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Abstract

In the unidimensional unfolding model, given m objects in general position there arise $1 + m(m-1)/2$ rankings. The set of rankings is called the ranking pattern of the m given objects. By changing these m objects, we can generate various ranking patterns. It is natural to ask how many ranking patterns can be generated and what is the probability of each ranking pattern when the objects are randomly chosen? These problems are studied by introducing a new type of arrangement called mid-hyperplane arrangement and by counting cells in its complement.

Key words and phrases. characteristic polynomial; ideal point; mid-hyperplane arrangement; preferential choice; ranking pattern; social choice; spherical tetrahedron; unfolding model.

1 Introduction

Various models have been developed for the analysis of ranking data. These include Thurstonian models, distance-based models, paired and higher-order comparison models, ANOVA-type loglinear models, multistage models and unfolding models, to name only a few. These models give a description of the ranking process and/or the population of rankers. For a comprehensive treatment of the methods for analyzing and modelling ranking data, see the excellent book by Marden [19].

The unfolding model was devised by Coombs [5, 6, 7] for the analysis of ranking data based on preferential choice behavior. According to De Soete, Feger and Klauer [11, p.1], "Historically, two of the most important contributions to psychological choice modelling are undoubtedly Thurstone's [27] Law of Comparative Judgment and Coombs' [5, 7] unfolding theory." This model has been widely used in practice in many fields beyond psychology: sociology, marketing science, voting theory, etc. In addition, the same mathematical structure can be found in Voronoi diagrams (Okabe, Boots, Sugihara and Chiu [21]), spatial competition models in urban economics (Hotelling [16], Eaton and Lipsey [12, 13]) and multiple discriminant analysis (Kamiya and Takemura [17]).

According to the unidimensional unfolding model, preferential choice is made in the following manner: all individuals evaluate m objects based on the objects' single common attribute. Each object is represented by a real number expressing the level of this attribute $x_i,\ i = 1, 2, \ldots, m$, or a point on the real line $\mathbb{R}$ (the "unidimensional underlying continuum"). At the same time, each individual is also represented by a point $y \in \mathbb{R}$ on the same line. The point y is considered the individual's favorite and is called his/her ideal point. In this model, the real line $\mathbb{R}$ containing both individuals and objects is thought of as the psychological space and is called the *joint scale* or the J scale. Here we identify individuals and objects with their corresponding points. The model assumes that individual y ranks the m objects $x_1, x_2, \ldots, x_m$ according to their distances from y, i.e., individual y prefers x_i to x_j iff $|y - x_i| < |y - x_j|$. Rankings generated by individuals in this way are sometimes called individual scales or I scales.

We say that the m points representing the objects are in general position if they and their midpoints are all distinct. Further, we do not consider partial rankings or ties in this paper, so we treat only those individuals whose ideal points do not coincide with any midpoint of two objects.

Let $x_1, \dots, x_m$ be m objects which satisfy these assumptions. By varying the location of the ideal point y throughout $\mathbb{R}$ except the midpoints, we can account for $\binom{m}{2} + 1$ kinds of rankings of $x_1, x_2, \dots, x_m$. The significance of using this model lies here: there are $m!$ potential rankings, but the psychological structure restricts the variety of rankings that can actually occur. These $\binom{m}{2} + 1$ rankings are called the *admissible rankings* of $\mathbf{x} = (x_1, \dots, x_m)$.

The unidimensional unfolding model was extended to the multidimensional case by Bennett and Hays [1] and Hays and Bennett [15]. As the dimension n of the psychological space gets large, the number of admissible rankings accounted for by this model increases, hence more rankings can be explained. This means that the psychological structure becomes looser as n increases. In fact, when $n \geq m - 1$, all $m!$ rankings are admissible, and the model in this case is not interesting at all. Thus finding an appropriate dimension is important in the actual analysis of ranking data. In this paper, we restrict our attention to the unidimensional unfolding model.

For a given set of m objects represented by $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^n$, we call the set of $\binom{m}{2} + 1$ admissible rankings of $\mathbf{x}$ the *ranking pattern* of $\mathbf{x}$. By considering different attributes, we can get different sets of m real numbers $x_1, x_2, \dots, x_m$ for the same m objects, and thus obtain different ranking patterns. Examination of the collected sample of ranking data can tell us what aspect of the m objects determines the present individuals' preferential choice behavior towards these m objects, thereby enabling some inference about the latent structure.

It is generally impossible to explain all ranking data by considering any single attribute. Van Blokland-Vogelesang [28] introduces an error structure into the unfolding model and makes it a probabilistic model. The error structure in her model is an extension of the Feigin and Cohen [14] model. Other types of error structures have also been studied by other authors (Brady [4], Böckenholt and Gaul [2], De Soete, Carroll and DeSarbo [10]). Moreover, for a set of ranking data which is not completely compatible with any joint scale, van Blokland-Vogelesang [28] proposes a method for finding the "best" joint scale based on Kendall's τ distance.

The arguments so far imply that it is important to know the variety of ranking patterns generated by the unfolding model. The significance of this problem can also be understood in the context of voting theory or social choice theory. It is well known (Coombs [7], Luce and Raiffa [18]) that the unfolding model avoids voting cycles by restricting the possible rankings.

Suppose three individuals A, B and C rank three objects labelled $1, 2$ and

3 as (123), (231) and (312), respectively, where objects 1, 2 and 3 are listed in order from best to worst in the expression $(i_1 i_2 i_3)$. Here two individuals A and C prefer 1 to 2, while B prefers 2 to 1, so by simple majority rule, 1 is preferred to 2 as a collective preference. In the same way, the simple majority rule yields the collective preference that 2 is preferred to 3 and that 3 is preferred to 1, entailing intransitivity called a voting cycle.

But if the individuals' preferences are limited to those determined by the unfolding model, we can see that the collective preference by simple majority rule coincides with the median individual's preference and thus in particular produces no voting cycles. Here the median individual means the individual $M \in \{A, B, C\}$ whose ideal point $y_M \in \mathbb{R}$ is the median of the individuals' ideal points $y_A, y_B, y_C \in \mathbb{R}$. The same holds true for any odd number of individuals and any number $m \geq 3$ of objects. Thus it is crucial to clarify how much restriction the unfolding model imposes on individuals' possible preferences.

We show next that it suffices to study the case where the m objects $x_1, \ldots, x_m$ are ordered as $x_1 < \cdots < x_m$. Consider two sets of m objects $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{x}' = (x'_1, \ldots, x'_m)$. If the rank orders of their midpoints from left to right on $\mathbb{R}$ are the same or the reverse of each other, then $\mathbf{x}$ and $\mathbf{x}'$ produce the same ranking pattern. Conversely, if $\mathbf{x}$ and $\mathbf{x}'$ induce the same ranking pattern, their midpoint orders are the same or the reverse of each other. These facts can be confirmed by using results of Kamiya and Takemura [17]. Here we agree to say that two rank orders of midpoints (or objects) are essentially different if one is different from the other as well as from the reverse of the other. These arguments imply that there is a one-to-one correspondence between the set of ranking patterns and the set of essentially different rank orders of midpoints of the objects. Since the rank order of objects is completely determined by the rank order of their midpoints, two sets of m objects having essentially different rank orders give rise to essentially different rank orders of their midpoints and thus different ranking patterns.

On the other hand, it is obvious that for any permutation $(i_1 \ldots i_m)$ of $\{1, 2, \ldots, m\}$, the set of ranking patterns generated by all $\mathbf{x} = (x_1, \ldots, x_m)$ satisfying $x_{i_1} < \cdots < x_{i_m}$ can be obtained from the set of ranking patterns generated by all $\mathbf{x} = (x_1, \ldots, x_m)$ with $x_1 < \cdots < x_m$ just by relabelling the objects (Lemma 2.2 in Section 2). These considerations tell us that it suffices to consider the case $x_1 < \cdots < x_m$.

Midpoint order depends on the distances between objects. The joint scale discussed so far is sometimes called the quantitative joint scale. Another type of joint scale is sometimes considered where we disregard the metric information of the quantitative joint scale and take into account only the order of its objects. In this case we obtain the so-called qualitative joint scale. The set of admissible rankings of the qualitative joint scale having objects $x_1, \ldots, x_m$ with $x_{i_1} < \cdots < x_{i_m}$ is, by definition, the union of the sets of admissible rankings of the quantitative joint scales whose objects are given by changing only the distances among $x_1, \ldots, x_m$ while keeping their rank order $x_{i_1} < \cdots < x_{i_m}$. Obviously, the number of qualitative joint scales is $m!/2$ and the number of admissible rankings of each qualitative joint scale is 2^{m-1} (Davison [9]). In this paper, we consider quantitative joint scales exclusively, hence a joint scale always means a quantitative joint scale.

Suppose the objects $x_1, \ldots, x_m$ are ordered as $x_1 < \cdots < x_m$. We want to know the number of possible rank orders of the midpoints $x_{ij} = (x_i + x_j)/2,\ 1 \leq i < j \leq m$. Any possible rank order of the midpoints $x_{ij},\ 1 \leq i < j \leq m$, must satisfy the condition that the rank $d(i,j)$ of x_{ij} from left to right on $\mathbb{R}$ be increasing in i for any fixed j as well as increasing in j for any fixed i. Consider the number g_m of functions $d : \{(i,j) \mid 1 \leq i < j \leq m\} \to \{1, 2, \ldots, m(m-1)/2\}$ satisfying this condition. Clearly g_m serves as an upper bound for the number of possible rank orders of the midpoints $x_{ij},\ 1 \leq i < j \leq m$. Thrall [26] obtained this number by considering a problem similar to that of counting the number of standard Young tableaux. However, g_m is only an upper bound, since the rank order of the midpoints meeting the above-mentioned condition does not necessarily satisfy other restrictions induced by the rank order of the objects. Van Blokland-Vogelesang [28] finds two kinds of such "intransitive" midpoint orders by way of "comparing intervals" and "merging intervals."

In this paper, we find the number of possible rank orders of midpoints and thereby obtain the number of ranking patterns generated by the unidimensional unfolding model. This is achieved by introducing a new type of arrangement called the mid-hyperplane arrangement. For the general theory of hyperplane arrangements, see Orlik and Terao [22]. Although we give a formula for the number of ranking patterns for all m in Theorem 2.5, we calculate this number only for $m \leq 8$ due to computational complexity. In addition to determining the number of ranking patterns, we may ask a further question of interest. Suppose the m objects are randomly determined. What is the probability that a given ranking pattern occurs? As will be seen in Section 6, this problem for $m = 5$ reduces to that of finding volumes of some spherical tetrahedra.

The organization of this paper is as follows. In Section 2, we define the mid-hyperplane arrangement and show that the number of ranking patterns can be obtained by counting the number of chambers of this arrangement. In Section 3, we reduce the problem to that of counting the number of points in certain finite sets. Based on these results, we actually obtain the number of ranking patterns for $m \leq 7$ in Section 4 and for $m = 8$ in Section 5. We also show in those sections that the characteristic polynomial of the mid-hyperplane arrangement is a product of linear factors in $\mathbb{Z}[t]$ if and only if $m \leq 7$. In Section 6, we consider the problem of the probabilities of ranking patterns and give the answer for $m \leq 5$ objects. In Section 7, we mention some open problems.

2 Arrangements and ranking maps

Let m be an integer with $m \geq 3$. Here we define two kinds of hyperplanes in the m-dimensional Euclidean space $\mathbb{R}^m$.

(I) $H_{ij} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i = x_j\} \quad (1 \leq i < j \leq m)$.

The hyperplane arrangement $\mathcal{B}_m := \{H_{ij} \mid 1 \leq i < j \leq m\}$ is called the *braid arrangement* [22, p.13]. It has $|\mathcal{B}_m| = \binom{m}{2}$ hyperplanes. Let

$$I_4 := \{(p, q, r, s) \mid 1 \leq p < q \leq m,\ p < r < s \leq m,\ p, q, r, s \text{ are distinct}\}.$$

(II) $H_{pqrs} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_p + x_q = x_r + x_s\} \quad (p, q, r, s) \in I_4$.

Define the *mid-hyperplane arrangement*

$$\mathcal{A}_m := \mathcal{B}_m \cup \{H_{pqrs} \mid (p, q, r, s) \in I_4\}.$$

Here $|\mathcal{A}_m| = \binom{m}{2} + 3\binom{m}{4}$. For an arbitrary arrangement $\mathcal{A}$ in $\mathbb{R}^m$, let

$$M(\mathcal{A}) := \mathbb{R}^m \setminus \bigcup_{H \in \mathcal{A}} H$$

be the complement of $\mathcal{A}$. The connected components of $M(\mathcal{A})$ are called *chambers* of $\mathcal{A}$. Let $\mathbf{Ch}(\mathcal{A})$ be the set of all chambers of $\mathcal{A}$.

Let $\mathbb{P}_m$ denote the set of all permutations of $\{1, 2, \ldots, m\}$. For $\pi = (i_1 \ldots i_m) \in \mathbb{P}_m$, let $\hat{\pi}$ denote the corresponding bijection from $\{1, \ldots, m\}$ to itself: $\hat{\pi}(k) = i_k$ $(1 \le k \le m)$. In this way we have a one-to-one correspondence between $\mathbb{P}_m$ and the symmetric group $\mathbb{S}_m$, which is defined to be the set of bijections from $\{1, \ldots, m\}$ to itself. The group $\mathbb{S}_m$ acts on the set $\mathbb{P}_m$ by

$$\sigma\pi := (\sigma(i_1) \ldots \sigma(i_m)) \in \mathbb{P}_m$$

for $\sigma \in \mathbb{S}_m$ and $\pi = (i_1 \ldots i_m) \in \mathbb{P}_m$. The action of $\mathbb{S}_m$ on $\mathbb{R}^m$ is defined by

$$\sigma(x_1, \ldots, x_m) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)}).$$

Then $\mathbb{S}_m$ acts on $M(\mathcal{A}_m)$ and $M(\mathcal{B}_m)$ and therefore on $\mathbf{Ch}(\mathcal{A}_m)$ and on $\mathbf{Ch}(\mathcal{B}_m)$.

It is well known (e.g., Bourbaki [3, Ch.5, §3, n°2, Th.1]) that the symmetric group $\mathbb{S}_m$ acts on $\mathbf{Ch}(\mathcal{B}_m)$ effectively and transitively. In other words, for any $C, C' \in \mathbf{Ch}(\mathcal{B}_m)$, there exists a unique $\sigma \in \mathbb{S}_m$ with $C' = \sigma C$. In particular, $|\mathbf{Ch}(\mathcal{B}_m)| = m!$. Let

$$C_0 := \{(x_1, x_2, \ldots, x_m) \mid x_1 < x_2 < \cdots < x_m\}$$

be a chamber of the braid arrangement $\mathcal{B}_m$. Then $\mathbf{Ch}(\mathcal{B}_m) = \{\sigma C_0 \mid \sigma \in \mathbb{S}_m\}$.

Fix $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in M(\mathcal{A}_m)$. Plot m points $x_1, x_2, \ldots, x_m$ on the real line $\mathbb{R}$. Let $R(\mathbf{x}) := \mathbb{R} \setminus \{x_{ij} \mid 1 \le i < j \le m\}$, where $x_{ij} := (x_i + x_j)/2$ is the midpoint. Define a map

$$\mathcal{R}_{\mathbf{x}} : R(\mathbf{x}) \longrightarrow \mathbb{P}_m$$

as follows:

$$\mathcal{R}_{\mathbf{x}}(y) = (i_1 i_2 \ldots i_m) \Longleftrightarrow |y - x_{i_1}| < |y - x_{i_2}| < \cdots < |y - x_{i_m}|,$$

where $y \in R(\mathbf{x})$ and $(i_1 i_2 \ldots i_m) \in \mathbb{P}_m$. The map $\mathcal{R}_{\mathbf{x}}$ is called the *ranking map*. The image of the ranking map $\mathcal{R}_{\mathbf{x}}$ is the *ranking pattern* of $\mathbf{x} \in M(\mathcal{A}_m)$.

Suppose $\mathbf{x} \in C_0 \cap M(\mathcal{A}_m)$. Then $x_1 < x_2 < \cdots < x_m$. For $y \in R(\mathbf{x})$ and $1 \le i < j \le m$, we have

$$y < x_{ij} \Longleftrightarrow |y - x_i| < |y - x_j| \Longleftrightarrow i \text{ precedes } j \text{ in } \mathcal{R}_{\mathbf{x}}(y),$$

$$y > x_{ij} \Longleftrightarrow |y - x_i| > |y - x_j| \Longleftrightarrow j \text{ precedes } i \text{ in } \mathcal{R}_{\mathbf{x}}(y).$$

Imagine that the point y moves on the real line $\mathbb{R}$ from left to right. When y is sufficiently small, $\mathcal{R}_{\mathbf{x}}(y) = (12 \ldots m)$. Every time y "passes" x_{ij}, the two integers i and j, which are adjacent in $\mathcal{R}_{\mathbf{x}}(y)$, switch their positions. When y is sufficiently large, $\mathcal{R}_{\mathbf{x}}(y) = (m \ldots 21)$.

Example 2.1. *Let $m = 3$ and $x_1 < x_2 < x_3$. Then*

$$\mathcal{R}_{\mathbf{x}}(y) = \begin{cases} (123) & \textit{if } y < x_{12}, \\ (213) & \textit{if } x_{12} < y < x_{13}, \\ (231) & \textit{if } x_{13} < y < x_{23}, \\ (321) & \textit{if } x_{23} < y. \end{cases}$$

Lemma 2.2. *Let $\sigma \in \mathbb{S}_m$, $\mathbf{x} \in M(\mathcal{A}_m)$ and $y \in R(\mathbf{x})$. Then*

$$\mathcal{R}_{\sigma\mathbf{x}}(y) = \sigma(\mathcal{R}_{\mathbf{x}}(y)).$$

Proof. Suppose $\mathbf{x} = (x_1, \dots, x_m)$. Then

$$\begin{aligned} \mathcal{R}_{\sigma\mathbf{x}}(y) = (i_1 \dots i_m) &\iff |y - x_{\sigma^{-1}(i_1)}| < \cdots < |y - x_{\sigma^{-1}(i_m)}| \\ &\iff \mathcal{R}_{\mathbf{x}}(y) = (\sigma^{-1}(i_1) \dots \sigma^{-1}(i_m)) \iff \sigma(\mathcal{R}_{\mathbf{x}}(y)) = (i_1 \dots i_m). \end{aligned}$$

□

Lemma 2.3. *Let $\sigma \in \mathbb{S}_m$ and $\mathbf{x}, \mathbf{x}' \in \sigma C_0 \cap M(\mathcal{A}_m)$. Then $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$ if and only if the following statement holds true:*

$$x_{pq} > x_{rs} \iff x'_{pq} > x'_{rs}$$

for each $(p, q, r, s) \in I_4$.

Proof. Each chamber of $\mathcal{A}_m$ inside σC_0 is equal to the intersection of σC_0 and half-spaces defined by either $2(x_{pq} - x_{rs}) = x_p + x_q - x_r - x_s > 0$ or $2(x_{pq} - x_{rs}) = x_p + x_q - x_r - x_s < 0$ for $(p, q, r, s) \in I_4$. □

Theorem 2.4. *Let $\sigma \in \mathbb{S}_m$ and $\mathbf{x}, \mathbf{x}' \in \sigma C_0 \cap M(\mathcal{A}_m)$. Then $\mathbf{x}$ and $\mathbf{x}'$ have the same ranking pattern if and only if $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$.*

Proof. Assume first that $\sigma = 1$, so $\mathbf{x}, \mathbf{x}' \in C_0 \cap M(\mathcal{A}_m)$.

Suppose that $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$. Write $\mathbf{x}' = (x'_1, x'_2, \dots, x'_m)$ and $x'_{ij} := (x'_i + x'_j)/2$ $(1 \le i < j \le m)$. By Lemma 2.3, we have

$$x_{i_1 j_1} < x_{i_2 j_2} < \cdots < x_{i_t j_t}, \quad x'_{i_1 j_1} < x'_{i_2 j_2} < \cdots < x'_{i_t j_t},$$

where $t = \binom{m}{2}$. This shows

$$\operatorname{im}\mathcal{R}_{\mathbf{x}} = \{\pi_0, \pi_1, \ldots, \pi_t\} = \operatorname{im}\mathcal{R}_{\mathbf{x}'},$$

where $\pi_0, \pi_1, \ldots \pi_t \in \mathbb{P}_m$ are defined inductively by

$$\begin{aligned} \pi_0 &= (12\ldots m), \\ \pi_s &= [i_s j_s]\pi_{s-1} \ \ (1 \leq s \leq t). \end{aligned}$$

Here $[ij] \in \mathbb{S}_m$ $(1 \leq i < j \leq m)$ denotes the transposition of i and j.

Conversely, assume $\operatorname{im}\mathcal{R}_{\mathbf{x}} = \operatorname{im}\mathcal{R}_{\mathbf{x}'}$. For $\pi = (i_1 i_2 \ldots i_m) \in \mathbb{P}_m$, let $\iota(\pi)$ denote the number of inversions in π:

$$\iota(\pi) := |\{(k_1, k_2) \mid k_1 < k_2, i_{k_1} > i_{k_2}\}|.$$

As the point y moves on the real line from left to right, $\iota(\mathcal{R}_{\mathbf{x}}(y))$ increases one by one. So we may write

$$\operatorname{im}\mathcal{R}_{\mathbf{x}} = \operatorname{im}\mathcal{R}_{\mathbf{x}'} = \{\pi_0, \pi_1, \ldots, \pi_t\}$$

such that $\iota(\pi_s) = s$, $(0 \leq s \leq t)$. Also there exists a unique transposition $[i_s j_s]$ such that $\pi_s = [i_s j_s]\pi_{s-1}$ $(1 \leq s \leq t)$. Thus $x_{i_1 j_1} < x_{i_2 j_2} < \cdots < x_{i_t j_t}$ and $x'_{i_1 j_1} < x'_{i_2 j_2} < \cdots < x'_{i_t j_t}$. It follows from Lemma 2.3 that $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$.

For a general $\sigma \in \mathbb{S}_m$, let $\mathbf{y} := \sigma^{-1}\mathbf{x} \in C_0 \cap M(\mathcal{A}_m)$ and $\mathbf{y}' := \sigma^{-1}\mathbf{x}' \in C_0 \cap M(\mathcal{A}_m)$. By Lemma 2.2,

$$\begin{aligned} \operatorname{im}\mathcal{R}_{\mathbf{x}} = \operatorname{im}\mathcal{R}_{\mathbf{x}'} &\Leftrightarrow \sigma^{-1}(\operatorname{im}\mathcal{R}_{\mathbf{x}}) = \sigma^{-1}(\operatorname{im}\mathcal{R}_{\mathbf{x}'}) \Leftrightarrow \operatorname{im}\mathcal{R}_{\sigma^{-1}\mathbf{x}} = \operatorname{im}\mathcal{R}_{\sigma^{-1}\mathbf{x}'} \\ &\Leftrightarrow \operatorname{im}\mathcal{R}_{\mathbf{y}} = \operatorname{im}\mathcal{R}_{\mathbf{y}'} \Leftrightarrow \mathbf{y} \text{ and } \mathbf{y}' \text{ lie in the same chamber of } \mathcal{A}_m \\ &\Leftrightarrow \mathbf{x} \text{ and } \mathbf{x}' \text{ lie in the same chamber of } \mathcal{A}_m. \end{aligned}$$

□

Let $r(m)$ denote the number of ranking patterns when $\mathbf{x}$ runs over the set $C_0 \cap M(\mathcal{A}_m)$:

$$r(m) := |\{\operatorname{im}\mathcal{R}_{\mathbf{x}} \mid \mathbf{x} \in C_0 \cap M(\mathcal{A}_m)\}|.$$

Note that for each $\sigma \in \mathbb{S}_m$, $|\{\operatorname{im}\mathcal{R}_{\mathbf{x}} \mid \mathbf{x} \in \sigma C_0 \cap M(\mathcal{A}_m)\}|$ is equal to $r(m)$ by Lemma 2.2.

Theorem 2.5. $r(m) = |\mathbf{Ch}(\mathcal{A}_m)|/(m!)$.

Proof. By Theorem 2.4, $r(m)$ is equal to the number of chambers of $\mathcal{A}_m$ which lie inside C_0. Thus we have $|\mathbf{Ch}(\mathcal{A}_m)| = r(m)|\mathbb{S}_m| = r(m)(m!)$. □

3 The number of chambers of $\mathcal{A}_m$

In this section, we study the number $|\mathbf{Ch}(\mathcal{A}_m)|$ of chambers of $\mathcal{A}_m$.

First let us review some general results about the number of chambers and the characteristic polynomial. Let $\mathbb{K}$ be a field and V an ℓ-dimensional vector space over $\mathbb{K}$. Assume that $\mathcal{A}$ is an arbitrary arrangement of hyperplanes in V. Let $L = L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. An element $X \in L$ is called an *edge* of $\mathcal{A}$. Define a *partial order* on L by $X \leq Y \iff Y \subseteq X$. Note that this is reverse inclusion. Thus V is the unique minimal element of L.

Let $\mu : L \to \mathbb{Z}$ be the Möbius function of L defined by $\mu(V) = 1$, and for $X > V$ by the recursion

$$\sum_{Y \leq X} \mu(Y) = 0.$$

The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) t^{\dim X}.$$

The next two theorems give geometric meaning to special values of the characteristic polynomial.

Theorem 3.1 (Zaslavsky [29]). *If* $\mathbb{K} = \mathbb{R}$, *then* $|\chi(\mathcal{A}, -1)| = |\mathbf{Ch}(\mathcal{A})|$.

An arrangement $\mathcal{A}$ is called *essential* if the dimension of a maximal element of $L(\mathcal{A})$ is zero. The mid-hyperplane arrangement $\mathcal{A}_m$ is not essential because the line $l = \operatorname{span}\{\mathbf{1}\} = \{\lambda \mathbf{1} \mid \lambda \in \mathbb{R}\} \subset \mathbb{R}^m$, where $\mathbf{1} \in \mathbb{R}^m$ is the vector of 1's is a maximal element. This implies that $\chi(\mathcal{A}_m, t)$ is divisible by t. The fact that l is contained in every hyperplane of $\mathcal{A}_m$ implies that $\chi(\mathcal{A}_m, t)$ is also divisible by $(t-1)$. Thus $\chi(\mathcal{A}_m, t)/t(t-1)$ is a monic polynomial of degree $m-2$.

Let H_0 be the hyperplane defined by $x_1 = 0$. Define $\mathcal{A}_m^* := \mathcal{A}_m \cup \{H_0\}$. Then $\mathcal{A}_m^*$ is essential and the lattice $L(\mathcal{A}_m)$ is isomorphic to the sublattice defined by $L(\mathcal{A}_m^*)_{\geq H_0} := \{X \in L(\mathcal{A}_m^*) \mid X \geq H_0\}$.

Theorem 3.2 (Crapo-Rota [8], Terao [24] (4.10)). *Let* $\mathbb{F}_q$ *be a finite field of* q *elements. If* $\mathbb{K} = \mathbb{F}_q$, *then* $\chi(\mathcal{A}, q) = |M(\mathcal{A})|$.

When $\mathbb{K} = \mathbb{F}_q$ and V is a finite set of q^ℓ elements, $\chi(\mathcal{A}, q)$ can be evaluated by counting the number of points not on any hyperplane $H \in \mathcal{A}$ in V. Let q

be a prime number greater than m. Let $\mathcal{A}^*_{m,q}$ be the modulo q reduction of $\mathcal{A}^*_m$ in $(\mathbb{Z}_q)^m$. In other words, the hyperplanes belonging to $\mathcal{A}^*_{m,q}$ are:

(0) $H_0 := \{(x_1, \ldots, x_m) \in (\mathbb{Z}_q)^m \mid x_1 = 0\}$,

(I_q) $H_{ij} := \{(x_1, \ldots, x_m) \in (\mathbb{Z}_q)^m \mid x_i = x_j\} \quad (1 \leq i < j \leq m)$, and

(II_q) $H_{pqrs} := \{(x_1, \ldots, x_m) \in (\mathbb{Z}_q)^m \mid x_p + x_q = x_r + x_s\} \quad (p, q, r, s \in I_4)$.

The arrangement $\mathcal{A}^*_{m,q}$ is essential. The modulo q reduction $\mathcal{A}_{m,q}$ of $\mathcal{A}_m$ is composed of the hyperplanes of type (I_q) and (II_q) above. Note that the lattice $L(\mathcal{A}_{m,q})$ is isomorphic to the sublattice defined by $L(\mathcal{A}^*_{m,q})_{\geq H_0} := \{X \in L(\mathcal{A}^*_{m,q}) \mid X \geq H_0\}$. Therefore, the intersection lattices $L(\mathcal{A}_m)$ and $L(\mathcal{A}_{m,q})$ are isomorphic if $L(\mathcal{A}^*_m)_{\geq H_0}$ and $L(\mathcal{A}^*_{m,q})_{\geq H_0}$ are isomorphic.

Let C be the coefficient matrix of $\mathcal{A}^*_m$. For example when $m = 4$,

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & -1 & -1 \end{pmatrix}.$$

Consider the m-minors of C. Each m-minor is parametrized by the set of m columns used for the minor. It is known that the intersection lattice of an essential arrangement is completely determined by the information which m-minors vanish and which do not [25, Proposition 3]. Thus we have

Theorem 3.3. *Define* $f(m) := \max\{|\det T| \mid T$ *is an* m*-minor of* C *and* T *contains the column* $(1, 0, \ldots, 0)^T$ *as its first column*$\}$. *Let* q *be a prime number greater than* $f(m)$. *Then* $L(\mathcal{A}_m)$ *and* $L(\mathcal{A}_{m,q})$ *are isomorphic.*

Next we will find an upper bound for $f(m)$. Let $m \geq 3$ as always. Consider the following three conditions concerning a matrix:

(i) every entry of the matrix is either -1, 0 or 1,

(ii) in every column 1 appears at most twice,

(iii) in every column -1 appears at most twice.

Define

$$g(m) := \max\{|\det A| \mid A \text{ is an } (m-1) \times (m-1)\text{-matrix satisfying (i,ii,iii)}\}.$$

It is clear that $f(m) \leq g(m)$.

Lemma 3.4.

$$g(m) = 2^{m-2} \textit{ for } m \leq 5, \quad g(m) \leq 8 \cdot 3^{m-5} \textit{ for } m \geq 6.$$

Proof. We argue by induction on m. For $m = 3, 4$ direct computation shows the result. The values are attained by

$$g(3) = 2 = \det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad g(4) = 4 = \det \begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Suppose $m = 5$. We show that $g(5) = 8$. Note that

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix} = 8.$$

Thus $g(5) \geq 8$. We must show $g(5) = \det A \leq 8$. Let $\mathbf{a}_i$ be the ith column of A. Denote the number of nonzero elements in $\mathbf{a}_i$ by $val(\mathbf{a}_i)$. If A has a column $\mathbf{a}_i$ with $val(\mathbf{a}_i) \leq 2$, then $g(5) \leq 2g(4) = 8$. So we may assume that $3 \leq val(\mathbf{a}_i) \leq 4$ for every i. Define $\mathbf{a}_i^*$ to be the uniquely determined four-dimensional column vector with two 1's and two -1's which satisfies the following property:

$$\begin{aligned} \text{if } val(\mathbf{a}_i) &= 4, \text{ then } \mathbf{a}_i^* = \mathbf{a}_i, \\ \text{if } val(\mathbf{a}_i) &= 3, \text{ then } \mathbf{a}_i^* \text{ is obtained from } \mathbf{a}_i \text{ by replacing} \\ & \qquad \text{the unique zero in } \mathbf{a}_i \text{ by either 1 or } -1. \end{aligned}$$

For example,

$$\text{if } \mathbf{a}_i = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} \text{ then } \mathbf{a}_i^* = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Since $\binom{4}{2}/2 = 3 < 4$, among the four vectors $\mathbf{a}_i^*$ $(1 \leq i \leq 4)$ at least two are either equal to or the negative of each other. Without loss of generality, we may assume $\mathbf{a}_1^* = \mathbf{a}_2^*$. Then $\mathbf{a}_{12} := \mathbf{a}_1 - \mathbf{a}_2$ is composed only of $0, -1$ and 1 with $val(\mathbf{a}_{12}) = 2$. We get

$$\begin{aligned} g(5) = \det A = \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) &= \det(\mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \\ &= \det(\mathbf{a}_{12}, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \leq val(\mathbf{a}_{12}) g(4) = 2g(4) = 8. \end{aligned}$$

For the induction step, assume $m \geq 6$. Choose an $(m-1) \times (m-1)$-matrix A satisfying (i), (ii) and (iii) with $\det A = g(m)$. Use the Laplace expansion formula along $\mathbf{a}_i$ to get

$$g(m) \leq val(\mathbf{a}_i) g(m-1)$$

for each i. If $val(\mathbf{a}_i) = 4$ for every column $\mathbf{a}_i$ of A, then all the rows sum up to the zero vector and thus $\det A = 0$. This is a contradiction. Therefore we may assume that there exists a column $\mathbf{a}_i$ with $val(\mathbf{a}_i) \leq 3$; so we obtain $g(m) \leq 3g(m-1)$. □

Theorem 3.3 and Lemma 3.4 imply

Theorem 3.5. *If a prime number q satisfies*

$$q > \begin{cases} 2^{m-2} & \text{if } m \leq 5, \\ 8 \cdot 3^{m-5} & \text{if } m \geq 6, \end{cases}$$

then the intersection lattices $L(\mathcal{A}_m)$ and $L(\mathcal{A}_{m,q})$ are isomorphic and

$$\chi(\mathcal{A}_m, q) = |M(\mathcal{A}_{m,q})|.$$

The following theorem shows that we can fix $x_1 = 0$, $x_2 = 1$ in counting $|M(\mathcal{A}_{m,q})|$.

Theorem 3.6. *Define*

$$M_1(m, q) := \{(0, 1, x_3, \ldots, x_m) \in M(\mathcal{A}_{m,q})\}.$$

Under the assumption of Theorem 3.5, we have

$$\frac{\chi(\mathcal{A}_m, q)}{q(q-1)} = |M_1(m, q)|.$$

Proof. Consider the action of the additive group $\mathbb{F}_q$ on $M(\mathcal{A}_{m,q})$ by

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1 + \alpha, x_2 + \alpha, \ldots, x_m + \alpha) \ \ (\alpha \in \mathbb{F}_q).$$

The set of orbits under this action is represented by the set

$$M_0 := \{(0, x_2, x_3, \ldots, x_m) \in M(\mathcal{A}_{m,q})\}.$$

Thus $|M_0| = \chi(\mathcal{A}_m, q)/q$. Next consider the action of the multiplicative group $\mathbb{F}_q^\times := \mathbb{F} \setminus \{0\}$ on M_0 by

$$(0, x_2, \dots, x_m) \mapsto (0, x_2\beta, \dots, x_m\beta) \ \ (\beta \in \mathbb{F}_q^\times).$$

The set of orbits under this action is represented by the set $M_1(m, q)$. Thus $|M_1(m, q)| = |M_0|/(q-1) = \chi(\mathcal{A}_m, q)/q(q-1)$. □

Our method to count the number of chambers of $\mathcal{A}_m$ is as follows: let q_i $(i = 1, \dots, m-2)$ be primes satisfying the conditions of Theorem 3.5. Count the number of points in the set $M_1(m, q_i)$ for each i. By Theorem 3.6, we have $\chi(\mathcal{A}_m, q_i)/q_i(q_i - 1) = |M_1(m, q_i)|$. Since $\chi(\mathcal{A}_m, t)/t(t-1)$ is a monic polynomial of degree $m-2$, the data $|M_1(m, q_i)|$ $(i = 1, \dots, m-2)$ determine the characteristic polynomial $\chi(\mathcal{A}_m, t)$ of $\mathcal{A}_m$. Theorem 3.1 asserts

$$|\mathbf{Ch}(\mathcal{A}_m)| = |\chi(\mathcal{A}_m, -1)|.$$

The number $r(m)$ of the ranking patterns when $\mathbf{x}$ runs over the set $C_0 \cap M(\mathcal{A}_m)$ is obtained by $r(m) = |\chi(\mathcal{A}_m, -1)|/(m!)$ by Theorem 2.5.

4 The number of ranking patterns for $m \leq 7$

In this section we determine $\chi(\mathcal{A}_m, t)$, $|\mathbf{Ch}(\mathcal{A}_m)|$ and $r(m)$ for $m \leq 7$. The case $m = 3$ is known because $\mathcal{A}_3 = \mathcal{B}_3$. Let $m = 4$. If $q > 4$ is a prime, then Theorem 3.6 gives

$$\frac{\chi(\mathcal{A}_4, q)}{q(q-1)} = |M_1(4, q)|.$$

Let

$$p(t) := \frac{\chi(\mathcal{A}_4, t)}{t(t-1)}.$$

Then $p(t)$ is a monic quadratic polynomial. We find

$$p(5) = |M_1(4, 5)| = 0 \ \text{ and } \ p(7) = |M_1(4, 7)| = 8.$$

Theorems 3.1 and 2.5 give

$$p(t) = t^2 - 8t + 15 = (t-3)(t-5), \ \chi(\mathcal{A}_4, t) = t(t-1)(t-3)(t-5),$$
$$|\mathbf{Ch}(\mathcal{A}_4)| = 48, \ \text{and} \ r(4) = 2.$$

Using the same method, computer calculations provide the following table:

Theorem 4.1.

m	$\chi(\mathcal{A}_m, t)$	$\lvert\mathbf{Ch}(\mathcal{A}_m)\rvert$	$r(m)$
3	$t(t-1)(t-2)$	6	1
4	$t(t-1)(t-3)(t-5)$	48	2
5	$t(t-1)(t-7)(t-8)(t-9)$	1440	12
6	$t(t-1)(t-13)(t-14)(t-15)(t-17)$	120960	168
7	$t(t-1)(t-23)(t-24)(t-25)(t-26)(t-27)$	23587200	4680

Corollary 4.2. *If $m \leq 7$, then the characteristic polynomial $\chi(\mathcal{A}_m, t)$ is a product of linear factors in $\mathbb{Z}[t]$.*

Remark. Define $a_n = n(n^{n-1}-1)((n-2)!)/(n-1)$. We note that $r(m) = a_{m-2}$ for $m = 3, 4, 5, 6, 7$ but we do not have any reasonable interpretation for the coincidence at this writing.

5 The number of ranking patterns for $m \geq 8$

We determine $r(8)$ first. Evaluating $r(m)$ for $m \geq 9$ is not feasible at present with our brute force counting method. Next we prove a theorem about the characteristic polynomial $\chi(\mathcal{A}_m, t)$ for $m \geq 8$.

For $m = 8$ we used a computer to count $|M_1(8, q)|$ with the primes $q =$ 223, 227, 229, 233, 239, 241, all greater than $8 \cdot 3^{8-5} = 216$. Theorem 3.6 implies:

Theorem 5.1.

$$\chi(\mathcal{A}_8, t) = t(t-1)(t-35)(t-37)(t-39)(t-41)(t^2-85t+1926),$$
$$|\mathbf{Ch}(\mathcal{A}_8)| = 9248117760,$$
$$r(8) = 229386.$$

Remark. The coincidence of $r(m)$ and a_{m-2} does not hold for $m = 8$. Here $r(8) = 229386 > a_6 = 223920$.

Write

$$\chi(\mathcal{A}_m, t) = \sum_{k=0}^{m} \mu_k t^{m-k}.$$

It is known that

$$\mu_0 = 1,\ \mu_1 = -|\mathcal{A}_m| = -\binom{m}{2} - 3\binom{m}{4},\ \mu_m = 0.$$

Although we do not have a general formula for μ_k, routine calculations yield a formula for μ_2:

Theorem 5.2.

$$\mu_2 = 2\binom{m}{3} + 15\binom{m}{4} + 120\binom{m}{5} + 375\binom{m}{6} + 630\binom{m}{7} + 315\binom{m}{8}.$$

Theorem 5.3. *The characteristic polynomial $\chi(\mathcal{A}_m, t)$ is a product of linear factors in $\mathbb{Z}[t]$ if and only if $m \leq 7$.*

Proof. This follows from Corollary 4.2 when $m \leq 7$. Let $m \geq 8$. Suppose that the characteristic polynomial is a product of linear factors in $\mathbb{Z}[t]$:

$$\chi(\mathcal{A}_m, t) = \sum_{k=0}^{m} \mu_k t^{m-k} = t(t-1)(t-b_2)\ldots(t-b_{m-1})$$

for $b_2, \ldots, b_{m-1} \in \mathbb{Z}$.

Applying Theorem 5.2, we have

$$\sum_{i=2}^{m-1} b_i = -\mu_1 - 1 = |\mathcal{A}_m| - 1 = -1 + \binom{m}{2} + 3\binom{m}{4},$$

$$\begin{aligned}\sum_{2\leq i<j\leq m-1} b_i b_j &= \mu_2 - \sum_{i=2}^{m-1} b_i \\ &= 1 - \binom{m}{2} + 2\binom{m}{3} + 12\binom{m}{4} \\ &\quad + 120\binom{m}{5} + 375\binom{m}{6} + 630\binom{m}{7} + 315\binom{m}{8}.\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=2}^{m-1}\left(b_i-\frac{\sum_{i=2}^{m-1}b_i}{m-2}\right)^2 &= \sum_{i=2}^{m-1}b_i^2-\frac{\left(\sum_{i=2}^{m-1}b_i\right)^2}{m-2}\\
&= \left(\sum_{i=2}^{m-1}b_i\right)^2-2\sum_{2\le i<j\le m-1}b_ib_j-\frac{\left(\sum_{i=2}^{m-1}b_i\right)^2}{m-2}\\
&= \frac{(m-3)\left(\sum_{i=2}^{m-1}b_i\right)^2}{m-2}-2\sum_{2\le i<j\le m-1}b_ib_j.
\end{aligned}$$

Compute

$$\begin{aligned}
h(m) &:= (m-2)\sum_{i=2}^{m-1}\left(b_i-\frac{\sum_{i=2}^{m-1}b_i}{m-2}\right)^2\\
&= (m-3)\left\{-1+\binom{m}{2}+3\binom{m}{4}\right\}^2-2(m-2)\left\{1-\binom{m}{2}+2\binom{m}{3}\right.\\
&\quad \left.+12\binom{m}{4}+120\binom{m}{5}+375\binom{m}{6}+630\binom{m}{7}+315\binom{m}{8}\right\}\\
&= 1+\frac{98m}{3}-\frac{1573m^2}{16}+\frac{5423m^3}{48}-\frac{12787m^4}{192}+\frac{527m^5}{24}-\frac{391m^6}{96}\\
&\quad +\frac{19m^7}{48}-\frac{m^8}{64}.
\end{aligned}$$

Thus $h(m)\ge 0$ for $m>2$. On the other hand, we may check by standard calculus techniques that $h(m)<0$ whenever $m\ge 8$. This is a contradiction. □

6 Probabilities of ranking patterns

We counted the number of possible ranking patterns in the preceding sections. Here we investigate the probabilities of ranking patterns when the objects $x_1,\dots,x_m$ are randomly determined. For $m=4$, the problem is trivial by symmetry considerations as long as the four objects are independently and identically distributed.

We consider the case $m=5$ and assume that $\mathbf{x}=(x_1,\dots,x_5)\in\mathbb{R}^5$ is distributed according to an arbitrary spherical distribution. Note that

$\mathbf{x} \in M(\mathcal{A}_5)$ with probability one. For $m = 5$, there are 1440 possible ranking patterns in all. By relabelling the indices it suffices to consider the case

$$x_1 < \cdots < x_5. \tag{1}$$

Furthermore, by replacing x_i by $-x_i$, it suffices to consider the case

$$x_1 < \cdots < x_5, \quad x_{24} < x_{15}. \tag{2}$$

Under restriction (2), we have $1440/(5! \cdot 2) = r(5)/2 = 6$ possible ranking patterns, which are characterized by the following midpoint orders (Lemma 2.3, Theorem 2.4):

$$\begin{aligned}
&\text{(I)} && x_{14} < x_{23} < x_{24} < x_{15} < x_{25} < x_{34},\\
&\text{(II)} && x_{14} < x_{23} < x_{24} < x_{15} < x_{34} < x_{25},\\
&\text{(III)} && x_{14} < x_{23} < x_{24} < x_{34} < x_{15} < x_{25},\\
&\text{(IV)} && x_{23} < x_{14} < x_{24} < x_{15} < x_{25} < x_{34},\\
&\text{(V)} && x_{23} < x_{14} < x_{24} < x_{15} < x_{34} < x_{25},\\
&\text{(VI)} && x_{23} < x_{14} < x_{24} < x_{34} < x_{15} < x_{25}.
\end{aligned}$$

We are interested in the conditional probabilities of the six midpoint orders above assuming (2). Recall that these midpoint orders represent chambers of $\mathcal{A}_5$ (Lemma 2.3). We argue next that our problem reduces to computing the spherical volumes of the restrictions of some chambers of $\mathcal{A}_5$ to the three-dimensional unit sphere.

We begin by recalling that all hyperplanes in $\mathcal{A}_5$ contain the line $l = \operatorname{span}\{\mathbf{1}\} = \{\lambda\mathbf{1} \mid \lambda \in \mathbb{R}\} \subset \mathbb{R}^5$, where $\mathbf{1} \in \mathbb{R}^5$ is the vector of 1's. The orthogonal projection of $\mathbf{x} = (x_1, \ldots, x_5) \in \mathbb{R}^5$ onto $H_0' = l^\perp = \{(x_1, \ldots, x_5) \in \mathbb{R}^5 \mid x_1 + \cdots + x_5 = 0\}$ will be denoted by $\mathbf{z} := (x_1 - \bar{x}, \ldots, x_5 - \bar{x})$, where $\bar{x} = (x_1 + \cdots + x_5)/5$. Since $\mathbf{x}$ is assumed to be distributed as a spherical distribution, the marginal distribution of the orthogonal projection $\mathbf{z}$ is a spherical distribution of one less dimension (Muirhead [20, p.34]). Now, any $\mathbf{x} \in M(\mathcal{A}_5)$ and its orthogonal projection $\mathbf{z}$ are on the same side of each hyperplane in $\mathcal{A}_5$, so for any chamber $C \in \mathbf{Ch}(\mathcal{A}_5)$, we have $\operatorname{Prob}(\mathbf{x} \in C) = \operatorname{Prob}(\mathbf{z} \in C_{H_0'})$ with $C_{H_0'} := C \cap H_0'$. This $C_{H_0'}$ can be regarded as a chamber of the arrangement $\mathcal{A}_5' := \{H \cap H_0' \mid H \in \mathcal{A}_5\}$ in H_0'.

Each hyperplane in $\mathcal{A}_5'$ contains the origin. Thus its chambers are the interiors of polyhedral cones in H_0'. As a result, for each $C_{H_0'} \in \mathbf{Ch}(\mathcal{A}_5')$, we have that $\mathbf{z} \in C_{H_0'}$ is equivalent to $\mathbf{z}/\|\mathbf{z}\| \in C_{\mathbb{S}^3} := C_{H_0'} \cap \mathbb{S}^3$, where $\mathbb{S}^3 := \{(x_1, \ldots, x_5) \in H_0' \mid x_1^2 + \cdots + x_5^2 = 1\}$ is the unit sphere in H_0'. Together with the uniformity of the distribution of $\mathbf{z}/\|\mathbf{z}\|$ on $\mathbb{S}^3$, this yields $\mathrm{Prob}(\mathbf{z} \in C_{H_0'}) = \mathrm{Prob}(\mathbf{z}/\|\mathbf{z}\| \in C_{\mathbb{S}^3}) = \mathrm{Vol}(C_{\mathbb{S}^3})/\mathrm{Vol}(\mathbb{S}^3)$.

We conclude that for any chamber C of $\mathcal{A}_5$,

$$\mathrm{Prob}\,(\mathbf{x} \in C) = \frac{\mathrm{Vol}(C_{\mathbb{S}^3})}{\mathrm{Vol}(\mathbb{S}^3)}$$

with $C_{\mathbb{S}^3} = C \cap \mathbb{S}^3$. Thus the probability of $\mathbf{x}$ being in chamber $C \in \mathbf{Ch}(\mathcal{A}_5)$ is proportional to the volume of $C_{\mathbb{S}^3} = C \cap \mathbb{S}^3$. Therefore, the desired conditional probabilities under (2) are given by the ratios of the volumes of the chambers $C_{\mathbb{S}^3}$ corresponding to the six midpoint orders to the volume of the union $T := \{(x_1, \ldots, x_5) \mid x_1 \leq \cdots \leq x_5,\ x_{24} \leq x_{15}\} \cap \mathbb{S}^3$ of their closures.

The binding inequalities of the spherical chambers associated with the six midpoint orders are

$$\begin{array}{cl}
\text{(I)} & x_{14} < x_{23},\ x_{25} < x_{34},\ x_3 < x_4,\ x_{24} < x_{15}, \\
\text{(II)} & x_{15} < x_{34},\ x_{14} < x_{23},\ x_{24} < x_{15},\ x_3 < x_4,\ x_{34} < x_{25}, \\
\text{(III)} & x_{14} < x_{23},\ x_2 < x_3,\ x_3 < x_4,\ x_{34} < x_{15}, \\
\text{(IV)} & x_1 < x_2,\ x_{25} < x_{34},\ x_{23} < x_{14},\ x_{24} < x_{15}, \\
\text{(V)} & x_{15} < x_{34},\ x_{23} < x_{14},\ x_{24} < x_{15},\ x_{34} < x_{25}, \\
\text{(VI)} & x_1 < x_2,\ x_2 < x_3,\ x_{23} < x_{14},\ x_{34} < x_{15}.
\end{array}$$

With the exception of (II), the closures of these chambers are spherical tetrahedra

$$\begin{array}{cl}
\text{(I)} & FBGH, \\
\text{(III)} & AFED, \\
\text{(IV)} & FBGC, \\
\text{(V)} & CGFE, \\
\text{(VI)} & AFCE
\end{array}$$

where

$$\begin{array}{ll}
A = (-1,-1,-1,-1,4)/\sqrt{20}, & B = (-3,-3,2,2,2)/\sqrt{30}, \\
C = (-2,-2,-2,3,3)/\sqrt{30}, & D = (-1,0,0,0,1)/\sqrt{2}, \\
E = (-7,-2,-2,3,8)/\sqrt{130}, & F = (-4,-4,1,1,6)/\sqrt{70}, \\
G = (-2,-1,0,1,2)/\sqrt{10}, & H = (-8,-3,2,2,7)/\sqrt{130};
\end{array}$$

Chamber (II) is a quadrilateral pyramid $FEDHG$, which can be divided into two tetrahedra, say, $FEDG$ and $FDGH$. Note that this observation implies that the closures of the chambers of the mid-hyperplane arrangement $\mathcal{A}_m$ are not necessarily simplices. See Figures 1 and 2.

The volumes of the seven spherical tetrahedra mentioned above can be computed as

$$\begin{array}{rl}
\text{(I)} & \mathrm{Vol}(FBGH) = 0.00628091, \\
\text{(II)} & \mathrm{Vol}(FEDG) = 0.00486715, \quad \mathrm{Vol}(FDGH) = 0.00481365, \\
\text{(III)} & \mathrm{Vol}(AFED) = 0.0189182, \\
\text{(IV)} & \mathrm{Vol}(FBGC) = 0.0146084, \\
\text{(V)} & \mathrm{Vol}(CGFE) = 0.00650684, \\
\text{(VI)} & \mathrm{Vol}(AFCE) = 0.0262516.
\end{array}$$

As an illustration, the calculation of the volume of $FBGH$ is given in the Appendix. Note that these values add up to the volume of the spherical tetrahedron $T = ABCD = \{(x_1, \ldots, x_5) \in \mathbb{S}^3 \mid x_1 \le \cdots \le x_5,\ x_{24} \le x_{15}\}$:

$$\mathrm{Vol}(T) = \frac{\mathrm{Vol}(\mathbb{S}^3)}{5! \cdot 2} = \frac{2\pi^2}{5! \cdot 2} = 0.0822467.$$

Let $S = \{(x_1, \ldots, x_5) \in \mathbb{S}^3 \mid x_1 \le \cdots \le x_5\}$: $\mathrm{Vol}(S) = 2\mathrm{Vol}(T)$. We use the values above to arrive at

$$\begin{aligned}
\mathrm{Prob}((\mathrm{I}) \mid S) &:= \mathrm{Prob}(x_{14} < x_{23} < x_{24} < x_{15} < x_{25} < x_{34} \mid \\
&\qquad\qquad\qquad x_1 < \cdots < x_5) \\
&= \frac{\mathrm{Vol}(FBGH)}{\mathrm{Vol}(S)} \\
&= \frac{0.00628091}{2 \times 0.0822467} = 0.0381834.
\end{aligned}$$

By replacing x_i by $-x_i$, we also consider the following cases:

$$\begin{array}{rl}
(\mathrm{I}') & x_{23} < x_{14} < x_{15} < x_{24} < x_{34} < x_{25}, \\
(\mathrm{II}') & x_{14} < x_{23} < x_{15} < x_{24} < x_{34} < x_{25}, \\
(\mathrm{III}') & x_{14} < x_{15} < x_{23} < x_{24} < x_{34} < x_{25}, \\
(\mathrm{IV}') & x_{23} < x_{14} < x_{15} < x_{24} < x_{25} < x_{34}, \\
(\mathrm{V}') & x_{14} < x_{23} < x_{15} < x_{24} < x_{25} < x_{34}, \\
(\mathrm{VI}') & x_{14} < x_{15} < x_{23} < x_{24} < x_{25} < x_{34}.
\end{array}$$

Using the symmetry we get

$$\begin{aligned}
\text{Prob}((\text{I}) \mid S) = \text{Prob}((\text{I}') \mid S) &= 0.0381834, \\
\text{Prob}((\text{II}) \mid S) = \text{Prob}((\text{II}') \mid S) &= 0.0588522, \\
\text{Prob}((\text{III}) \mid S) = \text{Prob}((\text{III}') \mid S) &= 0.1150086, \\
\text{Prob}((\text{IV}) \mid S) = \text{Prob}((\text{IV}') \mid S) &= 0.0888085, \\
\text{Prob}((\text{V}) \mid S) = \text{Prob}((\text{V}') \mid S) &= 0.0395569, \\
\text{Prob}((\text{VI}) \mid S) = \text{Prob}((\text{VI}') \mid S) &= 0.1595905.
\end{aligned}$$

We have confirmed that these values coincide with the result of our simulation study with $\mathbf{x} \sim N_5(\mathbf{0}, I_5)$, where I_5 denotes the 5×5-identity matrix.

7 Concluding remarks

In this paper, we have solved the problem of counting the number of ranking patterns in the unidimensional unfolding model although, due to computational complexity, at present we cannot determine the explicit number of ranking patterns for $m \geq 9$. Improving the bound in Lemma 3.4 might reduce the computational time. From some computer experiments, it seems that $L(\mathcal{A}_m)$ and $L(\mathcal{A}_{m,q})$ are isomorphic for much smaller q than the value guaranteed by Lemma 3.4.

The problem of counting the number of ranking patterns can be considered for the multidimensional unfolding model. Unlike the unidimensional case, the problem does not reduce to counting chambers of a hyperplane arrangement and the problem seems to be quite difficult at this stage.

8 Appendix

In this Appendix we illustrate the derivation of the volumes of the spherical tetrahedra in Section 6 by actually calculating the volume of $FBGH$. Take the following orthonormal basis of $H_0' = \{(x_1, \ldots, x_5) \in \mathbb{R}^5 \mid x_1 + \cdots + x_5 = 0\}$:

$$\begin{cases}
\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1, -1, 0, 0, 0), \\
\mathbf{e}_2 = \frac{1}{\sqrt{6}}(1, 1, -2, 0, 0), \\
\mathbf{e}_3 = \frac{1}{\sqrt{12}}(1, 1, 1, -3, 0), \\
\mathbf{e}_4 = \frac{1}{\sqrt{20}}(1, 1, 1, 1, -4).
\end{cases}$$

Let $z_1, \ldots, z_4$ be the coordinates of $\mathbf{x} = (x_1, \ldots, x_5) \in H'_0$ in terms of this basis:

$$\mathbf{x} = z_1\mathbf{e}_1 + \cdots + z_4\mathbf{e}_4 = \begin{pmatrix} \frac{z_1}{\sqrt{2}} + \frac{z_2}{\sqrt{6}} + \frac{z_3}{\sqrt{12}} + \frac{z_4}{\sqrt{20}} \\ -\frac{z_1}{\sqrt{2}} + \frac{z_2}{\sqrt{6}} + \frac{z_3}{\sqrt{12}} + \frac{z_4}{\sqrt{20}} \\ -\frac{2z_2}{\sqrt{6}} + \frac{z_3}{\sqrt{12}} + \frac{z_4}{\sqrt{20}} \\ -\frac{3z_3}{\sqrt{12}} + \frac{z_4}{\sqrt{20}} \\ -\frac{4z_4}{\sqrt{20}} \end{pmatrix}.$$

The conditions $x_{14} \leq x_{23},\ x_{25} \leq x_{34},\ x_3 \leq x_4,\ x_{24} \leq x_{15}$ are the binding inequalities of $FBGH$. They can be written as

$$\begin{cases} -\sqrt{3}z_1 - z_2 + \sqrt{2}z_3 \geq 0, \\ \sqrt{2}z_1 - \sqrt{6}z_2 - \sqrt{3}z_3 + \sqrt{5}z_4 \geq 0, \\ z_2 - \sqrt{2}z_3 \geq 0, \\ 2\sqrt{2}z_1 + \sqrt{3}z_3 - \sqrt{5}z_4 \geq 0 \end{cases}$$

in terms of $z_1, \ldots, z_4$.

Now consider the family of spherical tetrahedra $T(a),\ 0 \leq a \leq 1$, in $\mathbb{S}^3 = \{(z_1, \ldots, z_4) \in \mathbb{R}^4 \mid z_1^2 + \cdots + z_4^2 = 1\}$ determined by

$$(3) \qquad \begin{cases} HS_1: \ -a\sqrt{3}z_1 - z_2 + \sqrt{2}z_3 \geq 0, \\ HS_2: \ \sqrt{2}z_1 - \sqrt{6}z_2 - \sqrt{3}z_3 + \sqrt{5}z_4 \geq 0, \\ HS_3: \ z_2 - \sqrt{2}z_3 \geq 0, \\ HS_4: \ 2\sqrt{2}z_1 + \sqrt{3}z_3 - \sqrt{5}z_4 \geq 0. \end{cases}$$

We want to find $\mathrm{Vol}(T(1))$.

For each $a,\ 0 \leq a \leq 1$, let $e_{ij} = e_{ij}(a)$ be the edge of $T(a)$ determined by HS_i and $HS_j,\ 1 \leq i < j \leq 4$, and $v_{ijk} = v_{ijk}(a)$ the vertex of $T(a)$ determined by HS_i, HS_j and $HS_k,\ 1 \leq i < j < k \leq 4$. Furthermore, denote the length of e_{ij} and the dihedral angle along e_{ij} by $\theta_{ij} = \theta_{ij}(a)$ and $\lambda_{ij} = \lambda_{ij}(a),\ 1 \leq i < j \leq 4$, respectively.

The volume $\mathrm{Vol}(T(a))$ of $T(a)$ can be regarded as a function of its six dihedral angles $\lambda_{ij},\ 1 \leq i < j \leq 4$. The partial derivatives $\partial\mathrm{Vol}(T(a))/\partial\lambda_{ij},\ 1 \leq i < j \leq 4$, are given by the following lemma:

Lemma 8.1 (Schläfli [23]). *The partial derivatives of* $\mathrm{Vol}(T(a))$ *with respect to* $\lambda_{ij},\ 1 \leq i < j \leq 4$, *are given as*

$$\frac{\partial\mathrm{Vol}(T(a))}{\partial\lambda_{ij}} = \frac{\theta_{ij}}{2}, \qquad 1 \leq i < j \leq 4.$$

Making use of the above lemma, we will calculate $\mathrm{Vol}(T(1))$ in the following way:

$$
\begin{aligned}
\mathrm{Vol}(T(1)) &= \mathrm{Vol}(T(0)) + \int_0^1 \frac{d\mathrm{Vol}(T(a))}{da} da \\
&= \mathrm{Vol}(T(0)) + \int_0^1 \sum_{1 \le i < j \le 4} \frac{\partial \mathrm{Vol}(T(a))}{\partial \lambda_{ij}} \frac{d\lambda_{ij}}{da} da \qquad (4) \\
&= \mathrm{Vol}(T(0)) + \frac{1}{2} \sum_{1 \le i < j \le 4} \int_0^1 \theta_{ij}(a) \frac{d\lambda_{ij}}{da} da.
\end{aligned}
$$

First we calculate $\lambda_{ij} = \lambda_{ij}(a),\ 1 \le i < j \le 4$. By (3) we have

$$
\begin{aligned}
\lambda_{12} &= \arccos\left(\frac{-(-\sqrt{6}a))}{\sqrt{3a^2+3}\sqrt{16}} \right) = \arccos\left(\frac{\sqrt{2}a}{4\sqrt{a^2+1}} \right), \\
\lambda_{13} &= \arccos\left(\frac{-(-3)}{\sqrt{3a^2+3}\sqrt{3}} \right) = \arccos\left(\frac{1}{\sqrt{a^2+1}} \right), \\
\lambda_{14} &= \arccos\left(\frac{-(-2\sqrt{6}a+\sqrt{6})}{\sqrt{3a^2+3}\sqrt{16}} \right) = \arccos\left(\frac{\sqrt{2}(2a-1)}{4\sqrt{a^2+1}} \right)
\end{aligned}
$$

and that the other dihedral angles are constants.

Next we compute the lengths $\theta_{ij} = \theta_{ij}(a)$ of the edges $e_{ij} = e_{ij}(a),\ 1 \le i < j \le 4$. These lengths are obtained by $\theta_{ij} = \arccos(v_{ijk} \cdot v_{ijl}),\ 1 \le i < j \le 4,\ k \ne l,\ k, l \notin \{i, j\}$, where v_{ijk} is regarded as $v_{i'j'k'}$ with $\{i', j', k'\} = \{i, j, k\},\ 1 \le i' < j' < k' \le 4$. So we begin by finding the vertices $v_{ijk} = v_{ijk}(a),\ 1 \le i < j < k \le 4$ for $0 < a < 1$.

Vertex v_{123} is obtained by solving

$$
\begin{cases}
-\sqrt{3}az_1 - z_2 + \sqrt{2}z_3 = 0, \\
\sqrt{2}z_1 - \sqrt{6}z_2 - \sqrt{3}z_3 + \sqrt{5}z_4 = 0, \\
z_2 - \sqrt{2}z_3 = 0, \\
2\sqrt{2}z_1 + \sqrt{3}z_3 - \sqrt{5}z_4 \ge 0,
\end{cases}
$$

and the other vertices can be found in a similar manner. In this way, we obtain

$$
\begin{aligned}
v_{123} &= (0, -\sqrt{10}, -\sqrt{5}, -3\sqrt{3})/\sqrt{42}, \\
v_{124} &= (-\sqrt{10}, -\sqrt{30}, -\sqrt{15}(a+1), -3a-7)/\sqrt{8(3a^2+9a+13)}, \\
v_{134} &= (0, -\sqrt{10}, -\sqrt{5}, -\sqrt{3})/\sqrt{18}.
\end{aligned}
$$

Vertex v_{234} is not needed below.

Using these values of v_{123}, v_{124} and v_{134}, we get

$$\begin{aligned}\theta_{12} &= \arccos(v_{123}\cdot v_{124}) = \arccos\left(\frac{7a+18}{\sqrt{28(3a^2+9a+13)}}\right),\\ \theta_{13} &= \arccos(v_{123}\cdot v_{134}) = \arccos\left(\frac{4}{\sqrt{21}}\right),\\ \theta_{14} &= \arccos(v_{124}\cdot v_{134}) = \arccos\left(\frac{\sqrt{3}(4a+11)}{6\sqrt{3a^2+9a+13}}\right).\end{aligned}$$

Finally, we have $\mathrm{Vol}(T(0)) = 0$, since HS_1 with $a = 0$ and HS_3 in (3) imply $z_2 - \sqrt{2}z_3 = 0$. Consequently, by (4) and by numerical integration we can evaluate $\mathrm{Vol}(T(1))$ as

$$\begin{aligned}&\mathrm{Vol}(T(1))\\ &\quad= \frac{1}{2}\int_0^1 \theta_{12}(a)\frac{d\lambda_{12}}{da}da + \frac{1}{2}\int_0^1 \theta_{13}(a)\frac{d\lambda_{13}}{da}da + \frac{1}{2}\int_0^1 \theta_{14}(a)\frac{d\lambda_{14}}{da}da\\ &\quad= \frac{1}{2}\int_0^1 \arccos\left(\frac{7a+18}{\sqrt{28(3a^2+9a+13)}}\right)\cdot\frac{-\sqrt{2}}{(a^2+1)\sqrt{14a^2+16}}da\\ &\qquad\quad+\frac{1}{2}\arccos\left(\frac{4}{\sqrt{21}}\right)\cdot\{\lambda_{13}(1)-\lambda_{13}(0)\}\\ &\qquad\quad+\frac{1}{2}\int_0^1 \arccos\left(\frac{\sqrt{3}(4a+11)}{6\sqrt{3a^2+9a+13}}\right)\cdot\frac{-a-2}{(a^2+1)\sqrt{4a^2+4a+7}}da\\ &\quad= \frac{-0.0810845}{2} + \frac{\arccos\left(\frac{4}{\sqrt{21}}\right)\cdot\left(\frac{\pi}{4}-0\right)}{2} + \frac{-0.306702}{2}\\ &\quad= 0.00628091.\end{aligned}$$

References

[1] J. F. Bennett and W. L. Hays, Multidimensional unfolding: Determining the dimensionality of ranked preference data, *Psychometrika* **25** (1960), 27–43.

[2] U. Böckenholt and W. Gaul, Analysis of choice behavior via probabilistic ideal point and vector models, *Applied Stochastic Models and Data Analysis* **2** (1986), 209–226.

[3] N. Bourbaki, "Groupes et Algèbres de Lie," Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[4] H. Brady, Factor and ideal point analysis for interpersonally incomparable data, *Psychometrika* **54** (1989), 181–202.

[5] C. H. Coombs, Psychological scaling without a unit of measurement, *Psychol. Rev.* **57** (1950), 145–158.

[6] C. H. Coombs, A theory of psychological scaling, *Engng. Res. Inst. Bull.* No.34, University of Michigan Press, Ann Arbor, 1952.

[7] C. H. Coombs, "A Theory of Data," John Wiley & Sons, New York, 1964.

[8] H. Crapo and G.-C. Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," preliminary edition, MIT Press, Cambridge, MA, 1970.

[9] M. L. Davison, Testing a unidimensional, qualitative unfolding model for attitudinal or developmental data, *Psychometrika* **44** (1979), 179–194.

[10] G. De Soete, J. D. Carroll and W. S. DeSarbo, The wandering ideal point model: A probabilistic multidimensional unfolding model for paired comparison data, *J. Math. Psychol.* **30** (1986), 28–41.

[11] G. De Soete, H. Feger and K. C. Klauer, "New Developments in Psychological Choice Modeling," North-Holland, Amsterdam, 1989.

[12] B. C. Eaton and R. G. Lipsey, The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition, *Rev. Econ. Stud.* **42** (1975), 27–49.

[13] B. C. Eaton and R. G. Lipsey, The non-uniqueness of equilibrium in the Löschan location model, *Amer. Econ. Rev.* **66** (1976), 77–93.

[14] P. Feigin and A. Cohen, On a model of concordance between judges, *J. Roy. Statist. Soc. B* **40** (1978), 203–213.

[15] W. L. Hays and J. F. Bennett, Multidimensional unfolding: Determining configuration from complete rank order preference data, *Psychometrika* **26** (1961), 221–238.

[16] H. Hotelling, Stability in competition, *Economic Journal* **39** (1929), 41–57.

[17] H. Kamiya and A. Takemura, On rankings generated by pairwise linear discriminant analysis of m populations, *J. Multivariate Anal.* **61** (1997), 1–28.

[18] R. D. Luce and H. Raiffa, "Games and Decisions: Introduction and Critical Survey," John Wiley & Sons, New York, 1957.

[19] J. I. Marden, "Analyzing and Modeling Rank Data," Chapman & Hall, London, 1995.

[20] R. J. Muirhead, "Aspects of Multivariate Statistical Theory," John Wiley & Sons, New York, 1982.

[21] A. Okabe, B. Boots, K. Sugihara and S. N. Chiu, "Spatial Tessellations: Concepts and Applications of Voronoi Diagrams," 2nd ed., John Wiley & Sons, Chichester, 2000.

[22] P. Orlik and H. Terao, "Arrangements of Hyperplanes," Grundlehren der math. Wiss. **300**, Springer-Verlag, Berlin, 1992.

[23] L. Schläfli, On the multiple integral $\int^n dxdy \cdots dz$, whose limits are $p_1 = a_1x+b_1y+\cdots+h_1z > 0$, $p_2 > 0, \ldots, p_n > 0$, and $x^2+y^2+\cdots+z^2 < 1$, *Quart. J. Math.* **2** (1858), 269–300. Continued in **3** (1860), 54–68, 97–108.

[24] H. Terao, The Jacobians and the discriminants of finite reflection groups, *Tohoku Math. J.* **41** (1989), 237-247.

[25] H. Terao, Moduli space of combinatorially equivalent arrangements of hyperplanes and logarithmic Gauss-Manin connections, *Topology and Its Appl.* **118** (2002), 255–274.

[26] R. M. Thrall, A combinatorial problem, *Michigan Math. J.* **1** (1952), 81–88.

[27] L. L. Thurstone, A law of comparative judgment, *Psychol. Rev.* **34** (1927), 273–286.

[28] R. van Blokland-Vogelesang, "Unfolding and Group Consensus Ranking for Individual Preferences," DSWO Press, University of Leiden, 1991.

[29] T. Zaslavsky, Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* vol. 1, no. 154 (1975).

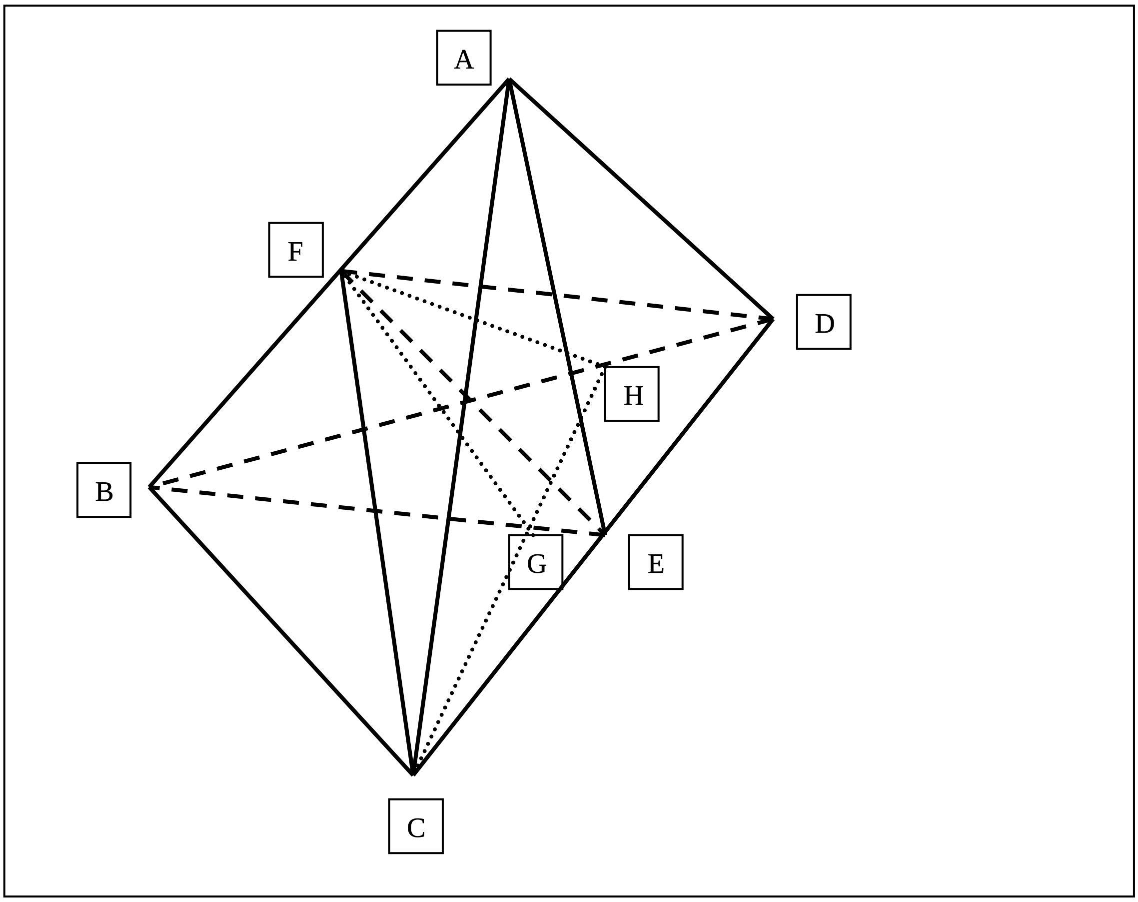

Figure 1

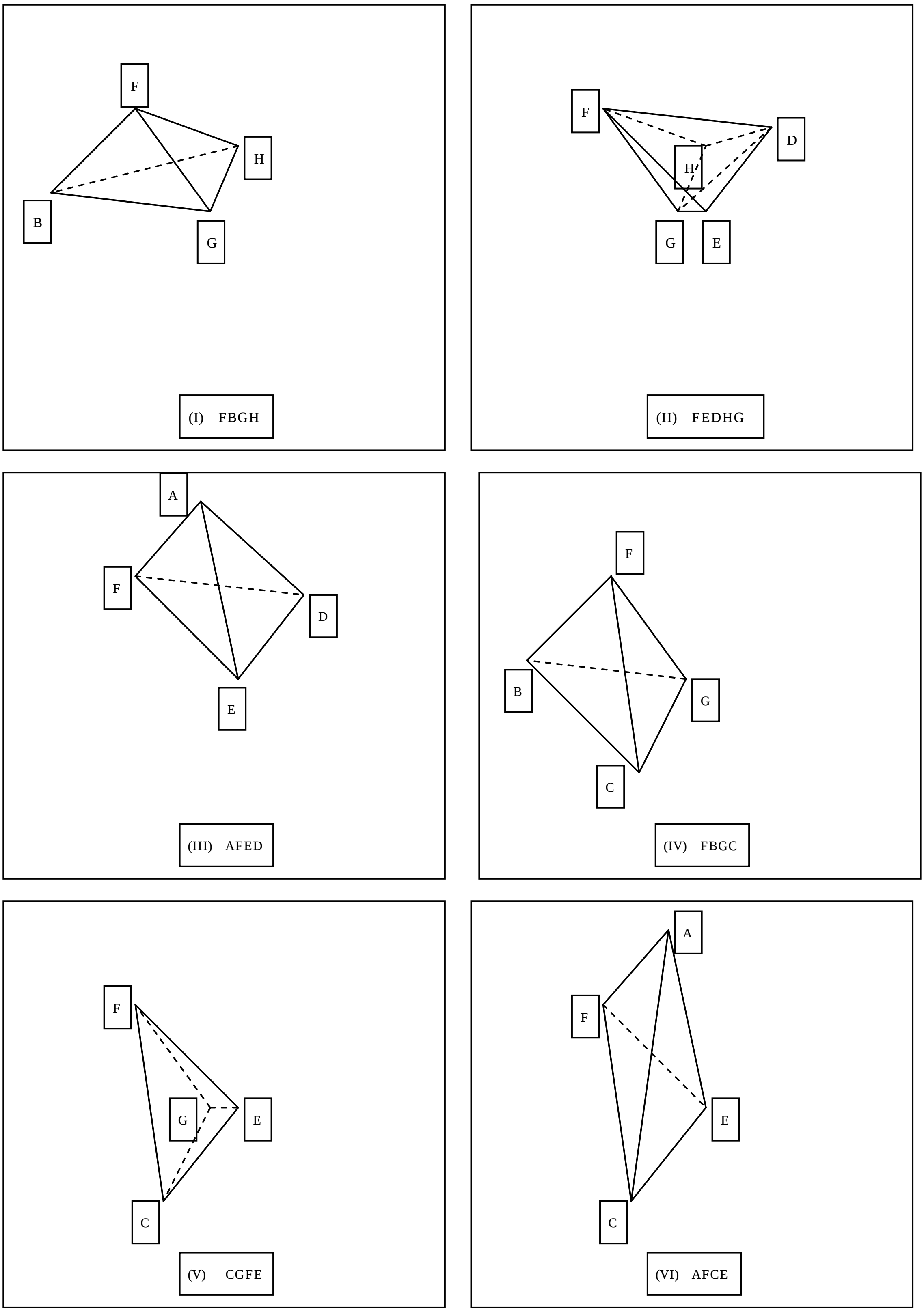
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Figure 2