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Abstract.
L-convex functions are nonlinear discrete functions on integer points that are computationally tractable in optimization. In this paper, a discrete Hessian matrix and a local quadratic expansion are defined for L-convex functions. We characterize L-convex functions in terms of the discrete Hessian matrix and the local quadratic expansion.

Key words. discrete optimization; discrete convex function; Hessian matrix

1 Introduction

Submodular functions are recognized as one of the most fundamental classes of set functions in many disciplines related to discrete optimization [2, 12, 13]. The concept of L-convex functions was proposed by Murota [8] as a natural extension of Lovász extensions of submodular set functions [6] and plays a central role in the theory of discrete convex analysis [9]. Multimodular functions introduced by Hajek [4] (see also [1]) may be understood as an equivalent variant of L-convex functions (see Remark 2.5). L-convex functions provide a nice framework of nonlinear combinatorial optimization; global optimality is guaranteed by local optimality and descent algorithms work for minimization.

The objective of this paper is to introduce a discrete Hessian matrix and a quadratic expansion for L-convex functions and give characterizations of L-convex functions in terms of these objects.

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In convex analysis of continuous variables, for a function g with appropriate differentiability, the most fundamental fact is as follows:

g is convex

$\iff$ The Hessian matrix of g is positive semidefinite at each point

$\iff$ The local quadratic approximation to g is convex at each point.

We shall construct a discrete version of this equivalence for L-convex functions.

In general terms, a matrix H may be regarded as a discrete analogue of the Hessian matrix associated with a discrete function g, if

- H is a symmetric matrix defined from the local behavior of g,
- H vanishes if g is affine,
- H is linear in g, and
- H coincides with the coefficient matrix A if $g(x) = \frac{1}{2}x^{\top}Ax$.

Given a class, say $\mathcal{C}$, of discrete functions, we would like to characterize the class of the discrete Hessian matrices by introducing a class, say $\mathcal{H}$, of matrices such that

- a matrix belongs to $\mathcal{H}$ if and only if it is the discrete Hessian matrix of some function in $\mathcal{C}$ around some point,
- a function belongs to $\mathcal{C}$ if and only if the associated discrete Hessian matrix belongs $\mathcal{H}$ around each point, and
- a function belongs to $\mathcal{C}$ if and only if the local quadratic approximation defined by the discrete Hessian matrix belongs to $\mathcal{C}$ around each point.

Our discrete Hessian matrix for L-convex functions satisfies these properties.

We put $V = \{1, \ldots, n\}$. For vectors $p, q \in \mathbf{Z}^V$, we write $p \vee q$ and $p \wedge p$ for their componentwise maximum and minimum. We write $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{Z}^V$. A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called L-convex [8, 9] if it satisfies

(SBF) $g(p) + g(q) \geq g(p \vee q) + g(p \wedge p) \quad (p, q \in \mathbf{Z}^V)$,

(TRF) $\exists r \in \mathbf{R}$ such that $g(p + \mathbf{1}) = g(p) + r \quad (p \in \mathbf{Z}^V)$,

where it is understood that the inequality (SBF) is satisfied if $g(p)$ or $g(q)$ is equal to $+\infty$.

A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called L$^\natural$-convex [3, 9] if it is obtained from an L-convex function $\tilde{g}(p_0, p_1, \ldots, p_n)$ by restriction, i.e., $g(p_1, \ldots, p_n) = \tilde{g}(0, p_1, \ldots, p_n)$.

Theorem 1.1 ([9, 11]). *A quadratic function*

$$g(p) = p^{\top}Ap = \sum_{i,j \in V} a_{ij} p_i p_j \quad (p \in \mathbf{Z}^V)$$

with $a_{ij} = a_{ji} \in \mathbf{R}$ $(i, j \in V)$ *is* $L^\natural$*-convex if and only if*

$$a_{ij} \leq 0 \quad (i, j \in V, i \neq j), \quad \sum_{j \in V} a_{ij} \geq 0 \quad (i \in V).$$

Accordingly, g *is L-convex if and only if*

$$a_{ij} \leq 0 \quad (i, j \in V, i \neq j), \quad \sum_{j \in V} a_{ij} = 0 \quad (i \in V).$$

The following characterization of $\mathrm{L}^\natural$-convexity is known.

Theorem 1.2 ([3]). *A function* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *is* $L^\natural$*-convex if and only if* g *satisfies discrete midpoint convexity*

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \quad (p, q \in \mathbf{Z}^V), \tag{1.1}$$

where $\lceil \cdot \rceil$ *and* $\lfloor \cdot \rfloor$ *mean the rounding-up and -down of a vector to the nearest integer vector.*

For L-convex functions in continuous variables, the ordinary Hessian matrix can be used for a characterization of L-convexity [10]. This, however, cannot be readily adapted for functions in discrete variables. The discrete Hessian matrix is considered for discrete convex functions by Yüceer [14] to introduce the concept of strong discrete convexity. This, however, is not suitable for L-convex functions. The discrete Hessian matrix for M-convex functions is defined in [5] with discussion on its relationship to tree metrics.

2 Results

For a discrete function $g : \mathbf{Z}^V \to \mathbf{R}$, any $p \in \mathbf{Z}^V$, and any $i, j \in V$ with $i \neq j$, we define

$$\begin{aligned} \eta_{ij}(g; p) &= -g(p + \chi_i + \chi_j) + g(p + \chi_i) + g(p + \chi_j) - g(p), \\ \eta_i(g; p) &= g(p) + g(p + \mathbf{1} + \chi_i) - g(p + \mathbf{1}) - g(p + \chi_i), \end{aligned}$$

where χ_i denotes the characteristic vector for $i \in V$. For a discrete function $g : \mathbf{Z}^V \to \mathbf{R}$ and $p \in \mathbf{Z}^V$, we define a discrete analogue $H(g; p) = (H_{ij}(g; p) \mid i, j \in V)$ of the Hessian matrix by

$$\begin{aligned} H_{ij}(g; p) &= -\eta_{ij}(g; p) \quad (i \neq j), \\ H_{ii}(g; p) &= \eta_i(g; p) + \sum_{j \neq i} \eta_{ij}(g; p) \;=\; \eta_i(g; p) - \sum_{j \neq i} H_{ij}(g; p). \end{aligned}$$

For each $p \in \mathbf{Z}^V$, we regard

$$U(p) = \{p + \chi_i + \chi_j \mid i, j \in V, i \neq j\}$$

as a kind of neighborhood in considering our local quadratic expansion. The local quadratic expansion of g, denoted $\hat{g}(q;p)$, is defined as

$$\begin{aligned}\hat{g}(q;p) &= \frac{1}{2}\sum_{i,j\in V} H_{ij}(g;p)(q-p)_i(q-p)_j \\ &\quad +\sum_{i\in V}(g(p+\chi_i)-g(p)-\frac{1}{2}H_{ii}(g;p))(q-p)_i+g(p).\end{aligned} \tag{2.1}$$

Note that an alternative expression

$$\begin{aligned}\hat{g}(q;p) &= \frac{1}{2}\sum_{i\neq j} H_{ij}(g;p)(q-p)_i(q-p)_j \\ &\quad +\sum_{i\in V}(g(p+\chi_i)-g(p))(q-p)_i+g(p)\end{aligned}$$

is good for $q\in U(p)$.

The following lemma shows that the matrix $H(g;p)$ may be qualified as a discrete version of the Hessian matrix and $\hat{g}(q;p)$ is a local quadratic expansion of $g(q)$ in the sense that $\hat{g}(q;p)$ interpolates $g(q)$ on $U(p)$. Recall the general view made in Introduction.

Lemma 2.1. *For $g,h:\mathbf{Z}^V\to\mathbf{R}$ and $p\in\mathbf{Z}^V$, we have the following.*

(1) $H(g+h;p)=H(g;p)+H(h;p)$.

(2) *If g is an affine function, then $H(g;p)=0$.*

(3) $g(q)=\hat{g}(q;p)$ *for* $q\in U(p)$.

(4) *If $g(q)=b+c^\top q+\frac{1}{2}q^\top Aq$ with $A=(a_{ij}\mid i,j\in V)$, then $H(g;p)=A$ and $g(q)=\hat{g}(q;p)$ for any $q\in\mathbf{Z}^V$.*

Proof. **(1)** This is immediate from the definition of $H(g;p)$.

(2) Let $g(p)=b+c^\top p$. Then $\eta_{ij}(g;p)=0$ $(i,j\in V, i\neq j)$ and $\eta_i(g;p)=0$ $(i,j\in V)$. Therefore $H_{ij}(g;p)=0$.

(3) For $q=p+\chi_i+\chi_j\in U(p)$, we have

$$\begin{aligned}\hat{g}(q;p) &= \frac{1}{2}(H_{ii}(g;p)+H_{jj}(g;p))+H_{ij}(g;p) \\ &\quad +g(p+\chi_i)-g(p)-\frac{1}{2}H_{ii}(g;p)+g(p+\chi_j)-g(p) \\ &\quad -\frac{1}{2}H_{jj}(g;p)+g(p) \\ &= g(p+\chi_i+\chi_j)-g(p+\chi_i)-g(p+\chi_j)+g(p) \\ &\quad +g(p+\chi_i)+g(p+\chi_j)-g(p) \\ &= g(p+\chi_i+\chi_j).\end{aligned}$$

(4) By (1) and (2), we have $H(g;p) = H(\frac{1}{2}p^\top Ap;p)$. For $i, j \in V$ with $i \neq j$, we have

$$\begin{aligned}
H_{ij}\left(\frac{1}{2}p^\top Ap;p\right) &= -\eta_{ij}\left(\frac{1}{2}p^\top Ap;p\right) \\
&= \frac{1}{2}(p+\chi_i+\chi_j)^\top A(p+\chi_i+\chi_j) \\
&\quad -\frac{1}{2}(p+\chi_i)^\top A(p+\chi_i) - \frac{1}{2}(p+\chi_j)^\top A(p+\chi_j) \\
&\quad +\frac{1}{2}p^\top Ap \\
&= \frac{1}{2}(p^\top Ap + 2p^\top A\chi_i + 2p^\top A\chi_j + a_{ii} + a_{jj} + 2a_{ij}) \\
&\quad -\frac{1}{2}(p^\top Ap + 2p^\top A\chi_i + a_{ii}) \\
&\quad -\frac{1}{2}(p^\top Ap + 2p^\top A\chi_j + a_{jj}) + \frac{1}{2}p^\top Ap \\
&= a_{ij}.
\end{aligned}$$

For $i \in V$, we have

$$\begin{aligned}
H_{ii}\left(\frac{1}{2}p^\top Ap;p\right) &= \eta_i\left(\frac{1}{2}p^\top Ap;p\right) + \sum_{j\neq i}\eta_{ij}\left(\frac{1}{2}p^\top Ap;p\right) \\
&= \frac{1}{2}p^\top Ap + \frac{1}{2}(p+\mathbf{1}+\chi_i)^\top A(p+\mathbf{1}+\chi_i) \\
&\quad -\frac{1}{2}(p+\mathbf{1})^\top A(p+\mathbf{1}) - \frac{1}{2}(p+\chi_i)^\top A(p+\chi_i) \\
&\quad -\sum_{j\neq i} a_{ij} \\
&= \mathbf{1}^\top A\chi_i - \sum_{j\neq i} a_{ij} = a_{ii}.
\end{aligned}$$

Hence, we obtain $H(g;p) = A$.

The latter statement is obtained by the following direct computations:

$$
\begin{aligned}
&\frac{1}{2}(q-p)^\top H(g;p)(q-p) = \frac{1}{2}q^\top Aq - q^\top Ap + \frac{1}{2}p^\top Ap,\\
&\sum_{i\in V}(g(p+\chi_i) - g(p) - \frac{1}{2}H_{ii}(g;p))(q-p)_i\\
&= \sum_{i\in V}\left\{\left(b + c^\top p + c_i + \frac{1}{2}p^\top Ap + \chi_i^\top Ap + \frac{1}{2}a_{ii}\right.\right.\\
&\qquad \left.\left. - b - c^\top p - \frac{1}{2}p^\top Ap - \frac{1}{2}a_{ii}\right)(q-p)_i\right\}\\
&= \sum_{i\in V}(c_i + \chi_i^\top Ap)(q-p)_i\\
&= c^\top(q-p) + q^\top Ap - p^\top Ap.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\hat{g}(q;p)\\
&= \left(\frac{1}{2}q^\top Aq - q^\top Ap + \frac{1}{2}p^\top Ap\right) + \left(c^\top(q-p) + q^\top Ap - p^\top Ap\right)\\
&\quad + \left(b + c^\top p + \frac{1}{2}p^\top Ap\right)\\
&= b + c^\top q + \frac{1}{2}q^\top Aq\\
&= g(q).
\end{aligned}
$$

□

The following theorems establish characterizations of the class of the discrete Hessian matrices for L$^\natural$-convex functions and L-convex functions.

Theorem 2.2. *For $g : \mathbf{Z}^V \to \mathbf{R}$, the following conditions are equivalent.*

(a) *g is an L$^\natural$-convex function.*

(b) *For each $p \in \mathbf{Z}^V$, $H(g;p)$ satisfies*

$$H_{ij}(g;p) \le 0 \quad (i, j \in V, i \ne j), \tag{2.2}$$

and

$$\sum_{j\in V} H_{ij}(g;p) \ge 0 \quad (i \in V). \tag{2.3}$$

(c) *For each $p \in \mathbf{Z}^V$, $\hat{g}(q;p)$ is an L$^\natural$-convex function in q.*

Proof. [(b) $\iff$ (c)] The equivalence between (b) and (c) is immediate from Theorem 1.1, where it is noted that the addition of a linear function preserves L$^\natural$-convexity.

[(a)$\Rightarrow$(b)] We have (2.2) from the submodularity of g. Discrete midpoint convexity (1.1) for p and $p+\mathbf{1}+\chi_i$ shows (2.3).

[(b)$\Rightarrow$(a)] Given a function $g(p)$ satisfying the condition (b), define $\tilde{g}$ by $\tilde{g}(p_0,p) = g(p - p_0\mathbf{1})$. We will show the L-convexity of $\tilde{g}$. First, $\tilde{g}$ is constant in the direction of $\tilde{\mathbf{1}} = (1,1,\ldots,1) \in \mathbf{Z}^{V\cup\{0\}}$. With the notation $\tilde{p} := (p_0,p) \in \mathbf{Z}^{V\cup\{0\}}$, the submodularity of $\tilde{g}$ is equivalent to

$$\tilde{g}(\tilde{p}+\tilde{\chi}_i) + \tilde{g}(\tilde{p}+\tilde{\chi}_j) \geq \tilde{g}(\tilde{p}+\tilde{\chi}_i+\tilde{\chi}_j) + \tilde{g}(\tilde{p}) \quad (\tilde{p} \in \mathbf{Z}^{V\cup\{0\}}), \tag{2.4}$$

where $\tilde{\chi}_i \in \mathbf{Z}^{V\cup\{0\}}$ denotes the characteristic vector for $i \in V \cup \{0\}$. From (2.2), for $p - p_0\mathbf{1} \in \mathbf{Z}^V$, we have

$$g(p-p_0\mathbf{1}+\chi_i) + g(p-p_0\mathbf{1}+\chi_j) \geq g(p-p_0\mathbf{1}+\chi_i+\chi_j) + g(p-p_0\mathbf{1}) \qquad (i,j \in V, i \neq j).$$

This shows (2.4) for the case of $i,j \neq 0$. From (2.3), we have $\eta_i(g;p) \geq 0$ for any $i \in V$ and $p \in \mathbf{Z}^V$. This, for $p-(p_0+1)\mathbf{1} \in \mathbf{Z}^V$, means

$$g(p-(p_0+1)\mathbf{1}) + g(p-p_0\mathbf{1}+\chi_j) \geq g(p-(p_0+1)\mathbf{1}+\chi_j) + g(p-p_0\mathbf{1}) \qquad (j \in V).$$

This shows (2.4) for the case of $i=0, j\neq 0$. □

Theorem 2.3. *For $g : \mathbf{Z}^V \to \mathbf{R}$, the following conditions are equivalent.*

(a) *g is an L-convex function.*

(b) *For each $p \in \mathbf{Z}^V$, $H(g;p)$ satisfies*

$$H_{ij}(g;p) \leq 0 \quad (i,j \in V, i \neq j), \tag{2.5}$$

and

$$\sum_{j\in V} H_{ij}(g;p) = 0 \quad (i \in V). \tag{2.6}$$

(c) *For each $p \in \mathbf{Z}^V$, $\hat{g}(q;p)$ is an L-convex function in q.*

Proof. [(b) $\iff$ (c)] The equivalence between (b) and (c) is immediate from Theorem 1.1, where it is noted that the addition of a linear function preserves L-convexity.

[(a) $\iff$ (b)] If we put $\Delta(p) = g(p+\mathbf{1}) - g(p)$, then we have $\eta_i(g;p) = \Delta(p+\chi_i) - \Delta(p)$. Theorem 2.2 and the following two observations prove Theorem 2.3.

(1) g is L-convex
$\iff$ g is L$^\natural$-convex and $\Delta(p)$ is constant for any p.
(2) $\Delta(p)$ is constant for any p
$\iff$ $\Delta(p+\chi_i) = \Delta(p)$ for any p and any i. □

The characterization of L$^\natural$-convexity by our discrete Hessian reveals that the function treated in the seminal paper of Miller [7] is L$^\natural$-convex.

Example 2.4 (Miller's example). The following function $f : \mathbf{Z}^n_+ \to \mathbf{R}$ is L$^\natural$-convex:

$$f(x) = \sum_{k=0}^{\infty}\left(1 - \prod_{j=1}^{n}\beta_j(x_j+k)\right) + \lambda\sum_{j=1}^{n} c_j x_j,$$

where $\lambda > 0, c_j > 0$, and $\beta_j(\cdot)$ is a cumulative distribution function of a discrete nonnegative random variable represented as

$$\beta_j(k) = \sum_{m=0}^{k}\varphi_j(m) \quad (k \in \mathbf{Z}_+)$$

with $\varphi_j(m) \geq 0$ for $m \geq 0$ and $1 \leq j \leq n$; we have

$$\varphi_j(m) = e^{-\lambda_j}\frac{\lambda_j^m}{m!} \quad (m \in \mathbf{Z}_+)$$

with $\lambda_j > 0$ in Miller's example.

The L$^\natural$-convexity of f can be shown by Theorem 2.2 as follows. The nonnegativity of $\eta_{ij}(f;x)$ is calculated as follows.

$$\begin{aligned}
\eta_{ij}(f;x) &= -\sum_{k=0}^{\infty}(1-\prod_{t=1}^{n}\beta_t((x+\chi_i+\chi_j)_t+k)) \\
&\quad +\sum_{k=0}^{\infty}(1-\prod_{t=1}^{n}\beta_t((x+\chi_i)_t+k)) \\
&\quad +\sum_{k=0}^{\infty}(1-\prod_{t=1}^{n}\beta_t((x+\chi_j)_t+k)) - \sum_{k=0}^{\infty}(1-\prod_{t=1}^{n}\beta_t(x_t+k)) \\
&= \sum_{k=0}^{\infty}((\beta_j(x_j+1+k)-\beta_j(x_j+k))\prod_{t\neq j}\beta_t((x+\chi_i)_t+k) \\
&\quad -\sum_{k=0}^{\infty}((\beta_j(x_j+1+k)-\beta_j(x_j+k))\prod_{t\neq j}\beta_t(x_t) \\
&= \sum_{k=0}^{\infty}\varphi_i(x_i+1+k)\varphi_j(x_j+1+k)\prod_{t\neq i,j}\beta_t(x_t+k) \geq 0.
\end{aligned}$$

It is noted that the nonnegativity of $\eta_{ij}(f;x)$ has been pointed out in Yüceer [14]. The nonnegativity of $\eta_i(f;x)$ can also be calculated as follows.

$$\begin{aligned}\eta_i(f;x) &= \sum_{k=0}^{\infty}\varphi_i(x_i+1+k)\prod_{j\neq i}\beta_j(x_j+k) \\ &\quad -\sum_{k=0}^{\infty}\varphi_i(x_i+2+k)\prod_{j\neq i}\beta_j(x_j+1+k) \\ &= \varphi_i(x_i+1)\prod_{j\neq i}\beta_j(x_j) \geq 0.\end{aligned}$$

The above observation implies, in particular, that we can make use of L$^\natural$-convex function minimization algorithms ([9], Section 10.3) to minimize $f(x)$ more efficiently than by the known algorithm [7].

Remark 2.5 (Multimodular function). A function $h : \mathbf{Z}^n \to \mathbf{R}\cup\{+\infty\}$ is said to be multimodular [1, 4] if the function $\tilde{g} : \mathbf{Z}^{n\cup\{0\}} \to \mathbf{R}\cup\{+\infty\}$ defined by

$$\tilde{g}(p_0,p) = h(p_1-p_0, p_2-p_1, \ldots, p_n-p_{n-1}) \quad (p_0\in\mathbf{Z}, p=(p_1,\ldots,p_n)\in\mathbf{Z}^n)$$

is submodular. This means that h is a multimodular function if and only if there exists an L$^\natural$-convex function $g : \mathbf{Z}^n \to \mathbf{R}\cup\{+\infty\}$ such that $h(x) = g(p)$ with

$$\begin{aligned}p_1 &= x_1, \\ p_2 &= x_1+x_2, \\ &\cdots \\ p_n &= x_1+x_2+\cdots+x_n.\end{aligned}$$

Then $-\eta_{ij}(g;p)$ and $\eta_i(g;p)$ are computed as follows:

$$\begin{aligned}-\eta_{ij}(g;p) &= h(x+\chi_i-\chi_{i+1}+\chi_j-\chi_{j+1})+h(x) \\ &\quad -h(x+\chi_i-\chi_{i+1})-h(x+\chi_j-\chi_{j+1}), \\ \eta_i(g;p) &= h(x)+h(x+\chi_1+\chi_i-\chi_{i+1}) \\ &\quad -h(x+\chi_1)-h(x+\chi_i-\chi_{i+1}),\end{aligned}$$

where $\chi_{n+1} = (0,\ldots,0)$.

This observation and the equivalence between (a) and (b) in Theorem 2.2 recover the original definition of multimodularity in [4]: f is said to be multimodular if for all x in $\mathbf{Z}^n$,

$$f(x+v)+f(x+w) \geq f(x)+f(x+v+w) \quad (v,w\in\mathcal{F}, v\neq w), \qquad (2.7)$$

where $\mathcal{F} = \{s_0, s_1, \ldots, s_n\}$ with

$$\begin{array}{rcl} s_0 & = & (-1, 0, \ldots, 0), \\ s_1 & = & (1, -1, 0, \ldots, 0), \\ s_2 & = & (0, 1, -1, 0, \ldots, 0), \\ & \cdots & \\ s_n & = & (0, \ldots, 0, 1). \end{array}$$

This is because $-\eta_{ij}(g; p) \leq 0$ means (2.7) for $v, w \in \{s_i \mid i = 1, \ldots, n\}$ and $\eta_i(g; p - \mathbf{1}) \geq 0$ means (2.7) for $v = s_0$.

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