{r_0}(D_0) - \zeta_{r_0}(D_0).$$

Then by Lemma 5.1 we have

$$\widehat{\psi}_k(D) \leq kr + \widehat{\delta}_{r_0}(D_0) - \widehat{\zeta}_{r_0}(D_0).$$

On the other hand, Theorems 2.3 and 5.3 imply $\psi_k(D) = rk + \widehat{\delta}_{r_0}(D_0) - \zeta_{r_0}(D_0)$ for $k \geq r_0$. Since $\psi_k(D_0) \leq \widehat{\psi}_k(D_0)$, we have $\psi_k(D) = \widehat{\psi}_k(D) = kr + \widehat{\delta}_{r_0}(D_0) - \widehat{\zeta}_{r_0}(D_0)$ for $k \geq r_0$. □

A similar argument applied to the vertical tail D_∞ leads to the following theorem.

THEOREM 7.2. *For a generic matrix pencil $D(s)$ of rank r, let r_∞ be the rank of the vertical tail D_∞. If $k \geq r_\infty$, we have*

$$\varphi_k(D) = \widehat{\varphi}_k(D) = kr + \widehat{\delta}_{r_\infty}(D_\infty) - \widehat{\zeta}_{r_\infty}(D_\infty).$$

□

For $k = 1, 2$, the ranks of the expanded matrices coincide with their term-ranks. This is immediate for $k = 1$ as Ψ_1 and Φ_1 are generic matrices. The following theorem deals with the case of $k = 2$.

THEOREM 7.3. *For a generic matrix pencil $D(s)$, we have $\psi_2(D) = \widehat{\psi}_2(D)$ and $\varphi_2(D) = \widehat{\varphi}_2(D)$.*

Proof. Let M^* be a maximum matching in $G(\Psi_2(D))$. The row set $\bar{R}$ and the column set $\bar{C}$ are given by $\bar{R} = R_1 \cup R_2 \cup R_3$ and $\bar{C} = C_1 \cup C_2$, where R_h and C_h are the copies of R and C. For $u \in R$ and $v \in C$, we denote their copies by $u_h \in R_h$ and $v_h \in C_h$. We also denote $X = \bar{R} \cap \partial M^*$ and $Y = \bar{C} \cap \partial M^*$. Then it suffices to show that $W = \Psi_2(D)[X, Y]$ is nonsingular.

Let M_1^* and M_2^* be the sets of edges in M^* incident to C_1 and C_2, respectively. We denote $X_1 = X \cap \partial M_1^*$, $X_2 = X \cap \partial M_2^*$, $Y_1 = C_1 \cap \partial M^*$, and $Y_2 = C_2 \cap \partial M^*$.

Let $\mathcal{P}$ denote the family of perfect matchings in $G(W)$. For each matching $M \in \mathcal{P}$, we denote $\pi(M) = \prod_{(u,v)\in M} W_{uv}$, where W_{uv} is the (u, v)-component of W. Recall that

$$\det W = \sum_{M \in \mathcal{P}} \sigma_M \pi(M),$$

where σ_M takes 1 or -1. We also denote by $\mathcal{P}^\bullet$ the family of perfect matchings $M = M_1 \cup M_2$ such that $\partial M_1 = \partial M_1^*$ and $\partial M_2 = \partial M_2^*$.

We now claim that $\pi(M^\bullet) \neq \pi(M')$ for any pair of $M^\bullet \in \mathcal{P}^\bullet$ and $M' \in \mathcal{P} \setminus \mathcal{P}^\bullet$. Suppose to the contrary that $\pi(M^\bullet) = \pi(M')$. The matching M' contains an edge (u_2, v_2) with $u_2 \in X_1$. Then we have $(u_1, v_1) \in M^\bullet \setminus M'$, and hence $(u_1, z_1) \in M'$ for some $z \in C \setminus \{v\}$, which implies $(u_2, z_2) \in M^\bullet$. This is a contradiction to $u_2 \in X_1$.

Since both $W[X_1, Y_1]$ and $W[X_2, Y_2]$ are generic matrices, we have

$$\sum_{M \in \mathcal{P}^\bullet} \sigma_M \pi(M) = \det W[X_1, Y_1] \cdot \det W[X_2, Y_2] \neq 0.$$

Then the claim above implies $\det W \neq 0$, which means W is nonsingular. □

For general k, however, the ranks of the expanded matrices may differ from their term-ranks. For instance, consider a generic matrix pencil

$$D(s) = sA + B = \begin{pmatrix} \beta_1 & s\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & s\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s\alpha_3 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & \beta_4 & 0 & s\alpha_4 & \beta_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s\alpha_5 + \beta_6 & s\alpha_6 & \beta_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_8 & s\alpha_7 \end{pmatrix}.$$

The Kronecker canonical form of $D(s)$ is $\bar{D}(s) = \text{block-diag}(L_4, L_2)$, which implies

$$\psi_k(D) = \begin{cases} 8k & (k \le 2) \\ 7k + 2 & (2 \le k \le 4) \\ 6k + 6 & (k \ge 4), \end{cases}$$

whereas

$$\widehat{\psi}_k(D) = \begin{cases} 8k & (k \le 3) \\ 6k + 6 & (k \ge 3). \end{cases}$$

Thus $\psi_k(D) = \widehat{\psi}_k(D)$ holds for $k \neq 3$. For $k = 3$, however, we have $\psi_3(D) = 23$ and $\widehat{\psi}_3(D) = 24$.

8. Generic matrix powers. As a byproduct of our combinatorial analysis in Section 6, we give a simple alternative proof for a theorem of Poljak [29] on the ranks of powers of generic square matrices.

Let A be an $n \times n$ generic matrix. We associate a directed graph $\vec{G}(A) = (R, \vec{E})$ with vertex set R identical with the row/column set of A. The arc set $\vec{E}$ is the set of nonzero entries of A, namely $\vec{E} = \{(u, v) \mid A_{uv} \neq 0\}$. A k-walk in $\vec{G}(A)$ is an alternating sequence $(v_0, e_1, v_1, \cdots, e_k, v_k)$ of vertices $v_h \in R$ and $e_h \in \vec{E}$ such that $e_h = (v_{h-1}, v_h)$ for $h = 1, \ldots, k$. A pair of k-walks $(v_0, e_1, v_1, \cdots, e_k, v_k)$ and $(v_0', e_1', v_1', \cdots, e_k', v_k')$ is called independent if $v_h \neq v_h'$ holds for $h = 0, 1, \ldots, k$. The following theorem characterizes the rank of A^k in terms of independent k-walks.

THEOREM 8.1 (Poljak [29]). *For a generic square matrix A, the rank of A^k is equal to the maximum number of mutually independent k-walks in $\vec{G}(A)$.*

Consider a regular matrix pencil $D(s) = sA + I$, where I denotes the unit matrix. Then a k-walk naturally corresponds to a path P in $G(\Theta_k(D))$ from C^1 to R^k. To be more specific, the path P is given by

$$P = \{({v_{h-1}}^h, \bar{v_h}^h) \mid h = 1, \ldots, k\} \cup \{({v_h}^h, \bar{v_h}^h) \mid h = 1, \ldots, k\},$$

where $\bar{v_h}$ denotes the column that is identical to the row $v_h \in R$. Then a pair of independent k-walks correspond to a pair of vertex-disjoint paths in $G(\Theta_k(D))$. Let $\bar{P} = P_1 \cup \cdots \cup P_\ell$ denotes the edge set of ℓ such vertex-disjoint paths that come from ℓ independent k-walks. Then the symmetric difference $\bar{P} \triangle \bar{F}$ forms a matching of size $\ell + n(k-1)$. Conversely, any periodic matching M° can be obtained in this way from a set of independent k walks. Therefore, Lemma 6.1 implies that the maximum number of independent k-walks is equal to $\widehat{\theta}_k(D) - (k-1)n$.

On the other hand, we have

$$\text{rank}\, A^k = \theta_k(D) - (k-1)n.$$

Therefore, in order to prove Theorem 8.1, it suffices to show that $\theta_k(D) = \widehat{\theta}_k(D)$. Since $D(s)$ is not a generic matrix, we can not directly apply Theorem 6.2. However, we can use essentially the same argument.

Let M° be a maximum periodic matching in $G(\Theta_k(D))$ that corresponds to a matching M in $G(D)$. Consider the submatrix $\Theta_k(D)[X, Y]$ with $X = \bar{R} \cap \partial M$ and $Y = \bar{C} \cap \partial M$. Then the expansion of $\det \Theta_k(D)[X, Y]$ contains a nonzero term

$$\prod_{(u,v) \in M \cap E} {A_{uv}}^k,$$

where A_{uv} denotes the (u, v)-component of A. Since each A_{uv} appears exactly k times in $\Theta_k(D)$, no other matching cancels this term in the expansion. Thus $\Theta_k(D)[X, Y]$ is a nonsingular submatrix of size $|M^\circ|$, which implies $\theta_k(D) = \widehat{\theta}_k(D)$ by Lemma 6.1.

9. Conclusion. This paper has investigated the Kronecker canonical form of generic matrix pencils. Even if the genericity assumption is not valid, we can efficiently compute the combinatorial estimates of the sums of the minimal row/column indices as well as the size of the strictly regular block. These estimates are correct in most cases unless there is an unlucky numerical cancellation. Hence we can use them for checking if the result of numerical computation is consistent with the combinatorial information.

Another way to use the combinatorial estimates is to design a numerical algorithm that exploits the combinatorial information. If one had an easier way to check the correctness of the estimates, it would lead to a new algorithm particularly efficient for sparse matrices. In fact, such algorithms of combinatorial relaxation type have been developed for the maximum degree of subdeterminants [17, 18, 25, 27]. It would be interesting to devise the same type of algorithms for minimal row/column indices.

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