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Multicoloring Unit Disk Graphs on Triangular Lattice Points

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Abstract

Given a pair of non-negative integers m and n, $P(m,n)$ denotes a subset of 2-dimensional triangular lattice points defined by $P(m,n) \stackrel{\text{def.}}{=} \{(x\boldsymbol{e}_1 + y\boldsymbol{e}_2) \mid x \in \{0,1,\ldots,m-1\},\ y \in \{0,1,\ldots,n-1\}\}$ where $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$, $\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2, \sqrt{3}/2)$. Let $T_{m,n}(d)$ be an undirected graph defined on vertex set $P(m,n)$ satisfying that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d. Given a non-negative vertex weight vector $\boldsymbol{w} \in \mathrm{Z}_+^{P(m,n)}$, a multicoloring of $(T_{m,n}(d), \boldsymbol{w})$ is an assignment of colors to $P(m,n)$ such that each vertex $v \in P(m,n)$ admits $w(v)$ colors and every adjacent pair of two vertices does not share a common color.

We propose a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d), \boldsymbol{w})$. Our algorithm is based on the well-solvable cases that the graph $T_{m,n}(d)$ is a perfect graph. We also showed a necessary and sufficient condition that $T_{m,n}(d)$ is perfect. For any $d \geq 1$, our algorithm finds a multicoloring which uses at most $\alpha(d)\omega + \mathrm{O}(d^3)$ colors, where ω denotes the weighted clique number. When $d = 1, \sqrt{3}, 2, \sqrt{7}, 3,$

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the approximation ratio $\alpha(d) = (4/3), (5/3), (7/4), (7/4)$, respectively. When $d > 1$, we showed that $\alpha(d) \leq \left(1 + \frac{2}{\sqrt{3}+\frac{2\sqrt{3}-3}{d}}\right)$.

We also showed the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d), \boldsymbol{w})$ with strictly less than $(4/3)\omega$ colors.

1 Introduction

Given a pair of non-negative integers m and n, $P(m,n)$ denotes the subset of 2-dimensional integer triangular lattice points defined by

$$P(m,n) \stackrel{\text{def.}}{=} \{(x\boldsymbol{e}_1 + y\boldsymbol{e}_2) \mid x \in \{0,1,2,\ldots,m-1\},\ y \in \{0,1,2,\ldots,n-1\}\}$$

where $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$, $\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2, \sqrt{3}/2)$. Given a finite set of 2-dimensional points $P \subseteq \mathrm{R}^2$ and a positive real d, a *unit disk graph*, denoted by (P,d), is an undirected graph with vertex set P such that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d. We denote the unit disk graph $(P(m,n),d)$ by $T_{m,n}(d)$.

Given an undirected graph H and a non-negative integer vertex weight $\boldsymbol{w}'$ of H, a *multicoloring* of $(H, \boldsymbol{w}')$ is an assignment of colors to vertices of H such that each vertex v admits $w'(v)$ colors and every adjacent pair of two vertices does not share a common color. A *multicoloring problem* on $(H, \boldsymbol{w}')$ finds a multicoloring of $(H, \boldsymbol{w}')$ which minimizes the required number of colors. The multicoloring problem is also known as weighted coloring [4], minimum integer weighted coloring [15] or w-coloring [12].

In this paper, we study weighted unit disk graphs on triangular lattice points $(T_{m,n}(d), \boldsymbol{w})$. First, we show a necessary and sufficient condition that $T_{m,n}(d)$ is a perfect graph. If the graph is perfect, we can solve the multicoloring problem easily. Next, we propose a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d), \boldsymbol{w})$. Our algorithm is based on the well-solvable case that the given graph is perfect. For any $d \geq 1$, our algorithm finds a multicoloring which uses at most

$$\left(1 + \frac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right)\omega + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right) \lfloor d+1 \rfloor^2$$

colors, where ω denotes the weighted clique number. Table 1 shows the values of the above approximation ratio in case that d is small.
We also show the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d), \boldsymbol{w})$ which uses strictly less than $(4/3)\omega$ colors.

The multicoloring problem has been studied in several context. When a given graph is the triangular lattice graph $T_{m,n}(1)$, the problem is related to the radio channel (frequency) assignment problem. McDiarmid and

Table 1: Approximation ratio

d	$1\cdots$	$\sqrt{3}\cdots$	$\sqrt{7}\cdots$	$2\sqrt{3}\cdots$	$\sqrt{13}\cdots$	$\sqrt{19}\cdots$
ratio	4/3	5/3	7/4	2	9/5	2
d	$\sqrt{21}\cdots$	$3\sqrt{3}\cdots$	$\sqrt{31}\cdots$	$\cdots$		∞
ratio	11/6	2	13/7	$\cdots$		$1+2/\sqrt{3}$

Reed [9] showed that the multicoloring problem on triangular lattice graphs is NP-hard. Some authors [9, 12] independently gave (4/3)-approximation algorithms for this problem. In case that a given graph H is a square lattice graph or a hexagonal lattice graph, the graph H becomes bipartite and so we can obtain an optimal multicoloring of $(H, \boldsymbol{w}')$ in polynomial time (see [9] for example). Halldórsson and Kortsarz [5] studied planar graphs and partial k-trees. For both classes, they gave a polynomial time approximation scheme (PTAS) for variations of multicoloring problem with min-sum objectives. These objectives appear in the context of multiprocessor task scheduling. For coloring (general) unit disk graphs, there exists a 3-approximation algorithm [6, 8, 14]. Here we note that the approximation ratio of our algorithm is less than $1 + 2/\sqrt{3} < 2.155$ for any $d \geq 1$.

2 Well-Solvable Cases and Perfectness

In this section, we discuss some well-solvable cases such that the multicoloring number is equivalent to the weighted clique number.

An undirected graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H, denoted by $\chi(H)$, is equal to its clique number $\omega(H)$. The following theorem is a main result of this paper.

Theorem 1 *The graph $T_{m,3}(d)$ is perfect, $\forall m \in \mathrm{Z}_+$, $\forall d \geq 1$.*
When $n \geq 4$ and $d \geq 1$, we have the following;
[$\forall m \in \mathrm{Z}_+$, $T_{m,n}(d)$ is perfect] if and only if $d \geq \sqrt{n^2 - 3n + 3}$.

Table 2 shows the perfectness and imperfectness of $T_{m,n}(d)$ for small n and d.

To show the above theorem, we introduce some definitions. We say that an undirected graph has a transitive orientation property, if each edge can be assigned a one-way direction in such a way that the resulting directed graph (V, F) satisfies that $[(a, b) \in F$ and $(b, c) \in F$ imply $(a, c) \in F]$. An undirected graph which is transitively orientable is called *comparability graph*. The complement of a comparability graph is called *co-comparability graph*. It is well-known that every co-comparability graph is perfect.

Lemma 1 *For any integer $n \geq 3$, if $d \geq \sqrt{n^2 - 3n + 3}$, then $T_{m,n}(d)$ is a co-comparability graph.*

Table 2: Perfectness and imperfectness

n \ d	$1 \cdots \sqrt{3} \cdots 2 \cdots$	$\sqrt{7} \cdots 3 \cdots 2\sqrt{3} \cdots$	$\sqrt{13} \cdots 4 \cdots \sqrt{19} \cdots$	$\sqrt{21} \cdots$
1				
2				
3				
4				Perfect
5				
6	Imperfect			
$\vdots$				

Proof: It suffices to show that the complement of $T_{m,n}(d)$ is a comparability graph. Let $\overline{T}_{m,n}(d)$ be the complement of $T_{m,n}(d)$. We direct each edge in $\overline{T}_{m,n}(d)$ as follows. For any edge $e = \{v_1, v_2\}$ in $\overline{T}_{m,n}(d)$, we direct the edge e from v_1 to v_2 when the x-coordinate of v_1 is strictly less than that of v_2. We show that the obtained directed graph, denoted by G', satisfies the transitivity.

Clearly, G' is acyclic. Assume that G' contains a pair of directed edges (v_1, v_2) and (v_2, v_3). We denote the position of v_i by (x_i, y_i) where x_i and y_i are the x-coordinate and the y-coordinate, respectively. The definition of G' implies that $x_1 < x_2 < x_3$. In the following, we show that $x_2 - x_1 > d/2$.
(Case 1) Consider the case that $|y_2 - y_1| < (\sqrt{3}/2)(n-1)$. Then, it is clear that $|y_2 - y_1| \leq (\sqrt{3}/2)(n-2)$. Since the distance between v_1 and v_2 is greater than d and $n \geq 3$, we have that $x_2 - x_1 > \sqrt{d^2 - |y_2 - y_1|^2} \geq \sqrt{d^2 - (3/4)(n-2)^2} > \sqrt{d^2 - (3/4)(n^2 - 3n + 3)} \geq \sqrt{d^2 - (3/4)d^2} = d/2$.
(Case 2) Assume that $|y_2 - y_1| \geq (\sqrt{3}/2)(n-1)$. Since $v_1, v_2 \in P(m, n)$, it is clear that $|y_2 - y_1| = (\sqrt{3}/2)(n-1)$. Without loss of generality, we can assume that $(x_2, y_2) = (x_1, y_1) + (x'\boldsymbol{e}_1 + (n-1)\boldsymbol{e}_2)$ for some integer x'. Since $0 < x_2 - x_1 = (n-1)/2 + x'$, we have $x' > -(n-1)/2$. If $x' \leq -1$, then $|x_2 - x_1|^2 + |y_2 - y_1|^2 = (x' + (n-1)/2)^2 + (3/4)(n-1)^2 \leq (-1 + (n-1)/2)^2 + (3/4)(n-1)^2 = n^2 - 3n + 3 \leq d^2$ and it contradicts with the non-adjacency between v_1 and v_2 on $T_{m,n}(d)$. Thus the integrality of x' implies that $x' \geq 0$ and $|x_2 - x_1| = (n-1)/2 + x' \geq (n-1)/2 = |y_2 - y_1|/\sqrt{3}$. Then the inequalities $d^2 < |x_2 - x_1|^2 + |y_2 - y_1|^2 \leq |x_2 - x_1|^2 + 3|x_2 - x_1|^2 = 4|x_2 - x_1|^2$ implies that $d/2 < x_2 - x_1$.

Similarly, we can show that $x_3 - x_2 > d/2$. Thus we have $x_3 - x_1 > d$ and the distance between v_1 and v_3 is greater than d. From the definition of G', the digraph G' contains the edge (v_1, v_3). ∎

The following lemma deals with the special case that $n = 3$, $d = 1$.

Lemma 2 *For any $m \in \mathrm{Z}_+$ and $1 \leq \forall d < \sqrt{3}$, the graph $T_{m,3}(d)$ is perfect.*

Proof: We only need to consider the case that $d = 1$, since $T_{m,n}(d) = T_{m,n}(1)$ when $1 \leq d < \sqrt{3}$. Let H be an induced subgraph of $T_{m,3}(1)$. When $\omega(H) \leq 2$, H has no 3-cycle. Then it is easy to show that H has no odd cycle and thus $\chi(H) = \omega(H)$, since H is bipartite. If $\omega(H) \geq 3$, then it is clear that $\omega(H) = 3$ and $\chi(H) \leq 3$, since $\omega(T_{m,3}(1)) = 3$ and $T_{m,3}(1)$ has a trivial 3-coloring. ∎

Note that though the graph $T_{m,3}(1)$ is perfect, the graph $T_{m,3}(1)$ is not co-comparability graph.

From the above, the perfectness of a graph satisfying the conditions of Theorem 1 is clear. In the following, we discuss the inverse implication. We say that an undirected graph G has an *odd-hole*, if G contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5. It is obvious that if a graph has an odd-hole, the graph is not perfect. In the following, we denote a point $(x\boldsymbol{e}_1 + y\boldsymbol{e}_2) \in P(m, n)$ by $\langle x, y\rangle$.

Lemma 3 *If* $1 \leq d < \sqrt{7}$*, then* $\forall m \geq 5$*,* $T_{m,4}(d)$ *has at least one odd-hole.*

Proof: If $1 \leq d < \sqrt{3}$, then a subgraph induced by

$$\{\ \langle 2,0\rangle,\ \langle 1,1\rangle,\ \langle 0,2\rangle,\ \langle 0,3\rangle,\ \langle 1,3\rangle,\ \langle 2,3\rangle,\ \langle 3,2\rangle,\ \langle 3,1\rangle,\ \langle 3,0\rangle\ \}$$

is a 9-hole. If $\sqrt{3} \leq d < 2$, then a subgraph induced by

$$\{\ \langle 3,0\rangle,\ \langle 1,1\rangle,\ \langle 0,2\rangle,\ \langle 1,3\rangle,\ \langle 2,3\rangle,\ \langle 4,2\rangle,\ \langle 4,1\rangle\ \}$$

is a 7-hole. If $2 \leq d < \sqrt{7}$, then a subgraph induced by

$$\{\ \langle 2,0\rangle,\ \langle 0,2\rangle,\ \langle 1,3\rangle,\ \langle 3,2\rangle,\ \langle 3,0\rangle\ \}$$

is a 5-hole. When $1 \leq d < \sqrt{7}$, $T_{5,4}(d)$ has at least one odd-hole, and hence the proof is completed. ∎

Lemma 4 *If* $1 \leq d < \sqrt{13}$*, then* $\forall m \geq 6$*,* $T_{m,5}(d)$ *has at least one odd-hole.*

Proof: If $1 \leq d < \sqrt{7}$, then odd-holes in the proof of Lemma 3 are induced subgraph of $T_{6,5}(d)$. If $\sqrt{7} \leq d < 3$, then a subgraph induced by

$$\{\ \langle 2,0\rangle,\ \langle 0,2\rangle,\ \langle 1,4\rangle,\ \langle 4,2\rangle,\ \langle 4,0\rangle\ \}$$

is a 5-hole. If $3 \leq d < \sqrt{13}$, then a subgraph induced by

$$\{\ \langle 3,0\rangle,\ \langle 0,3\rangle,\ \langle 2,4\rangle,\ \langle 5,3\rangle,\ \langle 5,0\rangle\ \}$$

is a 5-hole. When $1 \leq d < \sqrt{13}$, $T_{6,5}(d)$ has at least one odd-hole, and hence the proof is completed. ∎

Lemma 5 *For any integer* $n \geq 4$*, if* $1 \leq d < \sqrt{n^2 - 3n + 3}$*, then* $\exists m \in \mathrm{Z}_+$*,* $T_{m,n}(d)$ *is imperfect.*

Proof: In the following, we show that $\forall n \geq 4$, if $1 \leq d < \sqrt{n^2 - 3n + 3}$, then $\exists m \in \mathrm{Z}_+$, $T_{m,n}(d)$ has at least one odd-hole, by induction on n. When $n = 4, 5$, it is clear from Lemmas 3 and 4, respectively.

Now we consider the case that $n = n' \geq 6$ under the assumption that if $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3}$, then $\exists m' \in \mathrm{Z}_+$, $T_{m',n'-1}(d)$ has at least one odd-hole. If $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3} = \sqrt{n'^2 - 5n' + 7}$, then $T_{m',n'}(d)$ has at least one odd-hole, since $T_{m',n'-1}(d)$ is an induced subgraph of $T_{m',n'}(d)$. In the remained case that $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$, the set of points

$$\{ \ \langle n'-3, 0\rangle, \ \langle 0, n'-2\rangle, \ \langle n'-4, n'-1\rangle, \ \langle 2n'-7, n'-2\rangle, \ \langle 2n'-6, 0\rangle \ \}$$

is contained in $P(m'', n')$, if $m'' = 2n' - 5$. It is easy to see that the above five vertices induces a 5-hole of $T_{m'',n'}(d)$, when $n' \geq 6$ and $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$ (see Appendix: Lemma 9). ∎

Lemma 5 shows the imperfectness of every graph which violates a condition of Theorem 1. Thus, we completed a proof of Theorem 1. From the above lemmas, the following is immediate.

Corollary 1 *Let $d > 1$ be a real number. Then, $T_{m,n}(d)$ is a co-comparability graph, if and only if $n \leq \frac{3+\sqrt{4d^2-3}}{2}$.*

Next, we consider the multicoloring problems. Given an undirected graph $G = (V, E)$ and vertex weight vector $\boldsymbol{w} \in \mathrm{Z}_+^V$, the *multicoloring number* $\chi(G, \boldsymbol{w})$ is the least number of colors required in a multicoloring of $(G, \boldsymbol{w})$. The *weighted clique number* $\omega(G, \boldsymbol{w})$ is the weight of a maximum weight clique in $(G, \boldsymbol{w})$. It is clear that $\chi(G, \boldsymbol{w}) \geq \omega(G, \boldsymbol{w})$. First, we describe a property which helps the multicoloring case theoretically. We say that an undirected graph G has an *odd-antihole*, if the complement of G contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5.

Lemma 6 *Undirected graph $G = (V, E)$ is perfect if and only if $[\forall \boldsymbol{w} \in \mathrm{Z}_+^V, \chi(G, \boldsymbol{w}) = \omega(G, \boldsymbol{w})\ ]$ holds.*

Proof: Let $G_w = (V_w, E_w)$ be a graph obtained from G by replacing each vertex v with $w(v)$-clique. If two vertices u, v of G_w correspond to a unique vertex of G, we say that the pair $\{u, v\}$ belongs to a same class. It suffices to show that G is perfect if and only if G_w is perfect. Since G is an induced subgraph of G_w, the perfectness of G_w implies that G is perfect. Next we show that if G_w is not perfect then G is not perfect. In [1], it was shown that if a graph is not perfect, there exists an odd-hole or odd-antihole.

Consider the case that G_w contains an odd-hole C. If C contains a pair of vertices in a same class, it contradicts with the assumption that C is an induced subgraph. Otherwise, C is also an odd-hole of G and thus G is not perfect.

Consider the case that G_w contains an odd-antihole $\overline{C}$. We only need to consider the case that $\overline{C}$ contains a pair of vertices $\{u, v\}$ in a same class. Clearly, u and v are adjacent on G_w. Since the complement of $\overline{C}$ is an odd cycle, u has a pair of vertices u', u'' such that neither $\{u, u'\}$ nor $\{u, u''\}$ is adjacent pair. Since $\{u, v\}$ belongs to a same class, both $\{v, u'\}$ and $\{v, u''\}$ are non-adjacent pairs. Thus (u, u', v, u'') forms a 4-cycle in the complement of $\overline{C}$. It contradicts with the assumption that $\overline{C}$ is an odd-antihole. ∎

By combining Theorem 1 and Lemma 6 we have the following.

Theorem 2 *The equality $\chi(T_{m,3}(d), \boldsymbol{w}) = \omega(T_{m,3}(d), \boldsymbol{w})$ holds, $\forall m \in \mathrm{Z}_+$, $\forall d \in \mathrm{R}_+$, and $\forall \boldsymbol{w} \in \mathrm{Z}_+^{P(m,3)}$. When $n \geq 4$, the following property holds;*
[$\forall m \in \mathrm{Z}_+$ and $\forall \boldsymbol{w} \in \mathrm{Z}_+^{P(m,n)}$, $\chi(T_{m,n}(d), \boldsymbol{w}) = \omega(T_{m,n}(d), \boldsymbol{w})$]
if and only if $d \geq \sqrt{n^2 - 3n + 3}$.

Lastly, we discuss some algorithmic aspects. Assume that we have a co-comparability graph G and related digraph H which gives a transitive orientation of the complement of G. Then each independent set of G corresponds to a chain (directed path) of H. The multicoloring problem on G is essentially equivalent to the minimum size chain cover problem on H. Every clique of G corresponds to an anti-chain of H. Thus the equality $\omega(G) = \chi(G)$ is obtained from Dilworth's decomposition theorem [2]. It is well-known that the minimum size chain cover problem on an acyclic graph is solvable in polynomial time by using an algorithm for minimum-cost circulation flow problem (see [13] for example).

In the following, we describe an algorithm for multicoloring an weighted graph $(T_{m,3}(1), \boldsymbol{w})$. We denote the set of colors by $C^* = \{1, 2, \ldots, \omega^*\}$ where $\omega^* = \omega(T_{m,3}(1), \boldsymbol{w})$. The following algorithm finds an assignment of colors $c : P(m, n) \to 2^{C^*}$ such that $\forall v \in P(m, n)$, $|c(v)| = w(v)$ and for every edge $\{u, v\} \in T_{m,3}(1)$, $c(u) \cap c(v) = \emptyset$.

For any $x \in \{0, 1, \ldots, m-1\}$, we denote the points $x\boldsymbol{e}_1 + 2\boldsymbol{e}_2$, $x\boldsymbol{e}_1 + 1\boldsymbol{e}_1$, $x\boldsymbol{e}_1 + 0\boldsymbol{e}_2$ by t_{x+1}, u_{x+1}, v_x, respectively. Thus $\{t_1, t_2, \ldots, t_m\}$, $\{u_1, u_2, \ldots, u_m\}$ and $\{v_0, v_1, \ldots, v_{m-1}\}$ form a partition of $P(m, 3)$. Without loss of generality, we can assume that $w(v_0) = w(t_m) = w(u_m) = 0$. Our algorithm assigns colors to vertices in the following manner. Assume that we have a multicoloring $c : P' \to 2^C$ where $P' = \{t_1, t_2, \ldots, t_j\} \cup \{u_1, u_2, \ldots, u_j\} \cup \{v_0, v_1, \ldots, v_j\}$ satisfying that $\forall i \in \{1, 2, \ldots, j\}$, $c(t_i) \subseteq c(v_i)$ or $c(t_i) \supseteq c(v_i)$. Next, we assign colors to u_{j+1}. Since $w(v_0) = w(t_m) = w(u_m) = 0$, we can assume that $w(t_j) \geq w(v_j)$ without loss of generality. Since $\{u_j, t_j, u_{j+1}\}$ is a 3-clique, $|c(u_j)| + |c(t_j)| + w(u_{j+1}) \leq w^*$. Thus there exists a subset of colors C_1 with $|C_1| = w(u_{j+1})$ and C_1 is disjoint with $c(u_j) \cup c(t_j)$. Then we set $c(u_{j+1})$ to C_1. Next, we assign colors to t_{j+1}. there exists a set of colors $C_2 \subseteq C^* \setminus (c(t_j) \cup c(u_{j+1}))$ with $|C_2| = w(t_{j+1})$,

since $\{t_j, u_{j+1}, t_{j+1}\}$ is a 3-clique. Then set $c(t_{j+1})$ to C_2. Lastly we assign colors to v_{j+1}.
(Case 1) If $w(t_{j+1}) \geq w(v_{j+1})$, then set $c(v_{j+1})$ to a subset of $c(t_{j+1})$ whose cardinality is $w(v_{j+1})$.
(Case 2) Consider the case that $w(t_{j+1}) < w(v_{j+1})$ and $w(t_j) + w(t_{j+1}) \geq w(v_j) + w(v_{j+1})$. Then there exists a subset of colors $C_3 \subseteq c(t_j) \setminus c(v_j)$ whose cardinality is $w(v_{j+1}) - w(t_{j+1})$. We set $c(v_{j+1}) = c(t_{j+1}) \cup C_3$.
(Case 3) Consider the case that $w(t_{j+1}) < w(v_{j+1})$ and $w(t_j) + w(t_{j+1}) < w(v_j) + w(v_{j+1})$. Then we set $c(v_{j+1}) = c(t_{j+1}) \cup (c(t_j) \setminus c(v_j)) \cup C_4$ where C_4 and $v(t_j) \cup c(u_{j+1}) \cup c(t_{j+1})$ are disjoint and $w(v_{j+1}) = w(t_{j+1}) + (w(t_j) - w(v_j)) + |C_4|$. Since $\{v_j, u_{j+1}, v_{j+1}\}$ is a 3-clique, it is easy to see that there exists such a subset of colors.

A naive implementation of the above procedure gives a pseudo polynomial time algorithm, since the algorithm maintains the set of colors C^* explicitly. If we represent the assigned set of colors by the union of some intervals and implement the above procedure carefully, we can obtain a polynomial time algorithm with respect to m.

3 Approximation Algorithm

In this section, we propose an approximation algorithm for multicoloring the graph $(T_{m,n}(d), \boldsymbol{w})$. When $d = 1$, McDiarmid and Reed [9] proposed an approximation algorithm for $(T_{m,n}(1), \boldsymbol{w})$, which finds a multicoloring with at most $(4/3)\omega(T_{m,n}(1), \boldsymbol{w}) + 1/3$ colors.

In the following, we propose an approximation algorithm for $(T_{m,n}(d), \boldsymbol{w})$ when $d > 1$. The basic idea of our algorithm is similar to the shifting strategy [7].

Theorem 3 *When $d > 1$, there exists a polynomial time algorithm for multicoloring $(T_{m,n}(d), \boldsymbol{w})$ such that the number of required colors is bounded by*

$$\left(1 + \frac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right) \omega(T_{m,n}(d), \boldsymbol{w}) + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right) \chi(T_{m,n}(d)).$$

Proof: We describe an outline of the algorithm. For simplicity, we define $K_1 = \lfloor \frac{3+\sqrt{4d^2-3}}{2} \rfloor$ and $K_2 = \lfloor \frac{3+\sqrt{4d^2-3}}{2} \rfloor + \lfloor \frac{2}{\sqrt{3}} d \rfloor$.

First, we construct K_2 vertex weights $\boldsymbol{w}'_k$ for $k \in \{0, 1, \ldots, K_2 - 1\}$ by setting

$$w'_k(x, y) = \begin{cases} 0, & y \in \{k, k+1, \ldots, k + \lfloor \frac{2}{\sqrt{3}} d \rfloor - 1\} \pmod{K_2}, \\ K_1 \left\lfloor \frac{w(x,y)}{K_1} \right\rfloor, & \text{otherwise.} \end{cases}$$

Next, we exactly solve K_2 multicoloring problems defined by K_2 pairs $(T_{m,n}(d), \boldsymbol{w}'_k)$, $k \in \{0, 1, \ldots, K_2 - 1\}$ and obtain K_2 multicolorings. We can solve each problem exactly in polynomial time, since every connected component of the graph induced by the set of vertices with positive weight is a perfect graph discussed in the previous section. Thus $\chi(T_{m,n}(d), \boldsymbol{w}'_k) = \omega(T_{m,n}(d), \boldsymbol{w}'_k)$ for any $k \in \{0, 1, \ldots, K_2 - 1\}$. Put $\boldsymbol{w}'' = \boldsymbol{w} - \sum_{k=0}^{K_2-1} \boldsymbol{w}'_k$. Then each element of $\boldsymbol{w}''$ is less than or equal to $K_1 - 1$. Thus we can find a multicoloring of $(T_{m,n}(d), \boldsymbol{w}'')$ from the direct sum of $K_1 - 1$ trivial colorings of $T_{m,n}(d)$. The obtained multicoloring uses at most $(K_1 - 1)\chi(T_{m,n}(d))$colors. Lastly, we output the direct sum of K_2+1 multicolorings obtained above. The definition of the weight vector $\boldsymbol{w}'_k$ implies that $\forall k \in \{0, 1, \ldots, K_2 - 1\}$, $K_1\,\omega(T_{m,n}(d), \boldsymbol{w}'_k) \leq \omega(T_{m,n}(d), \boldsymbol{w})$. Thus, the obtained multicoloring uses at most $(K_2/K_1)\omega(T_{m,n}(d), \boldsymbol{w}) + (K_1 - 1)\chi(T_{m,n}(d))$ colors. ∎

The following lemma gives the chromatic number of $T_{m,n}(d)$.

Lemma 7 *If m, n are sufficiently large, then $\chi(T_{m,n}(d)) = \widehat{d}^2$ where $\widehat{d}$ is the minimum Euclidean distance between two points in $P(m, n)$ subject to that distance being greater than d. Clearly, $d < \widehat{d} \leq \lfloor d + 1 \rfloor$.*

Proof: See McDiarmid [9] for example. ∎

When d is small, Table 1 shows the approximation ratio. The following corollary gives a simple upper bound of the approximation ratio.

Corollary 2 *For any $d \geq 1$, we have* $1 + \frac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor} \leq 1 + \frac{2}{\sqrt{3} + \frac{2\sqrt{3}-3}{d}}$.

Here we note that if we apply our algorithm in the case that $d = 1$, then the algorithm finds a multicoloring which uses at most $(4/3)\omega(T_{m,n}(1), \boldsymbol{w})+6$ colors.

4 Hardness Result

In this section, we discuss the hardness of our problem. If we deal with a rational number d as an input data, then the problem for multicoloring $(T_{m,n}(d), \boldsymbol{w})$ is NP-hard, since $(T_{m,n}(1), \boldsymbol{w})$ is shown to be NP-hard by McDiarmid and Reed [9]. It is clear that even if we consider the case that d is integer and we define the input size of d is also d (not $O(\log d)$), the problem remains NP-hard. Thus we only need to consider the case that d is a constant. The following lemma gives an idea of our proof of hardness.

Lemma 8 *For any $d \geq 1$, the triangular lattice graph is an induced subgraph of $T_{m,n}(d)$.*

Table 3: Unit disk graphs on square lattice points

n \ d	$1\cdots$	$\sqrt{2}\cdots$	$2\cdots$	$\sqrt{5}\cdots 2\sqrt{2}\cdots$
1				
2				Perfect
3				
4			Imperfect	

Proof: Let $a\boldsymbol{e}_1 + b\boldsymbol{e}_2$ be a farthest neighbor of $(0,0)$ on $T_{m,n}(d)$. Put $\boldsymbol{e}'_1 = (a,b)$, $\boldsymbol{e}'_2 = (\frac{1}{2}a - \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$, and $P' = \{(x\boldsymbol{e}'_1 + y\boldsymbol{e}'_2) \in P(m,n) \mid x \in \mathrm{Z},\ y \in \mathrm{Z}\}$. Then the subgraph of $T_{m,n}(d)$ induced by P' is a triangular lattice graph. ∎

Theorem 4 *Let d be a constant rational number. Given a pair $(T_{m,n}(d), \boldsymbol{w})$, it is NP-complete to determine whether $(T_{m,n}(d), \boldsymbol{w})$ is multicolorable with strictly less than $(4/3)\omega(T_{m,n}(d), \boldsymbol{w})$ colors or not.*

Proof: It is known to be NP-complete to determine the 3-multicolorability of a given 4-colorable weighted triangular lattice graph $(T_{m,n}(1), \boldsymbol{w})$ [9]. We can reduce a 3-multicolorability problem on $T_{m,n}(1)$ to a 3-multicolorability problem on $T_{m',n'}(d)$, by copying the vertex weights of the original problem onto a sparse triangular lattice defined in the proof of Lemma 8. The obtained instance has $\mathrm{O}(mnd^2)$ vertices. Since d is a constant, the time complexity of the reduction procedure is bounded by a polynomial of the input size of $(T_{m,n}(1), \boldsymbol{w})$. ∎

5 Discussion

In this paper, we dealt with the triangular lattice. In the following, we discuss the square lattice. Given a pair of non-negative integers m and n, $Q(m,n) \stackrel{\text{def.}}{=} \{0,1,2,\ldots,m-1\} \times \{0,1,2,\ldots,n-1\}$ denotes the subset of 2-dimensional integer square lattice points. We denote the unit disk graph $(Q(m,n), d)$ by $S_{m,n}(d)$. In case that $d < \sqrt{2}$, it is clear that $S_{m,n}(d) = S_{m,n}(1)$ and the graph is bipartite for any m and n. If $d = \sqrt{2}$, we proposed a (4/3)-approximation algorithm for multicoloring $(S_{m,n}(\sqrt{2}), \boldsymbol{w})$ in our previous paper [11]. We also showed the NP-hardness of the problem.

Unfortunately, Theorem 1 is not extensible to the square lattice case. Table 3 shows the perfectness and imperfectness of unit disk graphs on the square lattice for small n and d. The perfectness of $T_{m,3}(\sqrt{2})$ was shown in [11]. The graph $S_{m,3}(2)$ contains a 5-hole: $\{(0,0),(2,0),(2,1),(1,2),(0.2)\}$.

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Appendix

Lemma 9 *Assume that* $\sqrt{n^2-5n+7} \le d < \sqrt{n^2-3n+3}$ *and* $n \ge 6$. *A subgraph of* $T_{m,n}(d)$ *induces by*

$$\{\ \langle n-3,0\rangle,\ \langle 0,n-2\rangle,\ \langle n-4,n-1\rangle,\ \langle 2n-7,n-2\rangle,\ \langle 2n-6,0\rangle\ \}$$

is a 5-hole.

Proof: Since the Euclidean distance between $\langle n-3,0\rangle$ and $\langle 0,n-2\rangle$ is

$$\begin{aligned}
&|(n-3)\boldsymbol{e}_1+(0)\boldsymbol{e}_2-\{(0)\boldsymbol{e}_1+(n-2)\boldsymbol{e}_2\}| \\
&=|(n-3)\boldsymbol{e}_1+(-n+2)\boldsymbol{e}_2| \\
&=\sqrt{(n-3)^2+(-n+2)^2+(n-3)(-n+2)} \\
&=\sqrt{n^2-5n+7} \\
&\le d,
\end{aligned}$$

the vertices $\langle n-3,0\rangle$ and $\langle 0,n-2\rangle$ are adjacent.
Since the Euclidean distance between $\langle 0,n-2\rangle$ and $\langle n-4,n-1\rangle$ is

$$\begin{aligned}
&|(0)\boldsymbol{e}_1+(n-2)\boldsymbol{e}_2-\{(n-4)\boldsymbol{e}_1+(n-1)\boldsymbol{e}_2\}| \\
&=|(-n+4)\boldsymbol{e}_1+(-1)\boldsymbol{e}_2| \\
&=\sqrt{(-n+4)^2+(-1)^2+(-n+4)(-1)} \\
&=\sqrt{n^2-7n+13} \\
&=\sqrt{\{n^2-5n+7\}-2(n-3)} \\
&\le\sqrt{n^2-5n+7}\ (\because n\ge 3) \\
&\le d,
\end{aligned}$$

the vertices $\langle 0,n-2\rangle$ and $\langle n-4,n-1\rangle$ are adjacent.
Since the Euclidean distance between $\langle n-4,n-1\rangle$ and $\langle 2n-7,n-2\rangle$ is

$$\begin{aligned}
&|(n-4)\boldsymbol{e}_1+(n-1)\boldsymbol{e}_2-\{(2n-7)\boldsymbol{e}_1+(n-2)\boldsymbol{e}_2\}| \\
&=|(-n+3)\boldsymbol{e}_1+\boldsymbol{e}_2| \\
&=\sqrt{(-n+3)^2+(1)^2+(-n+3)(1)} \\
&=\sqrt{n^2-7n+13} \\
&=\sqrt{\{n^2-5n+7\}-2(n-3)} \\
&\le\sqrt{n^2-5n+7}\ (\because n\ge 3) \\
&\le d,
\end{aligned}$$

the vertices $\langle n-4,n-1\rangle$ and $\langle 2n-7,n-2\rangle$ are adjacent.

Since the Euclidean distance between $\langle 2n-7, n-2\rangle$ and $\langle 2n-6, 0\rangle$ is

$$\begin{aligned}
&|(2n-7)\boldsymbol{e}_1 + (n-2)\boldsymbol{e}_2 - \{(2n-6)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2\}| \\
&= |(-1)\boldsymbol{e}_1 + (n-2)\boldsymbol{e}_2| \\
&= \sqrt{(-1)^2 + (n-2)^2 + (-1)(n-2)} \\
&= \sqrt{n^2 - 5n + 7} \\
&\leq d,
\end{aligned}$$

the vertices $\langle 2n-7, n-2\rangle$ and $\langle 2n-6, 0\rangle$ are adjacent.
Since the Euclidean distance between $\langle 2n-6, 0\rangle$ and $\langle n-3, 0\rangle$ is

$$\begin{aligned}
&|(2n-6)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2 - \{(n-3)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2\}| \\
&= |(n-3)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2| \\
&= \sqrt{(n-3)^2 + (0)^2 + (n-3)(0)} \\
&= \sqrt{n^2 - 6n + 9} \\
&= \sqrt{\{n^2 - 5n + 7\} - (n-2)} \\
&\leq \sqrt{n^2 - 5n + 7} \ (\because n \geq 2) \\
&\leq d,
\end{aligned}$$

the vertices $\langle 2n-6, 0\rangle$ and $\langle n-3, 0\rangle$ are adjacent.
Since the Euclidean distance between $\langle n-3, 0\rangle$ and $\langle n-4, n-1\rangle$ is

$$\begin{aligned}
&|(n-3)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2 - \{(n-4)\boldsymbol{e}_1 + (n-1)\boldsymbol{e}_2\}| \\
&= |(1)\boldsymbol{e}_1 + (-n+1)\boldsymbol{e}_2| \\
&= \sqrt{(1)^2 + (-n+1)^2 + (1)(-n+1)} \\
&= \sqrt{n^2 - 3n + 3} \\
&> d,
\end{aligned}$$

the vertices $\langle n-3, 0\rangle$ and $\langle n-4, n-1\rangle$ are not adjacent.
Since the Euclidean distance between $\langle n-4, n-1\rangle$ and $\langle 2n-7, n-2\rangle$ is

$$\begin{aligned}
&|(n-4)\boldsymbol{e}_1 + (n-1)\boldsymbol{e}_2 - \{(2n-7)\boldsymbol{e}_1 + (n-2)\boldsymbol{e}_2\}| \\
&= |(-n+4)\boldsymbol{e}_1 + (-n+2)\boldsymbol{e}_2| \\
&= \sqrt{(-n+4)^2 + (-n+2)^2 + (-n+4)(-n+2)} \\
&= \sqrt{3n^2 - 18n + 28} \\
&= \sqrt{\{n^2 - 3n + 3\} + (2n-5)(n-5)} \\
&\geq \sqrt{n^2 - 3n + 3} \ (\because n \geq 5) \\
&> d,
\end{aligned}$$

the vertices $\langle n-3, 0\rangle$ and $\langle 2n-7, n-2\rangle$ are not adjacent.

Since the Euclidean distance between $\langle 0, n-2\rangle$ and $\langle 2n-7, n-2\rangle$ is

$$\begin{aligned}
&|(0)\boldsymbol{e}_1 + (n-2)\boldsymbol{e}_2 - \{(2n-7)\boldsymbol{e}_1 + (n-2)\boldsymbol{e}_2\}| \\
&= |(-2n+7)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2| \\
&= \sqrt{(-2n+7)^2 + (0)^2 + (-2n+7)(0)} \\
&= \sqrt{4n^2 - 28n + 49} \\
&= \sqrt{\{n^2 - 3n + 3\} + 3(n-6)^2 + 11(n-6) + 4} \\
&\geq \sqrt{n^2 - 3n + 3} \ (\because n \geq 6) \\
&> d,
\end{aligned}$$

the vertices $\langle 0, n-2\rangle$ and $\langle 2n-7, n-2\rangle$ are not adjacent.
Since the Euclidean distance between $\langle 0, n-2\rangle$ and $\langle 2n-6, 0\rangle$ is

$$\begin{aligned}
&|(0)\boldsymbol{e}_1 + (n-2)\boldsymbol{e}_2 - \{(2n-6)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2\}| \\
&= |(-2n+6)\boldsymbol{e}_1 + (n-2)\boldsymbol{e}_2| \\
&= \sqrt{(-2n+6)^2 + (n-2)^2 + (-2n+6)(n-2)} \\
&= \sqrt{3n^2 - 18n + 28} \\
&= \sqrt{\{n^2 - 3n + 3\} + (2n-5)(n-5)} \\
&\geq \sqrt{n^2 - 3n + 3} \ (\because n \geq 5) \\
&> d,
\end{aligned}$$

the vertices $\langle 0, n-2\rangle$ and $\langle 2n-6, 0\rangle$ are not adjacent.
Since the Euclidean distance between $\langle n-4, n-1\rangle$ and $\langle 2n-6, 0\rangle$ is

$$\begin{aligned}
&|(n-4)\boldsymbol{e}_1 + (n-1)\boldsymbol{e}_2 - \{(2n-6)\boldsymbol{e}_1 + (0)\boldsymbol{e}_2\}| \\
&= |(-n+2)\boldsymbol{e}_1 + (n-1)\boldsymbol{e}_2| \\
&= \sqrt{(-n+2)^2 + (n-1)^2 + (-n+2)(n-1)} \\
&= \sqrt{n^2 - 3n + 3} \\
&> d,
\end{aligned}$$

the vertices $\langle n-4, n-1\rangle$ and $\langle 2n-6, 0\rangle$ are not adjacent. ∎